SICP 1.13 Solution

Xueqiao Xu

https://github.com/qiao/sicp-solutions

Problem:

$$Fib(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ Fib(n-1) + Fib(n-2) & \text{otherwise} \end{cases}$$

Prove that $Fib(n)$ is the closest integer to $\phi^n/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$.

Proof:

First, use induction to prove that

$$Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}} \quad \text{where } \psi = \frac{1-\sqrt{5}}{2} \tag{1}$$

For $n < 2$, we have:

$$Fib(0) = 0 = \frac{\phi^0 - \psi^0}{\sqrt{5}} \tag{2}$$

$$Fib(1) = 1 = \frac{\phi^1 - \psi^1}{\sqrt{5}} \tag{3}$$

Assume that (1) is true for k and $k+1$, where $k \in \mathrm{N}, k \geq 2$.
Then, for $n = k+2$, we have:

$$\begin{aligned} Fib(k+2) &= Fib(k) + Fib(k+1) \\ &= \frac{\phi^k - \psi^k}{\sqrt{5}} + \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}} \\ &= \frac{\phi^k(\phi+1) - \psi^k(\psi+1)}{\sqrt{5}} \end{aligned} \tag{4}$$

Note that ϕ and ψ satisfy the following equations:

$$\phi^2 = \phi + 1 \tag{5}$$

$$\psi^2 = \psi + 1 \tag{6}$$

According to (5) and (6), equation (4) can be further reduced into:

$$\begin{aligned} Fib(k+2) &= \frac{\phi^k(\phi^2) - \psi^k(\psi^2)}{\sqrt{5}} \\ &= \frac{\phi^{k+2} - \psi^{k+2}}{\sqrt{5}} \end{aligned} \tag{7}$$

Therefore, (1) is true for every $k \in \mathrm{N}$

Next, we need to prove:

$$|\frac{\psi^n}{\sqrt{5}}| < \frac{1}{2} \qquad (\forall n \in \mathrm{N}^*) \tag{8}$$

Note that

$$\frac{\sqrt{5}-1}{2} < \frac{1}{2} \tag{9}$$

$$|\psi^{n+1}| = |\psi^n| \cdot \frac{\sqrt{5}-1}{2} \tag{10}$$

Therefore,

$$|\psi^{n+1}| < |\psi^n| \qquad (\forall n \in \mathrm{N}^*) \tag{11}$$

Then, since

$$\frac{|\psi^0|}{\sqrt{5}} < \frac{1}{2} \tag{12}$$

It leads to the fact that (8) is true.

Combining (1) and (12), we have:

$$|\frac{\phi^n}{\sqrt{5}} - Fib(n)| = \frac{\psi^n}{\sqrt{5}} < \frac{1}{2} \tag{13}$$

Therefore, $Fib(n)$ is the closest integer to $\phi^n/\sqrt{5}$.

Q.E.D.