COMP-170: Homework #6

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Problem 4

Consider the following languages

$L_1 = \{\langle M, w\rangle \mid M \text{ is an oracle machine with access to } \mathrm{A_{TM}} \text{ and } M \text{ does not accept } w\}$

$L_2 = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are oracle machines with access to } \mathrm{A_{TM}} \text{ and } L(M_1) = L(M_2)\}$

Prove that if you can recognize L_2 you can recognize L_1, further argue that if you can recognize L_1 you can recognize $\overline{A_{TM}}$.

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To prove L_1 is recognizable if L_2 is recognizable, we can prove $L_1 \leq_m L_2$. In order to do this, we will construct a function $f : \Sigma_{L1}^* \to \Sigma_{L2}^*$ such that $x \in L_1 \Leftrightarrow f(x) \in L_2$. Once we show that $L_1 \leq_m L_2$, because we assumed L_2 to be recognizable, we will see that L_1 is also recognizable (if L_2 is recognizable).

To prove $\overline{A_{TM}}$ is recognizable if L_1 is recognizable, we can use similar methods to show $\overline{A_{TM}} \leq_m L_1$. This will show that if L_1 is recognizable, then A_{TM} is recognizable.

To start, we will claim $L_1 \leq_m L_2$. To prove this claim, we will first show how L_1 and L_2 relate to each other (using the "Tony Square").

	L_1	L_2
IN	M has oracle access and M doesn't accept w	M_1 and M_2 both have oracle access and $L(M_1) = L(M_2)$
OUT	M doesn't have oracle access or M accepts w	M_1 or M_2 do not have oracle access or $L(M_1) \neq L(M_2)$

Now that we've built our "Tony Square," we can define our function. Let $f : \Sigma_{L1}^* \to \Sigma_{L2}^*$ be defined as follows:

f on input $\langle M, w\rangle$ outputs $\langle M_1, M_2\rangle$, where M_1 and M_2 are defined by:

M_1 **on input** x**:**

Run M on w

If M accepts w, *ACCEPT*

If M rejects w, *LOOP*

M_2 **on input** x**:**

LOOP

Given our definition of f, we claim that f is computable. Because M_1 and M_2 are valid Turing machines and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in L_1 \Leftrightarrow f(x) \in L_2$. Consider the following two cases:

1. Suppose $\langle M, w \rangle \in L_1$ such that M doesn't accept w. Because M doesn't accept w, we can see that M_1 will loop on everything. We can also see that M_2 loops on everything always, so we can see $L(M_1) = \varnothing = L(M_2)$. Thus $\langle M_1, M_2 \rangle \in L_2$ if $\langle M, w \rangle \in L_1$.
2. Suppose $\langle M, w \rangle \notin L_1$ such that M accepts w. Because M accepts w, we can see that M_1 will accept everything. We can also see that M_2 again loops on everything. Thus, we can see that $L(M_1) \neq L(M_2)$. Thus $\langle M_1, M_2 \rangle \notin L_2$ if $\langle M, w \rangle \notin L_1$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in L_1 \Leftrightarrow f(x) \in L_2$. Therefore, we can see that $L_1 \leq_m L_2$. Thus, if L_2 is recognizable, we can see that L_1 is also recognizable.

We will now claim $\overline{A_{TM}} \leq_m L_1$. To prove this claim, we will first show how $\overline{A_{TM}}$ and L_1 relate to each other (using the "Tony Square").

	$\overline{A_{TM}}$	L_1
IN	M doesn't accept w	M has oracle access and M doesn't accept w
OUT	M accepts w	M doesn't have oracle access or M accepts w

Now that we've built our "Tony Square," we can define our function. Let $f : \Sigma^*_{L1} \to \Sigma^*_{L2}$ be defined as follows:

f on input $\langle M, w \rangle$ outputs $\langle M', w' \rangle$, where $M' = M$ and $w' = w$

Given our definition of f, we claim that f is computable. Because M' is a valid Turing machines (since M is) and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L_1$. Consider the following two cases:

1. Suppose $\langle M, w \rangle \in \overline{A_{TM}}$ such that M doesn't accept w. We therefore set $M' = M$ and $w' = w$, so we can see that $\langle M', w' \rangle \in L_1$.
2. Suppose $\langle M, w \rangle \notin L_1$ such that M accepts w. We therefore set $M' = M$ and $w' = w$, so we can see that $\langle M', w' \rangle \notin L_1$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L_1$. Therefore, we can see that $\overline{A_{TM}} \leq_m L_1$. Thus, if L_1 is recognizable, we can see that $\overline{A_{TM}}$ is also recognizable.