Intro to Algorithms, COMP-160, Homework #1

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4. **Prove or disprove that $\sum_{x=1}^{n}(8x^2 - 4x + 60) = \Theta(n^3)$.**

   We can prove $\sum_{x=1}^{n}(8x^2 - 4x + 60) = \Omega(n^3)$ by underestimating the summation.

   We will do this by removing the third term since this will be adding unnecessarily (we are trying to underestimate). We can also change the second term to be $4x^3$ because this will further subtract from the sum, again underestimating.

   This leads us to $\sum_{x=1}^{n}(8x^2 - 4x^2) = \sum_{x=1}^{n}(4x^2) \leq \sum_{x=1}^{n}(8x^2 - 4x + 60)$.

   To further underestimate, we can drop the leading coefficient of 4 leaving us with $\sum_{x=1}^{n}(x^2) \leq \sum_{x=1}^{n}(8x^2 - 4x + 60)$.

   Next, we can expand $\sum_{x=1}^{n}(x^2)$ to get $\sum_{x=1}^{n} x^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ and then further underestimate this to simply $\frac{1}{3}n^3$ since we know $\frac{1}{2}n^2 + \frac{1}{6}n \geq 0$. By removing these two terms further underestimates the initial summation.

   This leads us to $\frac{1}{3}n^3 \leq \sum_{x=1}^{n}(8x^2 - 4x + 60)$. This shows that for $c_1 \leq \frac{1}{3}$, $c_1 \cdot n^3 \leq \sum_{x=1}^{n}(8x^2 - 4x + 60)$, so we can see:

$$\sum_{x=1}^{n}(8x^2 - 4x + 60) = \Omega(n^3)$$

   Next we can prove $\sum_{x=1}^{n}(8x^2 - 4x + 60) = O(n^3)$ by overestimating this summation.

   We can do this by removing our second term because it is reducing our summation. Next we can raise the third term to $60x^2$ to further raise / overestimate the summation.

   This shows that $\sum_{x=1}^{n}(8x^2 - 4x + 60) \leq \sum_{x=1}^{n}(68x^2)$. To further overestimate this summation, we can simply multiply $(68n^2)$ by n to result in $n \cdot (68n^2) = 68n^3$. Since we have obtained $68n^3$ through gross overestimations, we can appropriately say $\sum_{x=1}^{n}(8x^2 - 4x + 60) \leq 68n^3$.

   This shows there exists a c_2 such that $\sum_{x=1}^{n}(8x^2 - 4x + 60) \leq c_2 \cdot n^3$ (more specifically, this is true if $c_2 \geq 68$). This proves:

$$\sum_{x=1}^{n}(8x^2 - 4x + 60) = O(n^3)$$

   Therefore, we have seen that $\sum_{x=1}^{n}(8x^2 - 4x + 60) = O(n^3)$ and $\sum_{x=1}^{n}(8x^2 - 4x + 60) = \Omega(n^3)$, we can say appropriately:

$$\sum_{x=1}^{n}(8x^2 - 4x + 60) = \Theta(n^3)$$