Medici Solutions (incomplete)

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Chapter 3

2. a) If $m = n$, and $\mathbf{AB}$ is invertible,
$(\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}$
$(\mathbf{B}^{-1}\mathbf{A}^{-1})\mathbf{AB} = \mathbf{I}$
$(\mathbf{A}^{-1}\mathbf{B}^{-1})(\mathbf{B}^{-1}\mathbf{A}^{-1})\mathbf{AB} = \mathbf{A}^{-1}\mathbf{B}^{-1}$
$(\mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1})\mathbf{AB}(\mathbf{BA}) = (\mathbf{A}^{-1}\mathbf{B}^{-1})(\mathbf{BA})$
Since any invertible square matrix $\mathbf{X}$ has the property $\mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = \mathbf{I}$, the left side simplifies to $\mathbf{I}$ so that $\mathbf{I} = (\mathbf{A}^{-1}\mathbf{B}^{-1})\mathbf{BA}$, i.e., $\mathbf{A}^{-1}\mathbf{B}^{-1}$ is the inverse of $\mathbf{BA}$.
b) Note that $m < n$ because if $m > n$ then by the Corollary to Theorem V, $\mathbf{Bx} = \mathbf{0}$ has nontrivial solutions and thus $\mathbf{ABx} = \mathbf{0}$ has nontrivial solutions, i.e., $\mathbf{AB}$ is not invertible. But if $m < n$, then $\mathbf{Ax} = \mathbf{0}$ has nontrivial solutions and thus $\mathbf{BAx} = \mathbf{0}$ also has nontrivial solutions, i.e., $\mathbf{BA}$ is not invertible.

3. If $m < n$ then $\mathbf{Ax} = \mathbf{0}$ has nontrivial solutions by the Corollary to Theorem V. But then $\mathbf{LAx} = \mathbf{0}$ also has nontrivial solutions, so $\mathbf{LA} = \mathbf{I}$ is impossible.

4. Yes
Putting the system in matrix form ($\mathbf{Ax} = \mathbf{b}$), we have
$\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$, so $\text{rref}\big[\mathbf{A}|\mathbf{b}\big] = \begin{bmatrix} 1 & 0 & -1/7 & 1 \\ 0 & 1 & -5/7 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which has a zero row and hence infinite solutions.

5. Yes
$\mathbf{A}$ is square and has a unique solution to $\mathbf{Ax} = \mathbf{b}$ for at least one $\mathbf{b} \in {}^3\mathbb{R}$, so by the Corollary to Theorem VI it is invertible, i.e., a product of elementary matrices.

6. No
$\mathbf{A}$ cannot be invertible because it does not have a unique solution to $\mathbf{Ax} = \mathbf{b}$ for every $\mathbf{b} \in {}^3\mathbb{R}$.

7. See 2b).

Chapter 4

1. No
Axiom MIV fails, since $1(x_1, x_2) \neq (x_1, x_2)$ when $x_2 \neq 0$.

Chapter 5

1. a) No
SII fails, consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
b) Yes
SI: $f = 0$ is the zero function.
SII: $\int_0^1 (f+g)(x)\,dx = 0$ for all $f, g \in U$.
SIII: If $\int_0^1 f(x)\,dx = 0$ then $\lambda\int_0^1 f(x)\,dx = 0$ for all $\lambda \in \mathbb{R}$ and $f \in U$.

2. $\implies$ If i) is true, U is not a subspace because SI fails. Hence U must be nonempty. Further, if ii) is true, we can set $\lambda = 1$ and note that all $\mathbf{u}_1 + \mathbf{u}_2$ are in U (i.e. SII holds), and we can also set $\mathbf{x}_1 = \mathbf{0}$ and note that all $\lambda\mathbf{u}_2$ are in U (i.e. SIII holds).
$\impliedby$ If U is a subspace, it contains the zero vector (SI) and hence is nonempty. Further, for all $\mathbf{u}_1, \mathbf{u}_2 \in U$, U must contain all $\mathbf{u}_1 + \lambda\mathbf{u}_2$ by SII and SIII.

4. No
SII fails, see 1a).

5. No
SII fails, consider $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$.

8. $\mathbf{u} = \frac{1}{2}((\mathbf{u}+\mathbf{v}) + -(\mathbf{v}+\mathbf{w}) + (\mathbf{w}+\mathbf{u}))$
$\mathbf{v} = \frac{1}{2}((\mathbf{u}+\mathbf{v}) + (\mathbf{v}+\mathbf{w}) + -(\mathbf{w}+\mathbf{u}))$
$\mathbf{w} = \frac{1}{2}(-(\mathbf{u}+\mathbf{v}) + (\mathbf{v}+\mathbf{w}) + (\mathbf{w}+\mathbf{u}))$
Thus, $\text{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\} \supseteq \text{span}\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$.
$\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{v}$
$\mathbf{v} + \mathbf{w} = \mathbf{v} + \mathbf{w}$
$\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{u}$
Thus, $\text{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\} \subseteq \text{span}\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$.

9. Yes
0(1,1) + 1(1, 2) = (1, 2).

10. No
Axiom SII fails, consider $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

11. $\mathbb{R}$: SI: $0 \in \mathbb{R}$
SII: For all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} \in \mathbb{R}$
SIII: For all $\mathbf{x} \in \mathbb{R}$ and any scalar $\lambda \in \mathbb{R}$, $\lambda\mathbf{x} \in \mathbb{R}$
$\{0\}$: SI: $0 \in \mathbb{R}$
SII: $0 + 0 \in \{0\}$
SIII: For any scalar $\lambda \in \mathbb{R}$, $\lambda 0 \in \{0\}$
For any other finite non-empty set of n elements $\{x_1, x_2, ..., x_n\}$ in $\mathbb{R}$, SIII fails because some scalar $\lambda \in \mathbb{R}$ exists such that λx_1 is not in the set.

Chapter 6

1. $\implies$ If $\{p, q, pq\}$ is linearly independent, $\lambda_1 p + \lambda_2 q + \lambda_3 pq = \mathbf{0}$ has only the trivial solution. But if $\deg p = 0$, $\lambda_1 p + \lambda_2 q + \lambda_3 pq = \mathbf{0}$ has the solution $\lambda_1 = 0, \lambda_2 = p, \lambda_3 = -1$ which is a nontrivial solution. Likewise if $\deg q = 0$, $\lambda_1 p + \lambda_2 q + \lambda_3 pq = \mathbf{0}$ has the solution $\lambda_1 = q, \lambda_2 = 0, \lambda_3 = -1$. Hence $\deg p$, $\deg q \neq 0$, i.e., $\deg p$, $\deg q \geq 1$ is required.
$\impliedby$ If $\deg p$, $\deg q \geq 1$, then $\deg(-\lambda_3 pq) > \deg(\lambda_1 p + \lambda_2 q)$ for any choice of $\lambda_1, \lambda_2, \lambda_3$, and since it was also given that $\{p, q\}$ is linearly independent, $\lambda_1 + \lambda_2 q + \lambda_3 pq = \mathbf{0}$ has only the trivial solution, i.e., $\{p, q, pq\}$ is linearly independent.

2. Reducing U to its basis, we have $U = \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $W = \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n, \mathbf{v}\}$, where $1 \leq n \leq k$.
If $\mathbf{v} \in \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\} = U$, then W can also be reduced to its basis so that $W = \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\}$, i.e., $U = W$ and thus $\dim U = \dim W$.
If $\mathbf{v} \notin \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\} = U$, then $\{\mathbf{v}_1, ..., \mathbf{v}_n, \mathbf{v}\}$ is a linearly independent set which spans W and is hence a basis for W. In this case, $\dim W$ = n + 1, i.e., $\dim W = \dim U + 1$.
As no possibilities other than the two described above exist, we conclude that either $\dim U = \dim W$ or $\dim U + 1 = \dim W$.

3. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent, $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \lambda_4\mathbf{v}_4 = \mathbf{0}$ has only the trivial solution.
But $\beta_1\mathbf{v}_1 + \beta_2(\mathbf{v}_1 + \mathbf{v}_2) + \beta_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + \beta_4(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) = \mathbf{0}$ is equivalent to
$(\beta_1 + \beta_2 + \beta_3 + \beta_4)\mathbf{v}_1 + (\beta_2 + \beta_3 + \beta_4)\mathbf{v}_2 + (\beta_3 + \beta_4)\mathbf{v}_3 + \beta_4\mathbf{v}_4 = \mathbf{0}$, which implies
$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$
$\beta_2 + \beta_3 + \beta_4 = 0$

$\beta_3 + \beta_4 = 0$
$\beta_4 = 0$, i.e., $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$. Thus, $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4\}$ is linearly independent.

5. If V is an n-dimensional vector space, where n is finite, then any nonzero vector $\mathbf{v}$ in V is linearly independent of $n-1$ other nonzero vectors in V. Hence a basis for V can be constructed by adding n - 1 nonzero vectors to the set $\{\mathbf{v}\}$ such that each vector in the set cannot be produced as a linear combination of other vectors in the set.

6. If $\mathbf{A}$ is not invertible, then its nullspace (the set of vectors such that $\mathbf{Av} = \mathbf{0}$) has a dimension of at least one. But since $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a linearly independent set containing n vectors, and $\mathbf{A}$'s rowspace (the orthogonal complement to its nullspace) has a dimension of less than n, there exists at least one $\mathbf{v}_i \in \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ such that $\mathbf{v}_i$ is in the nullspace of $\mathbf{A}$, so at least one of $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ will end up being $\mathbf{0}$. Since any set containing $\mathbf{0}$ is linearly dependent, we conclude that $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ must be linearly dependent.

7. No
The set is not a basis if $f_0 = 0$.

9. a) S is a subspace since it contains the zero vector and inherits the vector addition and scalar multiplication properties of V. Also, the maximum number of linearly independent vectors in S is k. Therefore, $\{\mathbf{v}_1 ... \mathbf{v}_k\}$ is a basis for S, and $S = \text{span}\{\mathbf{v}_1 ... \mathbf{v}_k\}$. It follows that $S \subseteq \text{span}\{\mathbf{v}_1 ... \mathbf{v}_k\}$.
b) $\{\mathbf{v}_1 ... \mathbf{v}_k, \mathbf{x}\}$ is a linearly independent set. Thus $\lambda_1 \mathbf{v}_1 + ... + \lambda_k \mathbf{v}_k + \lambda_{k+1} \mathbf{v}_{k+1} = 0$ has only the trivial solution. But $\beta_1(\mathbf{v}_1 + \mathbf{x}) + ... + \beta_k(\mathbf{v}_k + x) = 0$ is equivalent to $\beta_1 \mathbf{v}_1 + ... + \beta_k \mathbf{v}_k + (\beta_1 + ... + \beta_k)\mathbf{x} = 0$, so the latter equation must have only the trivial solution as well and hence be linearly independent.

10. If $U \cap W \neq \{\mathbf{0}\}$, $\dim(U \cap W)$ must be greater or equal to 1 since both U and W are subspaces and must contain the zero vector. If $\dim(U \cap W) = 2$ as well, then $U = W$, since U's basis has 2 linearly independent vectors, and W's basis contains 2 linearly independent vectors, and the set containing these four vectors is linearly dependent. But it was given in the question that $U \neq W$, so $\dim(U \cap W) = 1$.

11. No
$\sin^2 x + \cos^2 x = 1$.

12. No
On the interval $\boldsymbol{F}[0, 1]$, $\text{span}\{x, |x|\} = \text{span}\{x\}$.

13. Since a matrix with linearly independent columns has a unique solution for every $\mathbf{Ax} = \mathbf{b}$, and $\mathbf{A0} = \mathbf{0}$, we conclude that $\mathbf{x} = \mathbf{0}$ is the unique solution.

14. a) SI: $\mathbf{x} = \mathbf{0} \in W$
SII: $\mathbf{u}^T\mathbf{x} = \mathbf{0}$ can be rewritten as $u_1x_1 + ... + u_nx_n = 0$ where $\{u_1, ..., u_n\}$ and $\{x_1, ..., x_n\}$ are the individual entries of $\mathbf{u}^T$ and $\mathbf{x}$ respectively. If $\mathbf{x}$, $\mathbf{y} \in W$, then $u_1x_1 + ... + u_nx_n = 0$ and $u_1y_1 + ... + u_ny_n = 0$, so $u_1x_1 + ... + u_nx_n + u_1y_1 + ... + u_ny_n = u_1(x_1 + y_1) + ... + u_n(x_n + y_n) = 0$. But this equation can be rewritten as $\mathbf{u}^T(\mathbf{x} + \mathbf{y})$, so $\mathbf{u}^T(\mathbf{x} + \mathbf{y}) = \mathbf{0}$ for all $\mathbf{x}$, $\mathbf{y} \in W$.
SIII: If $\mathbf{x} \in W$, then $u_1x_1 + ... + u_nx_n = 0$, so $\lambda(u_1x_1 + ... + u_nx_n) = 0$ as well. But this equation can be rewritten as $\mathbf{u}^T\lambda\mathbf{x}$, so $\lambda\mathbf{x} \in W$ for all $\mathbf{x} \in W$.
b) $\mathbf{u}^T$ has n columns, so $n = \text{rank}\mathbf{u}^T + \text{nullity}\mathbf{u}^T$ and if $\mathbf{u} \neq 0$ then $\text{rank}\mathbf{u}^T$ is always equal to 1, so $\text{nullity}\mathbf{u}^T = \dim W = n - 1$.
c) If $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{u}^T\mathbf{u} \neq 0$ by Medici. Hence $\{\mathbf{u}, \mathbf{w}_1, ..., \mathbf{w}_{n-1}\}$ is a linearly independent set containing n vectors. But then this set is a basis for ${}^n\mathbb{R}$ since all of its elements are in ${}^n\mathbb{R}$.

15. Note that the equation $\lambda_1\mathbf{u}_1 + ... + \lambda_5\mathbf{u}_5 + \beta_1\mathbf{w}_1 + ... + \beta_10\mathbf{w}_{10} = 0$ is equivalent to $\lambda_1\mathbf{u}_1 + ... + \lambda_5\mathbf{u}_5 = -\beta_1\mathbf{w}_1 + ... + -\beta_{10}\mathbf{w}_{10}$. But since $U \cap W = \{\mathbf{0}\}$, only the trivial solution exists since it is impossible to create linear combinations of vectors in U out of vectors in W and vice versa. Hence $\{\mathbf{u}_1, ..., \mathbf{u}_5, \mathbf{w}_1, ..., \mathbf{w}_{10}\} \in V$ is a linearly independent set containing 15 vectors, which means its span has dimension 15.

16. Yes
If $\{\mathbf{x}\} \cup S$ is linearly dependent then $\mathbf{x}$ can be formed as a linear combination of elements in S, so $\mathbf{x} \in \text{span}S$.

18. No
Any set containing $\mathbf{0}$ is not linearly independent.

19. $\Longrightarrow$ If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is linearly dependent, then $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = 0$ has nontrivial solutions. Therefore some $\mathbf{v}_k \in \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ exists such that $\sum_{i=1}^{n-1} \lambda_i \mathbf{v}_i$, where $i \neq k$, $= -\lambda_k \mathbf{v}_k$, i.e., $\mathbf{v}_k = \sum_{i=1}^{n-1} (\lambda_i / -\lambda_k) \mathbf{v}_i$.
$\Longleftarrow$ If some $\mathbf{v}_k$ is a linear combination of the other members in the set, then $-\lambda_k \mathbf{v}_k = \sum_{i=1}^{n-1} \lambda_i \mathbf{v}_i$ where $i \neq k$ and λ_i, λ_k are not all 0, so $\sum_{i=1}^{n-1} \lambda_i \mathbf{v}_i$ (where $i \neq k$) $+ \lambda_k \mathbf{v}_k = 0$ has non-trivial solutions, i.e., the set $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is linearly dependent.

22. Since $\dim {}^n\mathbb{R}^n = n^2$, and there are $n^2 + 1$ terms in this equation, nontrivial solutions exist by FTOLA.

23. For $\dim(\text{span}S) = 3$, S must be a linearly independent set, i.e., $\lambda_1(1+x) + \lambda_2(1+kx+x^2) + \lambda_3(1+2x^2) = 0$ must only have the trivial solution. This equation is equivalent to $(\lambda_1+\lambda_2+\lambda_3)1 + (\lambda_1 + k\lambda_2)x + (\lambda_2 + 2\lambda_3)x^2 = 0$, and since $\{1, x, x^2\}$ is a linearly independent set,
$\lambda_1 + \lambda_2 + \lambda_3 = 0$,
$\lambda_1 + k\lambda_2 = 0$, and
$\lambda_2 + 2\lambda_3 = 0$ are required. Solving the system of equations, we see that k $= \frac{1}{2}$ is the only value which causes this set of equations to be linearly dependent. As nontrivial solutions for $\lambda_1, \lambda_2, \lambda_3$ will exist iff the set of equations is linearly dependent, we conclude that $k \neq \frac{1}{2}$ is the only condition for $\dim(\text{span}S) = 3$.

24. $\Longrightarrow$ If $\{\mathbf{e}_1 - \mathbf{v}, ..., \mathbf{e}_n - \mathbf{v}\}$ is a basis for V, then $\lambda_1(\mathbf{e}_1 - \mathbf{v}) + ... + \lambda_n(\mathbf{e}_n - \mathbf{v}) = 0$ has only the trivial solution. But this equation is equivalent to $\lambda_1 \mathbf{e}_1 + ... + \lambda_n \mathbf{e}_n = (\lambda_1 + ... + \lambda_n)\mathbf{v}$ which in turn is equivalent to $\frac{\lambda_1 \mathbf{e}_1 + ... + \lambda_n \mathbf{e}_n}{(\lambda_1 + ... + \lambda_n)} = \mathbf{v}$. If $\mathbf{v} = \alpha_1 \mathbf{e}_1 + ... + \alpha_n \mathbf{e}_n$ where $\alpha_1 + ... + \alpha_n = 1$, then infinite solutions exist for $\{\lambda_1 ... \lambda_n\}$ which means that $\{\mathbf{e}_1 - \mathbf{v}, ..., \mathbf{e}_n - \mathbf{v}\}$ cannot be a basis. Hence $\mathbf{v} \neq \alpha_1 \mathbf{e}_1 + ... + \alpha_n \mathbf{e}_n$ is required.
$\Longleftarrow$ If $\mathbf{v} \neq \alpha_1 \mathbf{e}_1 + ... + \alpha_n \mathbf{e}_n$ (where $\alpha_1 + ... + \alpha_n = 1$), then $\frac{\lambda_1 \mathbf{e}_1 + ... + \lambda_n \mathbf{e}_n}{(\lambda_1 + ... + \lambda_n)} = \mathbf{v}$ has no nontrivial solutions, i.e., only the trivial solution exists. It follows that $\{\mathbf{e}_1 - \mathbf{v}, ..., \mathbf{e}_n - \mathbf{v}\}$ is a basis for V.

Chapter 7

1. No
rank ≤ 4.

2. No
rank ≤ 7, so nullity ≥ 5.

3. No
Consider $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

4. Yes
n $=$ rank$\mathbf{A}$, so nullity$\mathbf{A} = 0$.

5. No
Any matrix whose columns are linearly independent has a nullity of 0.

6. Yes
It must: rank ≤ 34, so nullity ≥ 17.

7. $\Longrightarrow$ If null$\mathbf{A}$ = null$\mathbf{B}$ then rref$\mathbf{A}$ = rref$\mathbf{B}$ and so $\mathbf{E...EnA} = \mathbf{G..GnB}$ for some combinations of elementary matrices $\mathbf{e} = \mathbf{E...En}$ and $\mathbf{g} = \mathbf{G...Gn}$. But $\mathbf{A} = e^{-}1\mathbf{gB}$, and since $e^{-}1\mathbf{g}$ is invertible, we take $\mathbf{U} = e^{-}1\mathbf{g}$ and thereby conclude $\mathbf{U}$ must exist.
$\Longleftarrow$ If $\mathbf{A} = \mathbf{UB}$, then null$\mathbf{A}$ = null$\mathbf{UB}$. But since $\mathbf{U}$ is invertible, null$\mathbf{UB}$ = null$\mathbf{B}$ since the only solution to any $\mathbf{UBx} = \mathbf{0}$ is $\mathbf{Bx} = \mathbf{0}$, and null$\mathbf{B}$ contains all $\mathbf{x}$ such that $\mathbf{Bx} = \mathbf{0}$. Thus, null$\mathbf{A}$ = null$\mathbf{B}$.

8. col$\mathbf{AV}$ = col$\mathbf{A}$ [Prop I].
Thus, rank$\mathbf{AV}$ = rank$\mathbf{A}$, which implies nullity$\mathbf{AV}$ = nullity$\mathbf{A}$ since $\mathbf{A}$ and $\mathbf{V}$ both have n columns.

9. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & -2 \\ 0 & 5 & 5 \\ 3 & 1 & 7 \end{bmatrix}$. Then rref$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Taking the corresponding columns with leading ones from rref$\mathbf{A}$, we conclude that a basis for col$\mathbf{A}$ is $\left\{\begin{bmatrix}1\\-1\\0\\3\end{bmatrix},\begin{bmatrix}2\\1\\5\\1\end{bmatrix},\begin{bmatrix}4\\-2\\5\\7\end{bmatrix}\right\}$.

10. a) rref$\mathbf{A} = \begin{bmatrix}1&0&0&-4&-3\\0&1&0&1&2\\0&0&1&0&-1\\0&0&0&0&0\end{bmatrix}$.
Thus a basis for row$\mathbf{A}$ is $\{\begin{bmatrix}1&4&5&0&0\end{bmatrix},\begin{bmatrix}0&2&4&2&0\end{bmatrix},\begin{bmatrix}0&6&7&6&5\end{bmatrix}\}$.

b) Taking the corresponding columns with leading ones from rref$\mathbf{A}$, we conclude that a basis for col$\mathbf{A}$ is $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}4\\2\\6\\0\end{bmatrix},\begin{bmatrix}5\\4\\7\\0\end{bmatrix}\right\}$.

c) By looking at rref$\mathbf{A}$, we see that the system of equations needed to obtain solutions to $\mathbf{Ax} = \mathbf{0}$ is:
x1 - 4x4 - 3x5 = 0
x2 + x4 + 2x5 = 0
x3 - x5 = 0
Taking x4 and x5 as free variables, we obtain
null$\mathbf{A}$ = x4$\begin{bmatrix}4\\-1\\0\\1\\0\end{bmatrix} + x5\begin{bmatrix}3\\-2\\1\\0\\1\end{bmatrix}$

and thus a basis for null$\mathbf{A}$ is $\left\{\begin{bmatrix}4\\-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}3\\-2\\1\\0\\1\end{bmatrix}\right\}$.

14. a) Since $\mathbf{Ax} = \mathbf{0}$, $\mathbf{A}^T\mathbf{Ax} = \mathbf{0}$.
Since $\mathbf{A}^T\mathbf{Ax} = \mathbf{0}$, $\mathbf{x}^T\mathbf{A}^T\mathbf{Ax} = (\mathbf{Ax})^T\mathbf{Ax} = \mathbf{0}$, which implies $\mathbf{Ax} = \mathbf{0}$ [Lemma III].
Thus, null$\mathbf{A}$ = null$\mathbf{A}^T$.
b) By the above result, we have n - nullity$\mathbf{A}$ = rank$\mathbf{A}$ = n - nullity$\mathbf{A}^T$ = rank$\mathbf{A}^T$.

15. $\mathbf{LA} = \mathbf{I}$ implies $\mathbf{LAx} = \mathbf{0}$ has only $\mathbf{x} = \mathbf{0}$ as a solution.
But $\mathbf{A}$ has linearly dependent columns ($m < n$), which means its nullspace contains nontrivial solutions. It follows that $\mathbf{LAx} = \mathbf{0}$ must also have nontrivial solutions; hence, $\mathbf{LA} = \mathbf{I}$ is impossible.

18. $\Longrightarrow$ If $\mathbf{AB} = \mathbf{O}$, then $\mathbf{ABx} = \mathbf{0}$ for all $\mathbf{Bx}$, which implies that col$\mathbf{B} \subseteq$ null$\mathbf{A}$.
$\Longleftarrow$ If col$\mathbf{B} \subseteq$ null$\mathbf{A}$, then partitioning $\mathbf{B}$ into its columns we get:
$\mathbf{AB} = \mathbf{A}\begin{bmatrix}b1 & \dots & bn\end{bmatrix} = \begin{bmatrix}\mathbf{A}b1 & \dots & \mathbf{A}bn\end{bmatrix}$
and $\{b1, \dots, bn\} \in$ col$\mathbf{B} \subseteq$ null$\mathbf{A}$ so this equation becomes $\begin{bmatrix}\mathbf{0} & \dots & \mathbf{0}\end{bmatrix} = \mathbf{O}$ implying that $\mathbf{AB} = \mathbf{O}$.

21. $\Longrightarrow$ If null$\mathbf{A}$ = col$\mathbf{A}$, then rank$\mathbf{A}$ = n/2 because dimcol$\mathbf{A}$ = rank$\mathbf{A}$ and n = rank$\mathbf{A}$ + nullity$\mathbf{A}$, so the only value which satisfies rank$\mathbf{A}$ = nullity$\mathbf{A}$ is n/2.
If null$\mathbf{A}$ = col$\mathbf{A}$, $\mathbf{A}^2 = \mathbf{O}$ because, partitioning $\mathbf{A}$ into its columns $\{a1, \dots, an\}$, $\mathbf{A}^2 = \mathbf{A}\begin{bmatrix}a1 & \dots & an\end{bmatrix} = \begin{bmatrix}\mathbf{A}a1 & \dots & \mathbf{A}an\end{bmatrix}$ which becomes $\begin{bmatrix}\mathbf{0} & \dots & \mathbf{0}\end{bmatrix} = \mathbf{O}$ since $\{a1, \dots, an\} \in$ col$\mathbf{A} \subseteq$ null$\mathbf{A}$.
$\Longleftarrow$ If $\mathbf{A}^2 = \mathbf{O}$, then $\mathbf{A}^2\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$, so $\mathbf{A}(\mathbf{Ax}) = \mathbf{0}$ for all $\mathbf{x}$. But this means that col$\mathbf{A} \subseteq$ null$\mathbf{A}$, since all $\mathbf{b}$ such that $\mathbf{Ax} = \mathbf{b}$ are solutions to $\mathbf{Ab} = \mathbf{0}$. As col$\mathbf{A}$ and null$\mathbf{A}$ are both subspaces of ${}^n\mathbb{R}$, if col$\mathbf{A} \subseteq$ null$\mathbf{A}$, we can only conclude that col$\mathbf{A}$ = null$\mathbf{A}$ if dimcol$\mathbf{A}$ = nullity$\mathbf{A}$ [Theorem VI, Chapter 6]. This is only possible if rank$\mathbf{A}$ = nullity$\mathbf{A}$, i.e., rank$\mathbf{A}$ = n/2.

Chapter 8

2. $1 + x = 1(1) + 1x$
2 + 3x = $2(1) + 3x$

Taking the transpose of the coefficients, we get $\mathbf{T} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

3. a) The standard basis of P2 is $\{1, x, x^2\}$.
$(1/2)x^2$ - $(1/2)x = 0(1) - (1/2)x + (1/2)x^2$
1 - x^2 =1(1) + $0x$ + (-1)x^2
$(1/2)x$ + $(1/2)x^2$ = 0 + $(1/2)x$ + $(1/2)x^2$

Taking the transpose of the coefficients, we get $\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$. Since rref$\mathbf{T} = \mathbf{I}$, E is a linearly independent set in $\mathbb{P}^2$ containing three vectors, and hence a basis.

5. $2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = c\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + d\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
where $\begin{bmatrix} c \\ d \end{bmatrix}$ are the coordinates in F. Solving for c and d, we get coordinates of $\begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}$.

Chapter 9

1. No
det$\mathbf{B}$ = 6det$\mathbf{A}$.

2. No
Consider $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3. $\det(2\mathbf{A}^{-1}) = 2^n \det\mathbf{A}^{-1} = 2^n/\det\mathbf{A}$ = -4
det$\mathbf{A} = -(2^{n-2})$
-4 = $\det\mathbf{A}^3 \det\mathbf{B}^{-1} = \det\mathbf{A}\det\mathbf{A}\det\mathbf{A}/\det\mathbf{B} = (2^n/(-4))^3/\det\mathbf{B}$
det$\mathbf{B} = 2^{3n-8}$

4. No
Consider $\mathbf{A} = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

5. Since $\mathbf{A}^2$ = -$\mathbf{I}$ and $\mathbf{A}$ is square,
$\det\mathbf{A}^2 = \det\mathbf{A}\det\mathbf{A}$ = det(-$\mathbf{I}$) = $(-1)^n$.
Thus, $\det\mathbf{A} = (-1)^{n/2}$.

7. $\Longrightarrow$ Let $\mathbf{Ag} = \begin{bmatrix} a_1 & b_1 & g_1 \\ a_2 & b_2 & g_2 \\ a_3 & b_3 & g_3 \end{bmatrix}$, where $g_i = -c_i$. Note that this represents matrices $\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$ and $\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$ in augmented form. Also let $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. For one unique solution to exist for any $\mathbf{g}$ in $\mathbf{Av} = \mathbf{g}$, it is required that rank$\mathbf{A}$ = rank$\mathbf{Ag} = n$, where n is the number of columns in $\mathbf{A}$. This means that rank$\mathbf{Ag}$ must be 2, i.e. that $\mathbf{Ag}$ is singular and thus must have determinant 0. If $\mathbf{Ag}$ has determinant 0, then multiplying the third column of $\mathbf{Ag}$ by -1 will also yield a matrix (which is the matrix considered in the question) with determinant 0.
$\Longleftarrow$ If the matrix considered in the question has determinant 0, then det$\mathbf{Ag} = 0$, so rank$\mathbf{Ag} = 2$ because if rank$\mathbf{Ag} \neq 2$ then rank$\mathbf{A} \neq 2$ by necessity which implies nonuniqueness of any solution (if it exists) to $\mathbf{Av} = \mathbf{g}$. If rank$\mathbf{Ag} = 2$, then rank$\mathbf{A}$ = rank$\mathbf{Ag} = n$ and we can thereby conclude that a unique solution (i.e. one single point) exists as the intersection between the three given lines for any g_1, g_2, g_3. Since this conclusion holds for any g_1, g_2, g_3, it also holds for any c_1, c_2, c_3.

9. No
Any transformation matrix $\mathbf{T}$ must be invertible, so det$\mathbf{T} \neq 0$ is required.

10. No
Consider $\mathbf{x} = \begin{bmatrix} 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \end{bmatrix}$.

11. Let $\{a_1, ..., a_n\}$ be the rows of $\mathbf{A}$.

Since the determinant is multilinear along the rows of $\mathbf{A}$, $\mu^n \det\mathbf{A} = \det\begin{bmatrix}\mu a_1 \\ \vdots \\ \mu a_n\end{bmatrix} = \det\mu\mathbf{A}$.

13. Yes
If $\mathbf{A}$ is symmetric, then $\mathbf{C}^T = \mathbf{C} = \text{adj}\mathbf{A}$.

14. Yes
Let $\mathbf{A} = \begin{bmatrix}a & b \\ c & d\end{bmatrix}$. Then $(\text{tr}\mathbf{A})\mathbf{I}$ - $\mathbf{A} = \begin{bmatrix}d & -b \\ -c & a\end{bmatrix} = \text{adj}\mathbf{A}$.

18. $\implies$ If $\mathbf{A}$ is invertible, $\mathbf{A}^{-1}\mathbf{A}\text{adj}\mathbf{A} = \mathbf{A}^{-1}(\det\mathbf{A})\mathbf{I}$, so $\text{adj}\mathbf{A} = \mathbf{A}^{-1}\det\mathbf{A}$. Since $\mathbf{A}^{-1}\det\mathbf{A}$ is always invertible if $\det\mathbf{A} \neq 0$, adj$\mathbf{A}$ must also be invertible.
$\impliedby$ If adj$\mathbf{A}$ is invertible, $\mathbf{A}\text{adj}\mathbf{A}(\text{adj}\mathbf{A})^{-1} = (\det\mathbf{A})\mathbf{I}(\text{adj}\mathbf{A})^{-1}$, so $\mathbf{A} = (\det\mathbf{A})(\text{adj}\mathbf{A})^{-1}$. If $\mathbf{A}$ is not invertible, $\det\mathbf{A} = 0$ and so the only solution to this equation is $\mathbf{A} = \mathbf{O}$. But it was given that $\mathbf{A} \neq \mathbf{O}$, so $\mathbf{A}$ must also be invertible.

Chapter 10

1. Yes
Since all of $\mathbf{A}$'s eigenvalues are 0, $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{O}$. Multiplying both sides by $\mathbf{S}$ on the left and then $\mathbf{S}^{-1}$ on the right, we get $\mathbf{A} = \mathbf{O}$.

2. Yes
$\mathbf{A}^k = \mathbf{O}$ implies that $\det\mathbf{A} = 0$, since $\det(\mathbf{A}^k) = (\det\mathbf{A})^k = 0$. But then $\mathbf{A}$ is not invertible, so null$\mathbf{A}$ = null$(\mathbf{A} - 0\mathbf{I})$ contains nonzero vectors. Since $\lambda = 0$ has nonzero eigenvectors, it is an eigenvalue of $\mathbf{A}$.

3. No
Since $c_{\mathbf{A}}(\lambda) = \lambda^2$ - $3\lambda + 2 = \lambda^2$ - $(\text{tr}\mathbf{A})\lambda + \det\mathbf{A}$, $\text{tr}\mathbf{A} = 3$.

4. Yes
If $\mathbf{x}$ is a solution to $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, then $(\mathbf{A} + \mathbf{I})\mathbf{x} = \lambda\mathbf{x} + \mathbf{x}$ and so the eigenvectors of $\mathbf{A}$ and $\mathbf{A} + \mathbf{I}$ are the same.

5. Yes
If $\mathbf{x}$ is a solution to $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}$ and so the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^2$ are the same.

7. Yes
Since $\mathbf{A}$ is diagonalizable, the sum of the dimensions of all its eigenspaces must be equal to n. Thus E_d, the eigenspace corresponding to $\lambda = d$, must be equal to n, and since $E_d = \text{null}(d\mathbf{I} - \mathbf{A})$, we have by the Rank-Nullity theorem that $\text{rank}(d\mathbf{I} - \mathbf{A}) = 0$.

10. Yes
If $\mathbf{A}$ does not have 0 as an eigenvalue, then $E_0 = \text{null}(\mathbf{A} - 0\mathbf{I}) = \text{null}\mathbf{A}$ must have a dimension of 0 since the eigenspace associated with $\lambda = 0$ contains no eigenvectors. nullity$\mathbf{A} = 0$ is equivalent to $\mathbf{A}$ being invertible, i.e., a product of elementary matrices.

12. If $\det\mathbf{A} = 0$, $\det(\mathbf{A}\text{adj}\mathbf{A}) = 0$.
If $\det\mathbf{A} \neq 0$, $\mathbf{A}$ is invertible, so $\text{adj}\mathbf{A} = \mathbf{A}^{-1}(\det\mathbf{A})\mathbf{I}$
$\det(\text{adj}\mathbf{A}) = \det(\mathbf{A}^{-1}(\det\mathbf{A})\mathbf{I}) = (\det\mathbf{A})^n/\det\mathbf{A} = (\det\mathbf{A})^{n-1}$

13. a) rank$\mathbf{A} = 1$, nullity$\mathbf{A} = 3$
b) Since $E_0 = \text{null}\mathbf{A}$, $\lambda = 0$ exists and its eigenspace has dimension 3.
c) Since $c_{\mathbf{A}}(\lambda) = \lambda^3(\lambda - 4)$, another eigenvalue of $\mathbf{A}$ is $\lambda = 4$.

17. a) $\mathbf{A}\text{adj}\mathbf{A} = (\det\mathbf{A})\mathbf{I} = (\det\mathbf{A}^T)\mathbf{I} = \mathbf{A}^T\text{adj}\mathbf{A}^T$
b) $\implies$ If $\mathbf{A}$ is invertible, $\det\mathbf{A} \neq 0$, so $\mathbf{A}^T\text{adj}\mathbf{A}^T = (\det\mathbf{A})\mathbf{I}$
$\mathbf{A}^T(\text{adj}\mathbf{A}^T/\det\mathbf{A}) = \mathbf{I}$ which menas $\mathbf{A}^T$ has an inverse, $\text{adj}\mathbf{A}^T/\det\mathbf{A}$.
$\impliedby$ If $\mathbf{A}^T$ is invertible, $\det\mathbf{A}^T \neq 0$, so $\mathbf{A}\text{adj}\mathbf{A} = (\det\mathbf{A}^T)\mathbf{I}$
$\mathbf{A}(\text{adj}\mathbf{A}/\det\mathbf{A}^T) = \mathbf{I}$ which means $\mathbf{A}$ has an inverse, $\text{adj}\mathbf{A}/\det\mathbf{A}^T$.

20. Note that at least two of the λ_i's must be equal or else there are more distinct eigenvalues than columns of $\mathbf{A}$. Since 2 linearly independent vectors are in the repeated eigenvalue's eigenspace, this eigenspace has dimension 2. But then the other eigenvalue is not an eigenvalue if it is different from the two λ_i's, since then its eigenspace would have dimension 0. Thus, all three λ_i's are equal (call it λ), and since λ's eigenspace has dimension 2, all $\mathbf{x} \in {}^2\mathbb{R}$ are solutions to the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.