Question 4

a) The characteristic polynomial $p_A(\lambda)$ is

$$det(A-\lambda I) = det\begin{pmatrix} 3-\lambda & -9 & 1 \\ 2 & -8-\lambda & -2 \\ 3 & -3 & 1-\lambda \end{pmatrix}$$

which, using the Laplace expansion along the third column, becomes $p_A(\lambda) = (-1)^{3+1}(-6-3(-8-\lambda)) + (-1)^{3+2}(-2)((3-\lambda)(-3)+27) + (-1)^{3+3}(1-\lambda)((3-\lambda)(-8-\lambda)+18) = -(\lambda+6)(\lambda+2)(\lambda-4)$. The roots of this polynomial are $\lambda_1 = -6, \lambda_2 = -2, \lambda_3 = 4$.

b) To find the corresponding eigenvectors we must find vectors in $\mathrm{null}(A-\lambda_i I)$ for each $i \in \{1,2,3\}$. For λ_1, we have

$$\mathrm{null}(A-\lambda_1 I) = \mathrm{null}\begin{pmatrix} 9 & -9 & 1 \\ 2 & -2 & -2 \\ 3 & -3 & 7 \end{pmatrix} = \mathrm{null}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

after performing row reduction; hence we obtain the system of equations $x_1 = x_2, x_3 = 0$. Taking x_2 as the free variable, we find that an eigenvector corresponding to λ_1 is $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. For λ_2, we have

$$\mathrm{null}(A-\lambda_2 I) = \mathrm{null}\begin{pmatrix} 5 & -9 & 1 \\ 2 & -6 & -2 \\ 3 & -3 & 3 \end{pmatrix} = \mathrm{null}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

after performing row reduction; hence we obtain the system of equations $x_1 = -2x_3$ and $x_2 = -x_3$. Taking x_3 as the free variable, we find that an eigenvector corresponding to λ_2 is $v_2 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$. For λ_3, we have

$$\text{null}(A - \lambda_3 I) = \text{null} \begin{pmatrix} -1 & -9 & 1 \\ 2 & -12 & -2 \\ 3 & -3 & -3 \end{pmatrix} = \text{null} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after performing row reduction, hence we obtain the system of equations $x_1 = x_3$, $x_2 = 0$. Taking x_3 as the free variable, we find that an eigenvector corresponding to λ_3 is $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

c) The columns of a diagonalizing matrix C (where $C^{-1}AC = D$) are the eigenvectors of A in the order (left-to-right) that the eigenvalues appear in the diagonal matrix D; hence

$$C = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

is the diagonalizing matrix corresponding to the diagonal matrix

$$D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$