COMP-170: Homework #3

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Problem 2

Let M_1, M_2, and M_3 be machines that decide languages L_1, L_2, and L_3 respectively. Prove that $(L_1 - L_2) \cup L_3$ is decidable.

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We want to prove that language $(L_1 - L_2) \cup L_3$ is decidable, given L_1, L_2, and L_3 are decidable. To do this, we will use proof by construction. More specifically, we will construct a machine M that can decide $(L_1 - L_2) \cup L_3$ using a combination of M_1, M_2, and M_3 (the deciders for L_1, L_2, and L_3 respectively).

Given input x, we would say $x \in (L_1 - L_2) \cup L_3$ if $x \in L_1$ and $x \notin L_2$ or if $x \in L_3$. We can then use M_1, M_2, and M_3 to construct a machine M that is able to decide $(L_1 - L_2) \cup L_3$. Let's define M as follows:

M **on input** x**:**

Run M_3 on x

If M_3 accepts x, *ACCEPT*

Run M_1 on x and run M_2 on x

If M_1 accepts x and M_2 rejects, *ACCEPT*

Else, *REJECT*

END

We will now claim that M is a decider of $(L_1 - L_2) \cup L_3$. To show this, consider the following cases:

1. Let x be any element such that $x \in L_1$, $x \in L_2$, and $x \in L_3$. We can see that x should be accepted by M because $x \in L_3$ so $x \in (L_1 - L_2) \cup L_3$. Because $x \in L_3$, we know that M_3 will accept x, which will cause M to correctly accept x.
2. Let x be any element such that $x \notin L_1$, $x \in L_2$, and $x \in L_3$. We can see that, like case 1, because $x \in L_3$, we know that M_3 will accept x, which will cause M to correctly accept x.
3. Let x be any element such that $x \in L_1$, $x \notin L_2$, and $x \in L_3$. We can see that, like case 1, because $x \in L_3$, we know that M_3 will accept x, which will cause M to correctly accept x.
4. Let x be any element such that $x \notin L_1$, $x \notin L_2$, and $x \in L_3$. We can see that, like case 1, because $x \in L_3$, we know that M_3 will accept x, which will cause M to correctly accept x.

5. Let x be any element such that $x \in L_1$, $x \in L_2$, and $x \notin L_3$. We can see that x should be rejected by M because $x \in L_2$ and $x \notin L_3$ so $x \notin (L_1 - L_2) \cup L_3$. Because $x \notin L_3$ but $x \in L_2$, M_3 won't accept x and M_2 won't reject x. Thus, we can see that M would be forced to correctly reject x.

6. Let x be any element such that $x \notin L_1$, $x \in L_2$, and $x \notin L_3$. We can see that, like case 5, because $x \in L_2$ and $x \notin L_3$, we know that M will be forced to correctly reject x.

7. Let x be any element such that $x \in L_1$, $x \notin L_2$, and $x \notin L_3$. We can see that x should be accepted by M because $x \in L_1$ but $x \notin L_2$ so $x \in (L_1 - L_2) \cup L_3$. Because $x \in L_1$ and $x \notin L_2$, M_1 will accept x and M_2 will reject x. Thus, we can see that M would be forced to correctly accept x.

8. Finally, let x be any element such that $x \notin L_1$, $x \notin L_2$, and $x \notin L_3$. Because $x \notin L_1$ and $x \notin L_3$, M should reject x. Given this, neither M_3 nor M_1 would accept x so M would indeed be forced to correctly reject x.

Since these 8 cases account for all possibilities for any element, we can see that M does correctly decide the language $(L_1 - L_2) \cup L_3$, ultimately showing that the language is decidable if L_1, L_2, and L_3 are all decidable.