COMP-170: Homework #5

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Problem 3

Prove that $D5$ contains an undecidable subset.

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Let $L = \{\langle M \rangle \mid \langle M \rangle \text{ is divisible by 5 and } M \text{ accepts } \langle M \rangle\}$. Since every $x \in L$ is divisible by 5, we can see that $L \subseteq D5$. We will now show that L is undecidable by proving $A_{TM} \leq_m L$. In order to do this, we will construct a function $f : \Sigma^*_{A_{TM}} \to \Sigma^*_L$ that satisfies the condition $x \in A_{TM} \Leftrightarrow f(x) \in L$. Once we show that $A_{TM} \leq_m L$, because A_{TM} is undecidable, we will have shown that L is also undecidable.

First, we will claim $A_{TM} \leq_m L$. To prove this claim, we will first show how A_{TM} and L relate to each other (using the "Tony Square").

	A_{TM}	L
IN	M accepts w	$\langle M' \rangle \,\%\, 5 = 0$ and M' accepts $\langle M' \rangle$
OUT	M doesn't accept w	$\langle M' \rangle \,\%\, 5 \neq 0$ or M' doesn't accept $\langle M' \rangle$

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{A_{TM}} \to \Sigma^*_L$ be defined as follows:

f on input $\langle M, w \rangle$ outputs $\langle M' \rangle$, where M' is defined by:

M **on input** x

If x is divisible by 5, run M on w

If M accepts w, *ACCEPT*

If M rejects w, *LOOP*

Else, *LOOP*

Given our definition of f, we claim that f is computable. Because M' is a valid Turing machine and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Before proceeding with case analysis, we will assert that we use a binary encoding such that $\langle M \rangle$ is divisible by 5 for all M. We can do this because it is always possible to transform any binary encoding (which is a function) into one that encodes everything as being divisible by 5. Consider any encoding such that $\langle x \rangle \in \{0,1\}^*$ for all x. We can redefine this binary encoding by simply multiplying $\langle x \rangle$ by 5 for all x. If $\langle x \rangle \in \{0,1\}^*$, then $\langle x \rangle * 5 \in \{0,1\}^*$, so our encoding remains valid. Now, we can guarantee that $\langle x \rangle$ is divisible by 5 for all x.

Now, we can go through two cases to show that f correctly satisfies $x \in A_{TM} \Leftrightarrow f(x) \in L$. Consider the following two cases:

1. Suppose $\langle M, w\rangle \in A_{TM}$ such that M accepts w. Because $\langle M'\rangle$ is always divisible by 5, we can see that M' accepts $\langle M'\rangle$ if M accepts w. Since $\langle M, w\rangle \in A_{TM}$, then we can see that M' accepts $\langle M'\rangle$. Thus, $\langle M\rangle \in L$ if $\langle M, w\rangle \in A_{TM}$.

2. Suppose $\langle M, w\rangle \notin A_{TM}$ such that M doesn't accept w. Because M doesn't accept w, we can see that M' will loop on every input, including $\langle M'\rangle$. Thus, M' does not accept $\langle M'\rangle$ so $\langle M'\rangle \notin L$ if $\langle M, w\rangle \notin A_{TM}$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in A_{TM} \Leftrightarrow f(x) \in L$. Therefore, we can see that $A_{TM} \leq_m L$. Because we already know A_{TM} to be undecidable, we can see that L, which is a subset of $D5$, is also undecidable.