Question 4

Call the series given by the question a_n. Note that a_n is always greater than 0 $\forall n \geq 2$ because $\frac{1}{n} > 0$ and $\frac{1}{(\ln n)^p} > 0 \forall n \geq 2, \forall p$, so the product is also greater than 0 $\forall n \geq 2$. Also note that a_n is decreasing, which can be proven through induction: The base case is $\frac{1}{2p(\ln 2)} > \frac{1}{3(p \ln 3)}$, and the inductive case is $\frac{1}{np(\ln n)} > \frac{1}{(n+1)p(\ln (n+1))} \forall n \geq 2, \forall p$, which is true since the logarithm function is monotone increasing so the denominator will always have a higher value for higher values of n and fixed p. Since $a_n \geq 0 \forall n$ and a_n is decreasing, we may apply the Cauchy condensation test, which tells us that a_n converges iff

$$\sum_{n=2}^{\infty} \frac{2^n}{2^n(\ln 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{(n \ln 2)^p} = \frac{1}{p \ln 2} \sum_{n=2}^{\infty} \frac{1}{n^p}$$

converges, and using the fact from lecture that $\sum \frac{1}{n^p}$ only converges when $p > 1$, we conclude that a_n also only converges when $p > 1$ (since multiplying an infinite series by a fixed non-zero constant does not affect convergence/divergence of the series).