COMP-170: Homework #4

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Problem 1

Let $L = \{\langle M_1, M_2 \rangle \mid M_1 \text{ accepts 7 and } M_2 \text{ does not accept 7}\}$. Prove L is undecidable.

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To show L is undecidable, we will claim and prove $A_{TM} \leq_m L$. To do this, we will construct a function $f : \Sigma^*_{ATM} \rightarrow \Sigma^*_L$ such that $x \in A_{TM} \Leftrightarrow f(x) \in L$. Once we show $A_{TM} \leq_m L$, since A_{TM} is undecidable, we will then be able to show that L is also undecidable.

First, we claim $A_{TM} \leq_m L$. To prove this claim, we will first show how A_{TM} and L relate to each other (using the "Tony Square").

	A_{TM}	L
IN	M accepts w	M_1 accepts 7 and M_2 doesn't accept 7
OUT	M doesn't accept w	M_1 doesn't accept 7 or M_2 accept 7

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{ATM} \rightarrow \Sigma^*_L$ be defined as follows:

f on input $\langle M, w \rangle$ outputs $\langle M_1, M_2 \rangle$, where M_1 and M_2 are defined as:

M_1 **on input** x**:**

If $x = 7$, run M on w

If M accepts w, *ACCEPT*

If M rejects w, *LOOP*

Else, *LOOP*

M_2 **on input** x**:**

LOOP

Given our definition of f, we claim that f is computable. Because M_1 and M_2 are valid Turing machines and f is a finite function, we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in A_{TM} \Leftrightarrow f(x) \in L$. Consider the following two cases:

1. Suppose $\langle M, w \rangle \in A_{TM}$ such that M accepts w. Since M accepts w, we can see that M_1 loops on every input except for 7, which it accepts. We can also see that M_2 loops on every input so M_2 doesn't accept 7. Thus, since M_1 accepts 7 and M_2 doesn't accept (loops on) 7, we can see that $\langle M_1, M_2 \rangle \in L$.

2. Suppose $\langle M, w \rangle \notin A_{TM}$ such that M doesn't accept w. Since M doesn't accept w, we can see that M_1 will loop on all inputs, including 7. We can also again see that M_2 loops on every input, so again M_2 doesn't accept 7. Thus, since both M_1 and M_2 do not accept 7, we can see that $\langle M_1, M_2 \rangle \notin L$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in A_{TM} \Leftrightarrow f(x) \in L$. Therefore, we can see that $A_{TM} \leq_m L$. We know that A_{TM} is undecidable (previously proven) so, since $A_{TM} \leq_m L$, L is also undecidable. ⊠