Derivation of the modified KdV equation from the full velocity difference (FVD) model.

Let us consider the FVD model,

$$\frac{d^2\Delta x_n(t)}{dt^2}=a\left[V_F\left(\Delta x_n(t)\right)-\frac{d\Delta x_n(t)}{dt}\right]+\lambda.\left[\frac{d\Delta x_{n+1}(t)}{dt}-\frac{d\Delta x_n(t)}{dt}\right] \tag{1}$$

Equation (1) leads us to the reductive perturbation method. Around the critical point (a_c,h_c), a small positive scaling parameter ε is introduced. Let us then define the slow scales X and T [44] [45], where the space variable n and time variable t are transformed as follows:

$X=\varepsilon(n+bt)$ and $T=\varepsilon^3 t$ with $0<\varepsilon\le 1,$ (2)

Where b is a constant to be determined. The headway distance $\Delta x_n(t)$ can be defined as follows:

$$\Delta x_n(t)=h_c+\varepsilon R(X,T) \tag{3}$$

We know, $\Delta x_{n+1}(t)=h_c+\varepsilon R+\frac{\varepsilon^2}{1}\partial_X R+\frac{\varepsilon^3}{2}\partial_X^2 R+\frac{\varepsilon^4}{6}\partial_X^3 R+\frac{\varepsilon^5}{24}\partial_X^4 R$ (4)

By expanding each term in equation (1) to the fifth order of ε we get,

$$\frac{d\Delta x_n(t)}{dt}=b\varepsilon^2\partial_X R+\varepsilon^4\partial_T R \tag{5}$$

$$\frac{d^2\Delta x_n(t)}{dt^2}=b\varepsilon^3\partial_X^2 R+b\varepsilon^5\partial_X\partial_T R+b\varepsilon^5\partial_T\partial_X R+\varepsilon^7\partial_T^2 R=b\varepsilon^3\partial_X^2 R+2b\varepsilon^5\partial_X\partial_T R \tag{6}$$

$$\frac{d\Delta x_{n+1}(t)}{dt}=\varepsilon^2 b\partial_X R+\varepsilon^3 b\partial_X^2 R+\varepsilon^4\left[\frac{b}{2}\partial_X^3 R+\partial_T R\right]+\varepsilon^5\left[\frac{b}{6}\partial_X^4 R+\partial_X\partial_T R\right] \tag{7}$$

$$V_F\left(\Delta x_n(t)\right)=V_F(h_c)+\varepsilon V_F'(h_c)R+\frac{\varepsilon^3}{6}V_F'''(h_c)R^3 \tag{8}$$

$$\begin{aligned}V_F\left(\Delta x_{n+1}(t)\right)&=V_F(h_c)+\varepsilon V_F'(h_c)R+\varepsilon^2 V_F'(h_c)\partial_X R+\varepsilon^3\left(\frac{V_F'(h_c)}{2}\partial_X^2 R+\frac{V_F'''(h_c)}{6}R^3\right)+\\&\varepsilon^4\left(\frac{V_F'(h_c)}{6}\partial_X^3 R+\frac{V_F'''(h_c)}{6}\partial_X R^3\right)+\varepsilon^5\left(\frac{V_F'(h_c)}{24}\partial_X^4 R+\frac{V_F'''(h_c)}{12}\partial_X^2 R^3\right)\end{aligned} \tag{9}$$

$$\text{Where,}\, V_F'=V_F'(h_c)=\left.\frac{dV_F\left(\Delta x_n(t)\right)}{d\left(\Delta x_n(t)\right)}\right|_{\Delta x_n(t)=h_c},\ V_F'''=V_F'''(h_c)=\left.\frac{d^3V_F\left(\Delta x_n(t)\right)}{d\left(\Delta x_n(t)\right)^3}\right|_{\Delta x_n(t)=h_c} \tag{10}$$

$$\begin{aligned}V_F\left(\Delta x_{n+1}(t)\right)-V_F\left(\Delta x_n(t)\right)&=\varepsilon^2 V_F'(h_c)\partial_X R+\varepsilon^3\frac{V_F'(h_c)}{2}\partial_X^2 R+\varepsilon^4\left(\frac{V_F'(h_c)}{6}\partial_X^3 R+\frac{V_F'''(h_c)}{6}\partial_X R^3\right)\\&+\varepsilon^5\left(\frac{V_F'(h_c)}{24}\partial_X^4 R+\frac{V_F'''(h_c)}{12}\partial_X^2 R^3\right)\end{aligned} \tag{11}$$

$$\Rightarrow\frac{d\Delta x_{n+1}(t)}{dt}-\frac{d\Delta x_n(t)}{dt}=\varepsilon^3 b\partial_X^2 R+\varepsilon^4\frac{b}{2}\partial_X^3 R+\varepsilon^5\frac{b}{6}\partial_X^4 R+\varepsilon^5\partial_X\partial_T R \tag{12}$$

We next substitute Equations (2) to (12) into Equation (1), and then, we expand the equation using a Taylor expansion of ε up to the fifth order. We thus obtain the following nonlinear partial differential equation:

$$\varepsilon^3\frac{b^2}{a}\partial_X^2R+\frac{2b}{a}\varepsilon^5\partial_X\partial_TR=\left(\varepsilon^2V_F'\partial_XR+\frac{\varepsilon^3}{2}V_F'\partial_X^2R+\frac{\varepsilon^4}{6}\left(V_F'\partial_X^3R+V_F'''\,\partial_XR^3\right)+\frac{\varepsilon^5}{24}\left(V_F'\partial_X^4R+2V_F'''\,\partial_X^2R^3\right)\right)$$

$$-b\varepsilon^2\partial_XR-\varepsilon^4\partial_TR+\frac{\lambda}{a}\left(\varepsilon^3b\partial_X^2R+\varepsilon^4\frac{b}{2}\partial_X^3R+\varepsilon^5\frac{b}{6}\partial_X^4R+\varepsilon^5\partial_X\partial_TR\right)$$

$$\Rightarrow\varepsilon^2\left[b-b\right]\partial_XR+\varepsilon^3\left[\frac{b^2}{a}-\frac{1}{2}V_F'-\frac{\lambda b}{a}\right]\partial_X^2R+\varepsilon^4\left[\partial_TR-\left\{\frac{1}{6}V_F'+\frac{\lambda b}{2a}\right\}\partial_X^3R-\frac{1}{6}V_F'''\,\partial_XR^3\right]$$

$$+\varepsilon^5\left[\frac{2b-\lambda}{a}\partial_X\partial_TR-\left\{\frac{1}{24}\left(V_F'+2V_F'''\,\partial_X^2R^3\right)+\frac{\lambda b}{6a}\right\}\partial_X^4R-\frac{1}{12}V_F'''\,\partial_X^2R^3\right]=0 \tag{13}$$

Now, let us introduce $a_c=a\left(1+\varepsilon^2\right)$ as the neighbor to the critical point $\left(a_c,h_c\right)$ and consider $b=V_F'$. The terms in Equation (13) containing second and third orders of ε should be neglected; this allows us to simplify the equation as follows:

$$\Rightarrow\varepsilon^3\left[\frac{b^2}{a}-\frac{\lambda b}{a}-\frac{1}{2}V_F'\right]\partial_X^2R+\varepsilon^4\left[\partial_TR-g_1\partial_X^3R+g_2\partial_XR^3\right]+\varepsilon^5\left[g_3\partial_X^2R+g_4\partial_X^4R+g_5\partial_X^2R^3\right]=0 \tag{14}$$

where the values of g_i are given in Table 1.

To derive the regularized equation, the following transformations are applied to Equation (14):

$$T=\frac{1}{g_1}T' \text{ and } R=\sqrt{\frac{g_1}{g_2}}R'\left(X,T'\right), \tag{15}$$

Here,

$$T=\frac{1}{g_1}T'R=\sqrt{\frac{g_1}{g_2}}R'\left(X,T'\right),$$

$$\partial_TR=\frac{\partial R}{\partial T}=\frac{\partial R}{\partial T'}.\frac{\partial T'}{\partial R}=\sqrt{\frac{g_1}{g_2}}\frac{\partial R'}{\partial T'}g_1=\frac{g_1\sqrt{g_1}}{\sqrt{g_2}}\partial_{T'}R',\ g_1\partial_X^3R=g_1\frac{\partial}{\partial X}\left(\sqrt{\frac{g_1}{g_2}}R'\right)=\frac{g_1\sqrt{g_1}}{\sqrt{g_2}}\partial_X^3R'$$

$$g_2\partial_XR^3=g_2\frac{\partial}{\partial X}\left(\sqrt{\frac{g_1}{g_2}}R'\right)^3=\frac{g_1\sqrt{g_1}}{\sqrt{g_2}}\partial_XR'^3,\ g_3\partial_X^2R=g_3\frac{\partial^2}{\partial X}\left(\sqrt{\frac{g_1}{g_2}}R'\right)=\frac{g_3\sqrt{g_1}}{\sqrt{g_2}}\partial_X^2R'$$

$$g_4\partial_X^4R=g_4\frac{\partial^4}{\partial X}\left(\sqrt{\frac{g_1}{g_2}}R'\right)=\frac{g_4\sqrt{g_1}}{\sqrt{g_2}}\partial_X^4R',\ g_5\partial_X^2R^3=g_5\frac{\partial^2}{\partial X}\left(\sqrt{\frac{g_1}{g_2}}R'\right)^3=\frac{g_1g_5\sqrt{g_1}}{g_2\sqrt{g_2}}\partial_X^2R'^3$$

The standard mKdV equation with the correction term $\mathrm{O}\left(\varepsilon\right)$ is given as follows from (14):

$$\partial_{T'}R'\left(X,T'\right)-\partial^3{}_XR'\left(X,T'\right)+\partial_XR'^3\left(X,T'\right)+\varepsilon M\left[R'\left(X,T'\right)\right]=0, \tag{16}$$

Where $M\left[R'\left(X,T'\right)\right]=\left[\frac{g_3}{g_1}\partial^2{}_XR'+\frac{g_4}{g_1}\partial^4{}_XR'+\frac{g_5}{g_2}\partial^2{}_XR'^3\right]$.

Table 1. The coefficients g_i of the FVD model.

$g_1=\frac{1}{6}V_F'+\frac{\lambda b}{2a}$	$g_2=-\frac{1}{6}V_F'''$	$g_3=\frac{1}{2}V_F'$
$g_4=-\frac{1}{24}\left(V_F'+2V_F'''\,\partial_X^2R^3\right)+\frac{\lambda b}{6a}$		$g_5=-\frac{1}{12}V_F'''$