微积分 A(2)第二次习题课参考答案（第四周）

一、复合函数的微分,隐函数微分法

1.求解下列各题：

（1）. 设 $z=x^3f\left(xy,\dfrac{y}{x}\right)$，求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$。

解：$dz=f\cdot 3x^2dx+x^3df=3x^2fdx+x^3\left[f_1'd(xy)+f_2'd\left(\dfrac{y}{x}\right)\right]$

$$=3x^2fdx+x^3\left[f_1'(xdy+ydx+f_2'\frac{xdy-ydx}{x^2}\right]$$

$$=\left(3x^2f+x^3yf_1'-xyf_2'\right)dx+\left(x^4f_1'+x^2f_2'\right)dy$$

由微分形式不变性，

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\left(3x^2f+x^3yf_1'-xyf_2'\right)dx+\left(x^4f_1'+x^2f_2'\right)dy$$

故　$\dfrac{\partial z}{\partial x}=\left(3x^2f+x^3yf_1'-xyf_2'\right),\quad \dfrac{\partial z}{\partial y}=\left(x^4f_1'+x^2f_2'\right)$。

（2）. 已知 $y=(\dfrac{1}{x})^{-\frac{1}{x}}$，求 $\dfrac{dy}{dx}$.

解 考虑二元函数 $y=u^v$，$u=\dfrac{1}{x},v=-\dfrac{1}{x}$，应用推论得

$$\frac{dy}{dx}=\frac{\hat{o}y}{\hat{o}u}\frac{du}{dx}+\frac{\hat{o}y}{\hat{o}v}\frac{dv}{dx}.\ vu^{v-1}\frac{-1}{x^2}+(\ln u)u^v\frac{1}{x^2}=\left(\frac{1}{x}\right)^{2-\frac{1}{x}}(1-\ln x).$$

(3)　已知 $\dfrac{(x\ \text{H}\ ay)dx\ \text{H}\ ydy}{(x\ \text{H}\ y)^2}$ 为某个二元函数的全微分,则 a Z ___2___

2.求解下列各题

（1）. 已知函数 $y=f(x)$ 由方程 $ax+by=f\left(x^2+y^2\right),a,b$ 是常数，求导函数。

解：方程 $ax+by=f\left(x^2+y^2\right)$ 两边对 x 求导，

$$a+b\frac{dy}{dx}=f'(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=\frac{2xf'(x^2+y^2)-a}{b-2yf'(x^2+y^2)}$$

（2）. 已知函数 $z=z(x,y)$ 由参数方程: $\begin{cases}x=u\cos v\\ y=u\sin v\\ z=uv\end{cases}$, 给定，试求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解　这个问题涉及到复合函数微分法与隐函数微分法．x, y 是自变量，u, v 是中间变量（u, v 是 x, y 的函数），　先由 $z = uv$ 得到

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$$

u, v 是由方程 $\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$ 的 x, y 的隐函数，在这两个等式两端分别关于 x, y 求偏导数，得

$$\begin{cases} 1 = \cos v\dfrac{\partial u}{\partial x} - u\sin v\dfrac{\partial v}{\partial x} \\ 0 = \sin v\dfrac{\partial u}{\partial x} + u\cos v\dfrac{\partial v}{\partial x} \end{cases}, \qquad \begin{cases} 0 = \cos v\dfrac{\partial u}{\partial y} - u\sin v\dfrac{\partial v}{\partial y} \\ 1 = \sin v\dfrac{\partial u}{\partial y} + u\cos v\dfrac{\partial v}{\partial y} \end{cases}$$

得到
$$\frac{\partial u}{\partial x} = \cos v, \frac{\partial v}{\partial x} = \frac{-\sin u}{u}, \frac{\partial u}{\partial y} = \sin v, \frac{\partial v}{\partial x} = \frac{\cos v}{u}$$

将这个结果代入前面的式子，得到

$$\frac{\partial z}{\partial x} = v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} = v\cos v - \sin v$$

与
$$\frac{\partial z}{\partial y} = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} = v\sin v + \cos v$$

（3）. 设 $z = z(x, y)$ 二阶连续可微，并且满足方程

$$A\frac{\partial^2 z}{\partial x^2} + 2B\frac{\partial^2 z}{\partial x\partial y} + C\frac{\partial^2 z}{\partial y^2} = 0$$

若令 $\begin{cases} u = x + \sim y \\ v = x + \quad y \end{cases}$，试确定 $\sim$，　为何值时能变原方程为 $\dfrac{\partial^2 z}{\partial u\partial v} = 0$.

解　将 x, y 看成自变量，u, v 看成中间变量，利用链式法则得

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)z$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \sim\frac{\partial z}{\partial u} + \quad\frac{\partial z}{\partial v} = \left(\sim\frac{\partial}{\partial u} + \quad\frac{\partial}{\partial v}\right)z$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 z$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\sim\frac{\partial z}{\partial u} + \quad\frac{\partial z}{\partial v}\right) = \sim^2\frac{\partial^2 z}{\partial u^2} + 2\sim\quad\frac{\partial^2 z}{\partial u\partial v} + \quad^2\frac{\partial^2 z}{\partial v^2} = \left(\sim\frac{\partial}{\partial u} + \quad\frac{\partial}{\partial v}\right)^2 z$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\sim\frac{\partial z}{\partial u} + \quad\frac{\partial z}{\partial v}\right) = \sim\frac{\partial^2 z}{\partial u^2} + (\sim + \quad)\frac{\partial^2 z}{\partial u\partial v} + \quad\frac{\partial^2 z}{\partial v^2}$$

$$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\sim\frac{\partial}{\partial u} + \quad\frac{\partial}{\partial v}\right)z$$

由此可得，　$0 = A\dfrac{\partial^2 z}{\partial x^2} + 2B\dfrac{\partial^2 z}{\partial x\partial y} + C\dfrac{\partial^2 z}{\partial y^2} =$

$$= \left(A + 2B\sim + C\sim^2\right)\frac{\partial^2 z}{\partial u^2} + 2\left(A + B(\sim + \quad) + C\sim\quad\right)\frac{\partial^2 z}{\partial u\partial v} + \left(A + 2B\quad + C\quad^2\right)\frac{\partial^2 z}{\partial v^2} = 0$$

只要选取~，　使得 $\begin{cases} A+2B\sim+C\sim^{2}=0 \\ A+2B\quad+C\quad^{2}=0 \end{cases}$，可得 $\dfrac{\partial^2 z}{\partial u\partial v}=0$.

问题成为方程 $A+2B\,t+C\,t^2=0$ 有两不同实根，即要求：　$B^2-AC>0$.

令 $\sim=-B+\sqrt{B^2-AC}$，　$\quad=-B-\sqrt{B^2-AC}$，即可。

此时，$\dfrac{\partial^2 z}{\partial u\partial v}=0\Rightarrow\dfrac{\partial^2 z}{\partial u\partial v}=0\Rightarrow\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial v}\right)=0\Rightarrow\dfrac{\partial z}{\partial v}=\ddot{}\,(v)\Rightarrow z=\int\ddot{}\,(v)dv+f(u)$.

$$z=f(u)+g(v)=f(x+\sim y)+g(x+\quad y).$$

3. 求解下列各题

（1）. $z=z(x,y)$ 由 $x^2+y^2+z^2=a^2$ 决定，求 $\dfrac{\partial^2 z}{\partial x\partial y}$.

解：$2x+2z\dfrac{\partial z}{\partial x}=0$，$2y+2z\dfrac{\partial z}{\partial y}=0$

$$\frac{\partial z}{\partial x}=-\frac{x}{z},\quad \frac{\partial z}{\partial y}=-\frac{y}{z}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{y}{z^2}\cdot\frac{\partial z}{\partial x}=-\frac{xy}{z^3}$$

（2）设函数 $z=f(x,y)$ 是由方程 $\boldsymbol{xyz}\,Η\sqrt{\boldsymbol{x^2}\,Η\,\boldsymbol{y^2}\,Η\,\boldsymbol{z^2}}\;Ζ\;\sqrt{\mathbf{2}}$ 确定的, 则函数 $\boldsymbol{z}\;Ζ\;\boldsymbol{f(x,y)}$

在点 $(1,0,-1)$ 的微分 $\boldsymbol{dz}=$____________

答：$\boldsymbol{dz}\;Ζ\;\boldsymbol{dx}\;J\;\sqrt{\mathbf{2}}\boldsymbol{dy}$

（3）. 设函数 $x=x(z),\ y=y(z)$ 由方程组 $\begin{cases} x^2+y^2+z^2-1=0 \\ x^2+2y^2-z^2-1=0 \end{cases}$ 确定，求 $\dfrac{dx}{dz},\ \dfrac{dy}{dz}$.

解 $\begin{cases} x^2+y^2=-z^2+1 \\ x^2+2y^2=z^2+1 \end{cases}\Rightarrow\begin{cases} 2x\dfrac{dz}{dx}+2y\dfrac{dz}{dy}=-2z \\ 2x\dfrac{dz}{dx}+4y\dfrac{dz}{dy}=2z \end{cases}$ 解方程得：

$$\begin{bmatrix}\dfrac{dx}{dz}\\ \dfrac{dy}{dz}\end{bmatrix}=-\frac{1}{4xy}\begin{bmatrix}4y & -2y\\ -2x & 2x\end{bmatrix}\begin{bmatrix}2z\\ -2z\end{bmatrix}=-\frac{1}{4xy}\begin{bmatrix}12yz\\ -8xz\end{bmatrix}$$

由此得到　$\dfrac{dx}{dz}=\dfrac{3z}{x},\dfrac{dy}{dz}=-\dfrac{2z}{y}$.

(4) 设方程 $\begin{cases}\boldsymbol{F(y\;J\;x,y\;J\;z)\;Ζ\;0}\\ \boldsymbol{G(xy,\dfrac{z}{y})\;Ζ\;0}\end{cases}$ 可以确定隐函数 $\boldsymbol{x\;Ζ\;x(y),z\;Ζ\;z(y)}$，求 $\boldsymbol{\dfrac{dx}{dy},\dfrac{dz}{dy}}$.

（本题不用解出最终答案，会解题过程就可以.）

解：$$\begin{cases} F_1'\left(1-\dfrac{dx}{dy}\right)+F_2'\left(1-\dfrac{dz}{dy}\right)=0 \\ G_1'\left(x+y\dfrac{dx}{dy}\right)+G_2'\left(-\dfrac{z}{y^2}+\dfrac{1}{y}\dfrac{dz}{dy}\right)=0 \end{cases}$$

$$\frac{dx}{dy}=\frac{\frac{1}{y}F_1'G_2'+xF_2'G_1'+\left(\frac{1}{y}-\frac{z}{y^2}\right)F_2'G_2'}{\frac{1}{y}F_1'G_2'-yF_2'G_1'}$$

$$\frac{dz}{dy}=-\frac{(x+y)F_1'G_1'+yF_2'G_2'-\frac{z}{y^2}F_1'G_2'}{\frac{1}{y}F_1'G_2'-yF_2'G_1'}$$

4. 求解下列二阶偏导数问题

（1）. 设$z=f(xy,\dfrac{x}{y})$，f 二阶连续可微，求$\dfrac{\partial^2 z}{\partial x^2}$.

解　记　$u=xy, v=\dfrac{x}{y}$；　$f_1'=\dfrac{\partial f}{\partial u}, f_2'=\dfrac{\partial f}{\partial v}$，

$$f_{11}''=\frac{\partial^2 f}{\partial u^2}, f_{22}''=\frac{\partial^2 f}{\partial v^2}，f_{12}''=\frac{\partial^2 f}{\partial u\partial v}, f_{21}''=\frac{\partial^2 f}{\partial v\partial u}$$

则
$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=y f_1'+\frac{1}{y}f_2'，$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=y\frac{\partial f_1'}{\partial x}+\frac{1}{y}\frac{\partial f_2'}{\partial x}$$

因为　$f_1'=\dfrac{\partial f}{\partial u}, f_2'=\dfrac{\partial f}{\partial v}$ 都是以u,v为中间变量，以x,y为自变量的函数，所以

$$\frac{\partial f_1'}{\partial x}=f_{11}''\frac{\partial u}{\partial x}+f_{12}''\frac{\partial v}{\partial x}=y f_{11}''+\frac{1}{y}f_{12}''$$

$$\frac{\partial f_2'}{\partial x}=f_{21}''\frac{\partial u}{\partial x}+f_{22}''\frac{\partial v}{\partial x}=y f_{21}''+\frac{1}{y}f_{22}''$$

将以上两式代入前式得：　$\dfrac{\partial^2 z}{\partial x^2}=y^2 f''_{11}+2f''_{12}+\dfrac{1}{y^2}f''_{22}$.

（2）. 设$g(x)=f\left(x,\varphi\left(x^2,x^2\right)\right)$，其中函数 f 于φ 的二阶偏导数连续，求$\dfrac{d^2g(x)}{dx^2}$

解：$\dfrac{dg(x)}{dx}=f_1'+f_2'(\varphi_1'+\varphi_2')2x$，

$$\frac{d^2g(x)}{dx^2}=\frac{d}{dx}\left[f_1'+f_2'\left(\varphi_1'+\varphi_2'\right)2x\right]=f_{11}''+f_{12}''\left(\varphi_1'+\varphi_2'\right)2x$$
$$+2f_2'\left(\varphi_1'+\varphi_2'\right)+4x^2f_2'\left(\varphi_{11}''+2\varphi_{12}''+\varphi_{22}''\right)+2x\left(\varphi_1'+\varphi_2'\right)\left(f_{21}''+f_{22}''\left(\varphi_1'+\varphi_2'\right)2x\right)$$
$$=f_{11}''+4xf_{12}''\left(\varphi_1'+\varphi_2'\right)+4x^2f_{22}''\left(\varphi_1'+\varphi_2'\right)^2+2f_2'\left(\varphi_1'+\varphi_2'\right)+4x^2f_2'\left(\varphi_{11}''+2\varphi_{12}''+\varphi_{22}''\right)。$$

（3）设 $\boldsymbol{z = f(x,y)}$ 在点 $\boldsymbol{(a,a)}$ 可微，$\boldsymbol{f(a,a) = a, \frac{\partial f}{\partial x}\Big|_{(a,a)} = b, \frac{\partial f}{\partial y}\Big|_{(a,a)} = b}$.

令 $\boldsymbol{\varphi(x) = f(x, f(x, f(x,x)))}$，求 $\boldsymbol{\frac{d}{dx}\varphi^2(x)\Big|_{x=a}}$

分析：用 $\boldsymbol{f_1'}$ 和 $\boldsymbol{f_2'}$ 分别表示函数 $\boldsymbol{f}$ 对于第一个变量和第二个变量的偏导数. 理清函数的复合关系.

解　利用复合函数微分法则求导数：

$$\frac{d}{dx}\varphi^2(x)=2\varphi(x)\frac{d}{dx}\varphi(x)，\qquad \frac{d}{dx}\varphi(x)=f_1'+f_2'\frac{d}{dx}f(x,f(x,x))$$

其中 $\frac{d}{dx}f(x,f(x,x))=f_1'+f_2'(f_1'+f_2')$

于是 $\frac{d}{dx}\varphi^2(x)=2\varphi(x)[f_1'+f_2'(f_1'+f_2'(f_1'+f_2'))]$

当 $\boldsymbol{x = a, y = a}$ 时，代入题目条件：$\varphi(a)=f(a,f(a,f(a,a)))=a$，$f_1'(a,a)=b$，$f_2'(a,a)=b$. 得到 $\frac{d}{dx}\varphi^2(x)\Big|_{x=a}=2a(b+b^2+2b^3)$

（4）. 设 $u(x,y)\in C^2$，又 $\frac{\partial^2u}{\partial x^2}-\frac{\partial^2u}{\partial y^2}=0$，$u(x,2x)=x$，$u_x'(x,2x)=x^2$，求 $u_{xx}''(x,2x)$，$u_{xy}''(x,2x)$　$u_{yy}''(x,2x)$

解：
$$\frac{\partial u}{\partial x}(x,2x)=x^2,$$

两边对 x 求导，

$$\frac{\partial^2u}{\partial x^2}(x,2x)+\frac{\partial^2u}{\partial x\partial y}(x,2x)\cdot 2=2x. \qquad (1)$$

$$u(x,2x)=x,$$

两边对 x 求导，

$$\frac{\partial u}{\partial x}(x,2x)+\frac{\partial u}{\partial y}(x,2x)\cdot 2=1, \qquad \frac{\partial u}{\partial y}(x,2x)=\frac{1-x^2}{2}.$$

两再边对 x 求导，

$$\frac{\partial^2 u}{\partial x \partial y}(x,2x)+\frac{\partial^2 u}{\partial y^2}(x,2x)\cdot 2=-x. \qquad (2)$$

由已知 $$\frac{\partial^2 u}{\partial x^2}(x,2x)-\frac{\partial^2 u}{\partial y^2}(x,2x)=0, \qquad (3)$$

(1)，(2)，(3) 联立可解得：

$$\frac{\partial^2 u}{\partial x^2}(x,2x)=\frac{\partial^2 u}{\partial y^2}(x,2x)=-\frac{4}{3}x,\quad \frac{\partial^2 u}{\partial x \partial y}(x,2x)=\frac{5}{3}x$$

5. 设向量值函数 $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^n$ 满足：存在 $L:0<L<1$，对任意的 $X,Y\in\mathbb{R}^n$ 有 $\|\mathbf{f}(X)-\mathbf{f}(Y)\|\le L\|X-Y\|$. 证明：$\exists! X^*\in\mathbb{R}^n,\mathbf{f}(X^*)=X^*$.

证明：首先由 $\|\mathbf{f}(X)-\mathbf{f}(Y)\|\le L\|X-Y\|$，易之，向量值函数 $\mathbf{f}$ 是连续映射。

其次，构造序列 $\{X_n\}:X_{n+1}=\mathbf{f}(X_n)$，则

$\|X_{n+1}-X_n\|=\|\mathbf{f}(X_n)-\mathbf{f}(X_{n-1})\|\le L\|X_n-X_{n-1}\|\le\cdots\le L^{n-1}\|X_2-X_1\|$，

易之 $\|X_{n+p}-X_n\|\le\|X_{n+p}-X_{n+p-1}\|+\|X_{n+p-1}-X_{n+p-2}\|+\cdots+\|X_{n+1}-X_n\|$

$\le(L^{n+p-2}+L^{n+p-3}+\cdots+L^{n-1})\|X_2-X_1\|\le\dfrac{L^{n-1}(1-L^p)}{1-L}\|X_2-X_1\|\le\dfrac{L^{n-1}}{1-L}\|X_2-X_1\|$

容易证明：$\{X_n\}$ 为 $\mathbb{R}^n$ 中的 Cauchy 列，则 $\{X_n\}$ 收敛，$\lim\limits_{n\to\infty}X_n=X^*$，在 $X_{n+1}=\mathbf{f}(X_n)$ 两端取极限，且 $\mathbf{f}$ 是连续映射，有 $\mathbf{f}(X^*)=X^*$，唯一性易证，略。

6. 设 $\Omega\subset\mathbb{R}^n, X\in\mathbb{R}^n$，定义 $\circ(X,\Omega)=\inf\limits_{Y\in\Omega}\|X-Y\|_n$. 证明：

（1）$\circ(X,\Omega)$ 为 X 的连续函数；

（2）Ω 为有界闭集时，存在 $X_0\in\Omega$，使得 $\circ(X,\Omega)=\|X-X_0\|_n$

（3）$\Omega_1,\Omega_2\subset\mathbb{R}^n$，定义 $\circ(\Omega_1,\Omega_2)=\inf\limits_{X\in\Omega_1,Y\in\Omega_2}\|X-Y\|_n$，证明：当 Ω_1,Ω_2 为有界闭集时，存在 $X_0\in\Omega_1,Y_0\in\Omega_2$，使得 $\circ(\Omega_1,\Omega_2)=\|X_0-Y_0\|_n$.

证明：（1）$Z\in\Omega$，$\|X-Z\|_n\le\|X-Y\|_n+\|Z-Y\|_n$，因此有

$\circ(X,\Omega)\le\|X-Y\|_n+\|Z-Y\|_n$，固定 X,Y，有 $\circ(X,\Omega)\le\circ(X,Y)+\circ(Y,\Omega)$，

因此 $\circ(X,\Omega)-\circ(Y,\Omega)\le\circ(X,Y)$，类似有 $\circ(Y,\Omega)-\circ(X,\Omega)\le\circ(X,Y)$，总之有

$|\circ(Y,\Omega)-\circ(X,\Omega)|\le\circ(X,Y)$，从而 $\circ(X,\Omega)$ 为 X 的连续函数；

（2）固定 X ，$\| X-Y \|_n$ 是 $Y \in \Omega$ 的连续函数，Ω 为有界闭集，$\| X-Y \|_n$ 存在最小值，

即存在 $X_0 \in \Omega$ ，使得 $\| X - X_0 \|_n \ \min\limits_{Y \in \Omega} \| X-Y \|_n = \inf\limits_{Y \in \Omega} \| X-Y \|_n = {}^\circ(X,\Omega)$ ，

（3）首先证明：${}^\circ(\Omega_1,\Omega_2) \equiv \inf\limits_{X \in \Omega_1, Y \in \Omega_2} \| X-Y \|_n = \inf\limits_{X \in \Omega_1} {}^\circ(X,\Omega_2)$

先证明：$\inf\limits_{X \in \Omega_1, Y \in \Omega_2} \| X-Y \|_n \le \inf\limits_{X \in \Omega_1} {}^\circ(X,\Omega_2)$.

显然 $\inf\limits_{X \in \Omega_1, Y \in \Omega_2} \| X-Y \|_n \le \| X-Y \|_n$ ，左端为常数，右端对 $Y \in \Omega_2$ 取下确界，有 $\inf\limits_{X \in \Omega_1, Y \in \Omega_2} \| X-Y \|_n \le {}^\circ(X,\Omega_2)$ ，从而

$$\inf_{X \in \Omega_1, Y \in \Omega_2} \| X-Y \|_n \le \inf_{X \in \Omega_1} {}^\circ(X,\Omega_2) .$$

下证：$\inf\limits_{X \in \Omega_1, Y \in \Omega_2} \| X-Y \|_n \ge \inf\limits_{X \in \Omega_1} {}^\circ(X,\Omega_2)$

由于 $\| X-Y \|_n \ge {}^\circ(X,\Omega_2) \ge \inf\limits_{X \in \Omega_1} {}^\circ(X,\Omega_2)$ ，右端为常数，因此

$$\inf_{X \in \Omega_1, Y \in \Omega_2} \| X-Y \|_n \ge \inf_{X \in \Omega_1} {}^\circ(X,\Omega_2) ,$$

综上有，

$${}^\circ(\Omega_1,\Omega_2) \equiv \inf_{X \in \Omega_1, Y \in \Omega_2} \| X-Y \|_n = \inf_{X \in \Omega_1} {}^\circ(X,\Omega_2)$$

由（1）${}^\circ(X,\Omega)$ 为 X 的连续函数；因此存在 $X_0 \in \Omega_1$ ，使得

$${}^\circ(\Omega_1,\Omega_2) = \inf_{X \in \Omega_1} {}^\circ(X,\Omega_2) = {}^\circ(X_0,\Omega_2) ,$$

再由第（2）问，存在 $Y_0 \in \Omega_2$ ，使得

$${}^\circ(\Omega_1,\Omega_2) = {}^\circ(X_0,\Omega_2) = \| X_0 - Y_0 \|_n .$$

7. 设 $f(x,y,z)$ 可微，$\mathbf{l}_1,\mathbf{l}_2,\mathbf{l}_3$ 为 $\mathbb{R}^3$ 中互相垂直的三个单位向量，求证：

$$(\frac{\partial f}{\partial \mathbf{l}_1})^2 + (\frac{\partial f}{\partial \mathbf{l}_2})^2 + (\frac{\partial f}{\partial \mathbf{l}_3})^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + (\frac{\partial f}{\partial z})^2 .$$

证明：$\mathbf{l}_1,\mathbf{l}_2,\mathbf{l}_3$ 为 $\mathbb{R}^3$ 中互相垂直的三个单位向量，因此存在正交矩阵 $A=(a_{ij})_{3\times 3}$ 使得

$(\mathbf{l}_1,\mathbf{l}_2,\mathbf{l}_3) = (e_1,e_2,e_3)A$ ，其中 (e_1,e_2,e_3) 为 $\mathbb{R}^3$ 中的标准正交基。

$$(\frac{\partial f}{\partial \mathbf{l}_1})^2 + (\frac{\partial f}{\partial \mathbf{l}_2})^2 + (\frac{\partial f}{\partial \mathbf{l}_3})^2 = (\frac{\partial f}{\partial \mathbf{l}_1},\frac{\partial f}{\partial \mathbf{l}_2},\frac{\partial f}{\partial \mathbf{l}_3}) \cdot (\frac{\partial f}{\partial \mathbf{l}_1},\frac{\partial f}{\partial \mathbf{l}_2},\frac{\partial f}{\partial \mathbf{l}_3})^{\mathrm{T}}$$

而 $\dfrac{\partial f}{\partial \mathbf{l}_i}=\dfrac{\partial f}{\partial x}\cos\sim_i+\dfrac{\partial f}{\partial y}\cos\quad_i+\dfrac{\partial f}{\partial z}\cos¥_i$，其中

$\mathbf{l}_i=\cos\sim_i e_1+\cos\quad_i e_2+\cos¥_i e_3=a_{1i}e_1+a_{2i}e_2+a_{3i}e_3$，

即 $\dfrac{\partial f}{\partial \mathbf{l}_i}=\dfrac{\partial f}{\partial x}a_{1i}+\dfrac{\partial f}{\partial y}a_{2i}+\dfrac{\partial f}{\partial z}a_{3i}=(\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y},\dfrac{\partial f}{\partial z})\begin{pmatrix}a_{1i}\\a_{2i}\\a_{3i}\end{pmatrix}$

因此 $(\dfrac{\partial f}{\partial \mathbf{l}_1},\dfrac{\partial f}{\partial \mathbf{l}_2},\dfrac{\partial f}{\partial \mathbf{l}_3})=(\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y},\dfrac{\partial f}{\partial z})A$，因此

$$(\frac{\partial f}{\partial \mathbf{l}_1})^2+(\frac{\partial f}{\partial \mathbf{l}_2})^2+(\frac{\partial f}{\partial \mathbf{l}_3})^2=(\frac{\partial f}{\partial \mathbf{l}_1},\frac{\partial f}{\partial \mathbf{l}_2},\frac{\partial f}{\partial \mathbf{l}_3})\cdot(\frac{\partial f}{\partial \mathbf{l}_1},\frac{\partial f}{\partial \mathbf{l}_2},\frac{\partial f}{\partial \mathbf{l}_3})^{\mathrm{T}}$$

$$=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})AA^{\mathrm{T}}(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})^{\mathrm{T}}=(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2+(\frac{\partial f}{\partial z})^2.$$

8. 已知偏微分方程（输运方程）$\begin{cases}\dfrac{\partial z}{\partial t}=a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}\\ z(x,y,0)=z_0(x,y)\end{cases}$，证明它的解为 $z=z_0(x+at,y+bt)$.

证明：代入验证即可。

9. 求解下列问题.

（1）若 $f(x,y,z)$ 可微，则 $f(x,y,z)$ 为 k 次齐次函数（即 $f(tx,ty,tz)=t^k f(x,y,z),\forall t\neq 0$）的充要条件为 $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=kf(x,y,z)$.

（2）设函数 $u(x,y,z)=f(\sqrt{x^2+y^2+z^2})$，若 u 满足 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$，证明：$u=\dfrac{a}{\sqrt{x^2+y^2+z^2}}+b$（$a,b$ 为常数）.

证明：（1）必要性：$f(tx,ty,tz)=t^k f(x,y,z)$ 两端对 t 求导，

$$x\frac{\partial f}{\partial x}(tx,ty,tz)+y\frac{\partial u}{\partial y}(tx,ty,tz)+z\frac{\partial u}{\partial z}(tx,ty,tz)=kt^{k-1}f(x,y,z),$$

再令 $t=1$ 有 $x\dfrac{\partial f}{\partial x}(x,y,z)+y\dfrac{\partial u}{\partial y}(x,y,z)+z\dfrac{\partial u}{\partial z}(x,y,z)=kf(x,y,z)$.

充分性：只需证明 $g(t)=t^{-k}f(tx,ty,tz)$ 为常值函数即可。

$$g'(t)=(-k)t^{-k-1}f(tx,ty,tz)+t^{-k}(\frac{\partial f}{\partial x}(tx,ty,tz)+y\frac{\partial u}{\partial y}(tx,ty,tz)+z\frac{\partial u}{\partial z}(tx,ty,tz))\text{，}$$

由已知 $x\frac{\partial f}{\partial x}(x,y,z)+y\frac{\partial u}{\partial y}(x,y,z)+z\frac{\partial u}{\partial z}(x,y,z)=kf(x,y,z)$，用 tx,ty,tz 代替 x,y,z 有

$$tx\frac{\partial f}{\partial x}(tx,ty,tz)+ty\frac{\partial u}{\partial y}(tx,ty,tz)+tz\frac{\partial u}{\partial z}(tx,ty,tz)=kf(tx,ty,tz)\text{，}$$

所以 $x\frac{\partial f}{\partial x}(tx,ty,tz)+y\frac{\partial u}{\partial y}(tx,ty,tz)+z\frac{\partial u}{\partial z}(tx,ty,tz)=t^{-1}kf(tx,ty,tz)$，从而

$g'(t)\equiv 0$， $g(t)=g(1)$，即 $f(tx,ty,tz)=t^k f(x,y,z),\forall t\neq 0$.

（2）设 $t=\sqrt{x^2+y^2+z^2}$， $u(x,y,z)=f(\sqrt{x^2+y^2+z^2})$，

则 $\frac{\partial u}{\partial x}=f'(t)\frac{\partial t}{\partial x}=f'(t)\frac{x}{t}$， $\frac{\partial^2 u}{\partial x^2}=f''(t)\frac{x^2}{t^2}+f'(t)\frac{t-x\frac{x}{t}}{t^2}=f''(t)\frac{x^2}{t^2}+f'(t)\frac{t^2-x^2}{t^3}$

类似可以得到 $\frac{\partial^2 u}{\partial y^2},\frac{\partial^2 u}{\partial z^2}$，由 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$，得到

$$f''(t)+f'(t)\frac{2}{t}=0.$$

解得 $f(t)=\frac{a}{t}+b$，即 $u(x,y,z)=f(\sqrt{x^2+y^2+z^2})=\frac{a}{\sqrt{x^2+y^2+z^2}}+b$（$a,b$ 为常数）.