COMP-170: Homework #1

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Problem 2

Prove, by contradiction, that if $n-1$, n, and $n+1$ are consecutive positive integers, then the cube of the largest cannot be equal to the sum of the cubes of the other two.

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We want to show that if $n-1$, n, and $n+1$ are consecutive positive integers, then the cube of the largest cannot be equal to the sum of the cubes of the other two. In order to prove this, we will use proof by contradiction. More specifically, we will attempt (and fail) to show that if $n-1$, n, and $n+1$ are consecutive positive integers, then $(n+1)^3 = n^3 + (n-1)^3$.

Let $n-1$, n, and $n+1$ be consecutive positive integers and suppose that $(n+1)^3 = n^3 + (n-1)^3$. By expanding these, we get that $n^3 + 3n^2 + 3n + 1 = 2n^3 - 3n^2 + 3n - 1$, which can be simplified to get $n^3 - 6n^2 - 2 = 0$.

From here, we will find where the roots of this function $f(n) = n^3 - 6n^2 - 2$ are. In order to do this, we can first test the critical points of our function. By taking the derivative of our equation, we get $f'(n) = 3n^2 - 12n$, which is equal to zero at $n = 0, 4$. Using these critical points, we can see that $f(0) = (0)^3 - 6(0)^2 - 2 < 0$ and $f(4) = (4)^3 - 6(4)^2 - 2 = -34 < 0$, which means any real root of our equation must be greater than $n = 4$. Since there are no more critical points greater than 4, we know that there will only be one real root. By testing $n = 6$ and $n = 7$, we get that $f(6) = -2$ and $f(7) = 47$. This means that our only real root lies between 6 and 7, indicating it is not an integer.

Because the only real root that solves the equation $n^3 - 6n^2 - 2 = 0$ is not an integer, we can see that the only value of n for which $(n+1)^3 = n^3 + (n-1)^3$ is for a non-integer n. This contradicts our claim that $n-1$, n, and $n+1$ are consecutive positive integers such that $(n+1)^3 = n^3 + (n-1)^3$. As a result, we can see that if $n-1$, n, and $n+1$ are consecutive positive integers, then the cube of the largest cannot be equal to the sum of the cubes of the other two. $\boxtimes$