Question 1

a) Letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbf{R}$, we have

$$Av_1 = \lambda_1 v_1 \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Av_2 = \lambda_2 v_2 \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

which yields the systems of equations $a - b = 1, -2a + b = 4$ and $c - d = -1, -2c + d = -2$. Solving these we obtain $a = -5, b = -6, c = 3, d = 4$ so that

$$A = \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix}$$

b) Suppose A is not unique, i.e. that $A_i, A_j \in M_{2\times 2}(\mathbf{R})$ exist s.t. $A_i \neq A_j$ and

$$A_i v_1 = \lambda_1 v_1, A_j v_1 = \lambda_1 v_1, A_i v_2 = \lambda_2 v_2, A_j v_2 = \lambda_2 v_2$$

Then $(A_i - A_j)v_1 = (\lambda_1 - \lambda_1)v_1 = (A_i - A_j)v_2 = (\lambda_2 - \lambda_2)v_2 = 0$, so $v_1, v_2 \in \text{null}(A_i - A_j)$. But, since v_1 and v_2 are linearly independent vectors, this means dim null$(A_i - A_j) = 2$, i.e. rank$(A_i - A_j) = 0$ by the rank-nullity theorem (since dim $\mathbf{R}^2 =$ dim null$(A_i - A_j)$ + rank$(A_i - A_j)$). This is only possible if $A_i - A_j = O$, i.e. $A_i = A_j$. Thus a contradiction arises from our assumption that $A_i \neq A_j$ and it follows that A must be unique.

c) Plugging in A, we find that

$$A^3 - A^2 + 3I = \begin{pmatrix} -21 & -24 \\ 12 & 15 \end{pmatrix}$$

Denote this matrix Q. The eigenvalues are the roots of $det(Q - \lambda I)$:

$$0 = (-21 - \lambda)(15 - \lambda) - (-24)(12) \implies \lambda = 3, -9$$

Eigenvectors of Q are $v \neq 0$ such that $Qv = 3v$ or $Qv = -9v$. Let v satisfying the former case be v_1 and v satisfying the latter be v_2. Then $v_1 \in \text{null}(Q - 3I)$ and $v_2 \in \text{null}(Q + 9I)$. Letting $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$, we find that the systems of equations $-24a - 24b = 0$, $12a + 12b = 0$ and $-12c - 24d = 0$, $12c + 24d = 0$ must be satisfied. Then $a = -b$ and $c = -2d$, so taking b and d respectively as the free variables we have

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$