17/10/2013 (5)

SUMMARY OF LESSON ONE

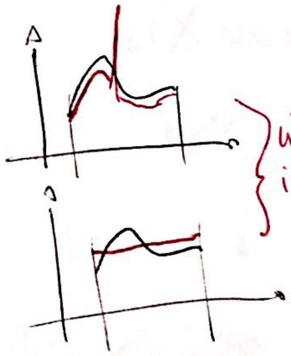
KEY CONCEPTS ; Pb: $F(x,d)=0$ Approximation $F_n(x_n,d_n)=0$

- Stability : i) $\forall d, \exists! x$ ii) $\|\delta x\| \leq k \|\delta d\|$
- Consistency : $F_n(x,d) \to 0$
- Convergence : $x_n \to x$
- Lax-Richtmyer : for CONSISTENT approximations

  STABILITY $\Leftrightarrow$ CONVERGENCE

Why is this important ? ERROR !

Sources of error :
- perturbation in data $\Rightarrow \delta d$
  (controlled by stability of cont. Pb.)
- bad approximation schemes
  (controlled by consistency of F_n to F)
- bad choices of approximating spaces
  (controlled by stability of discrete Pb.)
  $\Leftrightarrow$ convergence of " Pb.



which one is better?

How do we measure the ERROR ? Norms!

5 minutes introduction to Vector Spaces, Normed Spaces, Hilbert Spaces

Vector space X : set of elements, with two operations

- $\oplus$ sum : $X \times X \longrightarrow X$ $\qquad w = u+v$
- $\odot$ scale : $X \times \mathbb{R} \longrightarrow X$ $\qquad v = \alpha u$

Properties: $\forall u,v,w \in X$, $\forall \alpha \in \mathbb{R}$

- $u+v \in X$, $\alpha u \in X$
- $u+v = v+u$, $u+(v+w) = (u+v)+w$, $\alpha(u+v) = \alpha u + \alpha v$

Approximation in a subset of X, Banach

- $f \in X$, given
- $M \subseteq X$

$p \in M$ is best approximation of f in M when

$$\|f-p\| = E(f) := \inf_{q \in M} \|f-q\|$$

Q: - Does it exist?
- Is it unique?
- How do we find it?

Case 1 M is a finite dimensional subspace of X

$\exists$ v_n such that $M = \operatorname{span}\{v_n\}$

$\to \exists\, p$, B.A. of f in M finite dimensional

If X is strictly convex, p is unique

$$f \neq g, \quad \|f\| = \|g\| = 1, \quad 0 < \theta < 1$$

$$\|\theta f + (1-\theta) g\| < 1$$

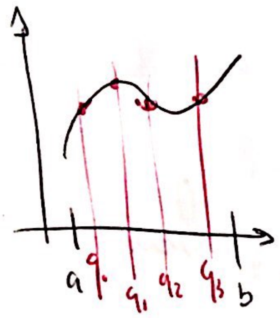
prove $\exists!\, p$.

Uniqueness: $p_1 \neq p_2$

$$\|f-p_1\| = \|f-p_2\| = E(f) \leq \left\|f - \tfrac{1}{2}(p_1+p_2)\right\|$$

$$= \left\|\tfrac{1}{2}(f-p_1) + \tfrac{1}{2}(f-p_2)\right\| < \tfrac{1}{2}\|f-p_1\| + \tfrac{1}{2}\|f-p_2\| = E(f)$$ impossible

Lagrange Interpolation



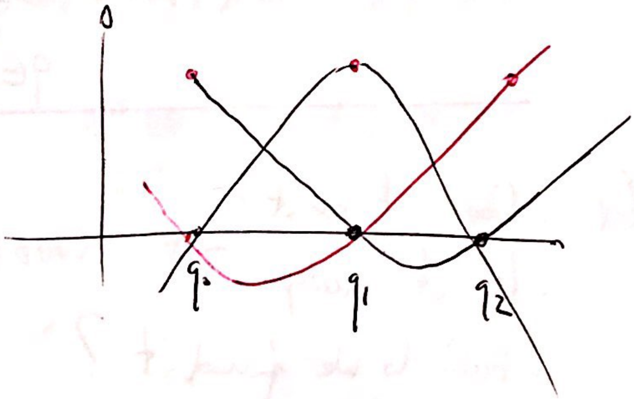
Given $f \in C^0([a,b])$, find

$p \in \mathbb{P}^n := \operatorname{span}\{x^i\}_{i=0}^n$ s.t.

$p(q_i) = f(q_i)$ for some $(n+1)$ points q_i

- Construction:

$$e_i(x) := \prod_{\substack{i \neq j \\ j=0}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$



$$e_i(q_j) = \delta_{ij}$$

$$\Rightarrow \quad p = \mathcal{L}^n f := \sum_{i=0}^{n} f(q_i) e_i$$

Theo $f \in C^{(n+1)}$, $x_i \in \mathring{I}, \forall i \Rightarrow \exists \xi \in \mathring{I}$ s.t.

$$(f - \mathcal{L}^n f)(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\, \omega(x) \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_\infty \|\omega(x)\|_\infty$$

where $\omega(x) = \prod_{i=0}^{n} (x - q_i)$

DIM $p = \mathcal{L}^n f$ $\quad G(t) = (f(t) - p(t))\, \omega(x) - (f(x) - p(x))\, \omega(t)$

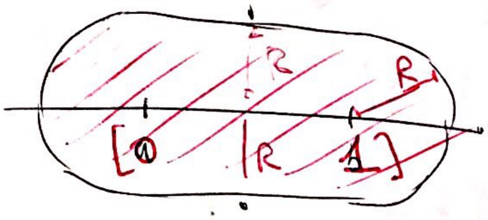
it has $n+2$ zeros: $t = q_i$, $t = x$

$$\Rightarrow \text{Rolle} \quad \exists \xi \text{ s.t. } \frac{d^{n+1}}{dt^{n+1}} G(\xi) = 0$$

$$f(x) - p(x) = \frac{f^{n+1}}{(n+1)!}(\xi)\, \omega(x)$$

$$\Rightarrow \| f(x) - p(x) \|_\infty \leq \| f^{(n+1)} \|_\infty \left\| \frac{\omega(x)}{(n+1)!} \right\|_\infty$$

THEO 2 Let f be analitically extendible in a oval $\mathcal{O}(a,b,R)$ with $R>0$.



q_i : $(n+1)$ points distinct in $[a,b]$

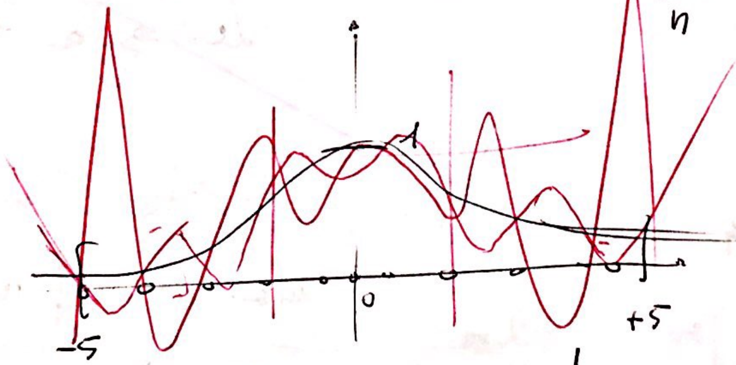
$p = L^n f$

$$\Rightarrow | f^{(n+1)}(\xi) | \leq \frac{(n+1)!}{R^{n+1}} \| f \|_{\infty, \bar{B}(\xi, R)}$$

that is $$\| f - p \|_{\infty, [a,b]} \leq \| f \|_{\infty, \mathcal{O}(a,b,R)} \left(\frac{|b-a|}{R} \right)^{n+1}$$

— Counter Example of Runge

Ex1 $I = [-5, 5]$. $x_i = -5 + \frac{10 i}{n}$ $\boxed{f = \frac{1}{1+x^2}}$



$\exists\, K \approx 3.6$ s.t. $\lim\limits_{n\to\infty} \left| (L^n f - f)(x) \right| = 0 \iff |x| < K$

Even Worse!

Ex 2 $L^n |x|$ does not converge $\forall x \neq -1, 0, 1$

How distant are we from B.A. ?

$$\| L^n \| := \sup_{\substack{f C(I) \\ \|f\|_\infty < 1}} \| L^n f \|_\infty$$

Then, since $\forall q \in P_n \quad L_n q = \cancel{Id} \; q$

$$\| f - L^n f \| = \| f - p + L^n (f-p) \| \leq$$

$$\| f - p \| + \| L^n (f-p) \|$$

$$\leq (1 + \| L^n \|) \| f - p \| \quad \forall p \in P^n$$

$$\leq (1 + \| L^n \|) \| f - B.A.(f) \|$$

$\| L^n \|$?

$$\Rightarrow \| L^n \| := \sup_{\substack{f \in C(I) \\ \|f\|_\infty \leq 1}} \| \sum_i l_i^n f(x_i) \|_\infty \leq \| \underbrace{\sum_i | l_i^n (x) |}_{\text{Lebesgue Function}} \|_\infty := \Lambda(X)$$

Erdos

q_0

$q_0 \quad q_1$

$q_0 \quad q_1 \quad q_2$

$q_0 \quad q_1 \quad q_2 \quad q_3$

$- \; - \; - \; - \; - \; - \; -$

X: Infinite triangular matrix of interpolation points

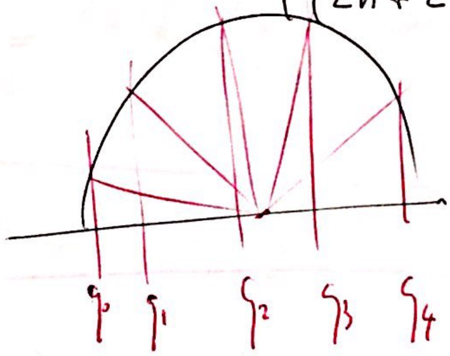
$\Lambda(X) \to \infty$

THEO: $\boxed{\forall X, \exists c > 0 \text{ s.t. } \| L^n \| \geq \frac{2}{\pi} \log(n+1) - c}$

Faber $\forall X, \exists f$ s.t. $L^n_X f \not\to f$

Chebishev points:

$$\underset{q_i}{\arg\min} \|\mathcal{L}_{q_i\,n}(x)\|_\infty \qquad I = [-1,1] \qquad x_i = \cos\left(\frac{(2i+1)\pi}{(2n+2)}\right)$$



$$\|\mathcal{L}_{n\,\text{chebishev}}\|_\infty \leq \frac{2}{\pi} \log(n+1) + 1$$

ovvero, $\exists c$ s.t.

$$\frac{2}{\pi} \log(n+1) - c \leq \|\mathcal{L}_n\|_\infty \leq \frac{2}{\pi} \log(n+1) + 1$$

Interpolation does not work well ...

Chebishev : $\Lambda \leq \frac{2}{\pi} \log(n+1) + 1$

Lagrange equispaced: $\Lambda \leq \frac{2^{n+1}}{e\,n \log n}$

is All lost ?

NO

Let's give one positive result Minimax Approx. Theorem

WEIERSTRASS Approx Theorem

Let $f \in C(I)$, $\varepsilon > 0$. Then there exist n and a polinomial $p \in \mathbb{P}^n$ such that $\|f - p\|_\infty \leq \varepsilon$

Proof : BERNSTEIN $\quad I = [0,1] \quad (1-x+x)^n$

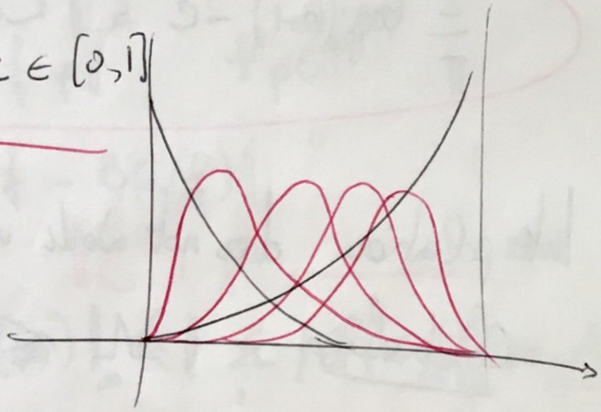
Define $\quad (B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$

$\forall x$, $(B_n f)(x)$ is a weighted average of the $(n+1)$ values $f\left(\frac{k}{n}\right)$

Properties of $\underbrace{N_i(x) := \binom{n}{i} x^i (1-x)^{n-i}}_{\text{BERNSTEIN POLYNOMIALS}}$

$$\sum_{i=0}^{n} N_i(x) = 1 \quad \forall x \in [0,1]$$

$$B_n f := \sum_{k=0}^{n} N_k(x)\, f\left(\frac{k}{n}\right)$$



$$N_i(x) \geq 0 \quad \forall x \in [0,1]$$

$N_i\left(\frac{j}{n}\right) \neq \delta_{ij}$!!! $\Leftarrow$ Different from Lagrange

- B_n is a linear positive operator : $B_n f \geq 0$ if $f \geq 0$
- $\forall p \in \mathbb{P}_2$, $B_n p \to p$!!! $L_n p \equiv p$ for $n \leq 2$

$f_i = x^i$

Direct computations... $\forall f_0 \in \mathbb{P}_0$, $f_1 \in \mathbb{P}_1$, $f_2 \in \mathbb{P}_2$

$n \geq 2 \quad B_n f_0 = f_0, \quad B_n f_1 = f_1, \quad B_n f_2 = \frac{n-1}{n} f_2$

$B_n 1 = 1, \quad B_n x = x, \quad B_n x^2 = \left(\frac{n-1}{n}\right)x^2 + \left(\frac{1}{n}\right)x$

$\neq x^2$

<u>...rem</u> Let B_n be a sequence of

i) linear positive operators

such that ii) $B_n f$ converges uniformly to f $\forall f \in P^2$

<u>Then</u> $B_n f$ converges uniformly to f $\forall f \in C(I)$

<u>Proof</u>

IDEA: $\forall f \in C(I)$, $\forall x_0 \in I$, we can find a quadratic function $q > f$ s.t. $|q(x_0) - f(x_0)| < \varepsilon$.

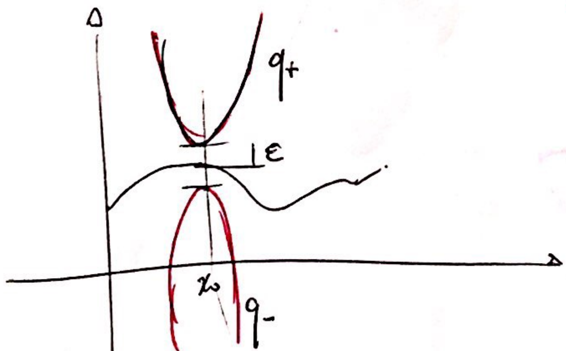
For $n > \bar{n}$, $|B_n q(x_0) - q(x_0)| < \varepsilon$, that is $|B_n f(x_0) - f(x_0)| < \varepsilon$

- f continuous on I (compact)

$\Rightarrow$ f is uniformly continuous.

$\forall \varepsilon > 0$, $\exists \delta$ s.t. $|f(x_1) - f(x_2)| \le \varepsilon$ if $|x_1 - x_2| < \delta$

$\Rightarrow$ for any x_0 set $q_{\pm}(x) = f(x_0) \pm \left(\varepsilon + \frac{2\|f\|_\infty}{\delta^2}(x - x_0)^2\right)$



$q(x) = a + bx + cx^2$ with

$|a|, |b|, |c| \le M$

where M depends on $\|f\|$, ε, δ but <u>not</u> on x_0

$q_+(x) \geq f(x)$ $\forall x$

$q_-(x) \le f(x)$ $\forall x$

Then choose N large enough s.t.

$\|B_n q_\pm^i - q_\pm^i\|_\infty \le \frac{\varepsilon}{M}$ for $i = 0, 1, 2$

$\Rightarrow \quad \| B_n q_\pm - q_\pm \| \leq 3\varepsilon$

Then $\quad B_n f(x_0) \leq B_n q_+(x_0) \leq q_+(x_0) + 3\varepsilon = f(x_0) + 4\varepsilon$

$\quad B_n f(x_0) \geq B_n q_-(x_0) \geq q_-(x_0) - 3\varepsilon = f(x_0) - 4\varepsilon$

That is $\quad -4\varepsilon \leq B_n f(x_0) - f(x_0) \leq 4\varepsilon \qquad \forall x_0$

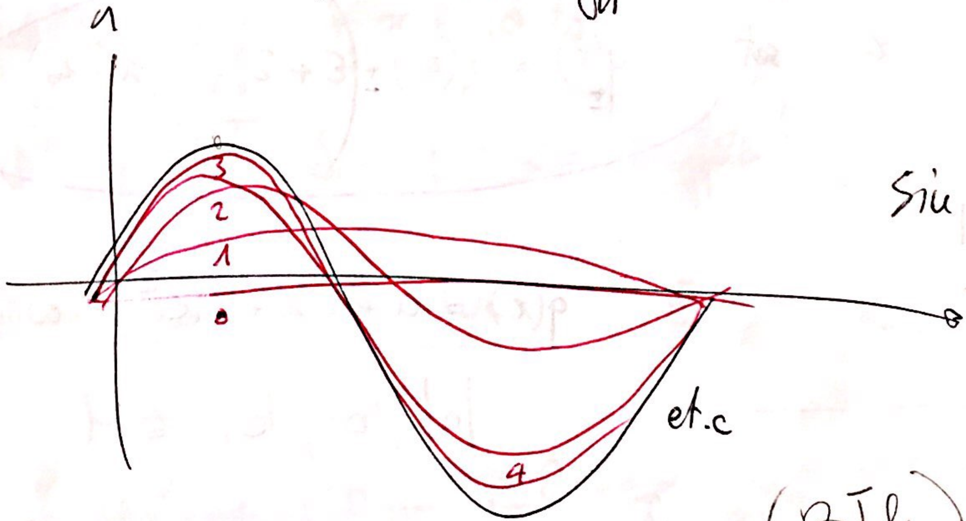
$\Rightarrow \quad \| B_n f(\cancel{x_0}) - f \|_\infty \leq 4\varepsilon$

Very robust but very slow

$$\| f - B_n f \|_\infty = O(1/n)$$

This holds also for all C^2 functions!

$$\| f - B_n f \|_\infty \leq \frac{1}{8n} \| f'' \|_\infty$$



$(B_1 f_1)(x_7) \neq f(x_7)$

$w_7 f_7 + w_0 f_0 + w_1 f$

$$e^i(e_j) \neq \delta^i_j$$