COMP-170: Homework #9

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Problem 5

In a graph, $G = (V, E)$, two nodes are allowed to marry if they are not connected by an edge, this marriage joins the two nodes into one new node with edges from both nodes.

Specifically, given $u, v \in V$ and $(u, v) \notin E$, marry(G, u, v) creates a new graph $G' = (V', E')$, where $V' = V \setminus \{u, v\} \cup \{w\}$ and $E' = E \setminus \{(u, x_i) \mid \forall x_i \in V\} \setminus \{(v, x_i) \mid \forall x_i \in V\} \cup \{(w, z_i) \mid \forall z_i \in \{(\text{neighborhood}(u) \cup \text{neighborhood}(v)\}$.

GraphMarriageEquality =

$\{\langle G, H\rangle \mid \exists$ a series of marriages which transform G into G', where G' is isomorphic to $H\}$

Prove that GraphMarriageEquality is NP-complete.

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In order to show that GraphMarriageEquality is NP-complete, we can show that GraphMarriageEquality = GME $\in$ NP and that 3COLOR $\leq_m^p$ GME. Since 3COLOR is NP-complete, this is sufficient to show that GME is also NP-complete.

First, we will show that GME is in NP. To do this, we can construct a polynomial time verifier V for GME. Consider the following:

V **on input** $\langle\langle G, H\rangle, M\rangle$**:**

// M is a list marriages to perform to transform G into G', which should be isomorphic to H

First, follow the marriages of M to transform G into G'. Then check G' and H are isomorphic (check if there are corresponding vertices of G' in H and if there are corresponding edges of G' in H).

If G' and H are isomorphic, *ACCEPT*. If not, *REJECT*.

We can see that V runs in polynomial time. Because we can only perform a linear number of marriages, we know that it takes $O(n)$ time to perform the marriages in M. Then it takes $O(n^2)$ time to check if H and G' are isomorphic. Thus, we can see that we can verify using V in polynomial time.

We can also see that V does correctly verify. If M is a list of valid marriages that transforms G into G' such that G' and H are isomorphic, then by definition, we would expect G' and H are isomorphic. V would then correctly verify $\langle G, H, M\rangle$. However, if the marriages in M are no sufficient to make G' isomorphic to H (either because M is a bad list of marriages or it is impossible), we can see that G' would not be isomorphic to H, so V would correctly reject. Thus, we can see that V is a valid polynomial verifier for GME. This shows that GME $\in$ NP.

Next, we will show 3COLOR $\leq_m^p$ GME. In order to do this, we will use the fact that it takes polynomial time to check if a graph G can be one or two colorable. Consider the following function $f : \Sigma^*_{\text{3COLOR}} \to \Sigma^*_{\text{GME}}$:

f on input $\langle G \rangle$ outputs $\langle G, H \rangle$, where H is defined as:

If G can is one-colorable, $H = K_1$ (a clique of size one, i.e., a single node)

If G cannot be one-colored but is two-colorable, $H = K_2$ (a clique of size two)

If G cannot be two-colored, $H = K_3$ (a clique of size three)

Output $\langle G, H \rangle$

Given our definition of f, we can see that f is computable in polynomial time. Since it takes polynomial time to check if a graph is one colorable or two colorable, and this is all we do in f (other than construct cliques of constant size), we can see that f is indeed computable in polynomial time.

Next, we can consider the following cases:

1. Suppose $G \in$ 3COLOR such that there exists a three coloring χ_3 for G but G is not two-colorable. Given this, we can see that we can use marriages to reduce G down to a clique of size three. Given the definition of G, $f(G) = \langle G, K_3 \rangle$. Since G can be reduced to a clique of size three, we can see that G can be transformed with marriages into G', which would be isomorphic to K_3. Thus, we can see that $\langle G, H \rangle \in$ GME.
2. Suppose $G \in$ 3COLOR such that there exists a two coloring χ_2 for G but G is not one-colorable. Given this, we can see that we can use marriages to reduce G down to a clique of size two. Given the definition of G, $f(G) = \langle G, K_2 \rangle$. Since G can be reduced to a clique of size two, we can see that G can be transformed with marriages into G', which would be isomorphic to K_2. Thus, we can see that $\langle G, H \rangle \in$ GME.
3. Suppose $G \in$ 3COLOR such that there exists a one coloring χ_1 for G. Given this, we can see that we can use marriages to reduce G down to a clique of size one. Given the definition of G, $f(G) = \langle G, K_1 \rangle$. Since G can be reduced to a clique of size one, we can see that G can be transformed with marriages into G', which would be isomorphic to K_1. Thus, we can see that $\langle G, H \rangle \in$ GME.
4. Suppose $G \notin$ 3COLOR such that there does not exist a three coloring for G. Suppose that G is k-colorable, where k is the fewest colors that can be used to color G ($k > 3$). We can see that G can be reduced to a clique of size k. However, given the definition of $f(G)$, we can see that G cannot be reduced by marriages to K_3. Therefore, we can see that $\langle G, K_3 \rangle \notin$ GME.

Given these cases, we can see that $\langle G \rangle \in$ 3COLOR $\Leftrightarrow f(G) = \langle G, H \rangle \in$ GME. Thus, we can see that 3COLOR $\leq_m^p$ GME.

Given that GME $\in$ NP, 3COLOR $\leq_m^p$ GME, and 3COLOR is NP-complete, we can see that GME is also NP-complete. $\boxtimes$