COMP-170: Homework #5

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Problem 5

Let $L = \{\langle M_1, M_2\rangle \mid M_1$ accepts fewer than 2 inputs and M_2 accepts greater than 2 inputs.$\}$ Prove that L and $\overline{L}$ are both unrecognizable.

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To show that L and $\overline{L}$ are both unrecognizable, we can prove $\overline{A_{TM}} \leq_m L$ and $\overline{A_{TM}} \leq_m \overline{L}$. We can prove $\overline{A_{TM}} \leq_m L$ by constructing a function $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_L$ such that $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. We can use similar methods to prove $\overline{A_{TM}} \leq_m \overline{L}$. Once we show $\overline{A_{TM}} \leq_m L$ and $\overline{A_{TM}} \leq_m \overline{L}$, because $\overline{A_{TM}}$ is unrecognizable, we will have shown that both L and $\overline{L}$ are also unrecognizable.

First, we will show $\overline{A_{TM}} \leq_m L$. To start, we claim $\overline{A_{TM}} \leq_m L$. To prove this claim, we will first show how $\overline{A_{TM}}$ and L relate to each other (using the "Tony Square").

	$\overline{A_{TM}}$	L
IN	M doesn't accept w	$\lvert L(M_1)\rvert < 2$ and $\lvert L(M_2)\rvert > 2$
OUT	M accepts w	$\lvert L(M_1)\rvert \geq 2$ or $\lvert L(M_2)\rvert \leq 2$

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_L$ be defined as follows:

f on input $\langle M, w\rangle$ outputs $\langle M_1, M_2\rangle$, where M_1 and M_2 are defined by:

M_1 **on input** x**:**

Run M on w

If M accepts w, *ACCEPT*

If M rejects w, *LOOP*

M_2 **on input** x**:**

ACCEPT

Given our definition of f, we claim that f is computable. Because M_1 and M_2 are valid Turing machines and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Consider the following two cases:

1. Suppose $\langle M, w\rangle \in \overline{A_{TM}}$ such that M doesn't accept w. Because M_1 loops on everything if M doesn't accept w, we can see that $|L(M_1)| = 0 < 2$. We can also see that M_2 accepts everything so $|L(M_2)| > 2$. Thus, we can see that if $\langle M, w\rangle \in \overline{A_{TM}}$, then $\langle M_1, M_2\rangle \in L$.
2. Suppose $\langle M, w\rangle \notin \overline{A_{TM}}$ such that M accepts w. Because M_1 accepts everything if M accepts w, we can see that $|L(M_1)| > 2$. We again can also see that M_2 accepts everything so $|L(M_2)| > 2$. Thus, we can see that if $\langle M, w\rangle \notin \overline{A_{TM}}$, then $\langle M_1, M_2\rangle \notin L$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Therefore, we can see that $\overline{A_{TM}} \leq_m L$.

Next, we will show $\overline{A_{TM}} \leq_m \overline{L}$. To start, we claim $\overline{A_{TM}} \leq_m \overline{L}$. To prove this claim, we will first show how $\overline{A_{TM}}$ and $\overline{L}$ relate to each other (using the "Tony Square").

	$\overline{A_{TM}}$	$\overline{L}$
IN	M doesn't accept w	$\lvert L(M_1)\rvert \geq 2$ or $\lvert L(M_2)\rvert \leq 2$
OUT	M accepts w	$\lvert L(M_1)\rvert < 2$ and $\lvert L(M_2)\rvert > 2$

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_{\overline{L}}$ be defined as follows:

f on input $\langle M, w\rangle$ outputs $\langle M_1, M_2\rangle$, where M_1 and M_2 are defined by:

M_1 **on input** x**:**

LOOP

M_2 **on input** x**:**

Run M on w

If M accepts w, *ACCEPT*

If M rejects w, *LOOP*

Given our definition of f, we claim that f is computable. Because M_1 and M_2 are valid Turing machines and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in \overline{L}$. Consider the following two cases:

1. Suppose $\langle M, w\rangle \in \overline{A_{TM}}$ such that M doesn't accept w. Because M_1 loops on everything, $|L(M_1)| = 0 < 2$. We can also see that M_2 loops on everything when M doesn't accept w so $|L(M_2)| = 0 < 2$. Thus, we can see that if $\langle M, w\rangle \in \overline{A_{TM}}$, then $\langle M_1, M_2\rangle \in \overline{L}$.

2. Suppose $\langle M, w\rangle \notin \overline{A_{TM}}$ such that M accepts w. Again, M_1 loops on everything so $|L(M_1)| = 0 < 2$. Now that M accepts w, we can see that M_2 accepts everything so $|L(M_2)| > 2$. Thus, we can see that if $\langle M, w\rangle \notin \overline{A_{TM}}$, then $\langle M_1, M_2\rangle \notin \overline{L}$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in \overline{L}$. Therefore, we can see that $\overline{A_{TM}} \leq_m \overline{L}$.

Because we know $\overline{A_{TM}}$ is unrecognizable and because we have proven $\overline{A_{TM}} \leq_m L$ and $\overline{A_{TM}} \leq_m \overline{L}$, we can see that both L and $\overline{L}$ are unrecognizable. $\boxtimes$