# COMP-170: Homework #4

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## Problem 5

Let $L = \{\langle M \rangle \mid M \text{ only accepts odd inputs } \}$. Prove $L \leq_m \overline{A_{TM}}$

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We want to prove $L \leq_m \overline{A_{TM}}$. To do this, we will construct a function $f : \Sigma_L^* \to \Sigma_{\overline{ATM}}^*$ such that $x \in L \Leftrightarrow f(x) \in \overline{A_{TM}}$. Once we show $f$ satisfies this criteria, we can see that $L \leq_m \overline{A_{TM}}$.

First, we will claim $L \leq_m \overline{A_{TM}}$. To prove this claim, we will first show how $L$ and $\overline{A_{TM}}$ relate to each other (using the "Tony Square").

| | $L$ | $\overline{A_{TM}}$ |
|---|---|---|
| IN | $M$ accepts only odd inputs | $M'$ doesn't accept $w$ |
| OUT | $M$ doesn't just accept odd inputs | $M'$ accepts $w$ |

Now that we've built our "Tony Square", we can define our function. In this function, we will use the recognizer $R$ for $\overline{L}$ that was constructed for problem 4. Let $f : \Sigma_L^* \to \Sigma_{\overline{ATM}}^*$ be defined as follows:

$f$ on input $\langle M \rangle$ outputs $\langle M', w \rangle$, where $M'$ and $w$ are defined as:

> **$M'$ on input $x$:**
>
> > Run $R$ on $\langle M \rangle$
> >
> > > If $R$ accepts $\langle M \rangle$, $ACCEPT$
>
> $w = 1$ // The specific value of $w$ doesn't matter

Given our definition of $f$, we claim that $f$ is computable. Because $M'$ is a valid Turing machine and $f$ is a finite function, we can see that $f$ is indeed computable.

Now, we can go through two cases to show that $f$ correctly satisfies $x \in L \Leftrightarrow f(x) \in \overline{A_{TM}}$. Consider the following two cases:

1. Suppose $\langle M \rangle \in L$ such that $M$ only accept odd inputs. Because $\langle M \rangle \in L$, we know that $\langle M \rangle \notin \overline{L}$ so we know that $R$ will not accept $\langle M \rangle$. Thus, because $R$ won't accept $\langle M \rangle$, we can see that $M'$ won't accept $w$. Thus, if $\langle M \rangle \in L$, $\langle M', w \rangle \in \overline{A_{TM}}$.
2. Suppose $\langle M \rangle \notin L$ such that $M$ does not just accept odd inputs (i.e. $M$ accepts at least one even input). Because $\langle M \rangle \notin L$, we know that $\langle M \rangle \in \overline{L}$. Thus, we know $R$ (as a recognizer of $\overline{L}$) would accept $\langle M \rangle$. This would then cause $M'$ to accept $w$. Thus, we can see that if $\langle M \rangle \notin L$, then $\langle M', w \rangle \notin \overline{A_{TM}}$.

Given these two cases, we can thus see that $f$ does indeed satisfy the claim $x \in L \Leftrightarrow f(x) \in \overline{A_{TM}}$. Therefore, we can see that $L \leq_m \overline{A_{TM}}$. $\boxtimes$