Question 3

Since

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -\lambda & 1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & -\lambda \end{pmatrix}$$

it follows that $p_A(\lambda) = det(A-\lambda I) = (-1)^{1+1}(-\lambda)(-\lambda)^{n-1} + (-1)^{n+1}(1)1^{n-1}$ through Laplace expansion of the determinant along the first column (and using the fact that the determinant of triangular matrices is the product of the diagonal entries). If n odd, then $n-1$ and $n+1$ are even, so

$$p_A(\lambda) = (-\lambda)\lambda^{n-1} + 1 = 1 - \lambda^n$$

and the roots of $p_A(\lambda)$ (i.e. the eigenvalues of A) are the solutions to $\lambda^n = 1$. Likewise, if n even then $n-1$ and $n+1$ are odd, so

$$p_A(\lambda) = (-\lambda)(-\lambda)^{n-1} - 1 = \lambda^n - 1$$

and the eigenvalues of A are the solutions to $\lambda^n = 1$. Thus we conclude that the eigenvalues of A are nth roots of unity, i.e. solve

$$\lambda^n = 1$$

when $A \in M_{n\times n}(\mathbf{C})$.

For eigenvectors, first note that left-multiplying A with any column vector $x = \begin{pmatrix} x_0 \\ x_1 \\ ... \\ x_{n-1} \end{pmatrix}$ produces a vector wherein all entries of x are shifted upward by 1 row (with the top entry being cycled down to the bottom):

$$Ax = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_0 \end{pmatrix}$$

Since $\lambda^n = 1$, we observe that $v = \begin{pmatrix} \lambda^0 \\ \lambda^1 \\ ... \\ \lambda^{n-1} \end{pmatrix}$ satisfies $Av = \lambda v$ and is thus an eigenvector. Since the n roots of $\lambda^n = 1$ are $e^{\frac{2\omega\pi i}{n}}$, where $\omega = 0, 1, ..., n-1$, we conclude that the eigenvectors of v are v_ω s.t.

$$v_\omega = \begin{pmatrix} (e^{\frac{2\omega\pi i}{n}})^0 \\ (e^{\frac{2\omega\pi i}{n}})^1 \\ ... \\ (e^{\frac{2\omega\pi i}{n}})^{n-1} \end{pmatrix}$$

where $\omega = 0, 1, ..., n-1$.