COMP-170: Homework #6

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Problem 2

Let ACCEPTMORE $= \{\langle M_1, M_2\rangle \mid |L(M_1)|$ and $|L(M_2)|$ are finite and $|L(M_1)| < |L(M_2)|\}$. Prove that ACCEPTMORE is unrecognizable.

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To show that ACCEPTMORE is unrecognizable, we can prove $\overline{A_{TM}} \leq_m$ ACCEPTMORE. In order to do this, we will construct a function $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_{\text{ACCEPTMORE}}$ such that $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in$ ACCEPTMORE. Once we show that $\overline{A_{TM}} \leq_m$ ACCEPTMORE, because $\overline{A_{TM}}$ is unrecognizable, we will have shown that ACCEPTMORE is also unrecognizable.

First, for ease, we will denote $L =$ ACCEPTMORE. Next, we will claim $\overline{A_{TM}} \leq_m L$. To prove this claim, we will first show how $\overline{A_{TM}}$ and L relate to each other (using the "Tony Square").

	$\overline{A_{TM}}$	L
IN	M doesn't accept w	$\lvert L(M_1)\rvert$ and $\lvert L(M_2)\rvert$ are finite and $\lvert L(M_1)\rvert < \lvert L(M_2)\rvert$
OUT	M accepts w	$\lvert L(M_1)\rvert$ and $\lvert L(M_2)\rvert$ are not finite or $\lvert L(M_1)\rvert \geq \lvert L(M_2)\rvert$

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_L$ be defined as follows:

f on input $\langle M, w\rangle$ outputs $\langle M_1, M_2\rangle$, where M_1 and M_2 are defined by:

> **M_1 on input x:**
>
> > Run M on w
> >
> > > If M accepts w, *ACCEPT*
> > >
> > > If M rejects w, *LOOP*
>
> **M_2 on input x:**
>
> > If $x = 1$, *ACCEPT*
>
> Else, *LOOP*

Given our definition of f, we claim that f is computable. Because M_1 and M_2 are valid Turing machines and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Consider the following two cases:

1. Suppose $\langle M, w \rangle \in \overline{A_{TM}}$ such that M doesn't accept w. Because M doesn't accept w, we can see that M_1 will loop on every input so $L(M_1) = \varnothing$. We can also see that M_2 always only accepts 1 so $L(M_2) = \{1\}$. Thus, $|L(M_1)| = 0 < |L(M_2)| = 1$, so we can see $\langle M_1, M_2 \rangle \in L$ when $\langle M, w \rangle \in \overline{A_{TM}}$.

2. Suppose $\langle M, w \rangle \notin \overline{A_{TM}}$ such that M accepts w. Because M accepts w, we can see that M_1 will accept every input so $|L(M_1)|$ is not finite. Additionally, since $L(M_2) = \{1\}$ again, we can see that $|L(M_1)| \not< |L(M_2)|$. Thus, we can see $\langle M_1, M_2 \rangle \notin L$ when $\langle M, w \rangle \notin \overline{A_{TM}}$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Therefore, we can see that $\overline{A_{TM}} \leq_m L$. Because we know $\overline{A_{TM}}$ is unrecognizable, we can see that $L =$ ACCEPTMORE is also unrecognizable. ⊠