Notes on Normalized Graph Cuts

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Let us consider there are 3 nodes such that 2 nodes are in group A and 1 node is in group B. So,

$$\begin{aligned} &A\ \ A\ \ B \\ x =&[1\ \ 1\ \ -1] \end{aligned}$$

$$\begin{aligned} \sum_{(x_i>0, x_j<0)} -w_{ij}x_ix_j \Rightarrow \\ -w_{13}x_1x_3 = w_{13} \\ -w_{23}x_2x_3 = w_{23} \end{aligned}$$

$$4N_{cut}(1^{st} part\ numerator) = 4(w_{13} + w_{23}) \tag{1}$$

$$(1+x)^T(D-w)(1+x) \Rightarrow$$

$$\begin{aligned} &\begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} w_{12}+w_{13} & -w_{12} & -w_{13} \\ -w_{21} & w_{21}+w_{23} & -w_{23} \\ -w_{31} & -w_{32} & w_{31}+w_{32} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \\ &= 4[w_{13}+w_{23}] \end{aligned} \tag{2}$$

$$(1) = (2)$$

$$k = \frac{\sum_{x_i>0} d_i}{\sum_i d_i} = \frac{w_{11}+w_{21}+w_{22}+w_{13}+w_{23}+w_{12}}{w_{11}+w_{12}+w_{13}+w_{21}+w_{22}+w_{23}+w_{31}+w_{32}+w_{33}}$$

$$\begin{aligned} &k\mathbf{1}^T D\mathbf{1} \\ &= \frac{w_{11}+w_{21}+w_{22}+w_{13}+w_{23}+w_{12}}{w_{11}+w_{12}+w_{13}+w_{21}+w_{22}+w_{23}+w_{31}+w_{32}+w_{33}}(w_{11}+w_{12}+w_{13}+w_{21}+w_{22}+w_{23}+w_{31}+w_{32}+w_{33}) \\ &= w_{11}+w_{21}+w_{22}+w_{13}+w_{23}+w_{12} \\ &= \sum_{x_i>0} d_i\ \text{[denominator part]} \end{aligned}$$

$$y^T D\mathbf{1} = \sum_{x_i>0} d_i - b\sum_{x_i<0} d_i = 0$$

$$b = \frac{k}{1-k} = \frac{w_{11}+w_{12}+w_{21}+w_{22}+w_{13}+w_{23}}{w_{31}+w_{32}+w_{33}}$$

$$\sum_{x_i>0} d_i = w_{11} + w_{12} + w_{13} + w_{22} + w_{23} + w_{21}$$

$$\sum_{x_i<0} d_i = w_{31} + w_{32} + w_{33}$$

$$\therefore y^T D\mathbf{1} = w_{11} + w_{12} + w_{13} + w_{22} + w_{23} + w_{21} - b(w_{31} + w_{32} + w_{33}) = 0$$

$$(D - w)y = \lambda D y \text{ [generalized eigen eystem]}$$

$$(D - w)D^{-\frac{1}{2}}z = \lambda D D^{-\frac{1}{2}}z \ \text{ (since } y = D^{-\frac{1}{2}}z)$$

$$D^{-\frac{1}{2}}(D - w)D^{-\frac{1}{2}}z = \lambda D^{-\frac{1}{2}} D D^{-\frac{1}{2}}z$$

$$D^{-\frac{1}{2}}(D - w)D^{-\frac{1}{2}}z = \lambda z \text{ [standard eigen system]}$$