COMP-170: Homework #3

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Problem 3

Let M_2 be the machine that decides the language L_2. Further, let R_1, and R_3 be the machines that recognize languages L_1, and L_3 respectively. Prove that $(L_1 - L_2) \cup L_3$ is recognizable.

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We want to prove that language $(L_1 - L_2) \cup L_3$ is recognizable, given R_1 and R_3 are recognizable while L_2 is decidable. To do this, we will use proof by construction. More specifically, we will construct a machine R that can recognize $(L_1 - L_2) \cup L_3$ using a combination of R_1, R_3, and M_2 (the recognizers for L_1 and L_3 and the decider for L_2 respectively).

Given input x, we would say $x \in (L_1 - L_2) \cup L_3$ if $x \in L_1$ and $x \notin L_2$ or if $x \in L_3$. Given this, let's define R as follows:

> R **on input** x**:**
>
> For $i = 1 \to \infty$
>
> Run R_1 on x for i steps and run R_3 on x for i steps
>
> If R_3 accepts x in i steps, $ACCEPT$
>
> If R_1 accepts x in i steps, run M_2 on x
>
> If M_2 rejects x, $ACCEPT$
>
> END

We will now claim that R is a recognizer of $(L_1 - L_2) \cup L_3$. To show this, consider the following cases:

1. Let x be any element such that $x \in L_1$, $x \in L_2$, and $x \in L_3$. We can see that x should be accepted by R because $x \in L_3$ so $x \in (L_1 - L_2) \cup L_3$. Because $x \in L_3$, we know that R_3 will accept x after some finite number of steps, which will cause R to correctly accept x.

2. Let x be any element such that $x \notin L_1$, $x \in L_2$, and $x \in L_3$. We can see that, like case 1, because $x \in L_3$, we know that R_3 will eventually accept x, which will cause R to correctly accept x.

3. Let x be any element such that $x \in L_1$, $x \notin L_2$, and $x \in L_3$. We can see that, like case 1, because $x \in L_3$, we know that R_3 will eventually accept x, which will cause R to correctly accept x.

4. Let x be any element such that $x \notin L_1$, $x \notin L_2$, and $x \in L_3$. We can see that, like case 1, because $x \in L_3$, we know that R_3 will eventually accept x, which will cause R to correctly accept x.

5. Let x be any element such that $x \in L_1$, $x \in L_2$, and $x \notin L_3$. We can see that x should not be accept by R because $x \in L_2$ and $x \notin L_3$ so $x \notin (L_1 - L_2) \cup L_3$. We can see that R will only accept x if M_2 rejects x or R_3 eventually accepts x. Since $x \notin L_3$, R_3 will never accept x and because $x \in L_2$, M_2 will accept x. Thus, we can see that R would correctly not accept x.

6. Let x be any element such that $x \notin L_1$, $x \in L_2$, and $x \notin L_3$. We can see that, like case 5, because $x \in L_2$ and $x \notin L_3$, we know that R will correctly never accept x.

7. Let x be any element such that $x \in L_1$, $x \notin L_2$, and $x \notin L_3$. We can see that x should be accepted by R because $x \in L_1$ but $x \notin L_2$ so $x \in (L_1 - L_2) \cup L_3$. Because $x \in L_1$ and $x \notin L_2$, R_1 will eventually accept x and then M_2 will reject x. Thus, we can see that R would be forced to correctly accept x.

8. Finally, let x be any element such that $x \notin L_1$, $x \notin L_2$, and $x \notin L_3$. Because $x \notin L_1$ and $x \notin L_3$, we know that $x \notin (L_1 - L_2) \cup L_3$ and we can then see that both R_1 and R_3 will never accept x. Thus, x would not be accepted by R.

Since these 8 cases account for all possibilities for any element, we can see that R does correctly recognize the language $(L_1 - L_2) \cup L_3$, ultimately showing that the language is recognizable if L_1 and L_3 are recognizable and L_2 is decidable.