COMP-170: Homework #3

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Problem 5

Prove that $L = \{\langle M \rangle \mid M \text{ accepts input } 1011\}$ is undecidable.

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We would like to show that the language $L = \{\langle M \rangle \mid M \text{ accepts input } 1011\}$ is undecidable. In order to do this, we will use proof by contradiction. More specifically, we will assume that L is decidable and then use this (incorrect) fact to claim A_{TM} is also decidable, which we know is false. This will give us our contradiction.

Let's begin our proof by assuming that L is in fact decidable. Given this, we will say there exists some machine D_L that is able to decide L. Using D_L, we can define another machine D_{ATM} as follows:

> **D_{ATM} on input $\langle M, w \rangle$:**
>
> > Define the following:
> >
> > > **M' on input x:**
> > >
> > > > Run M on w
> > > >
> > > > > If M accepts w, $ACCEPT$
> > > > >
> > > > > Otherwise, $REJECT$
> > > >
> > > > END
> >
> > Run D_L on $\langle M' \rangle$
> >
> > > If D_L accepts $\langle M' \rangle$, $ACCEPT$
> > >
> > > If D_L rejects $\langle M' \rangle$, $REJECT$
> >
> > END

We will now claim that D_{ATM} decides the language A_{TM}. To show this, suppose we have any machine M and we are given some input w. When we run D_{ATM} on $\langle M, w \rangle$, we will first construct our M' using $\langle M, w \rangle$ and run D_L on M'. This will tell us if M' accepts the input 1011. Because D_L promises to tell us if M' accepts 1011, we know that running M' on 1011 must also give us an answer. If M' on 1011 did not give us such an answer, D_L would never sufficiently know if M' accepts 1011. Therefore, we know M' gives some answer when given 1011.

Now consider the following cases:

1. Suppose M accepts the input w. This means that when we run D_{ATM} on $\langle M, w \rangle$, we will find out if M' accepts 1011 (through our use of D_L). In order to get this answer,

we will run M on w. Because running M' on 1011 must give us an answer, we can see that running M on w must either $ACCEPT$ or $REJECT$. Since we are considering the case that M accepts w, we can see that M will accept w, which will cause M' to accept 1011, which will cause D_L to accept $\langle M' \rangle$, which will cause D_{ATM} to correctly accept $\langle M, w \rangle$.

2. Suppose M rejects the input w. Using the same logic as case 1, we can see that since M rejects w, we will see the chain reaction of M rejects $w \to M'$ rejects $1011 \to D_L$ rejects $M' \to D_{ATM}$ correctly rejects $\langle M, w \rangle$.

3. Suppose M loops on w. Like case 2, we can see that since M loops on w, we will see the chain reaction of M loops on $w \to M'$ rejects $1011 \to D_L$ rejects $M' \to D_{ATM}$ correctly rejects $\langle M, w \rangle$.

Thus, through these cases, we can see that if M accepts w, D_{ATM} accepts $\langle M, w \rangle$ and if M doesn't accept w, D_{ATM} rejects $\langle M, w \rangle$. This shows that D_{ATM} decides $A_{TM} = \{\langle M, w \rangle \mid M \text{ accepts input } w\}$.

However, we have previously proven that A_{TM} is an undecidable language so there can be no such D_{ATM} to decide the language. This provides us with a contradiction. Thus, since there can exist no D_{ATM} to decide A_{TM}, we can see there can also not exist any decider D_L to decide our language L. Therefore, we can see that L must indeed be an undecidable language. ⊠