COMP-170: Homework #9

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Problem 1

A Hamiltonian Cycle is a simple path beginning and ending at the same vertex that visits every node exactly once. Remember that in a simple path repeated edges are not allowed.

DHC = $\{\langle D \rangle \mid D$ is a directed graph that contains a Hamiltonian Cycle $\}$

HC = $\{\langle G \rangle \mid G$ is a undirected graph that contains a Hamiltonian Cycle $\}$

Prove that DHC $\leq_m^p$ HC

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To prove DHC $\leq_m^p$ HC, we will construct a function $f : \Sigma^*_{\text{DHC}} \to \Sigma^*_{\text{HC}}$ such that $D \in$ DHC $\Leftrightarrow f(D) \in$ HC. Once we show that f satisfies this condition, we will have proven that DHC $\leq_m^p$ HC.

Given this, we can define $f : \Sigma^*_{\text{DHC}} \to \Sigma^*_{\text{HC}}$ as follows:

f on input D outputs G, where G is defined as follows:

Start with some empty graph G.

For every node v_i in the directed graph D, add three nodes, $\mathrm{i}_i, \mathrm{m}_i, \mathrm{o}_i$ to G. Also add undirected edges $(\mathrm{i}_i, \mathrm{m}_i)$ and $(\mathrm{m}_i, \mathrm{o}_i)$.

For every edge (u, v) in D, add an edge $(\mathrm{o}_u, \mathrm{i}_i)$ to G.

Output G.

Given our definition of f, we claim that f is computable in polynomial time. We can see that we could construct a Turing machine M that outputs G when given D. M would iterate over each of vertex and add three vertices as well as add two edges, which takes $O(1)$ time per vertex. Next, M would add one edge to G for each edge in D, which takes $O(n^2)$ time. Therefore, we get $O(n) + O(n^2)$ time overall, so we can see f is indeed computable in polynomial time.

Next, we must verify that f satisfies the condition that $D \in$ DHC $\Leftrightarrow f(D) \in$ HC. In order to do this, consider the following two cases:

1. Suppose $D \in$ DHC, such that there is a Hamiltonian cycle in the directed graph D. When we expand each vertex v_i into three vertices with two edges between them, we can see that the Hamiltonian cycle in G would still be maintained because v_i must have had an edge going into it and out of it. Thus, there would be an edge going into i_i and out of o_i. Thus, locally at each vertex, we maintain the same path. Therefore, we can see that since the path is just extended (when we add more vertices), we can

see that if D had a Hamiltonian cycle, $f(D) = G$ would also have a Hamiltonian cycle (just undirected). Thus, $G \in \text{HC}$.

2. Suppose $G \in \text{HC}$, such that G is an undirected graph and G has a Hamiltonian cycle. Since G was constructed from some other graph D (by definition of f), we know that we can convert G back into D. Specifically, for a three pairing of vertices $\mathrm{i}_i, \mathrm{m}_i, \mathrm{o}_i$, convert the pairing back into a single vertex v_i, where all the neighbors of o_i are now neighbors of v_i (pointing at v_i) and vice versa for i_i. Because there was a Hamiltonian cycle in G, we can see that there must have been at least one vertex i_j connected to o_i that wasn't m_i and the same for i_i. Therefore, we can see that locally, the Hamiltonian cycle is maintained, now with directed edges. Thus, since this is maintained for every vertex in G, we can see that if $G \in \text{HC}$, D had a Hamiltonian cycle, so $D \in \text{DHC}$.

Given these two cases, we can thus see that f does indeed satisfy the claim $D \in \text{DHC} \Leftrightarrow G \in \text{HC}$. Therefore, we can see $\text{DHC} \leq_m^p \text{HC}$. ⊠