Ends

A.P. Neate

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1 Ends

In this section I will introduce the notions of ends and coends. These are objects which can be thought of as generalisations of limits and colimits respectively. In particular ends are useful to us because they allow us another way to describe natural transformations in (enriched) categories. Ends can be thought of as a universal wedge so I will start by defining a wedge.

Wedges

Definition 1.1. Let C and D be categories with functor $T(-,-)\colon C^{\mathrm{op}} \times C \to D$. A ***wedge*** of T consists of an object $e \in \mathrm{ob}(D)$ and a collection of morphisms $\{\omega_a \colon e \to T(a,a)\}_{a \in \mathrm{ob}(C)}$ such that for every morphism $f\colon a \to b$ in C the following diagram commutes.

$$\begin{tikzcd} e \arrow[r, "\omega_a"] \arrow[d, "\omega_b"'] & T(a,a) \arrow[d, "T(\mathrm{id}_b, f)"] \\ T(b,b) \arrow[r, "T(f,\mathrm{id}_a)"'] & T(a,b) \end{tikzcd}$$

Example 1.2. Let T be the hom functor $\hom_C\colon C^{\mathrm{op}} \times C \to \mathbf{Set}$. For all morphisms $f\colon a \to b$, a wedge (e,ω) of T is required by definition to satisfy $\omega_a \circ \hom(\mathrm{id}_a, f) = \hom(f, \mathrm{id}_b) \circ \omega_b$. Re-written that is $\omega_a \circ f = f \circ \omega_b$. Note that this is exactly the condition for a collection of morphisms ω to comprise a natural transformation from id_C to itself. Hence, every object of e corresponds to some natural transformation $\mathrm{id}_C \to \mathrm{id}_C$.

Ends

Now we know what a wedge is we can provide the definition of an end.

Definition 1.3. Let $T(-,-)\colon C^{\mathrm{op}} \times C \to D$ be a functor of categories C and D. The ***end*** of T is a wedge (E,Ω) which, for any wedge (e,ω) and object $a \in \mathrm{ob}(C)$, satisfies the universal property such that the following diagram commutes.

$$\begin{tikzcd}
e \arrow[d, dashed, "!\exists"'] \arrow[dr, "\omega_a"] & \\
E \arrow[r, "\Omega_a"'] & T(a,a)
\end{tikzcd}$$

The object of an end of T is denoted $\int_{c \in C} T(c,c)$. Now we will take at look at some examples.

Example 1.4.

1. Consider T as the hom functor $\hom_C \colon C^{\mathrm{op}} \times C \to \mathbf{Set}$ as before. We know that the wedges correspond to sets of natural transformations (possibly with duplicates). The universal property ensures that we have all natural transformations and they only appear once and hence $\int_{c \in C} \hom(c,c) \cong \mathrm{NT}(\mathrm{id}_C, \mathrm{id}_C)$.

2. Let F and G be functors $C \to D$ then consider the functor $T = \hom_D(F^{\mathrm{op}}(-), G(-)) \colon C^{\mathrm{op}} \times C \to \mathbf{Set}$ where F^{op} is the functor $C^{\mathrm{op}} \to D^{\mathrm{op}}$ induced by F. For reasons similar to the first example, the end $\int_{c \in C} T(c,c)$ is the set of natural transformations $\mathrm{NT}(F,G)$.

3. Let T be a functor which does nothing in its first component and sends its second component to the image under a functor $F \colon J \to C$, i.e. $T(a,b) = F(b)$. To see what is happening here let us substitute this functor into the diagram which defines the end.

$$\begin{tikzcd}
e \arrow[d, "!\exists"] \arrow[dr, "\omega_a"] \arrow[dd, bend right=60, "\omega_b"'] & \\
E \arrow[r, "\Omega_a"'] \arrow[d, "\Omega_b"] & F(a) \arrow[d, "F(f)"] \\
F(b) \arrow[r, "F(\mathrm{id}_b)"'] & F(b)
\end{tikzcd}$$

Notice that if the above diagram commutes for any a, b, and morphism f between them then when we identify the objects $F(b)$ we have recovered the definition of a limit E of some diagram F over an indexing category J. It is in this sense which ends generalise limits.

4. Much like how some functors don't admit a limit, some ends do not exist. For example in ...

As with a many definitions in category theory, ends have a dual notion.

Definition 1.5. Let C and D be categories with functor $T(-,-)\colon C^{\mathrm{op}} \times C \to D$. A ***cowedge*** of T consists of an object $e \in \mathrm{ob}(D)$ and a collection of morphisms $\{\lambda_a \colon T(a,a) \to e\}_{a\in \mathrm{ob}(C)}$ such that for every morphism $f\colon b \to a$ in C the following diagram commutes.

$$\begin{tikzcd}
e & T(a,a) \arrow[l, "\lambda_a"'] \\
T(b,b) \arrow[u, "\lambda_b"] & T(a,b) \arrow[l, "T(f,\mathrm{id}_b)"] \arrow[u, "T(\mathrm{id}_a,f)"']
\end{tikzcd}$$

Much like with ends, a coend is a universal cowedge.

Definition 1.6. Let $T(-,-)\colon C^{\mathrm{op}} \times C \to D$ be a functor of categories C and D. The ***coend*** of T is a cowedge (E,Λ) which, for any cowedge (e,ω) and object $a \in \mathrm{ob}(C)$, satisfies the universal property such that the following diagram commutes.

$$\begin{tikzcd}
e & \\
E \arrow[u, dashed, "!\exists"] & T(a,a) \arrow[l, "\Lambda_a"] \arrow[ul, "\lambda_a"']
\end{tikzcd}$$

Example 1.7.

1. Much like how we can define limits as ends in Example 1.4 we can define a colimit of a functor $F\colon C \to D$ by defining $T(a,b) = F(b)$. In exactly the same way as the limit case we then have that the colimit of F is the coend $\int^{c\in c} T(c,c)$.

2. Lets consider the co-end of $\hom_{\mathbf{Set}}(a,b)\colon \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \to \mathbf{Set}$. Since the coend $E := \int^{c\in\mathbf{Set}} \hom(c,c)$ contains the image of all morphisms λ_a we can consider E as containing the disjoint union $\bigsqcup_{c\in C} \hom(c,c)$ under some equivalence relation $\sim$. Our cowedge diagram exactly says that for every morphism $f\colon b \to a$ we need $f \circ \lambda_a = \lambda_b \circ f$ and so we need $(\alpha\colon a \to a) \sim (\beta\colon b \to b)$ whenever there exists a morphism $f\colon b \to a$ such that $f\alpha = \beta f$. The universal property for a coend tells us gives us that $\sim$ is the smallest such relation where this is true and also that E cannot contain any extra elements and so $E = \bigsqcup_{c\in C} \hom_{\mathbf{Set}}(c,c)/\sim$.

3. For a group G, consider the one object category BG where the morphisms are group elements composed by multiplication. The coend of $\hom_{BG}(a,b) : BG^{\mathrm{op}} \times BG \to \mathbf{Set}$ as above is the set $\bigsqcup_{c\in C} \hom_{BG}(c,c)/\sim$ where $(\alpha\colon a \to a) \sim (\beta\colon b \to b)$ if and only if there exists $f : b \to a$ such that $f\alpha = \beta f$. For this category however, there is only one hom set and $f\alpha = \beta f$ implies $\alpha = f^{-1}\beta f$ which is exactly to say that $\alpha \sim \beta$ if they share a conjugacy class in G. Hence, the coend $\int^{c\in BG} \hom_{BG}(c,c)$ is the set of conjugacy classes of G.

2 Enriched Ends?