4.1.5

(page 124)

(a) If a and b are consecutive integers, show that $a^2+b^2+(ab)^2$ is a perfect square.

proof)

Let $b = a + 1$ and substituting into the original expression we arrive at:

$$a^2 + (a + 1)^2 + (a^2 + a)^2 = a^2 + a^2 + 2a + 1 + a^4 + 2a^3 + a^2$$

$$a^4 + 2a^3 + 2a^2 + a^2 + 2a + 1 = a^4 + a^2 + 1 + 2a^3 + 2a^2 + 2a$$

We recognize that $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$ thus,

$$a^4 + a^2 + 1 + 2a^3 + 2a^2 + 2a = (a^2 + a + 1)^2$$

which is a perfect square.

(b)If $2a$ is the harmonic mean of b and c (*i.e.* $2a = 2/(1/b + 1/c)$), show that the sum of the squares of the three numbers a,b, and c is the square of a rational number.

proof)

We are to prove that

$$a^2 + b^2 + c^2 = \frac{p^2}{q^2}, p, q \in \mathbb{Z}, q \neq 0$$

. We substitute $2a = \frac{1}{(\frac{1}{b}+\frac{1}{c})}$ into the expression, and we get:

$$a^2 + b^2 + c^2 = (\frac{1}{(\frac{1}{b} + \frac{1}{c})})^2 + b^2 + c^2$$

$$(\frac{bc}{bc(\frac{1}{b} + \frac{1}{c})})^2 + b^2 + c^2 = (\frac{bc}{c + b})^2 + b^2 + c^2$$

$$\frac{b^2c^2}{(c + b)^2} + b^2 + c^2 = \frac{b^2c^2}{(b + c)^2} + \frac{b^2(b + c)^2}{(b + c)^2} + \frac{c^2(b + c)^2}{(b + c)^2}$$

$$\frac{b^2(b + c)^2 + b^2c^2 + c^2(b + c)^2}{(b + c)^2} = \frac{b^2(b^2 + 2bc + c^2) + b^2c^2 + c^2(b^2 + +2bc + c^2)}{(b + c)^2}$$

$$\frac{b^4 + 2b^3c + b^2c^2 + b^2c^2 + b^2c^2 + 2bc^3 + c^4}{(b + c)^2} = \frac{b^4 + c^4 + b^2c^2 + 2b^3c + 2bc^3 + 2b^2c^2}{(b + c)^2}$$

We recognize that $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$ thus,

$$\frac{b^4 + c^4 + b^2c^2 + 2b^3c + 2bc^3 + 2b^2c^2}{(b + c)^2} = \frac{(b^2 + c^2 + bc)^2}{(b + c)^2}$$

(c)If N differs from two successive squares between which it lies by x and y respectively, prove that $N-xy$ is a square.

proof)

Let's assume the two successive squares are a^2 and $(a+1)^2$ and N is placed such that:

$$N - a^2 = x$$

$$(a+1)^2 - N = y$$

and we need to show that $N - xy$ is a square.

$$N - xy = N - (N - a^2)((a+1)^2 - N)$$

$$N-(N-a^2)(a^2+2a+1-N) = N-(a^2N+2aN+N-N^2-a^4-2a^3-a^2+a^2N)$$

$$N-(a^2N+2aN+N-N^2-a^4-2a^3-a^2+a^2N) = N-a^2N-2aN-N+N^2+a^4+2a^3+a^2-a^2N$$

$$N^2 + a^4 + a^2 - 2a^2N - 2aN + 2a^3$$

We recognize that $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$ thus,

$$N^2 + a^4 + a^2 - 2a^2N - 2aN + 2a^3 = (N - a^2 - a)^2$$

which is a square.

Equivalently, we can observe that $N = a^2 + x$ and thus,

$$N - xy = a^2 + x - xy$$

$$a^2 + x - xy = a^2 + x - x((a+1)^2 - N)$$

$$a^2 + x - x((a+1)^2 - N) = a^2 + x - x(a^2 + 2a + 1 - a^2 - x)$$

$$a^2 + x - x(a^2 + 2a + 1 - a^2 - x) = a^2 + x - x(2a + 1 - x)$$

$$a^2 + x - 2ax - x + x^2 = a^2 - 2ax + x^2$$

$$a^2 - 2ax + x^2 = (a - x)^2$$

which is a square.

Which we observe to be equivalent to the result of the first approach:

$$(N - a^2 - a)^2 = (a - x)^2$$

$$(a^2 + x - a^2 - a)^2 = (a - x)^2$$

$$(x - a)^2 = (a - x)^2$$