ENGG 5501: Foundations of Optimization	**2018–19 First Term**
Homework Set 1	
Instructor: Anthony Man–Cho So	Due: September 19, 2018

INSTRUCTIONS: Problems 1 and 2 are compulsory. The remaining problems are for practice and will not be graded.

Problem 1 (10pts). In Section 2.2 of Handout 1 we motivated the following sparse data fitting problem:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & \|x\|_0 \le K, \\ & x \in \mathbb{R}^n. \end{array} \tag{S}$$

Suppose we are given a constant $M > 0$ such that $\|x^*\|_\infty \le M$ for some optimal solution x^* to (S). Give a formulation of problem (S) that involves only linear and binary constraints. Justify your answer.

Problem 2 (20pts).

(a) **(10pts).** Let $S \subseteq \mathbb{R}$ be a set with the property that $(x + y)/2 \in S$ whenever $x, y \in S$. Is it true that S must be convex? Justify your answer.

(b) **(10pts).** Show that the set $S = \{x \in \mathbb{R}^n_+ : x_1 x_2 \cdots x_n \ge 1\}$ is convex. You may use the fact that for any $a, b \ge 0$ and $0 \le \theta \le 1$, we have $a^\theta b^{1-\theta} \le \theta a + (1 - \theta)b$.

Problem 3. Let $S = \left\{x \in \mathbb{R}^n : x^T Ax + b^T x + c \le 0\right\}$, where $A \in \mathcal{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$ are given.

(a) Show that S is convex if $A \succeq \mathbf{0}$. Is the converse true? Explain.

(b) Let $H = \left\{x \in \mathbb{R}^n : g^T x + h = 0\right\}$, where $g \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $h \in \mathbb{R}$. Show that $S \cap H$ is convex if $A + \lambda gg^T \succeq \mathbf{0}$ for some $\lambda \in \mathbb{R}$.

Problem 4.

(a) Consider the hyperplane $H(s, c) = \{x \in \mathbb{R}^n : s^T x = c\}$, where $s \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $c \in \mathbb{R}$ are given. Let $x \in \mathbb{R}^n$ be arbitrary. Find a formula for $\Pi_{H(s,c)}(x)$ in terms of s and c and prove its correctness.

(b) Consider the space $\mathcal{S}^n$ of $n \times n$ real symmetric matrices equipped with the inner product $\bullet$, where

$$A \bullet B = \sum_{i,j=1}^n A_{ij} B_{ij} \quad \text{for any } A, B \in \mathcal{S}^n.$$

Let $A \in \mathcal{S}^n$ be arbitrary and $A = U\Lambda U^T$ be its spectral decomposition. Prove that $\Pi_{\mathcal{S}^n_+}(A) = U\Lambda^+ U^T$, where Λ^+ is the $n \times n$ diagonal matrix given by

$$\Lambda^+_{ii} = \max\{\Lambda_{ii}, 0\} \quad \text{for } i = 1, \ldots, n.$$