COMP-170: Homework #9

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Problem 4

Let PARTITION $= \{\langle S \rangle \mid S$ is a set of non-negative integers, and $\exists A, \exists B$ where $A \subseteq S, B \subseteq S, A \cap B = \emptyset, A \cup B = S$ and $\forall a \in A, \forall b \in B, \Sigma a = \Sigma b\}$.

Here the problem is, can we partition the set S into two distinct pieces, where the sum of all elements in the two sets are equal?

Prove that $\text{SUBSET} - \text{SUM} \leq_m^p \text{PARTITION}$.

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To prove SUBSETSUM $\leq_m^p$ PARTITION, we will construct a function $f : \Sigma^*_{\text{SUBSETSUM}} \to \Sigma^*_{\text{PARTITION}}$ such that $\langle S, t \rangle \in \text{SUBSETSUM} \Leftrightarrow \langle S' \rangle \in \text{PARTITION}$. Once we show that f satisfies this condition, we will have proven that SUBSETSUM $\leq_m^p$ PARTITION.

For short hand, let SUBSETSUM = SS and PARTITION = P.

Given this, we can define $f : \Sigma^*_{\text{SS}} \to \Sigma^*_{\text{P}}$ as follows:

f on input $\langle S, t \rangle$ outputs $\langle S' \rangle$, where S' is defined as follows:

> Let $q = |2t - \Sigma_{s \in S} s|$
>
> If $q \notin S$, return $S' = S \cup \{q\}$
>
> If $q \in S$, loop for $i = i \to \infty$:
>
> > If $i \notin S$ and $q + i \notin S$, return $S' = S \cup \{q + i, i\}$
> >
> > If $i \in S$ or $q + i \in S$, repeat for $i = i + 1$

Given our definition of f, we claim that f is computable in polynomial time. We can see that we could construct a Turing machine M that outputs $\langle S \rangle$ when given $\langle S, t \rangle$. M would first compute q and check if q was in S. Then, if necessary, M would increment i until we find some combination of i and $q + i$ where neither are in S. Initially computing q takes $O(n)$ time (sum all the elements of S). We then know that i and $q + i$ can collide with n different values, so we can see that we would have to loop $O(n)$ times, where each iteration takes $O(1)$ time. Therefore, we get $O(n)$ time overall, so we can see f is indeed computable in polynomial time.

Next, we must verify that f satisfies the condition that $\langle S, t \rangle \in \text{SS} \Leftrightarrow \langle S \rangle \in \text{P}$. In order to do this, consider the following two cases:

1. Suppose $\langle S, t \rangle \in \text{SS}$, such that there exists some $I \subseteq S$ where $\Sigma_{i \in I} i = t$. Now, since I exists, let $J = S - I$. We know that $\Sigma_{j \in J} j = \Sigma_{s \in S} s - t$. Thus, given our definition of q, we know that if we add q to I, $\Sigma_{i \in I^+} = t + \Sigma_{s \in S} s - 2t = \Sigma_{s \in S} - t = \Sigma_{j \in J}$. Thus, by

adding q to I, we ensure there is some partition I and J where the $\Sigma_{i\in I} = \Sigma_{j\in J}$. In order to ensure that we add a distinct element (one not already in S, we can increment q with i until we find a value for $q+i$ and i that are distinct to S. If $i \geq 1$, we can then add i to J and maintain the equality relationship. Thus, since $I \cup J = S'$, we can see that $\langle S' \rangle \in$ P.

2. Suppose $\langle S, t\rangle \notin$ SS, such that there does not exist a $I \subseteq S$ where $\Sigma_{i\in I} = t$. Given this, if we were to add $q + i = ((\Sigma_{s\in S} s) - 2t) + i$ and i to S, we can see $\Sigma_{s\in S} s = \Sigma_{a\in A} a + \Sigma_{b\in B} b + ((\Sigma_{s\in S} s) - 2t) + 2i$, where A and B are partitions of S where neither A nor B add up to t. Given this, we can see that our total sum can not be evenly partitioned into any A or B where $\Sigma_{a\in A} a = \Sigma_{b\in B} b$. Thus, we can see that $\langle S' \rangle \notin$ P if $\langle S, t\rangle \notin$ SS.

Given these two cases, we can thus see that f does indeed satisfy the claim $\langle S, t\rangle \in \text{SS} \Leftrightarrow \langle S' \rangle \in \text{P}$. Therefore, we can see SS $\leq_m^p$ P. ⊠