Estimating Three-Dimensional Motion Parameters of a Rigid Planar Patch, II: Singular Value Decomposition

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Abstract—We show that the three-dimensional (3-D) motion parameters of a rigid planar patch can be determined by computing the singular value decomposition (SVD) of a 3×3 matrix containing the eight so called "pure parameters." Furthermore, aside from a scale factor for the translation parameters, the number of solutions is either one or two, depending on the multiplicity of the singular values of the matrix.

I. Introduction

The processing of image sequences involving motion has become increasingly important. Because of the key role motion estimation plays in image sequence processing, a considerable amount of effort has been devoted to this topic, for example, see [1]-[17]. However, except for [1]-[8], [16], and [17], most past work considers only two-dimensional (2-D) motion, especially translation. References [1]-[8], [16], and [17] were among the first to consider three-dimensional (3-D) motion and [1], [3], and [17] were among the first to consider the problem of uniqueness of solutions. In [1] and [3], the eight "pure parameters" were introduced for the case of a rigid planar patch undergoing general 3-D motion, and proved to be unique given two successive (in time) perspective views. The proof makes use of the Lie group theory of transformations. It was also shown that these eight pure parameters can be computed by solving a set of linear equations. Furthermore, once the pure parameters are determined, the actual motion parameters can be computed by solving a sixth-order polynomial equation of one variable if the motion is small. Theoretically, the number of solutions cannot exceed six aside from a common scale factor for the translation parameters; experimentally, the maximum number of solutions has been found to be two. In this paper, we show that whether the motion is small or not, once the eight pure parameters are computed using the method described in [1] and [3] the actual motion parameters can be estimated by computing the singular value decomposition (SVD) of a 3×3 matrix consisting of the eight pure parameters. Also, by using the rigidity constraint and the fact that a plane in 3-space can be oriented in at most two possible ways in order to intercept an ellipsoid at a circular cross section, we prove that the number of solutions is either one or two, depending on the multiplicity of the singular values. Physical description of the motion is stated and justified.

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II. The Eight Pure Parameters and the Motion of Planar Patches

The basic geometry of the problem considered in [1] and [3] is repeated here in Fig. 1. Throughout this paper, we shall assume that we work with only two frames at time t_1 and t_2 ($t_1 < t_2$). Consider a particular point P on the object. Let

(x, y, z) = object-space coordinates of a point P at time t_1.

(x', y', z') = object-space coordinates of P at time t_2.

(X, Y) = image-space coordinates of P at t_1.

(X', Y') = image-space coordinates of P at t_2.

It was shown in [1] and [3] that for a rigid planar patch undergoing 3-D motion [a rotation with a small angle θ around an axis through the origin with directional consines n_1, n_2, n_3 followed by a translation with translation vector (Δx, Δy, Δz)] the image-space coordinates before and after the motion are related by the following equations:

$$X' = \frac{a_1 X + a_2 Y + a_3}{a_7 X + a_8 Y + 1}$$

$$Y' = \frac{a_4 X + a_5 Y + a_6}{a_7 X + a_8 Y + 1} \tag{1}$$

where

$$a_1 = \frac{1 + a \cdot \Delta x}{1 + c \cdot \Delta z} \qquad a_2 = \frac{-n_3 \theta + b \cdot \Delta x}{1 + c \cdot \Delta z}$$

$$a_3 = \frac{(n_2 \theta + c \cdot \Delta x) F}{1 + c \cdot \Delta z} \qquad a_4 = \frac{n_3 \theta + a \cdot \Delta y}{1 + c \cdot \Delta z}$$

$$a_5 = \frac{1 + b \cdot \Delta y}{1 + c \cdot \Delta z} \qquad a_6 = \frac{(-n_1 \theta + c \cdot \Delta y) F}{1 + c \cdot \Delta z}$$

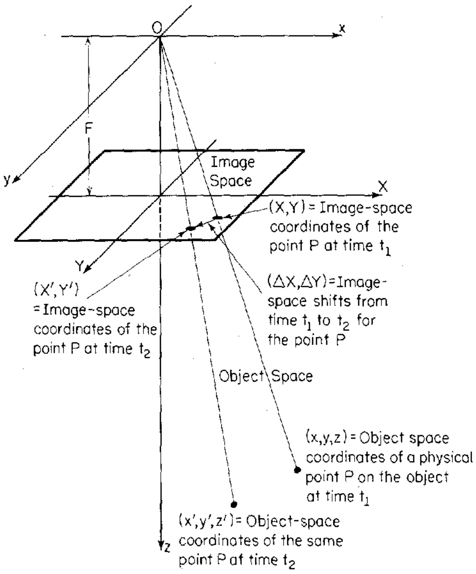


Fig. 1.

$$a_7 = \frac{-n_2\theta + a\cdot\Delta z}{(1 + c\cdot\Delta z)F} \qquad a_8 = \frac{n_1\theta + b\cdot\Delta z}{(1 + c\cdot\Delta z)F} \tag{2}$$

where a, b, and c are the parameters that appear in

$$ax + by + cz = 1 \tag{3}$$

which describes the surface of the object in the object-space coordinate system at time t_1.

The equations in (2) are applicable when the rotation angle is small. For general 3-D motion, it is well known in mechanics and computer graphics [18] that any 3-D rigid body motion is equivalent to a rotation followed by a translation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{4}$$

where R is a 3×3 orthonormal matrix

$$R = \begin{bmatrix} n_1^2 + (1 - n_1^2)\cos\theta & n_1n_2(1 - \cos\theta) - n_3\sin\theta & n_1n_3(1 - \cos\theta) + n_2\sin\theta \\ n_1n_2(1 - \cos\theta) + n_3\sin\theta & n_2^2 + (1 - n_2^2)\cos\theta & n_2n_3(1 - \cos\theta) - n_1\sin\theta \\ n_1n_3(1 - \cos\theta) - n_2\sin\theta & n_2n_3(1 - \cos\theta) + n_1\sin\theta & n_3^2 + (1 - n_3^2)\cos\theta \end{bmatrix}. \tag{5}$$

Following exactly the same procedure as in [1] and [3] one can show that (1) is again valid if (2) is replaced by

$$a_1 = \frac{n_1^2 + (1 - n_1^2)\cos\theta + a\cdot\Delta x}{n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_2 = \frac{n_1n_2(1 - \cos\theta) - n_3\sin\theta + b\cdot\Delta x}{n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_3 = \frac{n_1n_3(1 - \cos\theta) + n_2\sin\theta + c\cdot\Delta x}{n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_4 = \frac{n_1n_2(1 - \cos\theta) + n_3\sin\theta + a\cdot\Delta y}{n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_5 = \frac{n_2^2 + (1 - n_2^2)\cos\theta + b\cdot\Delta y}{n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_6 = \frac{n_2n_3(1 - \cos\theta) + n_1\sin\theta + c\cdot\Delta y}{n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_7 = \frac{n_1n_3(1 - \cos\theta) - n_2\sin\theta + a\cdot\Delta z}{n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_8 = \frac{n_2n_3(1 - \cos\theta) + n_1\sin\theta + b\cdot\Delta z}{n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z} \tag{6}$$

where for simplicity we have set $F = 1$.

It was shown in [1] and [3] using Lie group theory that given two perspective views at t_1 and t_2, the eight pure parameters $a_1, a_2, \cdots, a_8$ are unique and can be estimated by solving a system of linear equations.

III. Computing Actual Motion Parameters From Pure Parameters

By using the rigidity constraint and the fact that a plane in 3-space can be oriented in at most two possible directions in order to cut a circle in an ellipsoid, we shall prove that the number of possible solutions for the motion parameters can never exceed two aside from a scale factor for the translation parameters. The number of solutions depends upon the multiplicity of the singular values of the following matrix A consisting of the eight pure parameters a_i's:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix}. \tag{7}$$

The SVD of A is given by

$$A = U \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} V^T = U\Lambda V^T \tag{8}$$

where λ_i's are the singular values of A, and U, V are 3×3 orthonormal matrices.

Let $k = n_3^2 + (1 - n_3^2)\cos\theta + c\cdot\Delta z$; then it can be readily shown that

$$A = k^{-1}\left\{R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \;\; b \;\; c]\right\} \tag{9}$$

or

$$kA = R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \;\; b \;\; c].$$

From (3) and (4) it can be seen that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \quad b \quad c] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left\{ R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \quad b \quad c] \right\} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= kA \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{10}$$

If we transform the original coordinate system with the orthonormal matrix V in (8), as depicted in Fig. 2 where (x_n, y_n, z_n) is the new coordinate system after transformation, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = V \cdot \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{11}$$

and

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = V \cdot \begin{bmatrix} x'_n \\ y'_n \\ z'_n \end{bmatrix}. \tag{12}$$

Substituting (11) and (12) into (10) gives

$$V \begin{bmatrix} x'_n \\ y'_n \\ z'_n \end{bmatrix} = kAV \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}. \tag{13}$$

By taking the Euclidean norms of the vectors on both sides of (13), we obtain

$$[x'_n \quad y'_n \quad z'_n] \; V^T V \begin{bmatrix} x'_n \\ y'_n \\ z'_n \end{bmatrix}$$

$$= k^2 [x_n \quad y_n \quad z_n] \; V^T A^T A V \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}. \tag{14}$$

Since V is orthonormal, $V^T \cdot V$ on the left-hand side can be replaced by an identity matrix. Substituting (8) into (14) gives

$$x_n'^2 + y_n'^2 + z_n'^2 = k^2 [x_n \quad y_n \quad z_n] \; V^T V \Lambda U^T U \Lambda V^T V \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}. \tag{15}$$

Replacing $U^T U$ in (15) by an identity matrix gives

$$x_n'^2 + y_n'^2 + z_n'^2 = k^2 (\lambda_1^2 x_n^2 + \lambda_2^2 y_n^2 + \lambda_3^2 z_n^2). \tag{16}$$

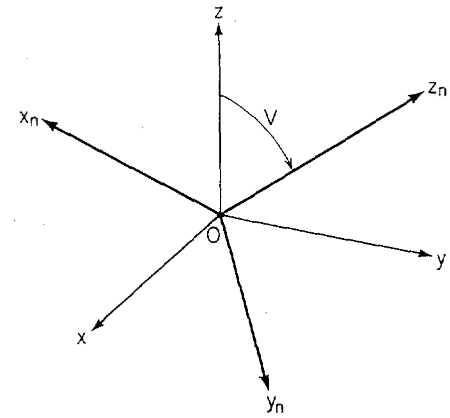


Fig. 2.

This is the key equation that will lead us to the solution of the uniqueness problem, as will be seen hereafter.

In the following, we state and prove three theorems regarding the uniqueness and computation of the motion parameters given the pure parameters, and the physical characterizations of the motion in the object space for different multiplicities of the singular values of the matrix A.

Theorem 1: If the multiplicity of the singular values of A is two, e.g., $\lambda_1 = \lambda_2 \neq \lambda_3$, then the solution for the motion and geometrical parameters is unique aside from a common scale factor for the translation parameters, and

$$R = \lambda_1^{-1} A - \left(\frac{\lambda_3}{\lambda_1} - s \right) U_3 V_3^T,$$

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = w^{-1} \left(\frac{\lambda_3}{\lambda_1} - s \right) U_3 \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = w V_3$$

where

$s = \det(U) \det(V)$,
w is a scale factor,
a, b, and c are the parameters in (3), which is the planar equation of the object surface at time t_1,
U_3 and V_3 are the third columns of U and V, respectively.

Furthermore, a necessary and sufficient condition for the multiplicity of the singular values of A to be two is that the motion can be realized by rotating the object around the origin and then translating it along the normal direction of the object surface.

Proof: The two sides of (16) can be equated to a collection of positive values corresponding to the range of values for x, y, z and x', y', z'. Let d be one such value. Then we have

$$d = x_n'^2 + y_n'^2 + z_n'^2 \tag{17}$$

and

$$d = k^2 (\lambda_1^2 x_n^2 + \lambda_2^2 y_n^2 + \lambda_3^2 z_n^2). \tag{18}$$

Clearly, (17) defines a sphere in the (x'_n, y'_n, z'_n) space, while (18) defines an ellipsoid in the (x_n, y_n, z_n) space. Since $\lambda_1 = \lambda_2$, two of the three principal axes of the ellipsoid are

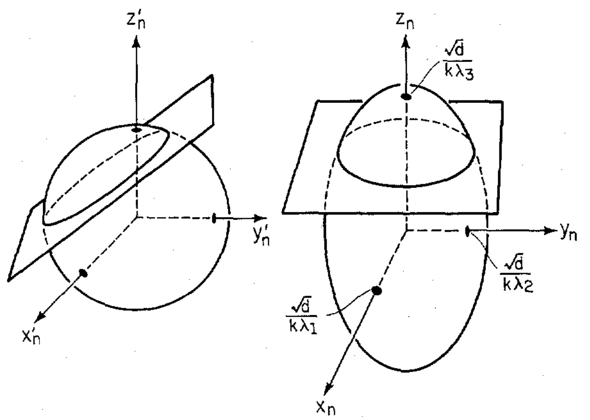


Fig. 3.

equally long. Since the object surface is assumed to be planar, the collection of the points on the object surface that also satisfy (17) must be the circle which lies on the intersection of the sphere and the object surface at time t_2 (see Fig. 3). Because of (16), (17), and (18) all the points on this circle at time t_2 must also satisfy (18) at time t_1, i.e., they must lie on the intersection of the object surface and the ellipsoid. Due to the rigidity constraint, this intersection should also be a circle. But the only possibility for a plane to cut a circle out of an ellipsoid with two of the three principal axes equally long is that the plane be perpendicular to the major axis (the longest one) of the ellipsoid, as depicted in Fig. 4. That is to say, there is only one possible orientation for the object surface before the motion. This is the key step that will lead us to the conclusion that the motion parameters are unique, as will be seen shortly.

Note that λ_1 (and λ_2) can never be zero since were this to be true, the ellipsoid defined by (18) would have degenerated into two parallel planes, and there is no way the object surface can intercept two parallel planes at a circular cross section.

Since, as depicted in Fig. 4, the object surface must be perpendicular to the z_n axis, and since the z_n axis is obtained by rotating the z axis with the orthonormal matrix V as in Fig. 2, it is seen that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = V \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} = wV_3 \tag{19}$$

where a, b, and c are the parameters in (3), V_3 is the third column of V in (8), and w is an arbitrary constant.

Substituting (19) and (8) into (9) gives

$$R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [0 \quad 0 \quad w] \cdot V^T = kU\Lambda V^T. \tag{20}$$

Premultiply (20) by U^T and postmultiply (20) by V to give

$$U^T R V + U^T \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [0 \quad 0 \quad w] \; V^T V = k\Lambda \tag{21}$$

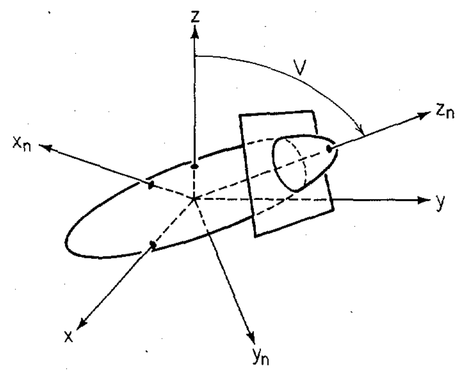


Fig. 4.

or

$$R' + \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} [0 \quad 0 \quad w] = k \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \tag{22}$$

where

$$R' = U^T R V \tag{23}$$

$$\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = U^T \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}. \tag{24}$$

Equation (22) gives

$$R' = \begin{bmatrix} k\lambda_1 & 0 & -w \cdot \Delta x' \\ 0 & k\lambda_1 & -w \cdot \Delta y' \\ 0 & 0 & k\lambda_3 - w \cdot \Delta z' \end{bmatrix}. \tag{25}$$

It will be shown now that $\Delta x'$ and $\Delta y'$ in (25) are zero and that therefore R' is diagonal.

Since U, V and R in (23) are all orthonormal, R' is also. Taking the inner product of the second and third columns of R' and equating it to zero gives

$$k\lambda_1 w \cdot \Delta y' = 0. \tag{26}$$

λ_1 and k cannot be zero because if they were, the first and second columns of (25) would be zero, which contradicts the fact that R' is orthonormal. Obviously, w cannot vanish either, because a, b and c would also vanish, which contradicts (3). Therefore, (26) implies that $\Delta y' = 0$.

Similarly, one can show that $\Delta x' = 0$. Thus, (25) is diagonal, which, when combined with the fact that R' is orthonormal, gives the following:

$$k\lambda_1 = +1 \quad \text{or} \quad -1 \tag{27}$$

$$k\lambda_3 - w\Delta z' = +1 \quad \text{or} \quad -1. \tag{28}$$

We show that k has to be positive.

From (10) we have

$$z' = k(a_7 x + a_8 y + z). \tag{29}$$

Because $x = 0$ and $y = 0$, $z' = kz$. Since the object must be in front of the camera, z and z' are both positive, which implies that k is positive.

Since λ_1 is nonnegative by definition, the right-hand side of (27) cannot be - 1. Therefore,

$$k = \frac{1}{\lambda_1}$$

and

$$R' = \begin{bmatrix} 1 & & \\ & 1 & \\ & & s \end{bmatrix} \tag{30}$$

where s is either +1 or -1.

Since $R' = URV^T$, we have $\det(R') = \det(U)\det(R)\cdot\det(V) = \det(U)\det(V)$. Thus $s = \det(U)\det(V)$. (28) gives the following:

$$\Delta z' = w^{-1}\left(\frac{\lambda_3}{\lambda_1} - s\right). \tag{31}$$

From (24), (31) and the fact that $\Delta x' = \Delta y' = 0$, we have

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = U \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = U \cdot \begin{bmatrix} 0 \\ 0 \\ w^{-1}\left(\frac{\lambda_3}{\lambda_1} - s\right) \end{bmatrix}$$

$$= w^{-1}\left(\frac{\lambda_3}{\lambda_1} - s\right) U_3. \tag{32}$$

Equations (19), (20) and (32) imply that

$$R = \lambda_1^{-1}A - \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \quad b \quad c]$$

$$= \lambda_1^{-1}A - \left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 V_3^T. \tag{33}$$

In the following, it will be shown that the solutions for the rotation matrix R in (33) is unique, and that aside from a scaling factor, the translation parameters Δx, Δy, and Δz in (32) and the geometrical parameters a, b, and c in (19) are also unique.

The first thing to show is that, once A is given, U_3 is fixed except for the sign. From (8),

$$AA^T U_3 = \lambda_3^2 U_3. \tag{34}$$

Let Q be any orthonormal eigenvector matrix of $A \cdot A^T$. Then

$$AA^T = Q^T \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix} Q. \tag{35}$$

From (34) and (35), we have

$$\left\{ Q^T \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix} Q - \lambda_3^2 I \right\} U_3 = 0$$

or

$$PU_3 = 0$$

where

$$P \triangleq AA^T - \lambda_3^2 I = Q^T \begin{bmatrix} \lambda_1^2 - \lambda_3^2 & & \\ & \lambda_1^2 - \lambda_3^2 & \\ & & 0 \end{bmatrix} Q. \tag{36}$$

P has rank 2 since $\lambda_1^2 - \lambda_3^2$ on the diagonal of the diagonal matrix in (36) is nonzero. Also, P is fixed once A is given since $P = AA^T - \lambda_3^2 I$. Therefore, U_3 is fixed except for the sign.

The next thing to prove is that (33) is unique, i.e.,

$$\left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 V_3^T = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \quad b \quad c]$$

is unique once A is given. Two cases are to be discussed. The first is when $\lambda_3 \neq 0$. In this case, $s = \operatorname{sgn}(\det(A))$ since $A = U\Lambda V^T$ and thus $s = \det(U)\det(V) = (\lambda_1\lambda_2\lambda_3)^{-1}\det(A) = \operatorname{sgn}(\det(A))$. Therefore, given A, s is fixed. The next thing to prove is that $U_3 V_3^T$ in (33) is unique.

Since U_3 and V_3 are fixed except for the sign, all one has to show is that when V_3 changes its sign, U_3 must also.

From (8) we have

$$AV = U\Lambda = [\lambda_1 U_1 \quad \lambda_2 U_2 \quad \lambda_3 U_3]$$

thus

$$AV_3 = \lambda_3 U_3.$$

Since A and λ_3 are fixed given two perspective views, we see that when V_3 changes its sign, U_3 must also. Therefore, $U_3 V_3^T$ has fixed sign. We have thus proved that the product

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \quad b \quad c]$$

and therefore R in (33) are all unique.

For the second case for which $\lambda_3 = 0$, we have from (33) that

$$R = \lambda_1^{-1}A + sU_3 V_3^T$$

$$= \lambda_1^{-1}A + \det(U) \cdot U_3 \cdot \det(V) \cdot V_3^T. \tag{37}$$

If U_3 changes its sign, $\det(U)$ will also. Thus the sign of $\det(U)\, U_3$ in (37) remains unchanged. Similarly, the sign of $\det(V)\, V_3^T$ also is fixed when V_3 changes its sign. Therefore, the uniqueness of (37) is not shaken by the ambiguity of the signs of U_3 and V_3.

Since $A^T A$ have double eigenvalues, the eigenvectors V_1 and V_2 that correspond to the multiple eigenvalues λ_1 $(= \lambda_2)$ are orthonormal to each other but may be anywhere in a certain fixed plane perpendicular to V_3 (note that we are now interpreting eigenvectors geometrically as some vectors in 3-space.) If the order of V_1 and V_2 on the plane are interchanged while keeping V_3 fixed, the sign of $\det(V)$ will change. We will now prove that when this does happen, the sign of det (U) will also change, thereby keeping (37) fixed. It is obvious from (8) that

$$(\lambda_1^{-1}A)\,V_1 = U_1$$
$$(\lambda_1^{-1}A)\,V_2 = U_2.$$

Since λ_1 and A are fixed, when V_1 and V_2 are interchanged, U_1 and U_2 will be also. Therefore, when det (V) changes sign, det (U) will also. Thus for the case when $\lambda_3 = 0$, the product

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix}[a \quad b \quad c].$$

as well as R in (33) are unique.

We will now prove that a necessary condition for $\lambda_1 = \lambda_2 \neq \lambda_3$ is that the translation vector

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix}$$

is parallel to the normal direction of the object surface at time t_2.

Since before the motion, i.e., at t_1,

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

is normal to the object surface. This vector is rotated by R at time t_2 and becomes

$$R\begin{bmatrix}a\\ b\\ c\end{bmatrix}.$$

It is only necessary to prove that there exists a scalar q such that

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix} = qR\begin{bmatrix}a\\ b\\ c\end{bmatrix}. \tag{38}$$

From (19), (32), (33), and (38), we have

$$\begin{aligned} w^{-1}\left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 &= q\left[\lambda_1^{-1}A - \left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 V_3^T\right] wV_3 \\ &= qw\lambda_1^{-1}AV_3 - qw\left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 V_3^T V_3 \end{aligned} \tag{39}$$

But $A = \lambda_1 U_1 V_1^T + \lambda_2 U_2 V_2^T + \lambda_3 U_3 V_3^T$. Thus AV_3 in (39) becomes

$$\begin{aligned} AV_3 &= \lambda_1 U_1 V_1^T V_3 + \lambda_2 U_2 V_2^T V_3 + \lambda_3 U_3 V_3^T V_3 \\ &= 0 + 0 + \lambda_3 U_3 \end{aligned} \tag{40}$$

Substituting (40) into (39) gives

$$w^{-1}\left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 = qw\left(\frac{\lambda_3}{\lambda_1} - \frac{\lambda_3}{\lambda_1} + s\right) U_3 = qwsU_3.$$

Therefore, if we take q as $w^{-2}s^{-1}(\lambda_3/\lambda_1 - s)$, then (38) will be satisfied. We have thus proved that the necessary condition for $\lambda_1 = \lambda_2 \neq \lambda_3$ is that the motion can be realized by first rotating the object around an axis passing through the origin, and then translating it along the normal direction of the object surface at time t_2.

In the proof of Theorem 2, it will be shown that if the translation is along the normal direction of the object surface at t_2, then the singular values of A can not be all distinct. This fact, together with Theorem 3, provide the sufficiency part. Q.E.D.

Theorem 2: If the singular values of A are all distinct, e.g., $\lambda_1 > \lambda_2 > \lambda_3$, then there are exactly two solutions for the motion and geometrical parameters aside from a scale factor for the translation and geometrical parameters, and that

$$R = U\begin{bmatrix}\alpha & 0 & \beta\\ 0 & 1 & 0\\ -s\beta & 0 & s\alpha\end{bmatrix}V^T$$

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix} = w^{-1}\left[-\beta U_1 + \left(\frac{\lambda_3}{\lambda_2} - s\alpha\right) U_3\right]$$

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix} = w[\delta V_1 + V_3]$$

where

$$\delta = \pm\left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_2^2 - \lambda_3^2}\right)^{1/2}$$

$$\alpha = \frac{\lambda_1 + s\lambda_3\delta^2}{\lambda_2(1+\delta^2)}$$

$$\beta = \pm\sqrt{1-\alpha^2}$$

$$s = \det(U)\det(V)$$

[in each of the two solutions, $\operatorname{sgn}(\beta) = -\operatorname{sgn}(\delta)$].

Furthermore, a necessary and sufficient condition for distinct singular values is that the motion can be decomposed into rotation around an axis through the origin, followed by translation along a direction different from the normal direction of the object surface at time t_2.

Proof: Since the three singular values are distinct, the three principal axes of the ellipsoid defined by (18) are of different lengths. By using the same argument as in Theorem 1, the object surface at t_1 must be oriented in such a way that it cuts a circle out of this ellipsoid. It is easy to verify using basic analytical geometry that a plane can be oriented in only two possible directions (see Fig. 5) in order to cut a circle out of an ellipsoid whose three principal axes are of different

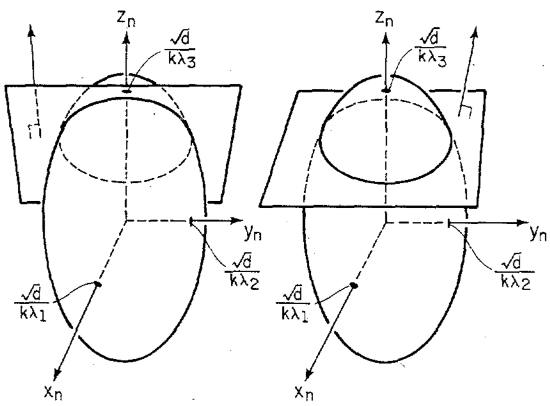


Fig. 5.

lengths. Since $\lambda_1 > \lambda_2 > \lambda_3$, the longest principal axis is aligned with z_n axis, and the vector normal to the object surface is

$$w\begin{bmatrix}\delta\\0\\1\end{bmatrix}, \quad \text{where } \delta = \pm\left(\frac{\lambda_1^2-\lambda_2^2}{\lambda_2^2-\lambda_3^2}\right)^{1/2}$$

in the (x_n, y_n, z_n) coordinate system, and w is some constant. Since, as shown in Fig. 2, the (x_n, y_n, z_n) axes are obtained by rotating the (x, y, z) axes with the orthonormal matrix V, we see that

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = wV\begin{bmatrix}\delta\\0\\1\end{bmatrix} \tag{41}$$

where w is some constant. Substituting (41) into (9) gives

$$kA = kU\Lambda V^T = R + w\begin{bmatrix}\Delta x\\\Delta y\\\Delta z\end{bmatrix}[\delta \quad 0 \quad 1]\ V^T. \tag{42}$$

Premultiplying (42) by U^T and postmultiplying (42) by V give

$$k\Lambda = U^TRV + wU^T\begin{bmatrix}\Delta x\\\Delta y\\\Delta z\end{bmatrix}[\delta \quad 0 \quad 1]\ V^TV.$$

Thus

$$R' = k\Lambda - \begin{bmatrix}\Delta x'\\\Delta y'\\\Delta z'\end{bmatrix}[\delta \quad 0 \quad 1] \tag{43}$$

where

$$R' \triangleq U^TRV \tag{44}$$

$$\begin{bmatrix}\Delta x'\\\Delta y'\\\Delta z'\end{bmatrix} \triangleq wU^T\begin{bmatrix}\Delta x\\\Delta y\\\Delta z\end{bmatrix}. \tag{45}$$

From (43)

$$R' = \begin{bmatrix}k\lambda_1 - \delta\cdot\Delta x' & 0 & -\Delta x'\\ -\delta\cdot\Delta y' & k\lambda_2 & -\Delta y'\\ -\delta\cdot\Delta z' & 0 & k\lambda_3 - \Delta z'\end{bmatrix}. \tag{46}$$

Since U, R and V are orthonormal, (46) implies that R' is also orthonormal. Taking the inner product of columns 2 and 3 and equating the result to zero gives

$$k\lambda_2 \cdot \Delta y' = 0. \tag{47}$$

Since $\lambda_1 > \lambda_2 > \lambda_3 \geqslant 0$, we have $\lambda_2 > 0$. Therefore, (47) implies that $\Delta y' = 0$. Thus (46) becomes

$$R' = \begin{bmatrix}k\lambda_1 - \delta\cdot\Delta x' & 0 & -\Delta x'\\ 0 & k\lambda_2 & 0\\ -\delta\cdot\Delta z' & 0 & k\lambda_3 - \Delta z'\end{bmatrix}. \tag{48}$$

The normality of column 2 implies that $k\lambda_2 = \pm 1$. But since $k > 0$ and $\lambda_2 > 0$, we have $k\lambda_2 = 1$, or $k = 1/\lambda_2$. Furthermore, from the fact that columns 1 and 3, as well as rows 1 and 3 of R' are mutually orthogonal, and that the norms of the rows and columns of R' are unity, it can be shown that

$$R' = \begin{bmatrix}\alpha & 0 & \beta\\ 0 & 1 & 0\\ -s\beta & 0 & s\alpha\end{bmatrix} \tag{49}$$

where

$$\alpha = \frac{\lambda_1}{\lambda_2} - \delta\cdot\Delta x' = s(k\lambda_3 - \Delta z') \tag{50}$$

$$\beta = -\Delta x' = s\delta\cdot\Delta z' = \pm\sqrt{1-\alpha^2} \tag{51}$$

$$s = \det(U)\det(V). \tag{52}$$

Since U and V are orthonormal, from (52), s is either +1 or −1. It will be shown that although det (U) and det (V) may be +1 or −1 for a particular A, s is unique once A is given.

Recall that U_1, U_2 and U_3 are the eigenvectors of AA^T corresponding to eigenvalues λ_1^2, λ_2^2, and λ_3^2, respectively. Since λ_1, λ_2, and λ_3 are distinct, U_1, U_2, U_3, V_1, V_2, and V_3 are all fixed except for the signs. However, as was seen in the proof of Theorem 1, we have

$$AV_1 = \lambda_1 U_1$$
$$AV_2 = \lambda_2 U_2$$
$$AV_3 = \lambda_3 U_3.$$

Therefore, when U_1 changes its sign, V_i will also, where $i = 1, 2, 3$. Hence the sign of det (U) det (V) remains fixed. Thus, s is unique.

From (50) and (51), we have

$$\alpha - \frac{\lambda_1}{\lambda_2} = \beta\cdot\delta \tag{53}$$

$$\delta\left(\alpha + s\frac{\lambda_3}{\lambda_2}\right) = \beta. \tag{54}$$

Cancelling β in (53) and (54) gives

$$\alpha = \frac{\lambda_1 + s\lambda_3\delta^2}{\lambda_2(1+\delta^2)}$$

where

$$\delta = \pm\left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_2^2 - \lambda_3^2}\right)^{1/2}.$$

From (50) and (51), and the fact that $\Delta y' = 0$, we have

$$\begin{bmatrix}\Delta x'\\ \Delta y'\\ \Delta z'\end{bmatrix} = \begin{bmatrix}-\beta\\ 0\\ \dfrac{\lambda_3}{\lambda_2} - s\alpha\end{bmatrix}. \tag{55}$$

From (45), (47), and (55)

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix} = w^{-1}U\begin{bmatrix}\Delta x'\\ \Delta y'\\ \Delta z'\end{bmatrix} = w^{-1}U\cdot\begin{bmatrix}-\beta\\ 0\\ \dfrac{\lambda_3}{\lambda_2} - s\alpha\end{bmatrix}$$

$$= w^{-1}\left[-\beta U_1 + \left(\frac{\lambda_3}{\lambda_2} - s\alpha\right)U_3\right]. \tag{56}$$

From (44) and (49)

$$R = U\begin{bmatrix}\alpha & 0 & \beta\\ 0 & 1 & 0\\ -s\beta & 0 & s\alpha\end{bmatrix}V^T. \tag{57}$$

From (41), (56), (57), and the fact that s is fixed, we see that there are exactly two solutions aside from a scaling factor for the translation and geometrical parameters.

It is to be shown that a necessary and sufficient condition for the singular values to be distinct is that the translation vector is not aligned with the normal direction of the object surface after rotation (or at time t_2). The sufficiency part was proved in Theorem 1. The necessity part is proved by contradiction. We shall show that if the translation vector is along the normal direction of the object surface at t_2, then the singular values cannot be distinct.

It was indicated in the proof of Theorem 1 that the normal direction of the object surface at t_2 is aligned with

$$R\begin{bmatrix}a\\ b\\ c\end{bmatrix}.$$

Suppose

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix}$$

is parallel to

$$R\begin{bmatrix}a\\ b\\ c\end{bmatrix}.$$

Then

$$\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix} = hR\begin{bmatrix}a\\ b\\ c\end{bmatrix} \tag{58}$$

for some constant h. With (41), (56), and (57), and (58), we have

$$w^{-1}U\begin{bmatrix}-\beta\\ 0\\ \dfrac{\lambda_3}{\lambda_2} - s\alpha\end{bmatrix} = hU\begin{bmatrix}\alpha & 0 & \beta\\ 0 & 1 & 0\\ -s\beta & 0 & s\alpha\end{bmatrix}V^TV\begin{bmatrix}\delta\\ 0\\ 1\end{bmatrix}$$

or

$$\begin{bmatrix}-\beta\\ 0\\ \dfrac{\lambda_3}{\lambda_2} - s\alpha\end{bmatrix} = wh\begin{bmatrix}\alpha & 0 & \beta\\ 0 & 1 & 0\\ -s\beta & 0 & s\alpha\end{bmatrix}\begin{bmatrix}\delta\\ 0\\ 1\end{bmatrix}$$

which implies that

$$-\beta = wh(\alpha\cdot\delta + \beta) \tag{59}$$

and

$$\frac{\lambda_3}{\lambda_2} - s\alpha = wh(-s\beta\cdot\delta + s\alpha). \tag{60}$$

Substituting (50) and (51) into (59) and (60) gives

$$\Delta x' = \frac{wh\delta}{1 + wh(1+\delta^2)}\,\frac{\lambda_1}{\lambda_2} \tag{61}$$

and

$$\Delta z' = \frac{wh}{1 + wh(1+\delta^2)}\,\frac{\lambda_3}{\lambda_2}. \tag{62}$$

But from (51)

$$-\Delta x' = s\delta\Delta z'. \tag{63}$$

Substituting (61) and (62) into (63) gives

$$\frac{wh\delta}{1 + wh(1+\delta^2)}\,\frac{\lambda_3}{\lambda_2} = \frac{wh\delta}{1 + wh(1+\delta^2)}\,\frac{-s\lambda_3}{\lambda_2}$$

which implies that $\lambda_1 = -s\lambda_3$. Since λ_1 and λ_3 are nonnegative by definition, we have $\lambda_1 = \lambda_3$. But this contradicts the assumption that $\lambda_1 \neq \lambda_3$. Therefore, the necessity part is proved. Q.E.D.

Theorem 3: The necessary and sufficient condition for the multiplicity for the singular values of A to be three, i.e., $\lambda_1 = \lambda_2 = \lambda_3$, is that the motion consists of rotation around an axis through the origin only, i.e., $\Delta x = \Delta y = \Delta z = 0$. Also, the rotation matrix is unique and $R = \lambda_1^{-1}A$. The object surface can be anywhere.

Proof: If $\lambda_1 = \lambda_2 = \lambda_3$, then (16) gives

$$x_n'^2 + y_n'^2 + z_n'^2 = k^2\lambda_1^2(x_n^2 + y_n^2 + z_n^2). \tag{64}$$

Since any 3-D rigid body motion can be decomposed into rotation followed by translation, we first rotate the object such that (x_n, y_n, z_n) becomes (x_n'', y_n'', z_n''). Then we carry out the translation which changes (x''_n, y''_n, z''_n) into (x'_n, y'_n, z'_n). That is,

$$\begin{bmatrix} x''_n \\ y''_n \\ z''_n \end{bmatrix} = R' \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{65}$$

and

$$\begin{bmatrix} x'_n \\ y'_n \\ z'_n \end{bmatrix} = \begin{bmatrix} x''_n \\ y''_n \\ z''_n \end{bmatrix} + \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} \tag{66}$$

where R', $\Delta x'$, $\Delta y'$, and $\Delta z'$ are the motion parameters in the (x_n, y_n, z_n) space as defined by (23) and (24). Equation (65) gives

$$x''^2_n + y''^2_n + z''^2_n = [x_n \;\; y_n \;\; z_n] \, R'^T R' \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$$

$$= x^2_n + y^2_n + z^2_n.$$

This, when combined with (66), gives

$$x^2_n + y^2_n + z^2_n = (x'_n - \Delta x')^2 + (y'_n - \Delta y')^2 + (z'_n - \Delta z')^2. \tag{67}$$

From (64) and (67), we have

$$\begin{aligned}(k^2\lambda_1^2 - 1)\, x'^2_n + (k^2\lambda_1^2 - 1)\, y'^2_n + (k^2\lambda_1^2 - 1)\, z'^2_n \\ - [2\Delta x' \cdot x'_n + 2\Delta y' \cdot y'_n + 2\Delta z' \cdot z'_n - \Delta x'^2 - \Delta y'^2 \\ - \Delta z'^2]\, k^2\lambda_1^2 = 0.\end{aligned} \tag{68}$$

Since (68) is true for all x'_n, y'_n, and z'_n, by equating the coefficients of all powers of x'_n, y'_n, and z'_n to zero, we have

$$\Delta x' = \Delta y' = \Delta z' = 0$$

and

$$k\lambda_1 = 1 \quad \text{or} \quad k = \frac{1}{\lambda_1}.$$

Therefore, from (24)

$$\Delta x = \Delta y = \Delta z = 0.$$

Then (9) gives

$$R + 0 = kA \quad \text{or} \quad R = \lambda_1^{-1} A.$$

Therefore, we have proved that if $\lambda_1 = \lambda_2 = \lambda_3$ then the motion consists of rotation around an axis through the origin only, and the solution for the rotation matrix is unique. The object surface can be anywhere. This proves the necessity part.

We now proceed to prove the sufficiency part. If the motion consists of rotation around an axis through the origin only, i.e., $\Delta x = \Delta y = \Delta z = 0$, then from (9)

$$A = k^{-1} R. \tag{69}$$

Let $U_A = R$, $V_A = I$ and $\Lambda_A = k^{-1} I$. Then (69) becomes

$$\begin{aligned} A &= U_A \Lambda_A V_A^T \\ &= U_A \begin{bmatrix} k^{-1} & & \\ & k^{-1} & \\ & & k^{-1} \end{bmatrix} V_A^T. \end{aligned} \tag{70}$$

Since U_A and V_A are orthonormal, (70) gives the SVD of A, with singular values k^{-1}, k^{-1}, and k^{-1}. Then from the fact that the singular values of any matrix are unique, we see that A has three identical singular values. This proves the sufficiency part. Q.E.D.

IV. Conclusions

Three theorems have been stated and proved regarding the uniqueness and the computation of the motion parameters, and the physical descriptions and classifications of the actual three-dimensional motion for a rigid planar patch. The motion parameters are unique aside from a scale factor for the translation parameters if the singular values of the 3×3 matrix consisting of the eight pure parameters are not all distinct; otherwise, the number of solutions is two. The distinction between the cases of multiplicity 1 and 2 lies in whether or not the translation vector coincides with the normal direction of the object surface at t_2. If there is no translation at all, then the singular values are all identical. In any case, once the eight pure parameters are estimated, which can be done by solving a system of linear equations, computing the singular value decomposition of a 3×3 matrix is all that it takes to obtain the 3-D motion parameters and the directional cosines of the normal direction of the planar patch.

References

[1] R. Y. Tsai and T. S. Huang, "Estimating three-dimensional motion parameters of a rigid planar patch," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 1147–1152, Dec. 1981.

[2] T. S. Huang and R. Y. Tsai, "Three-dimensional motion estimation from image-space shifts," in *Proc. IEEE Int. Conf. ASSP*, Atlanta, GA, Mar. 30–Apr. 1, 1981.

[3] R. Y. Tsai and T. S. Huang, "Estimating 3-D motion parameters of a rigid planar patch," in *Proc. 1981 IEEE Conf. Pattern Recognition, Image Processing*, Dallas, TX, Aug. 3–5, 1981.

[4] ——, "Three-dimensional motion estimation," in *Proc. 1st European Conf. Signal Processing*, Lausanne, Switzerland, Sept. 1980.

[5] T. S. Huang and R. Y. Tsai, "Image sequence analysis: Motion estimation," in *Image Sequence Analysis*, T. S. Huang, Ed. New York: Springer-Verlag, 1981, ch. 1.

[6] R. Y. Tsai and T. S. Huang, in "Moving image restoration and registration," in *Proc. IEEE Int. Conf. ASSP*, Denver, CO, Apr. 9–11, 1980.

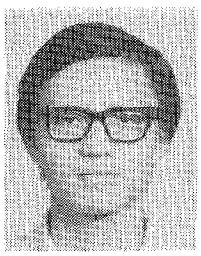
[7] T. S. Huang, Y. P. Hsu, and R. Y. Tsai, "Interframe coding with general two-dimensional motion compensation," presented at the 1981 Picture Coding Symp., Montreal, Canada, June 1–3, 1981.

[8] T. S. Huang and R. Y. Tsai, "3-D motion estimation from image-space shifts," Signal Processing Group, EPFL, Lausanne, Switzerland, Tech. Rep. 80-1.

[9] R. Y. Tsai and T. S. Huang, "Moving image restoration," presented at the IEEE Comput. Soc. Workshop, Comput. Analysis of Time Varying Imagery, Philadelphia, PA, Apr. 5–6, 1979.

[10] F. Rocca, "TV bandwidth compression utilizing frame-to-frame correlation and movement compensation," *Picture Bandwidth Compression*, T. S. Huang and O. J. Tretiak, Eds. London: Gordon and Breach, 1972.

[11] J. Limb and J. Murphy, "Estimating the velocity of moving images in TV signals," *Comput. Graphics Image Processing*, vol. 4, pp. 311–327, 1975.
[12] A. Netravali and J. Robbins, "Motion compensation TV coding: Part 1," *Bell Syst. Tech. J.*, no. 58, pp. 631–670, 1979.
[13] B. K. P. Horn and B. G. Schunck, "Determining optical flow," Massachusetts Inst. Technol., Cambridge, AI Memo. 572, Apr. 1980.
[14] C. Cafforio and F. Rocca, "Tracking moving objects in TV images," *Signal Processing*, vol. 1, pp. 133–140, 1979.
[15] R. J. Schalkoff, "Algorithms for a real-time automatic video tracking system," Ph.D. dissertation, Dep. Elec. Eng., Univ. Virginia, Charlottesville, 1979.
[16] J. W. Roach and J. K. Aggarwal, "Determining the movement of objects from a sequence of images," *IEEE Trans. Pattern Anal. Machine Intel.*, vol. PAMI-2, Nov. 1980.
[17] R. Y. Tsai and T. S. Huang, "On the uniqueness of 3-D motion parameters of rigid objects with curved surfaces," to be published.
[18] D. F. Rogers and J. A. Adams, *Mathematical Elements for Computer Graphics.* New York: McGraw-Hill, 1976.



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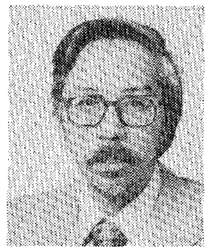

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