COMP-170: Homework #5

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Problem 2

Let $L = \{\langle M \rangle \mid L(M) \subseteq D5\}$. Prove L is unrecognizable.

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To show that L is unrecognizable, we can prove $\overline{A_{TM}} \leq_m L$. In order to do this, we will construct a function $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_L$ such that $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Once we show that $\overline{A_{TM}} \leq_m L$, because $\overline{A_{TM}}$ is unrecognizable, we will have shown that L is also unrecognizable.

First, we will claim $\overline{A_{TM}} \leq_m L$. To prove this claim, we will first show how $\overline{A_{TM}}$ and L relate to each other (using the "Tony Square").

	$\overline{A_{TM}}$	L
IN	M doesn't accept w	$L(M') \subseteq D5$
OUT	M accepts w	$L(M') \not\subseteq D5$

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_L$ be defined as follows:

f on input $\langle M, w \rangle$ outputs $\langle M' \rangle$, where M' is defined by:

M' **on input** x**:**

If $x = 1$, run M on w

If M accepts w, *ACCEPT*

If M rejects w, *LOOP*

Else, *LOOP*

Given our definition of f, we claim that f is computable. Because M' is a valid Turing machine and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Consider the following two cases:

1. Suppose $\langle M, w \rangle \in \overline{A_{TM}}$ such that M doesn't accept w. Because M doesn't accept w, we can see that M' will loop on every input, including 1. Thus, $L(M') = \varnothing$, which we know is a subset of $D5$ ($\varnothing$ is a subset of every set). Thus, we can see $\langle M' \rangle \in L$ when $\langle M, w \rangle \in \overline{A_{TM}}$.
2. Suppose $\langle M, w \rangle \notin \overline{A_{TM}}$ such that M accepts w. Because M accepts w, we can see that M' will loop on every input, except for 1, which M' accepts. Thus, $L(M') = \{1\}$.

Because 1 is not divisible by 5, we know that $\{1\} = L(M') \not\subseteq D5$. Thus, we can see $\langle M' \rangle \notin L$ when $\langle M, w \rangle \notin \overline{A_{TM}}$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Therefore, we can see that $\overline{A_{TM}} \leq_m L$. Because we know $\overline{A_{TM}}$ is unrecognizable, we can see that L is also unrecognizable. $\boxtimes$