1. Consider the sequence $\left\{\frac{1}{1}, \frac{3}{2}, \frac{7}{3}, \ldots, \frac{a_n}{b_n}, \ldots\right\}$ where $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$.

(a) Find a matrix A such that $A\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix}$.

Letting $A = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, we have:

$\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} a_n + 2b_n \\ a_n + b_n \end{bmatrix}$, so $ea_n + fb_n = a_n + 2b_n$ and $ga_n + hb_n = a_n + b_n$.

One solution to the above equation is $e = 1, f = 2, g = 1, h = 1$, i.e., $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

(b) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

If $A = SDS^{-1}$, then $S^{-1}A = S^{-1}SDS^{-1} = DS^{-1}$ and $S^{-1}AS = DS^{-1}S = D$, i.e., S is a matrix which diagonalizes A.

Since the eigenvalues of A are the roots of the characteristic polynomial

$c_A(\lambda) = \lambda^2$ - $(\text{tr}A)\lambda + \det A = \lambda^2$ - 2λ - 1, D becomes $\begin{bmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix}$.

Starting with $\lambda = 1 - \sqrt{2}$, we have A - $(1-\sqrt{2})I = \begin{bmatrix} -2+\sqrt{2} & 2 \\ 1 & -2+\sqrt{2} \end{bmatrix}$, and $\text{rref}(A - (1-\sqrt{2})I) = \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$.

Since $\text{null}\begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix} = t\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$, a basis for the eigenspace $E_{1-\sqrt{2}}$ corresponding to $\lambda = 1-\sqrt{2}$ is $\left\{\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}\right\}$.

Moving on to $\lambda = 1 + \sqrt{2}$, we have A - $(1+\sqrt{(2)})I = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix}$, and $\text{rref}((1+\sqrt{2})I - A) = \begin{bmatrix} 1 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}$.

Since $\text{null}\begin{bmatrix} 1 & -\sqrt{2} \\ 0 & 0 \end{bmatrix} = t\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$, a basis for the eigenspace $E_{1+\sqrt{2}}$ corresponding to $\lambda = 1+\sqrt{2}$ is $\left\{\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}\right\}$.

We have found two eigenvectors of A which are linearly independent of each other and hence a basis for $^2\mathbb{R}$.

By the Diagonalization Theorem, the columns of S are composed of these eigenvectors, i.e., $S = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$.

1. Consider the sequence $\left\{\frac{1}{1}, \frac{3}{2}, \frac{7}{3}, \ldots, \frac{a_n}{b_n}, \ldots\right\}$ where $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$.

(c) Use your answer from part (b) to find explicit formulas for a_n and b_n, and then show that $\lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{2}$.

We begin by noting that $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = A^{n-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (SDS^{-1})^{n-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = SD^{n-1}S^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Since $S^{-1} = \begin{bmatrix} -\sqrt{2}/4 & 1/2 \\ \sqrt{2}/4 & 1/2 \end{bmatrix}$,

$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix}^{n-1}\begin{bmatrix} -\sqrt{2}/4 & 1/2 \\ \sqrt{2}/4 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (1-\sqrt{2})^{n-1} & 0 \\ 0 & (1+\sqrt{2})^{n-1} \end{bmatrix}\begin{bmatrix} -\sqrt{2}/4 & 1/2 \\ \sqrt{2}/4 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} (2/4)((1-\sqrt{2})^n + (1+\sqrt{2})^n) \\ (\sqrt{2}/4)((1+\sqrt{2})^n - (1-\sqrt{2})^n) \end{bmatrix}$

and we conclude that formulas for a_n and b_n are

$a_n = (2/4)((1-\sqrt{2})^n + (1+\sqrt{2})^n)$ and

$b_n = (\sqrt{2}/4)((1+\sqrt{2})^n - (1-\sqrt{2})^n)$.

Since $\lim_{n\to\infty}(1-\sqrt{2})^n = 0$, $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{(2/4)(1+\sqrt{2})^n}{(\sqrt{2}/4)(1+\sqrt{2})^n} = \sqrt{2}$.

2. Let A be an $n \times n$ matrix, and suppose that the only eigenvalues of A are 0, 1, and 2.

(a) Prove that $\dim E_1(A) + \dim E_2(A) \leq \operatorname{rank} A$.

We know that $E_0(A)$, the eigenspace corresponding to $\lambda = 0$, is equal to null$A$ since $E_0(A)$ = null(A - $0I$) = null$A$.

But by the Rank-Nullity Theorem, n = rank$A$ + nullity$A$, so if dim$E_1(A)$+ dim$E_2(A)$ > rank$A$, then dim$E_0(A)$ + dim$E_1(A)$ + dim$E_2(A)$ > n which violates Theorem IV, Medici 10.4, which states that the sum of the dimension of all eigenspaces of A must be less than or equal to n. (We know that the set $\{E_0(A), E_1(A), E_2(A)\}$ constitutes all eigenspaces of A since it is given that the only eigenvalues of A are 0, 1, and 2). Therefore the highest possible value for dim$E_1(A)$ + dim$E_2(A)$ is rank$A$, which only occurs when the sum of the dimension of all eigenspaces of A is equal to n; it follows that dim$E_1(A)$ + dim$E_2(A)$ will be less than rank$A$ when the sum of the dimension of all eigenspaces of A is less than n.

Hence we conclude that dim$E_1(A)$ + dim$E_2(A) \leq$ rank$A$.

2. Let A be an $n \times n$ matrix, and suppose that the only eigenvalues of A are 0, 1, and 2.

(b) Prove that if $\dim E_1(A) + \dim E_2(A) = \operatorname{rank} A$, then A is diagonalizable.

Since the only eigenvalues of A are 0, 1, and 2, we have by the Corollary to Theorem IV, Medici 10.4, that if $\dim E_0(A) + \dim E_1(A) + \dim E_2(A) = n$, then A is diagonalizable. We also have that $E_0(A) = \text{null} A$ since $E_0(A) = \text{null}(A - 0I) = \text{null} A$. But then $\dim E_0(A) = \text{nullity} A$ and by the Rank-Nullity Theorem, $\text{rank} A + \text{nullity} A = n$, so A is diagonalizable if $\dim E_1(A) + \dim E_2(A) = \text{rank} A$.