ENGG 5501: Foundations of Optimization	**2018–19 First Term**
Homework Set 3	
Instructor: Anthony Man–Cho So	Due: October 12, 2018

INSTRUCTIONS: Problems 1 and 2 are compulsory. The remaining problems are for practice and will not be graded.

Problem 1 (15pts).

(a) **(5pts).** Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(x) = \begin{cases} x \ln x & \text{if } x > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Show that f is convex and compute f^*, the conjugate of f. Show all your work.

(b) **(10pts).** Let $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ be convex differentiable functions and define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \max_{i \in \{1,\ldots,m\}} f_i(x)$. Show that

$$\partial f(x) = \operatorname{conv} \{\nabla f_i(x) : f_i(x) = f(x)\}.$$

Problem 2 (15pts).

(a) **(5pts).** Let $A \in \mathbb{R}^{m \times n}$ be given. Use the Farkas lemma to show that exactly one of the following systems has a solution:

$$\begin{array}{ll} \text{(I)} & Ax \geq \mathbf{0},\ Ax \neq \mathbf{0}. \\ \text{(II)} & A^T y = \mathbf{0},\ y > \mathbf{0}. \end{array}$$

(b) **(10pts).** Consider the following LP:

$$\begin{array}{ll} \text{minimize} & -3x_1 + x_2 + 3x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 - x_3 + x_4 = 0, \\ & 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9, \\ & x_1 - x_2 + 2x_3 - x_4 = 6, \\ & x \geq \mathbf{0}. \end{array} \tag{1}$$

Let $\bar{x} = (1, 1, 3, 0)$. Determine whether $\bar{x}$ is optimal for Problem (1) by considering the dual of Problem (1).

Problem 3. Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, consider the polyhedron $P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0}\}$. Suppose that $P \neq \emptyset$. We say that P contains a *recession direction* $d \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ if for any $x_0 \in P$, we have $\{x \in \mathbb{R}^n : x = x_0 + \lambda d,\ \lambda \geq 0\} \subset P$. Show that the following statements are equivalent:

(i) P contains a recession direction $d \in \mathbb{R}^n$.

(ii) There exists a vector $d \in \mathbb{R}^n$ satisfying

$$Ad = \mathbf{0}, \;\; d \geq \mathbf{0}, \;\; d \neq \mathbf{0}.$$

(iii) P is unbounded.

Problem 4. Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^n$ be given. Define

$$C = \left\{ x \in \mathbb{R}^n : Ax \geq \mathbf{0} \right\}.$$

Suppose that $\mathbf{0}$ is a basic feasible solution of C. Consider the following LP:

$$v^* = \min_{x \in C} c^T x.$$

Show that $v^* = -\infty$ iff there exists a $d \in C \setminus \{\mathbf{0}\}$ such that there are $n-1$ linearly independent active constraints at d and $c^T d < 0$.