COMP-170: Homework #1

Ben Tanen - January 29, 2017

Problem 1

A function f is one-to-one if and only if for all real numbers x and y with $x \neq y$, $f(x) \neq f(y)$.

Prove, by contradiction, that if a, b, and c are real numbers with $a \neq 0$, then the function $f(x) = ax^2 + bx + c$ is not one-to-one.

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We want to show that if a, b, and c are real numbers with $a \neq 0$, then the function $f(x) = ax^2 + bx + c$ is not one-to-one. In order to prove this, we will use proof by contradiction. More specifically, we will show that if a, b, and c are real numbers with $a \neq 0$, then we can find some $x, y \in \mathbb{R}$ such that $x \neq y$ yet $f(x) = f(y)$, showing f cannot be one-to-one.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Let's assume f is a one-to-one function. Given f, we can define the x-coordinate of the vertex of the parabola represented by f simply as $v_x = -\frac{b}{2a}$. Now let x be any real number not equal to v_x and let $d = x - v_x = x + \frac{b}{2a}$. Now define $y = v_x - d = -\frac{b}{2a} - (x + \frac{b}{2a}) = -\frac{b}{a} - x$.

With x and y defined, we can see that $f(x) = ax^2 + bx + c$ and $f(y) = a(-\frac{b}{a} - x)^2 + b(-\frac{b}{a} - x) + c = \frac{b^2}{a} + 2bx + ax^2 - \frac{b^2}{a} - bx + c = ax^2 + (2bx - bx) + c + (\frac{b^2}{a} - \frac{b^2}{a}) = ax^2 + bx + c$. We can thus see that $f(x) = f(y)$. However, by construction of y, we know $y \neq x$. Thus, we arrive on a contradiction of our original claim that $f(x) = f(y) \implies x = y$, i.e. f is one-to-one. As a result, since we can always find this sort of pair x and y for any f (with any real coefficients a, b, c where $a \neq 0$), we can see through this contradiction that if a, b, c are real numbers where $a \neq 0$, then $f(x) = ax^2 + bx + c$ is not one-to-one. ⊠