Question 5

Suppose that $h \in \mathbf{N}$ is the largest index such that $\{v_1, ..., v_h\}$ is a linearly independent subset of $\{v_1, ..., v_h, ..., v_k\}$ and moreover let us impose $h < k$. It follows that $1 \leq h < j \leq k$ for some index j, so there exists a v_j that can be written as $v_j = \sum_{n=1}^{h} c_n v_n$ where c_n are scalars. As such,

$$Tv_j = \lambda_j v_j = \lambda_j \sum_{n=1}^{h} c_n v_n = \sum_{n=1}^{h} \lambda_n c_n v_n = T \sum_{n=1}^{h} c_n v_n$$

and manipulating the middle equality, we see that

$$\sum_{n=1}^{h} (\lambda_n - \lambda_j) c_n v_n = 0$$

which is impossible because $\lambda_n \neq \lambda_j \forall n \in [1, h] \in \mathbf{N}$ and we constructed $\{v_1, ..., v_h\}$ to be linearly independent. Hence a contradiction arises as a result of our imposing $h < k$, and we conclude that $h = k$ since that is the only remaining possible value of h.