微积分 A1 第 3 次习题课答案　　函数极限与数列极限

1. 试求下列极限:

(1) $\lim\limits_{x\to0}\dfrac{\cos3x-\cos7x}{x^2}$　　(2) $\lim\limits_{x\to0}\dfrac{1-\sqrt{\cos x}}{x-x\cos\sqrt{x}}$

(3) $\lim\limits_{x\to0}\dfrac{3\sin x+x^2\cos(1/x)}{(1+\cos x)\ln(1+x)}$　　(4) $\lim\limits_{x\to0}\dfrac{\sin2x+2\arctan3x+3x^2}{\ln(1+3x+\sin^2x)+xe^x}$

(5) $\lim\limits_{x\to+\infty}x\left[\left(1+\dfrac{a}{x}\right)^{1+1/x}-x^{-1/[x(x+a)]}\right]$　　(6) $\lim\limits_{x\to+\infty}\left(\cos\dfrac{1}{x}+\sin\dfrac{1}{x}\right)^x$

(7) $\lim\limits_{x\to0}\left(\sqrt{1+x}-x\right)^{1/x}$　　(8) $\lim\limits_{x\to0}\left[\ln(e+x)\right]^{\cot x}$

(9) $\lim\limits_{x\to0}\left(\dfrac{xe^x+1}{x\pi^x+1}\right)^{1/x^2}$　　(10) $\lim\limits_{x\to0}\left(\dfrac{a^{x+1}+b^{x+1}}{a+b}\right)^{1/x}$ $(a,b>0)$

证明:（1）$\lim\limits_{x\to0}\dfrac{\cos3x-\cos7x}{x^2}=\lim\limits_{x\to0}\dfrac{\cos3x-1}{x^2}-\lim\limits_{x\to0}\dfrac{\cos7x-1}{x^2}$

$$=\lim_{x\to0}\frac{\cos3x-1}{\frac{1}{2}(3x)^2}\cdot\lim_{x\to0}\frac{\frac{1}{2}(3x)^2}{x^2}-\lim_{x\to0}\frac{\cos7x-1}{\frac{1}{2}(7x)^2}\cdot\lim_{x\to0}\frac{\frac{1}{2}(7x)^2}{x^2}=-\frac{9}{2}+\frac{49}{2}=20.$$

（2）$\lim\limits_{x\to0}\dfrac{1-\sqrt{\cos x}}{x-x\cos\sqrt{x}}=\lim\limits_{x\to0}\dfrac{1-e^{\frac{1}{2}\ln\cos x}}{x(1-\cos\sqrt{x})}$

$$=\frac{1}{2}\lim_{x\to0}\frac{1-e^{\frac{1}{2}\ln\cos x}}{\frac{1}{2}\ln\cos x}\cdot\lim_{x\to0}\frac{\ln(1+\cos x-1)}{\cos x-1}\cdot\lim_{x\to0}\frac{\cos x-1}{\frac{1}{2}x^2}\cdot\lim_{x\to0}\frac{\frac{1}{2}x}{(1-\cos\sqrt{x})}$$

$$=\frac{1}{2}\cdot(-1)\cdot1\cdot(-1)\cdot1=\frac{1}{2}.$$

（3）$\lim\limits_{x\to0}\dfrac{3\sin x+x^2\cos(1/x)}{(1+\cos x)\ln(1+x)}=\lim\limits_{x\to0}\dfrac{1}{(1+\cos x)}\cdot\lim\limits_{x\to0}\dfrac{x}{\ln(1+x)}\cdot\lim\limits_{x\to0}\dfrac{3\sin x+x^2\cos(1/x)}{x}$

$$=\frac{1}{2}\lim_{x\to0}\frac{3\sin x}{x}+\lim_{x\to0}x\cos(1/x)=\frac{3}{2}+0=\frac{3}{2}.$$

（4）$\lim\limits_{x\to0}\dfrac{\sin2x+2\arctan3x+3x^2}{\ln(1+3x+\sin^2x)+xe^x}=\lim\limits_{x\to0}\dfrac{\dfrac{\sin2x}{x}+2\dfrac{\arctan3x}{x}+3x}{\dfrac{\ln(1+3x+\sin^2x)}{x}+e^x}$

$$=\frac{\lim\limits_{x\to0}\frac{\sin 2x}{x}+2\lim\limits_{x\to0}\frac{\arctan 3x}{x}+\lim\limits_{x\to0}3x}{\lim\limits_{x\to0}\frac{\ln(1+3x+\sin^2 x)}{x}+\lim\limits_{x\to0}e^x}$$

$$=\frac{2+6}{1+\lim\limits_{x\to0}\frac{\ln(1+3x+\sin^2 x)}{3x+\sin^2 x}\cdot\lim\limits_{x\to0}\frac{3x+\sin^2 x}{x}}=\frac{8}{1+1\cdot3}=2.$$

（5） $\lim\limits_{x\to+\infty} x\left[\left(1+\frac{a}{x}\right)^{1+1/x}-x^{-1/[x(x+a)]}\right]$

$$=\lim_{x\to+\infty}\left[x\left(e^{(1+1/x)\ln(1+a/x)}-1\right)-x\left(e^{\frac{-\ln x}{x(x+a)}}-1\right)\right]$$

$$=\lim_{x\to+\infty}\left(\frac{e^{(1+1/x)\ln(1+a/x)}-1}{(1+1/x)\ln(1+a/x)}\cdot a(1+1/x)\cdot\frac{\ln(1+a/x)}{a/x}\right)-\lim_{x\to+\infty}\left(\frac{e^{\frac{-\ln x}{x(x+a)}}-1}{\frac{-\ln x}{x(x+a)}}\cdot\frac{-\ln x}{x+a}\right)$$

$$=a.$$

（6） $\lim\limits_{x\to+\infty}\left(\cos\frac{1}{x}+\sin\frac{1}{x}\right)^x=\lim\limits_{x\to+\infty}\left(1+\cos\frac{1}{x}+\sin\frac{1}{x}-1\right)^{\frac{1}{\cos\frac{1}{x}+\sin\frac{1}{x}-1}\cdot x\left(\cos\frac{1}{x}+\sin\frac{1}{x}-1\right)}$

$$=e^{\lim\limits_{x\to+\infty}x\left(\cos\frac{1}{x}+\sin\frac{1}{x}-1\right)}=e^{\lim\limits_{x\to+\infty}x\left(\cos\frac{1}{x}-1\right)+\lim\limits_{x\to+\infty}\frac{\sin(1/x)}{1/x}}=e^{1+\lim\limits_{x\to+\infty}x\left(-\frac{1}{2x^2}\right)}=e.$$

（7） $\lim\limits_{x\to0}\left(\sqrt{1+x}-x\right)^{1/x}=\lim\limits_{x\to0}(1+x)^{\frac{1}{2x}}\left(1-\frac{x}{\sqrt{1+x}}\right)^{1/x}=e^{1/2}\lim\limits_{x\to0}\left(1-\frac{x}{\sqrt{1+x}}\right)^{1/x}$

$$=e^{1/2}\cdot e^{\lim\limits_{x\to0}\frac{-x}{x\sqrt{1+x}}}=e^{-1/2}.$$

（8） $\lim\limits_{x\to0}[\ln(e+x)]^{\cot x}=\lim\limits_{x\to0}\left[1+\ln(1+\frac{x}{e})\right]^{\cot x}=\lim\limits_{x\to0}e^{\cot x\cdot\ln(1+x/e)}=\lim\limits_{x\to0}e^{\frac{x}{\tan x}\cdot\frac{\ln(1+x/e)}{x/e}\cdot\frac{1}{e}}=e^{1/e}.$

（9） $\lim\limits_{x\to0}\left(\frac{xe^x+1}{x\pi^x+1}\right)^{1/x^2}=\lim\limits_{x\to0}\left(1+\frac{x(e^x-\pi^x)}{x\pi^x+1}\right)^{1/x^2}=\lim\limits_{x\to0}e^{\frac{e^x-\pi^x}{x(x\pi^x+1)}}$

$$=\lim_{x\to0}e^{\frac{\pi^x}{x\pi^x+1}\frac{(e/\pi)^x-1}{x}}=e^{\ln(e/\pi)}=\frac{e}{\pi}.$$

（10）$\lim\limits_{x\to 0}\left(\dfrac{a^{x+1}+b^{x+1}}{a+b}\right)^{1/x}=\lim\limits_{x\to 0}\left(1+\dfrac{a(a^x-1)+b(b^x-1)}{a+b}\right)^{1/x}$

$$=\lim_{x\to 0}e^{\frac{a(a^x-1)+b(b^x-1)}{(a+b)x}}=e^{\frac{a\ln a+b\ln b}{(a+b)}}.$$

2. 已知$\lim\limits_{x\to 0^+}\dfrac{\ln(1+\dfrac{f(x)}{\tan x})}{2^x-1}=1$,求$\lim\limits_{x\to 0^+}\dfrac{f(x)}{x^2}$.

解：$\lim\limits_{x\to 0^+}\dfrac{\ln(1+\dfrac{f(x)}{\tan x})}{x}=\lim\limits_{x\to 0^+}\dfrac{\ln(1+\dfrac{f(x)}{\tan x})}{2^x-1}\cdot\lim\limits_{x\to 0^+}\dfrac{2^x-1}{x}=\ln 2,$

$\lim\limits_{x\to 0^+}\ln(1+\dfrac{f(x)}{\tan x})=0,\qquad \lim\limits_{x\to 0^+}\dfrac{f(x)}{\tan x}=0,$

$\lim\limits_{x\to 0^+}\dfrac{f(x)}{x^2}=\lim\limits_{x\to 0^+}\dfrac{\tan x}{x}\cdot\lim\limits_{x\to 0^+}\dfrac{\dfrac{f(x)}{\tan x}}{\ln(1+\dfrac{f(x)}{\tan x})}\cdot\lim\limits_{x\to 0^+}\dfrac{\ln(1+\dfrac{f(x)}{\tan x})}{x}=1\cdot 1\cdot\ln 2=\ln 2.$

3. 设$0<m\le\dfrac{f(x)}{x^\alpha}\le M,\lim\limits_{x\to 0^+}g(x)=0,\lim\limits_{x\to 0^+}g(x)\ln x^\alpha=l$，求证：$\lim\limits_{x\to 0^+}f(x)^{g(x)}=e^l$.

证明：$0<m\le\dfrac{f(x)}{x^\alpha}\le M$,则$\ln\dfrac{f(x)}{x^\alpha}$有界. 又$\lim\limits_{x\to 0^+}g(x)=0$,则

$$\lim_{x\to 0^+}g(x)\ln\frac{f(x)}{x^\alpha}=0.$$

$$\lim_{x\to 0^+}f(x)^{g(x)}=\lim_{x\to 0^+}e^{g(x)\ln f(x)}=\lim_{x\to 0^+}e^{g(x)\ln x^\alpha+g(x)\ln\frac{f(x)}{x^\alpha}}=e^l.\square$$

4. 当$x\to+\infty$时,$f(x)=1-\cos(1-\cos\dfrac{1}{x})$与$\alpha x^\beta$是等价无穷小. 求$\alpha,\beta$.

解：当$x\to+\infty$时,

$$f(x)=1-\cos(1-\cos\frac{1}{x})\sim\frac{1}{2}(1-\cos\frac{1}{x})^2\sim\frac{1}{2}\left(\frac{1}{2x^2}\right)^2=\frac{1}{8x^4},$$

因此，$\alpha=\dfrac{1}{8},\beta=-4.$ □

5. 试求下列极限:

(1) $\lim\limits_{n\to+\infty}\left(\dfrac{2+\sqrt[n]{a}}{3}\right)^{2n-2}\quad (a>0)$ (2) $\lim\limits_{n\to+\infty} n(\sqrt[n]{x}-\sqrt[2n]{x})\ \ (x>0)$

(3) $\lim\limits_{n\to+\infty}\dfrac{1}{n}\sum\limits_{k=1}^{n}e^{k/n}$ (4) $\lim\limits_{n\to+\infty} n^p\sin(\sqrt{2}+1)^n\pi\ \ (p>0)$

解:（1） $\lim\limits_{n\to+\infty}\left(\dfrac{2+\sqrt[n]{a}}{3}\right)^{2n-2}=\lim\limits_{n\to+\infty}\left(1+\dfrac{\sqrt[n]{a}-1}{3}\right)^{\frac{3}{\sqrt[n]{a}-1}\cdot\frac{2(n-1)(\sqrt[n]{a}-1)}{3}}=\lim\limits_{n\to+\infty}e^{\frac{2(n-1)(\sqrt[n]{a}-1)}{3}}$

$=\lim\limits_{n\to+\infty}e^{\frac{2(n-1)\ln a}{3n}\cdot\frac{\sqrt[n]{a}-1}{\frac{1}{n}\ln a}}=e^{\frac{2}{3}\ln a}=a^{2/3}.$

（2） $\lim\limits_{n\to+\infty} n(\sqrt[n]{x}-\sqrt[2n]{x})=\lim\limits_{n\to+\infty}\sqrt[2n]{x}\cdot\lim\limits_{n\to+\infty} n(\sqrt[2n]{x}-1)=\lim\limits_{n\to+\infty}\dfrac{\ln x}{2}\cdot\dfrac{\sqrt[2n]{x}-1}{\dfrac{1}{2n}\ln x}=\dfrac{\ln x}{2}.$

法二: $\lim\limits_{n\to+\infty} n(\sqrt[n]{x}-\sqrt[2n]{x})=\lim\limits_{n\to+\infty}\dfrac{\sqrt[n]{x}-1}{1/n}-\lim\limits_{n\to+\infty}\dfrac{1}{2}\dfrac{\sqrt[2n]{x}-1}{1/(2n)}=\ln x-\dfrac{1}{2}\ln x=\dfrac{1}{2}\ln x.$

（3） $\lim\limits_{n\to+\infty}\dfrac{1}{n}\sum\limits_{k=1}^{n}e^{k/n}=\lim\limits_{n\to+\infty}\dfrac{e^{1/n}(e-1)}{n(e^{1/n}-1)}=\dfrac{e-1}{\lim\limits_{n\to+\infty}\dfrac{(e^{1/n}-1)}{1/n}}=e-1.$

（4） $(\sqrt{2}+1)^n+(1-\sqrt{2})^n$ 为整数, 所以

$$\left|\sin(\sqrt{2}+1)^n\pi\right|=\left|\sin[(\sqrt{2}+1)^n+(1-\sqrt{2})^n-(1-\sqrt{2})^n]\pi\right|=\sin(\sqrt{2}-1)^n\pi.$$

$$\lim_{n\to+\infty}\left|n^p\sin(\sqrt{2}+1)^n\pi\right|=\lim_{n\to+\infty}\frac{\sin(\sqrt{2}-1)^n\pi}{(\sqrt{2}-1)^n\pi}\cdot\lim_{n\to+\infty}\frac{n^p\pi}{(\sqrt{2}+1)^n}=1\cdot 0=0.$$

$$\lim_{n\to+\infty} n^p\sin(\sqrt{2}+1)^n\pi=0.$$

6. 设 $-1<a_0<1,a_n=\sqrt{\dfrac{1+a_{n-1}}{2}},b_n=4^n(1-a_n)$. 求 $\lim\limits_{n\to\infty}b_n$.

解: 记 $a_0=\cos\theta(0<\theta<\pi)$, 则 $a_1=\sqrt{\dfrac{1+\cos\theta}{2}}=\cos\dfrac{\theta}{2},a_n=\cos\dfrac{\theta}{2^n},$

$$b_n = 4^n(1-a_n) = 4^n(1-\cos\frac{\theta}{2^n}) = 2^{2n+1}\sin^2\frac{\theta}{2^{n+1}} \to \frac{\theta^2}{2}(n\to\infty).$$

7. 设$b_1 > a_1 > 0, a_{n+1} = (a_n + b_n)/2, b_{n+1} = \sqrt{a_{n+1}b_n}$. 求$\lim\limits_{n\to\infty} a_n, \lim\limits_{n\to\infty} b_n$.

解： 记$a_1 = b_1\cos\theta(0<\theta<\pi/2)$, 归纳可证

$$a_{n+1} = b_1\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^{n-1}}\cos^2\frac{\theta}{2^n},$$
$$b_{n+1} = b_1\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^n}.$$

因此,
$$\lim_{n\to\infty} b_n = b_1\lim_{n\to\infty}\frac{\frac{1}{2^n}\sin\theta}{\sin\frac{\theta}{2^n}} = \frac{b_1\sin\theta}{\theta},$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n\cos\frac{\theta}{2^n} = \lim_{n\to\infty} b_n = \frac{b_1\sin\theta}{\theta}. \ \square$$

8. 若$2a^{1/n} - b^{1/n} - c^{1/n} = o(1/n)(n\to\infty)$, 试问正数$a,b,c$满足什么关系?

解： $1 = \lim\limits_{n\to\infty}\dfrac{2a^{1/n} - b^{1/n} - c^{1/n}}{1/n} = \lim\limits_{n\to\infty}\dfrac{2(a^{1/n}-1)}{1/n} - \lim\limits_{n\to\infty}\dfrac{b^{1/n}-1}{1/n} - \lim\limits_{n\to\infty}\dfrac{c^{1/n}-1}{1/n}$

$$= 2\ln a - \ln b - \ln c = \ln\frac{a^2}{bc}.$$

故$a^2 = bc$. $\square$

9. 求$\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\sqrt[3]{1+\dfrac{k}{n^2}} - 1\right)$.

解： $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{k}{3n^2} = \lim\limits_{n\to\infty}\dfrac{n+1}{6n} = \dfrac{1}{6}$. $\forall 0<\varepsilon<1, \exists N_1\in\mathbb{N}$, 当$n>N_1$时, 有

$$\frac{1}{6}(1-\varepsilon) < \sum_{k=1}^{n}\frac{k}{3n^2} < \frac{1}{6}(1+\varepsilon).$$

对此$\varepsilon>0$, 因$\lim\limits_{x\to 0}\dfrac{(1+x)^{1/3}-1}{x/3} = 1$, $\exists\delta>0$, 当$0<x<\delta$时, 有

$$1-\varepsilon < \frac{(1+x)^{1/3}-1}{x/3} < 1+\varepsilon.$$

对此$\delta > 0$, $\exists N > N_1$,使得$1/N < \delta$. 因此，当$n > N$时,有

$$\frac{1}{6}-\frac{\varepsilon}{3} < \frac{(1-\varepsilon)^2}{6} < (1-\varepsilon)\sum_{k=1}^{n}\frac{k}{3n^2} < \sum_{k=1}^{n}\left(\sqrt[3]{1+\frac{k}{n^2}}-1\right) < (1+\varepsilon)\sum_{k=1}^{n}\frac{k}{3n^2} < \frac{(1+\varepsilon)^2}{6} < \frac{1}{6}+\frac{\varepsilon}{2},$$

由极限定义，有$\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\sqrt[3]{1+\frac{k}{n^2}}-1\right)=\frac{1}{6}$. □