COMP-170: Homework #1

Ben Tanen - January 29, 2017

Problem 5

Prove, by the contrapositive method, that if u is a least upper bound for a set S of real numbers, then $\forall$ real numbers $\epsilon > 0$, $\exists$ an element $x \in S$, such that $x > u - \epsilon$.

Recall that the definition of least upper bound:

Given a non-empty set of real numbers, S, u is a *least upper bound* if:

- $u \geq x$, $\forall x \in S$
- $\forall k$, $k \in \mathbb{R}$, if k is an upper bound for S, $k \geq u$

$* * *$

Let u be a real number. We want to show that if u is a least upper bound for a set S of real numbers, then there exists an element $x \in S$ such that $x > u - \epsilon$ for all $\epsilon > 0$. In order to prove this, we will use the contrapositive. More specifically, we will show that if, for all $x \in S$, there exists some real number $\epsilon > 0$ such that $x \leq u - \epsilon$, then u is not a least upper bound.

Let's define our set $S = \{x_0, x_1, \ldots, x_n\}$. Given some real number u, let's suppose that we have found some real number $\epsilon > 0$ for each of the elements of S such that $x_i \leq u - \epsilon_i$ for $0 \leq i \leq n$. Given that we can find some $\epsilon_i > 0$ such that $u - \epsilon_i \geq x_i$ for all $0 \leq i \leq n$, we can see that $u > x$ for all $x \in S$. This means u is an upper bound for S.

Without loss of generality, let x_n be the largest element of S such that $x_n \geq x$ for all $x \in S$. Given the statements above, we know that there exists some real ϵ_n such that $u - \epsilon_n \geq x_n$. Now, let's define $u_0 = u - \frac{\epsilon_n}{2}$. Through this definition, we know that $u_0 \geq u - \epsilon_n \geq x_n$ and $x_n \geq x$ for all $x \in S$. As a result, we know $u_0 \geq x$ for all $x \in S$ so we know u_0 is an upper bound for S. Since $u_0 = u - \epsilon_n < u$, we can see that $u_0 < u$, showing that there exists some upper bound of S that is less than u. As a result, we know that u is not a least upper bound of S.

We can use the above statement as the contrapositive to also show that if u is a least upper bound for a set S of real numbers, then there exists an element $x \in S$ such that $x > u - \epsilon$ for all $\epsilon > 0$. $\boxtimes$