COMP-170: Homework #5

Ben Tanen - March 5, 2017

Problem 4

Let $L = \{\langle M \rangle \mid L(M) = D5\}$. Prove L and $\overline{L}$ are both unrecognizable.

* * *

To show that L and $\overline{L}$ are both unrecognizable, we can prove $\overline{A_{TM}} \leq_m L$ and $\overline{A_{TM}} \leq_m \overline{L}$. We can directly prove $\overline{A_{TM}} \leq_m L$ by constructing a function $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_L$ such that $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. To prove $\overline{A_{TM}} \leq_m \overline{L}$, we can alternatively prove $A_{TM} \leq_m L$ (an equivalent statement) by constructing a function $f : \Sigma^*_{A_{TM}} \to \Sigma^*_L$ such that $x \in A_{TM} \Leftrightarrow f(x) \in L$. Once we show $\overline{A_{TM}} \leq_m L$ and $\overline{A_{TM}} \leq_m \overline{L}$, because $\overline{A_{TM}}$ is unrecognizable, we will have shown that both L and $\overline{L}$ are also unrecognizable.

First, we will show $\overline{A_{TM}} \leq_m L$ directly. To start, we claim $\overline{A_{TM}} \leq_m L$. To prove this claim, we will first show how $\overline{A_{TM}}$ and L relate to each other (using the "Tony Square").

	$\overline{A_{TM}}$	L
IN	M doesn't accept w	$L(M') = D5$
OUT	M accepts w	$L(M') \neq D5$

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{\overline{A_{TM}}} \to \Sigma^*_L$ be defined as follows:

f on input $\langle M, w \rangle$ outputs $\langle M' \rangle$, where M' is defined by:

> M' **on input** x**:**
>
> > If x is divisible by 5, $ACCEPT$
> >
> > Else, run M on w
> >
> > > If M accepts w, $ACCEPT$
> > >
> > > If M rejects w, $LOOP$

Given our definition of f, we claim that f is computable. Because M' is a valid Turing machine and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Consider the following two cases:

1. Suppose $\langle M, w \rangle \in \overline{A_{TM}}$ such that M doesn't accept w. We can see that M' always accepts everything in $D5$ so $D5 \subseteq L(M')$. Additionally, because M doesn't accept w,

we can see that M' will loop on every other input, so M' doesn't accept $x \notin D5$. Thus, since M' accepts all $x \in D5$ and rejects all $y \notin D5$, we can see $L(M') = D5$. Thus, we can see $\langle M' \rangle \in L$ when $\langle M, w \rangle \in \overline{A_{TM}}$.

2. Suppose $\langle M, w \rangle \notin \overline{A_{TM}}$ such that M accepts w. We again can see that M' always accepts everything in $D5$ so $D5 \subseteq L(M')$. However, because M does accept w, we can see that M' will additionally accept every other input. Thus, since M' accepts everything, we can see $L(M') \neq D5$. Thus, we can see $\langle M' \rangle \notin L$ when $\langle M, w \rangle \notin \overline{A_{TM}}$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in \overline{A_{TM}} \Leftrightarrow f(x) \in L$. Therefore, we can see that $\overline{A_{TM}} \leq_m L$.

Next, we will show $\overline{A_{TM}} \leq_m \overline{L}$ by proving $A_{TM} \leq_m L$ (an equivalent statement). To start, we claim $A_{TM} \leq_m L$. To prove this claim, we will first show how A_{TM} and L relate to each other (using the "Tony Square").

	A_{TM}	L
IN	M accepts w	$L(M') = D5$
OUT	M doesn't accept w	$L(M') \neq D5$

Now that we've built our "Tony Square", we can define our function. Let $f : \Sigma^*_{A_{TM}} \to \Sigma^*_L$ be defined as follows:

f on input $\langle M, w \rangle$ outputs $\langle M' \rangle$, where M' is defined by:

M' **on input** x**:**

If x is divisible by 5, run M on w

If M accepts w, *ACCEPT*

If M rejects w, *LOOP*

Else, *LOOP*

Given our definition of f, we claim that f is computable. Because M' is a valid Turing machine and f is a finite function (made up of finite steps), we can see that f is indeed computable.

Now, we can go through two cases to show that f correctly satisfies $x \in A_{TM} \Leftrightarrow f(x) \in L$. Consider the following two cases:

1. Suppose $\langle M, w \rangle \in A_{TM}$ such that M accepts w. Since M accepts w, we can see that M' will accept any input that is divisible by 5 and loop on everything else. Thus, $L(M') = D5$. Thus, we can see $\langle M' \rangle \in L$ when $\langle M, w \rangle \in A_{TM}$.

2. Suppose $\langle M, w \rangle \notin A_{TM}$ such that M doesn't accept w. Because M doesn't accept w, we can see that M' loops on all inputs. Thus, $L(M') = \varnothing \neq D5$. Thus, we can see

$\langle M' \rangle \notin L$ when $\langle M, w \rangle \notin A_{TM}$.

Given these two cases, we can thus see that f does indeed satisfy the claim $x \in A_{TM} \Leftrightarrow f(x) \in L$. Therefore, we can see that $A_{TM} \leq_m L$ and thus that $\overline{A_{TM}} \leq_m \overline{L}$.

Because we know $\overline{A_{TM}}$ is unrecognizable and because we have proven $\overline{A_{TM}} \leq_m L$ and $\overline{A_{TM}} \leq_m \overline{L}$, we can see that both L and $\overline{L}$ are unrecognizable. ⊠