4.1.6

(page 124)

Prove that there are infinitely many natural numbers a with the following property: The number $n^4 + a$ is not prime for any natural number n.

proof)

We need to show that $n^4 + a$ can be factorized into p and q such that $p, q \neq 1$ First, since a and n are natural numbers then $a, n \geq 1$, then we need to recognize that $x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$ (Sophie Germain's identity). Thus, if $a = 4k^4, k \in \mathbb{N}$ then $n^4 + 4k^4$ can be factored out as following:

$$n^4 + 4k^4 = n^4 + 4k^4 + 4n^2k^2 - 4n^2k^2$$

$$n^4 + 4k^4 + 4n^2k^2 - 4n^2k^2 = (n^2 + 2k^2)^2 - (2nk)^2$$

$$(n^2 + 2k^2)^2 - (2nk)^2 = (n^2 + 2nk + 2k^2)(n^2 - 2nk + 2k^2)$$

$$(n^2 + 2nk + 2k^2)(n^2 - 2nk + 2k^2) = ((n + k)^2 + k^2)((n - k)^2 + k^2)$$

Let $p = (n + k)^2 + k^2$, $q = (n - k)^2 + k^2$ then $p, q > 0$ thus the factors are always positive. Also, natural numbers are closed under the used operations which proves that the factors are natural numbers. Then we need to prove that no factor can be equal to one (i.e. $p \neq 1, q \neq 1$). Since $n, k \in \mathbb{N}$ then $p \geq 3$ and $q \geq 1$ thus $p \neq 1, \forall k \in \mathbb{N}$, thus we need to set a constraint on k such that $q \neq 1$ or equivalently $k^2 \geq 4$ (Since k^2 is the smallest possible q) thus, $k \geq 2$ must hold for $q \neq 1, \forall n \in \mathbb{N}$.