Question 2

We require an $N \in \mathbf{N}$ s.t.

$$b_n = |\frac{a_1 + a_2 + ... + a_n}{n} - a| < \epsilon$$

for all $n \geq N$ and $\epsilon > 0$. Note that $a = \frac{na}{n}$, so the inequality simplifies to

$$b_n = |\frac{(a_1 - a) + (a_2 - a) + ... + (a_n - a)}{n}| < \epsilon$$

and by the triangle inequality, we get

$$b_n \leq |\frac{a_1 - a}{n}| + |\frac{a_2 - a}{n}| + ... + |\frac{a_n - a}{n}| =$$

$$\frac{1}{n}(|a_1 - a| + |a_2 - a| + ... + |a_{\mathcal{N}} - a| + |a_{\mathcal{N}+1} - a| + ... + |a_n - a|) = c_n$$

with the last simplification coming because $\lim_{n\to\infty} a_n = a$, i.e. there exists $\mathcal{N} \in \mathbf{N}$ s.t. $|a_n - a| < \epsilon$ for all $n \geq \mathcal{N}, \epsilon > 0$. All convergent sequences must be bounded by some finite number; as such a_n is bounded by some finite M and it follows that we can bound c_n by choosing

$$C = \frac{|M - a|(\mathcal{N})}{n} + \frac{(n - \mathcal{N})\frac{\epsilon}{k}}{n} \geq c_n$$

where $k > 1$ is an arbitrary constant. Observe we can select $\mathfrak{N} \in \mathbf{N}$ s.t.$\epsilon(1 - \frac{1}{k}) > \frac{|M-a|(\mathcal{N})}{\mathfrak{N}}$ because $\mathcal{N}, M, a$ are fixed. Now consider $\mathcal{M} = \text{MAX}(\mathcal{N}, \mathfrak{N})$. For all $n \geq \mathcal{M}$, $b_n \leq c_n \leq C = \frac{|M-a|(\mathcal{N})}{n} + \frac{(n-\mathcal{N})\frac{\epsilon}{k}}{n} < \epsilon(1 - \frac{1}{k}) + \frac{\epsilon}{k} = \epsilon \forall \epsilon > 0$, which means $N = \mathcal{M}$ meets the requirements we imposed at the beginning of this proof. Hence we have proven that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = a$$

by using the definition of the limit.