1. Let $V = \{(x_1, x_2) \mid x_1, x_2 > 0, \text{ and } x_1 + x_2 = 1\}$. Is V a real vector space with respect to the usual entry-wise vector addition and scalar multiplication? Why or why not?

Consider $\alpha = 69420$

and $(x_1, x_2) = (0.5, 0.5)$

This satisfies requirements for V, as $x_1 + x_2 = 0.5 + 0.5 = 1$, x_1 and x_2 both > 0, and $\alpha \in \mathbb{R}$

However, $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2) = (69420(0.5), 69420(0.5)) = (34710, 34710)$ according to the usual definition of scalar multiplication, and

$(34710, 34710)$ is **not** in V because $\alpha x_1 + \alpha x_2 = 34710 + 34710 = 69420 \neq 1$

Therefore, V is not a real vector space because it fails MI: Closure

$(\alpha(x_1, x_2) \in V \ \forall \ (x_1, x_2) \in V)$

2 Let $V = \{(x_1, x_2) \mid x_1, x_2 > 0, \text{ and } x_1 + x_2 = 1\}$. Define vector addition in V by

$$(x_1, x_2) + (y_1, y_2) = \frac{(x_1y_1, x_2y_2)}{x_1y_1 + x_2y_2}$$

and scalar multiplication in V by

$$c(x_1, x_2) = \frac{(x_1^c, x_2^c)}{x_1^c + x_2^c}$$

Then V is a real vector space.

(a) Verify that axiom AIII. (Medici, pp104) holds in V.

AIII: Zero states

$\exists\, \underset{\sim}{0} \in V$ s.t. $\underset{\sim}{u} + \underset{\sim}{0} = \underset{\sim}{u}\ \forall \underset{\sim}{u} \in V$

<-We need to find a zero vector that satisfies this axiom.

Prop V (Medici) states $0\underset{\sim}{x} = \underset{\sim}{0}\ \forall \underset{\sim}{x} \in V$. Therefore, $0(x_1,x_2) = \frac{(x_1^0, x_2^0)}{x_1^0 + x_2^0} = \frac{(1,1)}{1+1} = (0.5, 0.5) = \underset{\sim}{0}$

We verify this is in fact the zero vector by adding it to an arbitrary vector in V (x1, x2).->

$$(x_1, x_2) + (0.5, 0.5) = \frac{(0.5x_1, 0.5x_2)}{0.5x_1 + 0.5x_2} = \frac{(0.5x_1, 0.5x_2)}{0.5(x_1 + x_2)}$$

← Distributivity of $\mathbb{R}$

$$= \frac{(0.5x_1, 0.5x_2)}{0.5(1)} = \left(\frac{0.5}{0.5}x_1, \frac{0.5}{0.5}x_2\right)$$

Property of V →

$$= (x_1, x_2)$$

Therefore, $\underset{\sim}{0} = (0.5, 0.5)$ and AIII holds in V

(b) Verify that axiom AIV. (Medici, pp104) holds in V.

AIV: Inverse states

$\exists\, -(\underset{\sim}{x_1, x_2}) \in V$ s.t. $(\underset{\sim}{x_1, x_2}) + (-(\underset{\sim}{x_1, x_2})) = \underset{\sim}{0}\ \forall (\underset{\sim}{x_1, x_2}) \in V$

→ For this question, $\underset{\sim}{0} = (0.5, 0.5)$

According to the given definition of scalar multiplication, $-1(x_1, x_2) = \frac{(x_1^{-1}, x_2^{-1})}{x_1^{-1} + x_2^{-1}} = \left(\frac{x_1^{-1}}{x_1^{-1} + x_2^{-1}}, \frac{x_2^{-1}}{x_1^{-1} + x_2^{-1}}\right)$ *

To prove this is the inverse $\forall (x_1, x_2) \in V$, we add it to (x_1, x_2)

According to the given definition of vector addition, $(x_1, x_2) + (-(x_1, x_2)) = \dfrac{\left(\frac{x_1x_1^{-1}}{x_1^{-1} + x_2^{-1}}, \frac{x_2x_2^{-1}}{x_1^{-1} + y_2^{-1}}\right)}{\frac{x_1x_1^{-1}}{x_1^{-1} + x_2^{-1}} + \frac{x_2x_2^{-1}}{x_1^{-1} + x_2^{-1}}}$

$$= \frac{(x_1x_1^{-1}, x_2x_2^{-1})}{x_1x_1^{-1} + x_2x_2^{-1}}$$

$$= \frac{(1,1)}{1+1} \quad (\text{property of } \mathbb{R})$$

$$= (0.5, 0.5)$$

$$= \underset{\sim}{0}$$

→ Therefore, AIV holds in V

* As mentioned in lecture, multiplying a vector in V by the scalar -1 always yields its inverse if V is a real vector space.

2. Let $V = \{(x_1, x_2) \mid x_1, x_2 > 0, \text{ and } x_1 + x_2 = 1\}$. Define vector addition in V by

$$(x_1, x_2) + (y_1, y_2) = \frac{(x_1 y_1, x_2 y_2)}{x_1 y_1 + x_2 y_2}$$

and scalar multiplication in V by

$$c(x_1, x_2) = \frac{(x_1^c, x_2^c)}{x_1^c + x_2^c}$$

Then V is a real vector space.

(c) Verify that axiom MIII. (Medici, pp104) holds in V.

My solution for part b) of this subquestion flows better if you start at the bottom and work up to the top. If you read it that way, all the comments I've added on the side will need to be shifted up one line to make sense

MIII:

a)

$$\alpha(x_1, x_2) + \beta(x_1, x_2)$$

$$= \frac{(x_1^\alpha, x_2^\alpha)}{x_1^\alpha + x_2^\alpha} + \frac{(x_1^\beta, x_2^\beta)}{x_1^\beta + x_2^\beta}$$ [By the definition of scalar multiplication.]

$$= \left(\frac{x_1^\alpha}{x_1^\alpha + x_2^\alpha}, \frac{x_2^\alpha}{x_1^\alpha + x_2^\alpha}\right) + \left(\frac{x_1^\beta}{x_1^\beta + x_2^\beta}, \frac{x_2^\beta}{x_1^\beta + x_2^\beta}\right)$$

$$= \frac{\left(\frac{x_1^\alpha x_1^\beta}{(x_1^\alpha + x_2^\alpha)(x_1^\beta + x_2^\beta)}, \frac{x_2^\alpha x_2^\beta}{(x_1^\alpha + x_2^\alpha)(x_1^\beta + x_2^\beta)}\right)}{\frac{x_1^\alpha x_1^\beta}{(x_1^\alpha + x_2^\alpha)(x_1^\beta + x_2^\beta)} + \frac{x_2^\alpha x_2^\beta}{(x_1^\alpha + x_2^\alpha)(x_1^\beta + x_2^\beta)}}$$ [By the definition of vector addition.]

$$= \frac{(x_1^\alpha x_1^\beta, x_2^\alpha x_2^\beta)}{x_1^\alpha x_1^\beta + x_2^\alpha x_2^\beta}$$

$$= \frac{(x_1^{\alpha+\beta}, x_2^{\alpha+\beta})}{x_1^{\alpha+\beta} + x_2^{\alpha+\beta}}$$ [Property of R.]

$$= (\alpha+\beta)(x_1, x_2)$$ [By the definition of scalar multiplication.]

b)

$$\alpha((x_1, x_2) + (y_1, y_2))$$

$$= \alpha\left(\frac{(x_1 y_1, x_2 y_2)}{x_1 y_1 + x_2 y_2}\right)$$ [By the definition of vector addition.]

$$= \frac{((x_1 y_1)^\alpha, (x_2 y_2)^\alpha)}{(x_1 y_1)^\alpha + (x_2 y_2)^\alpha}$$ [By the definition of scalar multiplication.]

$$= \frac{(x_1^\alpha y_1^\alpha, x_2^\alpha y_2^\alpha)}{x_1^\alpha y_1^\alpha + x_2^\alpha y_2^\alpha}$$ [Property of R.]

$$= \frac{\left(\frac{x_1^\alpha y_1^\alpha}{(x_1^\alpha + x_2^\alpha)(y_1^\alpha + y_2^\alpha)}, \frac{x_2^\alpha y_2^\alpha}{(x_1^\alpha + x_2^\alpha)(y_1^\alpha + y_2^\alpha)}\right)}{\frac{x_1^\alpha y_1^\alpha}{(x_1^\alpha + x_2^\alpha)(y_1^\alpha + y_2^\alpha)} + \frac{x_2^\alpha y_2^\alpha}{(x_1^\alpha + x_2^\alpha)(y_1^\alpha + y_2^\alpha)}}$$

$$= \left(\frac{x_1^\alpha}{x_1^\alpha + x_2^\alpha}, \frac{x_2^\alpha}{x_1^\alpha + x_2^\alpha}\right) + \left(\frac{y_1^\alpha}{y_1^\alpha + y_2^\alpha}, \frac{y_2^\alpha}{y_1^\alpha + y_2^\alpha}\right)$$ [By the definition of vector addition.]

$$= \frac{(x_1^\alpha, x_2^\alpha)}{x_1^\alpha + x_2^\alpha} + \frac{(y_1^\alpha, y_2^\alpha)}{y_1^\alpha + y_2^\alpha}$$

$$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$$ [By the definition of scalar multiplication.]

Therefore: $\alpha(x_1, x_2) + \beta(x_1, x_2) = (\alpha+\beta)(x_1, x_2)$

$\alpha(x_1, x_2) + \alpha(y_1, y_2) = \alpha((x_1, x_2) + (y_1, y_2))$

$\rightarrow$ MIII holds in V

3. Recall that $P_3(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, the set of polynomials of degree at most 3 with real coefficients, is a real vector space with respect to the usual polynomial addition and scalar multiplication.

(a) Give an example of a subset S of $P_3(\mathbb{R})$ that is closed under vector addition but not under scalar multiplication. You should both state clearly your subset S and demonstrate that S satisfies the requirement of the question.

Let S = {ax^2 | a >= 0, a ∈ R}.

If a is positive, then any negative scalar B will yield a polynomial (B*a)x^2 which lies outside of S. Therefore, S is not closed under scalar multiplication.

Closure holds under vector addition--consider any subset {(a1)x^2, (a2)x^2 … anx^2} of polynomials in S; since {a1, a2, … an} are all either positive or zero, then all possible combinations of {a1, a2, … an}--the possibilities for the "new" value of a once vector addition is performed--must also be either positive or zero. Hence, vector addition must always yield a polynomial in S.

3. Recall that $P_3(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, the set of polynomials of degree at most 3 with real coefficients, is a real vector space with respect to the usual polynomial addition and scalar multiplication.

(b) Give an example of a subset S of $P_3(\mathbb{R})$ that is closed under scalar multiplication but not under vector addition. You should both state clearly your subset S and demonstrate that S satisfies the requirement of the question.

Let S be the set of all polynomials of degree 2 with real coefficients, as well as the zero vector from P3(R). (S = {0, all polynomials of degree 2 with real coefficients})

Consider two members of S, (x^2) and (-x^2 + x). When these polynomials are added under standard vector addition, the result is x which has a degree of 1. This violates closure under vector addition.

Closure holds under standard scalar multiplication because multiplying any polynomial in S by a real scalar cannot change its degree--unless that scalar is 0, in which case the zero vector from P3(R) would be produced, which still satisfies the requirements for closure.