4.1.5

(page 124)

(a) If a and b are consecutive integers, show that $a^2+b^2+(ab)^2$ is a perfect square.

proof)

Let $b = a + 1$ and substituting into the original expression we arrive at:

$$a^2 + (a+1)^2 + (a^2+a)^2 = a^2 + a^2 + 2a + 1 + a^4 + 2a^3 + a^2$$

$$a^4 + 2a^3 + 2a^2 + a^2 + 2a + 1 = a^4 + a^2 + 1 + 2a^3 + 2a^2 + 2a$$

We recognize that $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$ thus,

$$a^4 + a^2 + 1 + 2a^3 + 2a^2 + 2a = (a^2 + a + 1)^2$$

which is a perfect square.

(b)If $2a$ is the harmonic mean of b and c (*i.e.* $2a = 2/(1/b + 1/c)$), show that the sum of the squares of the three numbers a,b, and c is the square of a rational number.

proof)

We are to prove that

$$a^2 + b^2 + c^2 = \frac{p^2}{q^2}, p, q \in \mathbb{Z}, q \neq 0$$

. We substitute $2a = \frac{1}{(\frac{1}{b}+\frac{1}{c})}$ into the expression, and we get:

$$a^2 + b^2 + c^2 = (\frac{1}{(\frac{1}{b}+\frac{1}{c})}) + b^2 + c^2$$