COMP-170: Homework #4

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Problem 2

Let $L = \{\langle M_1, M_2 \rangle \mid \text{ Everything } M_1 \text{ accepts } M_2 \text{ doesn't accept}\}$. Prove L is undecidable.

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To show L is undecidable, we will claim and prove $\overline{A_{TM}} \leq_m L$. To do this, we will construct a function $f : \Sigma^*_{\overline{ATM}} \to \Sigma^*_L$ such that $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Once we show $\overline{A_{TM}} \leq_m L$, since $\overline{A_{TM}}$ is undecidable, we will then be able to show that L is also undecidable.

First, we claim $\overline{A_{TM}} \leq_m L$. To prove this claim, we will first show how $\overline{A_{TM}}$ and L relate to each other (using the "Tony Square").

	$\overline{A_{TM}}$	L
IN	M doesn't accept w	$\forall x \in L(M_1)$, $x \notin L(M_2)$
OUT	M accepts w	$\exists x \in L(M_1)$ where $x \in L(M_2)$

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{\overline{ATM}} \to \Sigma^*_L$ be defined as follows:

f on input $\langle M, w \rangle$ outputs $\langle M_1, M_2 \rangle$, where M_1 and M_2 are defined as:

> M_1 **on input** x**:**
>
> > $ACCEPT$
>
> M_2 **on input** x**:**
>
> > If $x = 1$, run M on w
> >
> > > If M accepts w, $ACCEPT$
> > >
> > > Else, $LOOP$
> >
> > Else, $LOOP$

Given our definition of f, we claim that f is computable. Because M_1 and M_2 are valid Turing machines and f is a finite function, we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Consider the following two cases:

1. Suppose $\langle M, w \rangle \in \overline{A_{TM}}$ such that M does not accept w. We can see that M_1 accepts everything, regardless of $\langle M, w \rangle$. We can also see that, because M does not accept w,

M_2 will loop on everything. Thus, since $L(M_2) = \varnothing$, we can see that everything M_1 accepts, M_2 doesn't accept. Thus $\langle M_1, M_2 \rangle \in L$.

2. Suppose $\langle M, w \rangle \notin \overline{A_{TM}}$ such that M accepts w. We can again see that M_1 accepts everything. However, this time, because M accepts w, M_2 loops on everything except for 1, which it accepts. Thus, $L(M_2) = \{1\}$. Because M_1 accepts everything, we know $1 \in L(M_1)$ and $1 \in L(M_2)$, so we can see there does exist an input that both M_1 and M_2 accept. Thus, $\langle M_1, M_2 \rangle \notin L$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Therefore, we can see that $\overline{A_{TM}} \leq_m L$. We know that $\overline{A_{TM}}$ is undecidable (since A_{TM} is undecidable) so $\overline{A_{TM}} \leq_m L$ implies L is also undecidable. $\boxtimes$