COMP-170: Homework #8

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Problem 1

4SAT is the problem where given a formula in conjunctive normal form with exactly four literals in each clause and you want to know if it can be satisfied by some assignment.

Prove that 4SAT $\leq_m^p$ 3SAT

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To prove 4SAT $\leq_m^p$ 3SAT, we will construct a function $f : \Sigma_{\text{4SAT}}^* \to \Sigma_{\text{3SAT}}^*$ such that $\phi \in \text{4SAT} \Leftrightarrow f(\phi) \in \text{3SAT}$. Once we show that f satisfies this condition, we will have proven that 4SAT $\leq_m^p$ 3SAT.

Given this, we can define $f : \Sigma_{\text{4SAT}}^* \to \Sigma_{\text{3SAT}}^*$ as follows:

f on input ϕ outputs ϕ', where ϕ' is defined as follows:

Suppose $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ where $C_1, C_2, \ldots, C_m$ are all four-literal clauses. Given this, initially set $\phi' = \phi$.

For each clause C_i from ϕ', where $C_i = a_i \vee b_i \vee c_i \vee d_i$ for some literals a_i, b_i, c_i, d_i, replace C_i with $(a_i \vee b_i \vee x_i) \wedge (c_i \vee d_i \vee \overline{x_i})$, where x_i is a dummy variable that does not appear anywhere else in ϕ'.

Output ϕ'

Given our definition of f, we claim that f is computable in polynomial time. We can see that we could construct a Turing machine M that outputs ϕ' when given ϕ. M would iterate through each of our clauses and replace each four-literal clause with two three-literal clauses, which takes $O(1)$ time per clause and takes $O(m)$ time overall. Thus, we can see f is indeed computable in polynomial time.

Next, we must verify that f satisfies the condition that $\phi \in \text{4SAT} \Leftrightarrow f(\phi) \in \text{3SAT}$. In order to do this, consider the following two cases:

1. Suppose $\phi \in$ 4SAT such that there exists some satisfying assignment that makes every clause in ϕ evaluate to true, and thus makes ϕ, which is in conjunctive normal form with exactly four literals in each clause, evaluate to true. Let $\mathscr{A}$ be any assignment that satisfies ϕ and C_i be any clause from ϕ such that $C_i = a_i \vee b_i \vee c_i \vee d_i$. Because we have a satisfying assignment for ϕ, we know that at least one of a_i, b_i, c_i, or d_i is true in $\mathscr{A}$. Now consider C_i', the clause pair in ϕ' that C_i was transformed into, where $C_i' = (a_i \vee b_i \vee x_i) \wedge (c_i \vee d_i \vee \overline{x_i})$. Because we know at least one of a_i, b_i, c_i, or d_i is true, we know that either $(a_i \vee b_i \vee x_i)$ or $(c_i \vee d_i \vee \overline{x_i})$ evaluates to true, regardless of what x_i is. We can also see if $(a_i \vee b_i \vee x_i)$ evaluates to true, setting x_i to false will also make $(c_i \vee d_i \vee \overline{x_i})$ evaluate to true, and if $(c_i \vee d_i \vee \overline{x_i})$ evaluates to true, setting

x_i to true will also make $(a_i \vee b_i \vee \overline{x_i})$ evaluate to true. Because x_i only appears in C_i', it is free to be set to any value. Therefore, we can see it is always possible to assign x_i a value that satisfies C_i' given $\mathscr{A}$. Given this, we can see that we can find a valid assignment to satisfy all C_i' in ϕ'. Thus, we can see $\phi' \in$ 3SAT.

2. Suppose $\phi \notin$ 4SAT such that there exists no assignment that makes every clause in ϕ evaluate to true. This means given any assignment, there exists some clause C_i that will evaluate to false. Let $\mathscr{A}$ be any assignment for ϕ and C_i be any clause in ϕ that evaluates to false given $\mathscr{A}$ where $C_i = a_i \vee b_i \vee c_i \vee d_i$. Because C_i evaluates to false, we know that a_i, b_i, c_i, and d_i are all assigned to false in $\mathscr{A}$. Now consider C_i', the clause pair in ϕ' that C_i was transformed into, where $C_i' = (a_i \vee b_i \vee x_i) \wedge (c_i \vee d_i \vee \overline{x_i})$. Because a_i, b_i, c_i, and d_i are all false, we can see that C_i' can only evaluate to true if x_i is set to true and $\overline{x_i}$ sets to true. However, this is impossible, so we can see that C_i' must evaluate to false if C_i evaluates to false. Therefore, if all assignments of ϕ leave at least one clause evaluating as false, we know that all assignments of ϕ' will also leave at least one clause pair evaluating as false. Thus, we can see if $\phi \notin$ 4SAT, $\phi' \notin$ 3SAT.

Given these two cases, we can thus see that f does indeed satisfy the claim $\phi \in$ 4SAT $\Leftrightarrow$ $\phi' \in$ 3SAT. Therefore, we can see 4SAT $\leq_m^p$ 3SAT. ⊠