第八次习题课补充答案

2. (2) 补面 S_0: $\begin{cases} z=0 \\ x^2+y^2\le a \end{cases}$

$$I + \iint_{S_0^-} \overset{\text{Gauss}}{=} \iiint_\Omega (3x^2+3y^2+3z^2)\,dx\,dy\,dz$$

$$= \iiint_\Omega 3r^4\,dr \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_0^{2\pi} d\theta = \frac{6}{5}\pi a^5$$

$$\iint_{S_0^+} = \iint_{S_0} (z^3+ay^2)\,dx\,dy = \iint_{S_0} ay^2\,dx\,dy = \frac{1}{4}a^5\pi$$

$$I = \frac{6}{5}\pi a^5 + \frac{1}{4}a^5\pi = \frac{29}{20}\pi a^5$$

8. (1) $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos(n,x) + \frac{\partial u}{\partial y}\cos(n,y)$

$$\begin{cases} \cos(n,x)\,dl = \cos\beta\, dl = dy \\ \cos(n,y)\,dl = -\cos\alpha\, dl = -dx \end{cases}$$

$$\oint_{\partial D} \frac{\partial u}{\partial n}\,dl = \oint_{\partial D} \frac{\partial u}{\partial x}\,dy - \frac{\partial u}{\partial y}\,dx \overset{\text{Green}}{=} \iint_D \Delta u\,dx\,dy$$

(2) $\oint_{\partial D} v\frac{\partial u}{\partial n}\,dl = \oint_{\partial D} v\cdot\frac{\partial u}{\partial x}\,dy - v\cdot\frac{\partial u}{\partial y}\,dx$

$$= \iint_D \frac{\partial}{\partial x}\left(v\cdot\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\cdot\frac{\partial u}{\partial y}\right)\,dx\,dy$$

$$= v\cdot\Delta u - \nabla u\cdot\nabla v$$

(3) $\oint_{\partial D} \left(v\cdot\frac{\partial u}{\partial n} - u\cdot\frac{\partial v}{\partial n}\right)dl$

$$= \iint_D (v\cdot\Delta u - \nabla u\cdot\nabla v - u\cdot\Delta v + \nabla u\cdot\nabla v)\,dx\,dy$$

$$= \iint_D (v\cdot\Delta u - u\cdot\Delta v)\,dx\,dy$$

9. $\begin{cases} \cos(n,x)\,dS = \cos\alpha\, dS = dy\,dz \\ \cos(n,y)\,dS = \cos\beta\, dS = dz\,dx \\ \cos(n,z)\,dS = \cos\gamma\, dS = dx\,dy \end{cases}$

其余与8题一致

12. 证明：设 $a = (a_1, a_2, a_3)$

$$a\times r = \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{pmatrix}$$

则 $\int_{\partial S^+} a\times r\,dl =$

$$\int_{\partial S^+} (a_2 z - a_3 y)\,dx + (a_3 x - a_1 z)\,dy + (a_1 y - a_2 x)\,dz$$

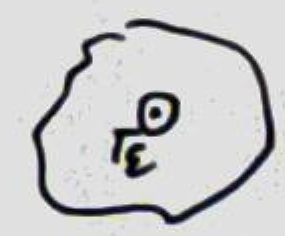
$$= \iint_{S^+} 2a_1\,dy\,dz + 2a_2\,dz\,dx + 2a_3\,dx\,dy$$

$$= 2\iint_{S^+} a\cdot n^0\,dS$$

19. 利用8(3)结论

$$\oint_{\partial D} \left(v\cdot\frac{\partial u}{\partial n} - u\cdot\frac{\partial v}{\partial n}\right)dl = \iint_D (v\Delta u - u\cdot\Delta v)\,dx\,dy$$

(4) 设 Γ_ε 为以 (x_0, y_0) 为圆心，ε 为半径的无穷小圆



则有 $\iint_{D\setminus D_\varepsilon} (u\cdot\Delta(\ln r) - \ln r\,\Delta u)\,dS$

$$= \oint_{\partial D\cup\Gamma_\varepsilon} \left(u\cdot\frac{\partial \ln r}{\partial n} - \ln r\cdot\frac{\partial u}{\partial n}\right)dl$$

$\Delta u = 0$，$\Delta(\ln r) = 0$，故上式 $=0$

$$\oint_{\Gamma_\varepsilon} \left(u\cdot\frac{\partial \ln r}{\partial n}\right)dl = -\frac{1}{\varepsilon}\oint_{\Gamma_\varepsilon} u\,dl = -2\pi u^*\cdot\varepsilon\cdot\frac{1}{\varepsilon} = -2\pi u^*$$

（其中 $\frac{\partial \ln r}{\partial n} = -\frac{\partial \ln r}{\partial r} = -\frac{1}{r}$，$u^*$ 为 Γ_ε 圆周上平均值）

$\oint_{\Gamma_\varepsilon}(\ln r\cdot\frac{\partial u}{\partial n})dl = \ln\varepsilon\oint_{\Gamma_\varepsilon}\frac{\partial u}{\partial n}dl = 0$ 式

(应用8(3)，令$v=1$可证)

故 $\oint_{\partial D}(u\cdot\frac{\partial \ln r}{\partial n} - \ln r\cdot\frac{\partial u}{\partial n})dl = 2\pi u^*$

$\varepsilon\to 0$时，$u^*\to u(x_0,y_0)$

得证

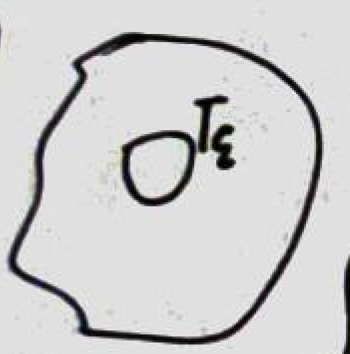
(2) $u(x_0,y_0) = \frac{1}{2\pi}\oint_L(u\cdot\frac{\partial \ln r}{\partial n} - \ln R\cdot\frac{\partial u}{\partial n})dl$

$= \frac{1}{2\pi}\oint_L(u\cdot\frac{1}{R} - \ln R\cdot\frac{\partial u}{\partial n})dl$

$= \frac{1}{2\pi R}\oint_L u\,dl$

9. 应用 $\iiint_\Omega(u\cdot\Delta v - v\cdot\Delta u)d\Omega = \iint_\Gamma(u\cdot\frac{\partial v}{\partial n} - v\cdot\frac{\partial u}{\partial n})dS$

(1)



K_ε：以r_0为圆心，ε为半径的小球

Γ_ε为其球面

$\iiint_{\Omega\setminus K_\varepsilon}(u\cdot\Delta\frac{1}{r} - \frac{1}{r}\cdot\Delta u)d\Omega = \iint_{\Gamma\cup\Gamma_\varepsilon}(u\frac{\partial}{\partial n}(\frac{1}{r}) - \frac{1}{r}\frac{\partial u}{\partial n})dS$

$\Delta u=0$，$\Delta\frac{1}{r}=0$，故上式为0.

球面Γ_ε上，$\frac{\partial}{\partial n}(\frac{1}{r}) = -\frac{\partial}{\partial r}(\frac{1}{r}) = \frac{1}{r^2} = \frac{1}{\varepsilon^2}$

$\iint_{\Gamma_\varepsilon} u\cdot\frac{\partial}{\partial n}(\frac{1}{r})dS = \frac{1}{\varepsilon^2}\iint_{\Gamma_\varepsilon} u\,dS = 4\pi u^*$

u^*是u在Γ_ε上的平均值

$\iint_{\Gamma_\varepsilon}\frac{1}{r}\frac{\partial u}{\partial n}dS = \frac{1}{\varepsilon}\iint_{\Gamma_\varepsilon}\frac{\partial u}{\partial n}dS = 4\pi\varepsilon(\frac{\partial u}{\partial n})^*$

$(\frac{\partial u}{\partial n})^*$是$\frac{\partial u}{\partial n}$在球$\Gamma_\varepsilon$上平均值

$\iint_\Gamma(u\cdot\frac{\partial}{\partial n}(\frac{1}{r}) - \frac{1}{r}\frac{\partial u}{\partial n})dS + 4\pi u^* - 4\pi\varepsilon\cdot(\frac{\partial u}{\partial n})^* = 0$

$\varepsilon\to 0$，$u^*\to \cancel{u(x_0,y_0)}\ r_0$

得 $u(r_0) = -\frac{1}{4\pi}\iint_\Gamma[u\frac{\partial}{\partial n}(\frac{1}{r}) - \frac{1}{r}\cdot\frac{\partial u}{\partial n}]dS$

(2) $\iint_\Gamma\frac{\partial u}{\partial n}dS = 0$ (令$v=0$即证)

可得 $u(r_0) = -\frac{1}{4\pi}\iint_{\Gamma_0}(u\frac{\partial}{\partial n}(\frac{1}{r}) - \underbrace{\frac{1}{r}\frac{\partial u}{\partial n}}_{\downarrow\,0})dS$

$\frac{\partial}{\partial n}(\frac{1}{r})\Big|_{\Gamma_0} = -\frac{1}{R_0^2}$

得 $u(r_0) = \frac{1}{4\pi R_0^2}\iint_{\Gamma_0}u\,dS$

即证