COMP-170: Homework #9

Ben Tanen - April 17, 2017

Problem 2

Given a universe U and a subset of the power set of U, $S \subseteq \mathcal{P}(U)$, we are interested in determining if S contains an exact covering of U, where an exact covering is a set $I, I \subseteq S$ such that every element of U belongs to *exactly* one set in I.

EXACT_COVER $= \{\langle U, S\rangle \mid S \subseteq \mathcal{P}(U)$ and S contains an exact cover of $U\}$

Prove EXACT_COVER is NP-complete. Do a reduction from 3_COLOR

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To prove EXACT_COVER is NP-complete, we will show that EXACT_COVER $\in$ NP and that 3_COLOR $\leq_m^p$ EXACT_COVER. Given that 3_COLOR is NP-complete, we will then be able to see that EXACT_COVER is NP-complete.

First, we will prove EXACT_COVER $\in$ NP. In order to prove this, we can construct a polynomial time verifier V for EXACT_COVER. Once we show that V does indeed verify in polynomial time, we will then be able to see that EXACT_COVER $\in$ NP. We can define V as follows:

V on input $\langle\langle U, S\rangle, I\rangle$:

// I is an subset of S, where S is a subset of $\mathcal{P}(U)$, to be verified

1. For each set in S, check said set is a subset of U. If there exists a set in S that is not a subset of U, *REJECT*
2. For each set in I, check said set is in S. If there exists a set in I that is not in S, *REJECT*
3. For each $u \in U$, check that u is in exactly one set of I. If u doesn't appear in a set in I, *REJECT*. If u appears in more than one set in I, *REJECT*.
4. Return *ACCEPT*.

Given our definition of V, we claim V runs in polynomial time. We can see that step 1 takes $O(n^2)$ time, where $n = |U|$, step 2 takes $O(n^3)$ time, step 3 takes $O(n^2)$ time. Thus, V takes polynomial time overall.

Finally, to show that V correctly verifies solutions, consider the following cases:

1. Suppose $\langle U, S\rangle \in$ EXACT_COVER and I is an exact covering of U. We then know that V would not halt on step 1, 2, or 3, by definition of U, S, and I. Thus, V would run until step 4, thus properly accepting.
2. Suppose $\langle U, S\rangle \notin$ EXACT_COVER where S is not a collection of subsets of U. We can then see that there would be some set in S that would halt on step 1, resulting in

V properly rejecting.

3. Suppose $\langle U, S\rangle \in$ EXACT_COVER but I (our witness) is not a subset of S. Given this, there must exist some set in I that is not in S. Therefore, we can see there is some set that would halt V on step 2, causing V to properly reject.

4. Suppose $\langle U, S\rangle \in$ EXACT_COVER but I (our witness) is not an exact covering of S. Given this, there must exist some element u that either doesn't appear in any of the sets of I or appears in two sets of I. Therefore, we can see that V would halt after finding this u and checking against all of the sets of I. Therefore, V would correctly reject.

Given these cases, we can see that V correctly verifies solutions, thus showing V is indeed a polynomial time verifier for EXACT_COVER. Thus, the existence and validity of V shows that EXACT_COVER $\in$ NP.

Next, we will prove 3COLOR $\leq_m^p$ EXACTCOVER. In order to prove this, we can construct a function $f : \Sigma^*_{\text{3COLOR}} \rightarrow \Sigma^*_{\text{EXACTCOVER}}$ such that $G \in$ 3COLOR $\Leftrightarrow f(G) \in$ EXACTCOVER. Once we show that f satisfies this condition, we will have proven that 3COLOR $\leq_m^p$ EXACTCOVER.

Given this, we can define $f : \Sigma^*_{\text{3COLOR}} \rightarrow \Sigma^*_{\text{EXACTCOVER}}$ as follows:

f on input G outputs $\langle U, S\rangle$, where U and S are defined as follows:

For each node v in G, add four elements to our universe U: v, R_v, G_v, and B_v.

For each node v in G, add three sets to S (one for each color), where each set contains v and C_{uv} for each $(u, v) \in G$ (in which $C \in \{R, G, B\}$).

For each edge (u, v) in G, add three elements to our universe U: R_{uv}, G_{uv}, and B_{uv}.

For individual element e added to G (excluding the elements added corresponding to the vertices of G), add $\{e\}$ to S.

Given our definition of f, we claim that f is computable in polynomial time. We can see that we could construct a Turing machine M that outputs $\langle U, S\rangle$ when given G. M would simply walk through G and add a linear number of elements for each vertex. M also does the same for each edge. This would take $O(E + V)$ time overall, so we can see that f is indeed computable in polynomial time.

Next, we must verify that f satisfies the condition that $G \in$ 3COLOR $\Leftrightarrow f(G) \in$ EXACTCOVER. In order to do this, consider the following:

1. *Case analysis to prove this TBD*

Given this, we can thus see that f does indeed satisfy the claim $G \in$ 3COLOR $\Leftrightarrow f(G) \in$ EXACTCOVER. Therefore, we can see that 3COLOR $\leq_m^p$ EXACTCOVER.

Given that EXACTCOVER $\in$ NP, 3COLOR $\leq_m^p$ EXACTCOVER, and 3COLOR is NP-complete, we can see that EXACTCOVER is also NP-complete. $\boxtimes$