

Section 12.4 The Cross Product(Ex.6 제외)

■ Definition

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - b_2 a_3, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2 \rangle$$

or

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\ &= (a_2 b_3 - b_2 a_3) \mathbf{i} - (a_1 b_3 - b_1 a_3) \mathbf{j} + (a_1 b_2 - b_1 a_2) \mathbf{k} \\ &= \langle a_2 b_3 - b_2 a_3, a_1 b_3 - b_1 a_3, a_1 b_2 - b_1 a_2 \rangle\end{aligned}$$

Note 1 The cross product $\mathbf{a} \times \mathbf{b}$ is defined only when \mathbf{a} and \mathbf{b} are 3-dimensional vectors.

Example 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\ &= \langle -15 - 28, -(-5 - 8), 7 - 6 \rangle = \langle -43, 13, 1 \rangle\end{aligned}$$

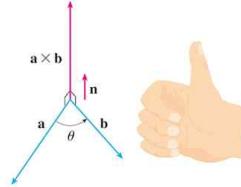
$$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 7 & -5 \\ 1 & 3 & 4 \end{vmatrix} =$$

Note 2 $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$

Example 2 Show that $\mathbf{a} \times \mathbf{a} = \overrightarrow{0}$ for any vector \mathbf{a} in V_3 .

■ Theorem 1

- (1) The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .
- (2) The direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule*.



The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

■ Theorem 2

If θ is the angle between \mathbf{a} and \mathbf{b} ($0 \leq \theta \leq \pi$), then

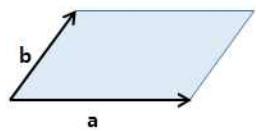
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

■ Corollary Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel $\Leftrightarrow \mathbf{a} \times \mathbf{b} = \overrightarrow{0}$

ex.) If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle -2, -6, a \rangle$ are parallel. then $a = \underline{\hspace{2cm}}$

Answer. $a = 8$

■ Formula 1(평행사변형의 넓이)



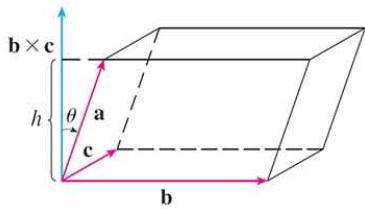
the **area** of the parallelogram determined by the vector \mathbf{a} and \mathbf{b}
 $= |\mathbf{a} \times \mathbf{b}|$

설명)

Example 3 Find a vector perpendicular to the plane through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Example 4 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

■ Formula 2(평행육면체의 부피)



the **volume** of the parallelepiped determined by the vector \mathbf{a} , \mathbf{b} , \mathbf{c}
 $= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$

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Properties of the Cross Product

1. $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$
2. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

Note 2 If the volume of the parallelepiped determined by the vector \mathbf{a} , \mathbf{b} , \mathbf{c} is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

Example 5 Show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

연습문제 33. Find the volume of the parallelepiped determined by the vectors $\mathbf{a} = \langle 6, 3, -1 \rangle$, $\mathbf{b} = \langle 0, 1, 2 \rangle$, $\mathbf{c} = \langle 4, -2, 5 \rangle$. Answer 82