2020년 1학기 수학2 Lecture Note2(동영상2) 학과: ____________ 학번: ____________ 성명: ________

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Section 12.3 The Dot Product

■ **Definition**

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then, the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Similarly, the dot product of two-dimension vectors is defined:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

Example 1 (1) $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle =$ ____________________

(2) $\langle -1, 7, 4 \rangle \cdot \left\langle 6, 2, -\dfrac{1}{2} \right\rangle =$ ____________________

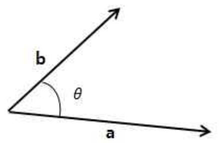
(3) $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) =$ ____________________

■ **Properties of the dot product**

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (교환법칙)
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{a} \cdot \mathbf{0} = 0$

■ **Theorem**(a geometric interpretation of **the dot product)**



$(0 \le \theta \le \pi)$

If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

Example 2 If the vectors $\mathbf{a}$ and $\mathbf{b}$ have length 4 and 6, and the angle between them is $\dfrac{\pi}{3}$. find $\mathbf{a} \cdot \mathbf{b}$.

Solution.

■ **Corollary** If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example 3 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

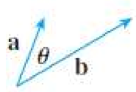
Solution.

■ **Definition** Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are called **perpendicular**(or **orthogonal**) if the angle between them is $\theta = \dfrac{\pi}{2}$.

■ **Theorem** $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

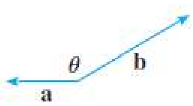
Example 4 Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

■ **Remark** (1)



If $0 \le \theta < \dfrac{\pi}{2}$, then $\mathbf{a} \cdot \mathbf{b} > 0$.

(2)



If $\dfrac{\pi}{2} < \theta \le \pi$, then $\mathbf{a} \cdot \mathbf{b} < 0$.

■ Projections

For the given vectors $\mathbf{a}$ and $\mathbf{b}$ with the same initial point P:

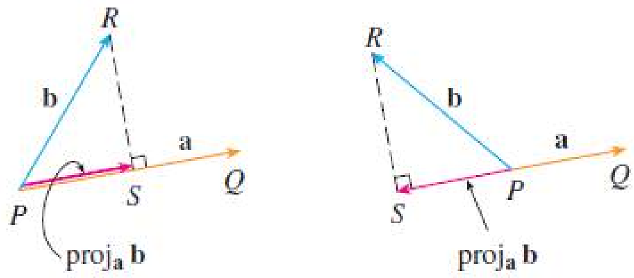


Figure 4 Vector projection

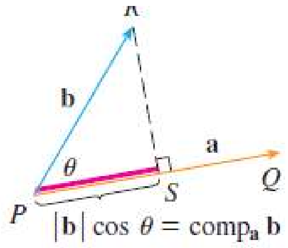


Figure 5 Scalar projection

Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \text{comp}_{\mathbf{a}}\mathbf{b} = |\mathbf{b}|\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

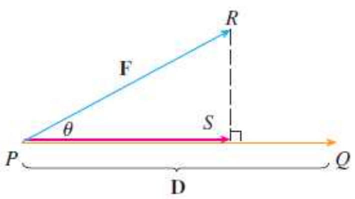
Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

(설명)

Example 6 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

■ Work

Suppose that the constant **force** is a vector $\mathbf{F} = \overrightarrow{PR}$ and the force moves the object from P to Q. Let $\mathbf{D} = \overrightarrow{PQ}$ be the **displacement** vector. Then we defined:



$$\begin{aligned}\text{the } \textbf{work} \text{ done by the force } \mathbf{F} &= \text{comp}_{\mathbf{D}}\mathbf{F} \times \text{the distance moved}\\ &= (|\mathbf{F}|\cos\theta)\,|\mathbf{D}|\\ &= \mathbf{F}\cdot\mathbf{D}\end{aligned}$$

Example 8 A force is given by a vector $\mathbf{F} = \langle 3, 4, 5 \rangle$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.