1 Rayleigh's Quotient

We want to prove that if A is a real symmetric $n \times n$ matrix (more generally complex Hermitian matrix, in which case we need to put proper complex conjugates for scalar product) then the Rayleigh's quotient

$$G(\mathrm{y}) = \frac{(\mathrm{y}, A\mathrm{y})}{(\mathrm{y}, \mathrm{y})} = \frac{\sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} y_i y_k}{\sum_{i=1}^{n} y_i^2}, \qquad A_{ik} = A_{ki}. \tag{1}$$

has a stationary value for $\mathrm{y} \neq 0$ if y is an eigen vector of A.

Obviously, each vector can be normalized by such way that

$$\|\mathrm{y}\|^2 = (\mathrm{y}, \mathrm{y}) = 1. \tag{2}$$

So the problem is to prove that eigen vectors of A provide stationary value for

$$(\mathrm{y}, A\mathrm{y}) = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} y_i y_k, \tag{3}$$

subject to constraint (2).

This can be proven by considering the following functional of y (method of Lagrange multipliers):

$$F(y_1, ..., y_n) = F(\mathrm{y}) = (\mathrm{y}, A\mathrm{y}) - \lambda(\mathrm{y}, \mathrm{y}) = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} y_i y_k - \lambda \sum_{i=1}^{n} y_i^2, \tag{4}$$

and its stationary values. In fact stationary values of $F(\mathrm{y})$ will provide stationary values of $G(\mathrm{y})$. These values can be found as

$$\frac{\partial F}{\partial y_j} = 0, \qquad j = 1, ..., n. \tag{5}$$

Using Eq. (4) we can see that

$$\begin{aligned} \frac{\partial F}{\partial y_j} &= \frac{\partial}{\partial y_j} \left(\sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} y_i y_k - \lambda \sum_{i=1}^{n} y_i^2 \right) = \qquad (6) \\ &= \sum_{k=1}^{n} A_{jk} y_k + \sum_{i=1}^{n} A_{ij} y_i - 2\lambda y_j = 2 \left(\sum_{k=1}^{n} A_{jk} y_k - \lambda y_j \right). \end{aligned}$$

The last equality holds due to symmetry of matrix A, $A_{ik} = A_{ki}$. Therefore the stationary values (5) satisfy the following relation

$$\sum_{k=1}^{n} A_{jk} y_k = \lambda y_j, \qquad j = 1, ..., n. \tag{7}$$

or in vector form

$$A\mathrm{y} = \lambda \mathrm{y}. \tag{8}$$

which show that y should be an eigen vector of A and λ is a corresponding eigen value. This eigen value also yields stationary value for $G(y)$, since

$$G(y) = \frac{(y, Ay)}{(y, y)} = \frac{\lambda (y, y)}{(y, y)} = \lambda. \tag{9}$$

Note that we proved also that if A has n eigen values $\lambda_1, ..., \lambda_n$, and $y_1, ..., y_n$ are corresponding eigen values, then $G(y)$ has n stationary values $\lambda_1, ..., \lambda_n$ and $y_1, ..., y_n$ are points at which these stationary values are reached.

Some generalizations follow immediately from this proof. E.g. we can consider stationary values of

$$G(y) = \frac{(y, Ay)}{(y, By)}, \tag{10}$$

where A is a Hermitian operator, and B is Hermitian and positive definite. This solves the "generalized" eigen value problem

$$Ay = \mu By. \tag{11}$$

For proof it is only necessary replace the definition of the scalar product

$$(y, y)_B = (y, By). \tag{12}$$