Question 1

a) Notice that

$$a_n = \frac{n!}{n^n} = \frac{(n)(n-1)(n-2)...(2)(1)}{(n)(n)(n)...(n)(n)} \leq \frac{1}{n}$$

due to each $n, n-1$, $n-2$, ..., 1 in the numerator being smaller than or equal to the corresponding n in the denominator. Since $0 \leq a_n \leq \frac{1}{n} \forall n \in \mathbf{N}$, and $\lim_{n\to\infty} \frac{1}{n} = 0$ (result from lecture), we can apply the squeeze theorem:

$$0 = \lim_{n\to\infty} 0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence $\lim_{n\to\infty} a_n = 0$.

b) Multiply by 1 and simplify:

$$a_n = \frac{(\sqrt{n^2+4n-3}-n)(\sqrt{n^2+4n-3}+n)}{\sqrt{n^2+4n-3}+n} = \frac{n^2+4n-3-n^2}{\sqrt{n^2+4n-3}+n}$$

Multiply by 1 again and simplify further:

$$a_n = \frac{(4n-3)\frac{1}{n}}{(\sqrt{n^2+4n-3}+n)\frac{1}{n}} = \frac{4-\frac{3}{n}}{\frac{\sqrt{n^2+4n-3}}{n}+1} = \frac{4-\frac{3}{n}}{\sqrt{\frac{n^2+4n-3}{n^2}}+1}$$

Using the algebraic limit laws of addition and multiplication,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4-\frac{1}{n}}{\sqrt{1+\frac{4}{n}-\frac{3}{n^2}}+1} = \frac{\lim_{n\to\infty}(4-\frac{3}{n})}{\lim_{n\to\infty}(\sqrt{1+\frac{4}{n}-\frac{3}{n^2}}+1)}$$

$$= \frac{\lim_{n\to\infty}(4) - 3\lim_{n\to\infty}\frac{1}{n}}{\sqrt{\lim_{n\to\infty} 1 + 4\lim_{n\to\infty}\frac{1}{n} - 3(\lim_{n\to\infty}\frac{1}{n})(\lim_{n\to\infty}\frac{1}{n})} + \lim_{n\to\infty} 1}$$

and by the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ (result from lecture), we have

$$\lim_{n\to\infty} a_n = \frac{4-0}{1+1} = 2$$

Note that when we took the limit under the square root this does not follow directly from the algebraic limit laws, but nevertheless is a valid technique for taking limits of sequences in this context due to the fact that the quantity under the square root $(1 + \frac{4}{n} - \frac{3}{n^2})$ is always ≥ 0 for all n.