Intro to Algorithms, COMP-160, Homework #6
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1. A group of k vikings independently set out to make a new home. Each viking has a copy of the same map, showing n islands. Each viking decides to set sail for some random island. If two or more vikings land on the same island, they have a battle (no matter how many vikings land on that island, it counts as one battle).

   (a) **How many islands do we expect will be visited by the vikings?**

   For the following, $1 \leq i \leq k$ and $1 \leq j \leq n$

   Consider $x =$ the number of islands visited by all vikings. Ultimately, we are looking for $E(x)$. In order to compute this, we can use indicator random variables x_j where:

   $$x_j = \begin{cases} 1 & \text{island } j \text{ is visited by one or more vikings} \\ 0 & \text{island } j \text{ is not visited by any vikings} \end{cases}$$

   Thus, $x = x_1 + x_2 + \ldots + x_n = \sum_{j=1}^{n} x_j$
   $\frac{1}{n} =$ the probability of viking i visiting island j
   $1 - \frac{1}{n} =$ the probability of viking i not visiting island j
   $(1 - \frac{1}{n})^k =$ the probability of no viking visiting island j
   $1 - (1 - \frac{1}{n})^k =$ the probability of one or more vikings visit island $j = E(x_j)$
   Since $x = \sum_{j=1}^{n} x_j$, we can see that $E(x) = E(\sum_{j=1}^{n} x_j)$. Because of linearity of expectation, we can then turn this into:

   $$E(x) = \sum_{j=1}^{n} E(x_j) = \sum_{j=1}^{n} (1 - (1 - \frac{1}{n})^k) = n - n(1 - \frac{1}{n})^k$$

   (b) **How many battles do we expect will occur?**

   Similar to part (a), for the following, $1 \leq j \leq n$

   Consider the variable $y =$ the number of battles that happen among n islands. Ultimately, we are looking for $E(y)$. In order to compute this, we can use indicator random variables y_j where:

   $$y_j = \begin{cases} 1 & \text{island } j \text{ has a battle on it} \\ 0 & \text{island } j \text{ doesn't have a battle on it} \end{cases}$$

   Thus, $y = y_1 + y_2 + \ldots + y_n = \sum_{j=1}^{n} y_j$
   For a battle to occur on island j, there must be more than one viking on that island. Thus, $y_j = 0$ if one or zero vikings visit island j.
   $(1 - \frac{1}{n})^k =$ the probability of no vikings visiting island j (see part a)
   $(\frac{1}{n})(1 - \frac{1}{n})^{k-1} =$ the probability of exactly one viking visiting island j
   $E(y_j) = 1 -$ (probability of 0 vikings) $-$ (probability of 1 viking)

$E(y_j) = 1 - (1 - \frac{1}{n})^k - (\frac{1}{n})(1 - \frac{1}{n})^{k-1}$
Since $y = \sum_{j=1}^{n} y_j$, we can see that $E(y) = E(\sum_{j=1}^{n} y_j)$. Because of linearity of expectation, we can then turn this into:

$$E(y) = \sum_{j=1}^{n} E(y_j) = \sum_{j=1}^{n}(1-(1-\frac{1}{n})^k-(\frac{1}{n})(1-\frac{1}{n})^{k-1}) = n-n(1-\frac{1}{n})^k-(1-\frac{1}{n})^{k-1})$$

(c) **Do your solutions confirm the intuitive answer for the special case where there is only one island on the map? Or what if there's only one viking? What answers do you get for 400 vikings and 100 islands?**

For the special case where there is one island ($n = 1$), we would expect $x = 1$ and $y = 1$ for all $k \geq 1$ (if there are 0 vikings, there would be no settling). To confirm this, we can check the answers from part (a) and part (b):

$$E(x) = n - n(1 - \frac{1}{n})^k = 1 - 1(1 - \frac{1}{1})^k = 1 - 0 = 1$$

$$E(y) = n - n(1 - \frac{1}{n})^k - (1 - \frac{1}{n})^{k-1}) = 1 - 1(1 - \frac{1}{1})^k - (1 - \frac{1}{1})^{k-1} = 1 - 0 - 0 = 1$$

For the special case where there is one viking ($k = 1$), we would expect $x = 1$ and $y = 0$ for all $n \geq 1$ (if there are 0 islands, the viking cannot settle anywhere). To confirm this, we can check the answers from part (a) and part (b):

$$E(x) = n - n(1 - \frac{1}{n})^k = n - n(1 - \frac{1}{n})^1 = n - n + 1 = 1$$

$$E(y) = n-n(1-\frac{1}{n})^k-(1-\frac{1}{n})^{k-1} = n-n(1-\frac{1}{n})^1-(1-\frac{1}{n})^{1-1} = n-n+1-1 = 0$$

For the final special case where there is 400 vikings ($k = 400$) and 100 islands ($n = 100$), we get the following:

$$E(x) = n - n(1 - \frac{1}{n})^k = 100 - 100(1 - \frac{1}{100})^{400} = 98.205$$

$$E(y) = n-n(1-\frac{1}{n})^k-(1-\frac{1}{n})^{k-1} = 100-100(1-\frac{1}{100})^{400}-(1-\frac{1}{100})^{399} = 98.187$$

Since all of these special conditions return valid expected values, we can see that the results from part (a) and part (b) are valid.