coding of moving images. The algorithm has not been compared to adaptive quantization, since we feel that both should be cooperating rather than competing schemes. Further interests of future research are general stability conditions for the correction factor and the optimum choice of the window for the short-time expectation operator.

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References

[1] A. Kikuchi, S. Omatu, and T. Soeda, "Applications of adaptive digital filtering to the data processing for environmental systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-27, pp . 790–803, 1979.

[2] M. G. Larimore, J. R. Treichler, and C. R. Johnson, Jr., "SHARF: An algorithm for adapting IIR digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 428–440, Aug. 1980.

[3] N. J. Bershad, P. L. Feintuch, F. A. Reed, and B. Fisher, "Tracking characteristics of the adaptive line enhancer-response to a linear chirp signal in noise," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 504–516, Oct. 1980.

[4] J. B. O'Neal, "Predictive quantizing systems (differential pulse code modulation) for the transmission of television signals," *Bell Syst. Tech. J.*, vol. 45, pp. 689–721, May–June 1966.

[5] D. J. Connor, R. F. W. Pease, and W. G. Scholes, "Television coding using two-dimensional spatial prediction," *Bell Syst. Tech. J.*, vol. 50, pp. 1049–1061, 1971.

[6] H. G. Musmann, "A comparison of extended coding schemes for video signals," in *Proc. 1974 IEEE Int. Zurich Sem. Digital Commun.*, Mar. 12–15, 1974, pp. C1(1)–C1(7).

[7] J. D. Gibson, "Adaptive prediction in speech differential encoding systems," *Proc. IEEE*, vol. 68, pp. 488–525, 1980.

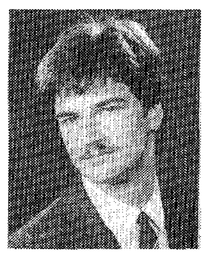
[8] A. N. Netravali and J. O. Limb, "Picture coding: A review," *Proc. IEEE*, vol. 68, pp. 366–406, 1980.

[9] B. Girod, "Untersuchung ueber adaptive Praediktion fuer die DPCM-Codierung von Bildsignalen," Thesis, Institut für Theoretische Nachrichtentechnik und Informationsverarbeitung, Universität Hannover, Hannover, Germany, 1979.

[10] A. Habibi, "Comparison of *N*th-order DPCM encoder with linear transformations and block quantization techniques," *IEEE Trans. Commun. Technol.*, vol. COM-19, pp. 948–956, 1971.

[11] P. Jung and R. Lippmann, "Error response of DPCM decoders," *Nachrichtentech. Z.*, vol. 28, pp. 431–436, Dec. 1975.

[12] J. Max, "Quantizing for minimum distortion," *IRE Trans. Inform. Theory*, vol. IT-6, pp. 7–12, Mar. 1960.



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Estimating Three-Dimensional Motion Parameters of a Rigid Planar Patch

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***Abstract*—We present a new direct method of estimating the three-dimensional motion parameters of a rigid planar patch from two time-sequential perspective views (image frames). First, a set of eight pure parameters are defined. These parameters can be determined uniquely from the two given image frames by solving a set of linear equations. Then, the actual motion parameters are determined from these pure parameters by a method which requires the solution of a sixth-order polynomial of one variable only, and there exists a certain efficient algorithm for solving a sixth-order polynomial. Aside from a scale factor for the translation parameters, the number of real solutions never exceeds two. In the special case of three-dimensional translation, the motion parameters can be expressed directly as some simple functions of the eight pure parameters. Thus, only a few arithmetic operations are needed.**

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I. Introduction

In the past, most work on motion estimation has been restricted to two-dimensional translation. Recently, Roach and Aggarwal [1] and Huang and Tsai [2]–[4] presented methods of estimating three-dimensional motion parameters of rigid bodies based on image-space shifts. The method of Roach and Aggarwal requires the solution of a set of 18 simultaneous nonlinear equations; that of Huang and Tsai requires five simultaneous nonlinear equations. Huang and Tsai [4], [5] also described a direct method of estimating three-dimensional motion parameters of rigid planar patches based on the rela-

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tionship between temporal and spatial differentials of image intensity. This method results in the solution of eight simultaneous nonlinear equations. In none of the above works was the question of the uniqueness of the solution to the nonlinear equations investigated.

In this paper, we present a new direct method of estimating three-dimensional motion parameters of rigid planar patches. We define a set of eight pure parameters and demonstrate, using the theory of the Lie transformation group, that given two pictures, these pure parameters are unique. As for the estimation procedure, we first show, using the converse of the second Lie theorem [10]-[13], that these eight pure parameters can serve as the coordinate system of a certain Lie transformation group. Then, we use the result in [10]-[15] to show that these eight pure parameters must satisfy a set of linear equations. Furthermore, the real motion parameters can be computed from these pure parameters by solving a sixth-order polynomial.

Our new direct method has several advantages. First, it requires the solution of a single sixth-order polynomial of one variable only. Second, it demonstrates that more than one solution may exist, and therefore answers the uniqueness question. Third, in the special case of three-dimensional translation, the motion parameters can be expressed directly as some simple functions of the eight pure parameters. Therefore, only a few arithmetic operations are needed.

II. The Basic Motion Equations

We are interested in estimating three-dimensional motion parameters of rigid and deformable bodies from time-sequential perspective views (frames). Throughout this paper, we shall assume that we work with only two frames at times t_1 and t_2 $(t_1 < t_2)$.

The basic geometry of the problem is sketched in Fig. 1. Consider a particular point P on an object. Let

(x, y, z) = object-space coordinates of a point P at time t_1
(x', y', z') = object-space coordinates of P at time t_2
(X, Y) = image-space coordinates of P at t_1
(X', Y') = image-space coordinates of P at t_2.

It is obvious from Fig. 1 that

$$X = F\frac{x}{z} \qquad X' = F\frac{x'}{z'}$$
$$Y = F\frac{y}{z} \qquad Y' = F\frac{y'}{z'}. \tag{1}$$

Assume that, from time t_1 to t_2, the three-dimensional object has undergone translation, rotation, and linear deformation [8]. Then, we have

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = S\begin{bmatrix} x \\ y \\ z \end{bmatrix} + R\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{2}$$

where

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \tag{3}$$

$$R = \begin{bmatrix} 0 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 0 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 0 \end{bmatrix} \tag{4}$$

$$\varphi_1 = n_1\theta,\ \varphi_2 = n_2\theta,\ \varphi_3 = n_3\theta \tag{5}$$

$$n_1^2 + n_2^2 + n_3^2 = 1. \text{ (Small rotation assumed.)} \tag{6}$$

Note that $(\Delta x, \Delta y, \Delta z)$ is the amount of translation, S is the linear deformation matrix, and $(R + I)$, where I is a 3×3 unit matrix, is the rotation matrix. The rotation is around an axis through the origin and with directional cosines (n_1, n_2, n_3). The amount of rotation is θ. Therefore, the φ_1, φ_2, φ_3 defined in (5) are the x, y, and z components of the rotation vector with length θ and directional cosines (n_1, n_2, n_3).

Clearly, (2) represents an affine transformation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}. \tag{7}$$

Conversely, any affine transformation can be decomposed as in (2).

III. Motion of Planar Patches

We now restrict ourselves to points on a planar patch with equation

$$ax + by + cz = 1 \tag{8}$$

at time t_1. Then, it is readily shown from (1) and (8) that

$$z = \frac{F}{aX + bY + cF} \tag{9}$$

and from (1), (9), and (7) that

$$X' = \frac{a_1X + a_2Y + a_3}{a_7X + a_8Y + 1} \triangleq T_1(X, Y)$$
$$Y' = \frac{a_4X + a_5Y + a_6}{a_7X + a_8Y + 1} \triangleq T_2(X, Y) \tag{10}$$

where

$$a_1 = \frac{b_{11} + a\Delta x}{b_{33} + c\Delta z} \qquad a_2 = \frac{b_{12} + b\Delta x}{b_{33} + c\Delta z}$$
$$a_3 = \frac{(b_{13} + c\Delta x)F}{b_{33} + c\Delta z} \qquad a_4 = \frac{b_{21} + a\Delta y}{b_{33} + c\Delta z}$$
$$a_5 = \frac{b_{22} + b\Delta y}{b_{33} + c\Delta z} \qquad a_6 = \frac{(b_{23} + c\Delta y)F}{b_{33} + c\Delta z}$$
$$a_7 = \frac{b_{31} + a\Delta z}{(b_{33} + c\Delta z)F} \qquad a_8 = \frac{b_{32} + b\Delta z}{(b_{33} + c\Delta z)F}. \tag{11}$$

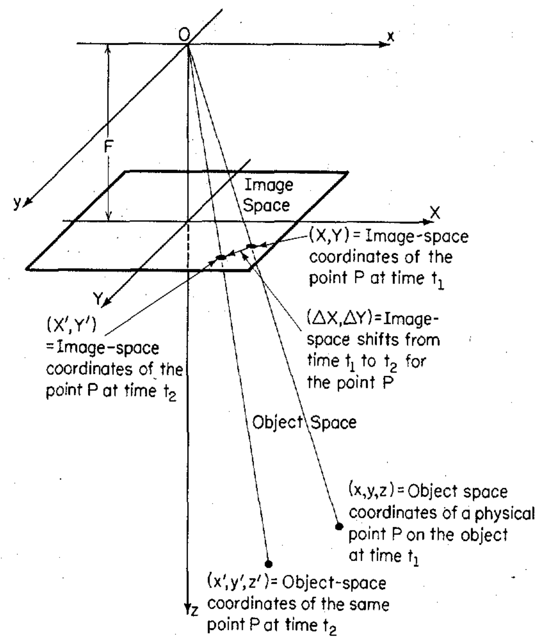

Fig. 1. Basic geometry for three-dimensional motion estimation.

We now specialize to the case of a rigid planar patch. Then (11) becomes

$$a_1 = \frac{1 + a\Delta x}{1 + c\Delta z} \qquad a_2 = \frac{-\varphi_3 + b\Delta x}{1 + c\Delta z}$$

$$a_3 = \frac{(\varphi_2 + c\Delta x)F}{1 + c\Delta z} \qquad a_4 = \frac{\varphi_3 + a\Delta y}{1 + c\Delta z}$$

$$a_5 = \frac{1 + b\Delta y}{1 + c\Delta z} \qquad a_6 = \frac{(-\varphi_1 + c\Delta y)F}{1 + c\Delta z}$$

$$a_7 = \frac{-\varphi_2 + a\Delta z}{(1 + c\Delta z)F} \qquad a_8 = \frac{\varphi_1 + b\Delta z}{(1 + c\Delta z)F}. \tag{12}$$

Equation (10) defines a mapping from the two-space (X, Y) onto the two-space (X', Y'). It will be shown in Section III that corresponding to any specified mapping between the two-spaces, there can be only one set of values for the parameters $a_1, \cdots, a_8$. We call them the eight *pure parameters.* In Section III we shall also describe a method of determining these pure parameters from the two given image frames.

Once we have determined the pure parameters $a_1, a_2, \cdots, a_8$, we can attempt to find the actual motion parameters Δx, Δy, Δz, φ_1, φ_2, φ_3, a, b, and c by using (12). First of all it is obvious from looking at the right-hand sides of (12) that Δz is a scale factor which cannot be determined. We therefore let

$$a'' \triangleq a\Delta z \qquad b'' = b\Delta z \qquad \bar{c} \triangleq c\Delta z$$

$$\Delta x'' \triangleq \frac{\Delta x}{\Delta z} \qquad \Delta y'' \triangleq \frac{\Delta y}{\Delta z}. \tag{13}$$

The unknown motion parameters are now

$\varphi_1, \varphi_2, \varphi_3, \Delta x'', \Delta y'', a'', b''$, and $\bar{c}$.

Thus, we have eight nonlinear equations with eight unknowns. This, however, does not mean that the solution is necessarily unique. In fact, it turns out that it is not.

After some tedious manipulations, we get from (12)

$$d_6 \Delta x''^6 + d_5 \Delta x''^5 + d_4 \Delta x''^4 + d_3 \Delta x''^3 + d_2 \Delta x''^2 + d_1 \Delta x'' + d_0 = 0 \tag{14}$$

where

$$\begin{aligned}
d_6 &= a_{50} h_3^2 - h_6 (h_3 h_2 - h_6 a_{10}) \\
d_5 &= h_3 (h_2^2 + h_6^2 + h_3^2 - 4a_{10} a_{50} \\
d_4 &= -h_3^2 (a_{50} + 5a_{10}) + a_{10} (2h_6^2 - h_2^2) - 3h_2 h_3 h_6 + 4a_{10} a_{50} \\
d_3 &= 2h_3 (h_2^2 + h_6^2 - h_3^2 + 4a_{10}^2) \\
d_2 &= (-h_3^2 + 4a_{10}^2) a_{50} + (6h_3^2 - 4a_{10}^2 - 2h_2^2 + h_6^2) a_{10} \\
&\quad - 3h_2 h_3 h_6 \\
d_1 &= h_3 (h_2^2 + h_6^2 + h_3^2 - 4a_{10}^2 + 4a_{10} a_{50}) \\
d_0 &= (a_5 - a_1) h_3^2 - h_2 (h_2 a_{10} + h_3 h_6)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
h_2 &= a_2 + a_4 \\
h_3 &= \frac{a_3}{F} + a_7 F \\
h_6 &= \frac{a_6}{F} + a_8 F \\
a_{10} &= a_1 - 1 \\
a_{50} &= a_5 - 1.
\end{aligned} \tag{16}$$

And, furthermore,

$$\Delta y'' = \frac{-h_6 \Delta x''^3 + h_2 \Delta x''^2 - h_6 \Delta x'' + h_2}{-h_3 \Delta x''^2 + 2a_{10} \Delta x'' + h_3}$$

$$\bar{c} = \frac{h_3 \Delta x'' - a_{10}}{\Delta x''^2 - h_3 \Delta x'' + a_1} = \frac{h_6 \Delta y'' - a_{50}}{\Delta y''^2 - h_6 \Delta y'' + a_5}$$

$$b'' = \bar{c}\,(h_6 - \Delta y'') + h_6$$

$$a'' = \bar{c}\,(h_3 - \Delta x'') + h_3$$

$$\varphi_1 = b'' - (\bar{c} + 1)\, a_8 F = (\bar{c} + 1) \frac{a_6}{F} - \Delta y'' \bar{c}$$

$$\varphi_2 = -a'' + (\bar{c} + 1)\, a_7 F = -(\bar{c} + 1) \frac{a_3}{F} + \Delta x'' \bar{c}$$

$$\varphi_3 = \Delta y'' a'' - a_4 (\bar{c} + 1) = -\Delta x'' b'' + a_2 (\bar{c} + 1). \tag{17}$$

To find the motion parameters, we first solve (14) for $\Delta x''$. Then the others are obtained from (17). Since (14) is a sixth-order polynomial equation, we can have potentially six real roots which give us six solutions for the motion parameters. For all the numerical examples we have tried, only two real roots are found for (14). One such numerical example follows.

$a_1 = 0.976 \quad a_2 = 0.058$

$a_3 = 0.059 \quad a_4 = 0.027$

$a_5 = 0.976 \quad a_6 = 0.059$

$a_7 = 0.047 \quad a_8 = 0.047.$

Solution 1:

$\Delta x = 0.9$

$\Delta y = 0.9$

$\Delta z = 1$

$\theta = 1°$

$n_1 = \cos 90°$

$n_2 = \cos 90°$

$n_3 = \cos 0°$

$a/A = \cos 60°$

$b/A = \cos 60°$

$c/A = \cos 45°$

$A = \sqrt{a^2 + b^2 + c^2} = \frac{1}{10}$

Solution 2:

$\Delta x = 0.707$

$\Delta y = 0.707$

$\Delta z = 1$

$\theta = 1.59°$

$n_1 = \cos 58.4°$

$n_2 = \cos 121.6°$

$n_3 = \cos 47.9°$

$a/A = \cos 56.2°$

$b/A = \cos 56.2°$

$c/A = \cos 51.8°$

$A = \sqrt{a^2 + b^2 + c^2} = \frac{1}{8.74}.$

We mention in passing that an efficient iterative method for finding the real roots of a sixth-order polynomial equation is given in [7].

For the special case of three-dimensional translation, the results are considerably simpler. From (12), we get

$$\Delta x'' = \frac{a_1 (a_1 a_8 F - a_1 + 1) + (a_1 a_8 - a_2 a_7) F}{a_7 F (a_1 a_8 F - a_1 + 1)}$$

$$\Delta y'' = \frac{a_5 (a_5 a_7 F - a_5 + 1) + (a_5 a_7 - a_4 a_8) F}{a_8 F (a_5 a_7 F - a_5 + 1)}$$

$$a'' = \frac{(a_5 a_7 F - a_5 + 1)\, a_7}{a_5 a_7 - a_4 a_8} = \frac{(a_1 a_8 F - a_1 + 1)\, a_7}{a_1 a_8 - a_2 a_7}$$

$$b'' = \frac{(a_5 a_7 F - a_5 + 1)\, a_8}{a_5 a_7 - a_4 a_8} = \frac{(a_1 a_8 F - a_1 + 1)\, a_8}{a_1 a_8 - a_2 a_7}$$

$$\bar{c} = \frac{a_2 a_7 F - a_1 + 1}{(a_1 a_8 - a_7 a_2) F} = \frac{a_4 a_8 F - a_5 + 1}{(a_5 a_7 - a_8 a_4) F}. \tag{18}$$

Therefore, only a few simple arithmetic operations are needed.

IV. Determining the Eight Pure Parameters

We now go back and examine (10). For a particular set of values for the parameters $(a_1, a_2, \cdots, a_8)$, the equations represent a transformation which maps the two-space (X, Y) (the coordinate space of our image frame at time t_1) onto the two-space (X', Y') (the coordinate space of our image frame at time t_2). Let us consider the collection G of transformations corresponding to all $(a_1, a_2, \cdots, a_8) \in R^8$. We shall show that it is a continuous (Lie) group of dimension eight, and that to any given mapping from the (X, Y)-space onto (X', Y')-space corresponds only one set of values for $(a_1, a_2, \cdots, a_8)$. Furthermore, we shall describe a method of determining the pure parameters $(a_1, a_2, \cdots, a_8)$ from a given pair of image frames at times t_1 and t_2.

In classical continuous group theory, it is known [13] that G satisfies the four group axioms, namely, closure, existence of inverse and identity, and associativity. Furthermore, the composition function for the group parameters a_i's are continuous. It is also known [13] that the a_i's in (10) are essential parameters in the sense that the a_i's are functionally independent. However, it is not known whether the a_i's in (10) are unique, i.e., whether there can be two different sets of values of a_i's such that (10) gives the same mapping $(X, Y) \to (X', Y')$. Because of this reason, it is not easy to verify whether G is a Lie group according to the modern definition since, in modern definition, in addition to the properties satisfied by the classical continuous group according to the classical definition [12], [13], several topological properties have to be satisfied, and these properties can not be easily verified unless we are certain that the group parameters a_i's are unique.

In the following, we prove that G is strictly a Lie group and that the a_i's are unique.

Before we give the proof, we motivate it by the following considerations. Let us assume that G is indeed a Lie group and that $(a_1, a_2, \cdots, a_8)$ is a coordinate representation for the group G. The identity element of the group is obviously $e = (1, 0, 0, 0, 1, 0, 0, 0)$. Then the operators of the Lie algebra associated with the group G are given by

$$X_j = \frac{\partial T_1}{\partial a_j}\bigg|_{g=e} \frac{\partial}{\partial X} + \frac{\partial T_2}{\partial a_j}\bigg|_{g=e} \frac{\partial}{\partial Y} \tag{19}$$

where g is used to represent a member of G. From (10) we readily get

$$\begin{aligned}
X_1 &= X\frac{\partial}{\partial X}\\
X_2 &= Y\frac{\partial}{\partial X}\\
X_3 &= \frac{\partial}{\partial X}\\
X_4 &= X\frac{\partial}{\partial Y}\\
X_5 &= Y\frac{\partial}{\partial Y}\\
X_6 &= \frac{\partial}{\partial Y}\\
X_7 &= -X^2\frac{\partial}{\partial X} - XY\frac{\partial}{\partial Y}\\
X_8 &= -XY\frac{\partial}{\partial X} - Y^2\frac{\partial}{\partial Y}.
\end{aligned} \tag{20}$$

Now we start our proof. Consider the set of vector fields on the differentiable manifold (X, Y) as given by (20). It can be easily verified that none of the X_j's can be expressed as a linear combination of the others, i.e., $\{X_j: j = 1, 2, \cdots, 8\}$ are linearly independent; and, furthermore, for any i, j

$$[X_i, X_j] \triangleq X_iX_j - X_jX_i = \sum_k c_{ij}^k X_k \tag{21}$$

where c_{ij}^k are constants. From these two properties of $\{X_i\}$, we conclude from the converse of Lie's second fundamental theorem [10]-[13] that there is a unique Lie group of transformation of order eight which has $\{X_i\}$ as its Lie algebra. We proceed now to show that G is that group.

From [15], [10], [11], one can generate the finite equations of a Lie group of transformation with the λ_i's as the canonical coordinates of the second kind as follows.

$$\begin{aligned}
X' &= \exp(\lambda_8 X_8)\exp(\lambda_7 X_7)\cdots\exp(\lambda_1 X_1)X\\
Y' &= \exp(\lambda_8 X_8)\exp(\lambda_7 X_7)\cdots\exp(\lambda_1 X_1)Y.
\end{aligned} \tag{22}$$

It is proved in [6] that (22) is equivalent to the following.

$$X' = \frac{a_1X + a_2Y + a_3}{a_7X + a_8Y + 1} \qquad Y' = \frac{a_4X + a_5Y + a_6}{a_7X + a_8Y + 1} \tag{23}$$

where

$$\begin{aligned}
a_1 &= e^{\lambda_1}\lambda\\
a_2 &= \lambda_2\lambda\\
a_3 &= (\lambda_3 e^{\lambda_1} + \lambda_6\lambda_2)\lambda\\
a_4 &= e^{\lambda_1}\lambda_4 e^{\lambda_5}\lambda\\
a_5 &= e^{\lambda_5}(1 + \lambda_2\lambda_4)\lambda\\
a_6 &= [\lambda_6 e^{\lambda_5} + \lambda_4 e^{\lambda_5}(\lambda_3 e^{\lambda_1} + \lambda_6\lambda_2)]\lambda\\
a_7 &= -(\lambda_7 e^{\lambda_1} + \lambda_8 e^{\lambda_1}\lambda_4 e^{\lambda_5})\lambda\\
a_8 &= -[\lambda_7\lambda_2 + e^{\lambda_5}\lambda_8(1 + \lambda_2\lambda_4)]\lambda\\
\lambda &= [1 - \lambda_7(\lambda_3 e^{\lambda_1} + \lambda_6\lambda_2) - \lambda_6 e^{\lambda_5}\lambda_8 - \lambda_4 e^{\lambda_5}\lambda_8\\
&\quad\cdot(\lambda_3 e^{\lambda_1} + \lambda_6\lambda_2)]^{-1}.
\end{aligned} \tag{24}$$

Comparing (10) and (23) shows that G is indeed a Lie group of transformation, and that since the λ_i's are the canonical coordinates, they are unique, and therefore from (24), the pure parameters $a_1 \cdots a_8$ are also unique.

Let f be any function defined on the two-space (X, Y) (in our case, f will be the intensity of the picture elements); then from Lie group theory [10]-[15], we have

$$\Delta f = \sum_{i=1}^{8} \beta_i X_i f \tag{25}$$

where

$\beta_i = a_i - e_i$

e_i = ith component of the group parameters at the identity

$\Delta f = f(X', Y') - f(X, Y)$ = frame difference.

(The implicit assumption here is that the intensities of the picture elements in the image frames corresponding to the same physical object point are the same.) Clearly

$$\begin{aligned}
a_1 &= \beta_1 + 1\\
a_5 &= \beta_5 + 1\\
a_i &= \beta_i, \qquad i = 2, 3, 4, 6, 7, 8.
\end{aligned}$$

Equation (25) is used to determine the β_i's and, therefore, the eight pure parameters a_i's. We choose eight or more points (X, Y), calculate at each point Δf and $X_i f$ $(i = 1, 2, \cdots, 8)$, and substitute into (25) to obtain eight or more equations which are linear in the eight unknowns β_i's. Then we find the least-square solution.

V. Discussions

In this paper we have investigated the problem of estimating three-dimensional motion parameters of a rigid planar patch from two image frames. The following results have been established.

1) The fact that we can define eight pure parameters $a_1, a_2, \cdots, a_8$ which are unique for any given mapping from the two-space (X, Y) (the coordinate space of the image frame at t_1) onto the two-space (X', Y') (the coordinate space of the image frame at t_2).

2) A method of determining the actual motion parameters $\varphi_1, \varphi_2, \varphi_3, \Delta x'', \Delta y'', a'', b''$, and $\bar{c}$ from the pure parameters $a_1, a_2, \cdots, a_8$ which requires the solution of a sixth-order polynomial of one variable only. Aside from a scale factor in the translation parameters, the number of real solutions never exceeds two.

3) A method of determining the eight pure parameters $a_1, a_2, \cdots, a_8$ from the two given image frames. This requires the solution of a set of linear equations only.

It is to be noted that 1) and 2) are independent of 3). The pure parameters can be determined by other methods. For example, if one can identify four or more corresponding point pairs in the two image frames, then the a_i's can be determined from (10) by solving a set of linear equations.

Recently, an alternative way of analyzing the uniqueness problem and estimating the three-dimensional motion parameters has been developed which stems from the results contained in this paper, and requires computing the singular value decomposition (SVD) of a certain 3×3 matrix only. The eight pure parameters defined by the authors in this work will again be used. It is briefly mentioned in [6], and the detailed paper has been submitted for publication.

Finally, the conclusion that the motion parameters are generally not unique is, of course, independent of the method of determining these parameters. The question arises: Are the motion parameters unique, aside from the scaling factor, if the rigid patch is nonplanar? We have solved this problem recently [7], and will publish it in the near future.

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References

[1] J. W. Roach and J. K. Aggarwal, "Determining the movement of objects from a sequence of images," *IEEE Trans. Pattern Anal. Machine Intell.*, Nov. 1980.

[2] T. S. Huang and R. Y. Tsai, "3-D motion estimation from image-space shifts," in *Proc. 1980 IEEE Int. Conf. ASSP*, Atlanta, GA, Mar. 1981.

[3] —, "Three-dimensional motion estimation from image-space shifts," Signal Processing Group, Dep. Elec. Eng., Swiss Fed. Inst. Technol., Lausanne, Switzerland, Tech. Rep. 80-1, Aug. 1980.

[4] —, "Image sequence analysis: Motion estimation, *Image Sequence Analysis*, ch. 1, T. S. Huang, Ed. New York: Springer-Verlag, 1981.

[5] R. Y. Tsai and T. S. Huang, "Three-dimensional motion estimation," in *Proc. 1st European Conf. Signal Processing*, Lausanne, Switzerland, Sept. 1980.

[6] —, "Estimating three-dimensional motion parameters of a rigid planar patch," in *Proc. 1981 Int. Conf. Pattern Recognition, Image Processing*, Dallas, TX, Aug. 3-5, 1981.

[7] —, "Uniqueness and estimation of 3-D motion parameters of a moving object with curved surfaces," to be published.

[8] A. Summerfeld, *Mechanics of Deformable Bodies*. New York: Academic, 1964.

[9] M. A. Aizerman, *Theory of Automatic Control*. New York: Pergamon, 1963, p. 240.

[10] C. Loewner, *Theory of Continuous Groups*. Cambridge, MA: MIT Press, 1971.

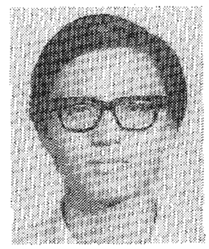
[11] L. S. Pontryagin, *Topological Groups*. New York: Gordon and Breach, 1966.

[12] E. Cartan, *La Theorie des Groupes Finis et Continus*. Paris, France: Librairie-Imprimerie Gauthier-Villars, 1937.

[13] L. P. Eisenhart, *Continuous Groups of Transformations*. New York: Dover, 1933.

[14] T. G. Newman and D. A. Demus, "Lie theoretic methods in video tracking," in *Proc. Workshop Auto. Missile Guidance*, Redstone Arsenal, AL, Nov. 1979.

[15] Y. Matsushima, *Differentiable Manifolds*. New York: Marcel Dekker, 1972.



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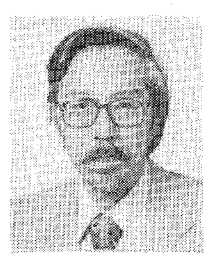

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