Question 3

We first prove by induction that a_n is monotone increasing. Since $a_1 = \sqrt{2} < a_2 = \sqrt{2+\sqrt{2}}$, the base case is evident. It remains to show that $a_{n+2} > a_{n+1}$ follows from $a_{n+1} > a_n$ for any n. When $a_{n+1} = \sqrt{2+a_n} > a_n$, we have $a_{n+2} = \sqrt{2+\sqrt{2+a_n}} > a_{n+1}$, so a_n is indeed monotone increasing. Next we will prove by induction that a_n is bounded by 2. Since $a_1 = \sqrt{2} < 2$, the base case is evident. It remains to show that $a_{n+1} < 2$ whenever $a_n < 2$. When $a_n < 2$, $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$, so a_n is indeed bounded by 2. We have shown that a_n is bounded and increasing; thus it is convergent upon some finite number by the Monotone Convergence Theorem; thus a_{n+1} also converges upon the same number. Thus we observe that as n approaches ∞, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+\lim_{n\to\infty} a_n}$. Now letting $A = \lim_{n\to\infty} a_n$, we obtain the algebraic equation $A = \sqrt{2+A}$ which yields $A = 2$ as the solution. Hence $\lim_{n\to\infty} a_n = 2$.