COMP-170: Homework #10

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Problem 4

For languages A and B, let the *shuffle* of A and B be the language L, $L = \{w \mid w = a_1b_1a_2b_2 \ldots a_kb_k \text{ where } a_1, a_2, \ldots, a_k \in A \text{ and } b_1, b_2, \ldots, b_k \in B, \text{ and each } a_i, b_j \in \Sigma^*\}$.

Prove that if A and B are regular then L is regular.

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To prove L is regular if A and B are regular, we can use a proof by construction. Specifically, we can use DFAs for A and B to construct a NFA for L.

Suppose A and B are both regular languages. Given this, we know that there exists DFAs for both A and B. Let d_a and d_b be DFAs for A and B respectively. With d_a and d_b, we can construct a new NFA d by combining d_a and d_b. Specifically, we can connect the accepting state of d_a to the start state of d_b with an epsilon transition and vice versa. Finally, if we remove the accepting state of d_a (simply make it a normal non-accepting state) and just leave the accepting state of d_b, we are left with a NFA d for L (the shuffle of A and B).

Now consider any $w \in L$, where $w = a_1b_1a_2b_2 \ldots a_kb_k$ for $a_1, a_2, \ldots, a_k \in A$ and $b_1, b_2, \ldots, b_k \in B$. Given the individual substrings of w, we know that d_a would accept any a_i and d_b would accept any b_j (for $1 \leq i, j \leq k$). Thus, because w is constructed as a shuffle of A and B, we can see that d would validly consume a_1, epsilon transition, validly consume b_1, and then epsilon transition back to the start state of d (if $k > 1$). This loop would continue over for each pair of a_i and b_i of $i < k$. It would similarly occur one last time for the pair a_k and b_k, but because d would have consumed all of w, it wouldn't epsilon transition back to the start state but instead it would stay in the accepting state, thus accepting w.

If we consider any $w \notin L$, we can see that d would not accept w because, by definition, w would not be made of a shuffle between A and B strings. Thus, there must be some substring w_i of w that should have been either of an element of A or B but is not. In the former case, d_a wouldn't accept w_i. In the latter case, d_b wouldn't accept w_i. In either case, we can see that d would halt / reject on w_i, so we can see that d correctly wouldn't accept w.

Thus, since d accepts all $w \in L$ and rejects all $w \notin L$, we can see that d is indeed a NFA of L. Since such an NFA exists, we can thus see that L is indeed regular if A and B are regular. $\boxtimes$