

1 Scalar operations

2 Vector operations

1. Norm $\alpha = \|\mathbf{x}\|_2$

- Forward: $\dot{\alpha} = \frac{\mathbf{x}^T \dot{\mathbf{x}}}{\|\mathbf{x}\|_2}$
- Reverse: $\bar{\mathbf{x}} = \frac{\bar{\alpha}}{\|\mathbf{x}\|_2} \mathbf{x}$

2. Scale $\mathbf{v} = \alpha \mathbf{x}$

- Forward: $\dot{\mathbf{v}} = \dot{\alpha} \mathbf{x} + \alpha \dot{\mathbf{x}}$
- Reverse: $\bar{\alpha} = \bar{\mathbf{v}}^T \mathbf{x}, \bar{\mathbf{x}} = \alpha \bar{\mathbf{v}}$

3. AXPY $\mathbf{v} = \alpha \mathbf{x} + \mathbf{y}$

- Forward: $\dot{\mathbf{v}} = \dot{\alpha} \mathbf{x} + \alpha \dot{\mathbf{x}} + \dot{\mathbf{y}}$
- Reverse: $\bar{\alpha} = \bar{\mathbf{v}}^T \mathbf{x}, \bar{\mathbf{x}} = \alpha \bar{\mathbf{v}}, \bar{\mathbf{y}} = \bar{\mathbf{v}}$

4. Dot-product $\alpha = \mathbf{x}^T \mathbf{y}$

- Forward: $\dot{\alpha} = \dot{\mathbf{x}}^T \mathbf{y} + \mathbf{x}^T \dot{\mathbf{y}}$
- Reverse: $\bar{\mathbf{x}} = \bar{\alpha} \mathbf{y}, \bar{\mathbf{y}} = \bar{\alpha} \mathbf{x}$

5. Cross-product $\mathbf{v} = \mathbf{x} \times \mathbf{y}$,

- Forward: $\dot{\mathbf{v}} = \dot{\mathbf{x}} \times \mathbf{y} + \mathbf{x} \times \dot{\mathbf{y}}$
- Reverse: $\bar{\mathbf{x}} = \mathbf{y} \times \bar{\mathbf{v}}, \bar{\mathbf{y}} = \bar{\mathbf{v}} \times \bar{\mathbf{x}},$

6. Normalize $\mathbf{v} = \frac{\mathbf{x}}{\|\mathbf{x}\|_2}$

- Forward: $\dot{\mathbf{v}} = \frac{\dot{\mathbf{x}}}{\|\mathbf{x}\|_2} - \frac{(\dot{\mathbf{x}}^T \mathbf{x}) \mathbf{x}}{\|\mathbf{x}\|_2^3}$
- Reverse: $\bar{\mathbf{x}} = \frac{\bar{\mathbf{v}}}{\|\mathbf{x}\|_2} - \frac{(\bar{\mathbf{v}}^T \mathbf{x}) \mathbf{x}}{\|\mathbf{x}\|_2^3}$

7. Symmetric outer-product $\mathbf{S} = \alpha \mathbf{x} \mathbf{x}^T$

- Forward: $\dot{\mathbf{S}} = \dot{\alpha} \mathbf{x} \mathbf{x}^T + \alpha (\dot{\mathbf{x}} \mathbf{x}^T + \mathbf{x} \dot{\mathbf{x}}^T)$
- Reverse: $\bar{\alpha} = \mathbf{x}^T \bar{\mathbf{S}} \mathbf{x}, \bar{\mathbf{x}} = 2\alpha \bar{\mathbf{S}} \mathbf{x}$

3 Matrix-vector operations

1. Matrix-vector product $\mathbf{v} = \mathbf{A} \mathbf{x}$

- Forward: $\dot{\mathbf{v}} = \dot{\mathbf{A}} \mathbf{x} + \mathbf{A} \dot{\mathbf{x}}$
- Reverse: $\bar{\mathbf{x}} = \mathbf{A}^T \bar{\mathbf{y}}, \bar{\mathbf{A}} = \bar{\mathbf{v}} \mathbf{x}^T$

2. Matrix-vector product $\mathbf{v} = \mathbf{A}^T \mathbf{x}$

- Forward: $\dot{\mathbf{v}} = \dot{\mathbf{A}}^T \mathbf{x} + \mathbf{A}^T \dot{\mathbf{x}}$
- Reverse: $\bar{\mathbf{x}} = \mathbf{A} \bar{\mathbf{y}}, \bar{\mathbf{A}} = \mathbf{x} \bar{\mathbf{v}}^T$

3. Matrix inner product $\alpha = \mathbf{x}^T \mathbf{A} \mathbf{y}$

- Forward: $\dot{\alpha} = \dot{\mathbf{x}}^T \mathbf{A} \mathbf{y} + \mathbf{x}^T \dot{\mathbf{A}} \mathbf{y} + \mathbf{x}^T \mathbf{A} \dot{\mathbf{y}}$
- Reverse: $\bar{\mathbf{x}} = \bar{\alpha} \mathbf{A} \mathbf{y}, \bar{\mathbf{y}} = \bar{\alpha} \mathbf{A}^T \mathbf{x}, \bar{\mathbf{A}} = \bar{\alpha} \mathbf{x} \mathbf{y}^T$

4 Matrix-matrix operations

1. Trace $\alpha = \text{tr}(\mathbf{A})$
 - Forward $\dot{\alpha} = \text{tr}(\dot{\mathbf{A}})$
 - Reverse $\bar{\mathbf{A}}_{ii} = \bar{\alpha}$
2. Symmetric trace $\alpha = \text{tr}(\mathbf{S})$
 - Forward $\dot{\alpha} = \text{tr}(\dot{\mathbf{S}})$
 - Reverse $\bar{\mathbf{S}}_{ii} = \bar{\alpha}$
3. Determinant $\alpha = \det(\mathbf{A})$
4. Symmetric determinant $\alpha = \det(\mathbf{A})$
5. Inverse $\mathbf{B} = \mathbf{A}^{-1}$
 - Forward: $\dot{\mathbf{B}} = -\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}$
 - Reverse $\bar{\mathbf{A}} = -\mathbf{A}^{-T}\bar{\mathbf{B}}\mathbf{A}^{-T}$
6. Symmetric Inverse $\mathbf{T} = \mathbf{S}^{-1}$
 - Forward: $\dot{\mathbf{T}} = -\mathbf{S}^{-1}\dot{\mathbf{S}}\mathbf{S}^{-1}$
 - Reverse $\bar{\mathbf{S}} = -\mathbf{S}^{-T}\bar{\mathbf{T}}\mathbf{S}^{-T}$
7. Symmetric part $\mathbf{S} = \alpha(\mathbf{A} + \mathbf{A}^T)$
8. Anti-symmetric part $\mathbf{B} = \alpha(\mathbf{A} - \mathbf{A}^T)$
9. Matrix addition $\mathbf{C} = \mathbf{A} + \mathbf{B}$
 - Forward: $\dot{\mathbf{C}} = \dot{\mathbf{A}} + \dot{\mathbf{B}}$
 - Reverse: $\bar{\mathbf{A}} = \bar{\mathbf{C}}, \bar{\mathbf{B}} = \bar{\mathbf{C}}$
10. Matrix-matrix product $\mathbf{C} = \mathbf{A}\mathbf{B}$
 - Forward $\dot{\mathbf{C}} = \dot{\mathbf{A}}\mathbf{B} + \mathbf{A}\dot{\mathbf{B}}$
 - Reverse $\bar{\mathbf{A}} = \bar{\mathbf{C}}\mathbf{B}^T, \bar{\mathbf{B}} = \mathbf{A}^T\bar{\mathbf{C}},$
11. Matrix-matrix product $\mathbf{C} = \mathbf{A}^T\mathbf{B}$
 - Forward $\dot{\mathbf{C}} = \dot{\mathbf{A}}^T\mathbf{B} + \mathbf{A}^T\dot{\mathbf{B}}$
 - Reverse $\bar{\mathbf{A}} = \mathbf{B}\bar{\mathbf{C}}^T, \bar{\mathbf{B}} = \mathbf{A}\bar{\mathbf{C}},$
12. Matrix-matrix product $\mathbf{C} = \mathbf{A}\mathbf{B}^T$
 - Forward $\dot{\mathbf{C}} = \dot{\mathbf{A}}\mathbf{B}^T + \mathbf{A}\dot{\mathbf{B}}^T$
 - Reverse $\bar{\mathbf{A}} = \bar{\mathbf{C}}\mathbf{B}, \bar{\mathbf{B}} = \bar{\mathbf{C}}^T\mathbf{A}$
13. Matrix-matrix product $\mathbf{C} = \mathbf{A}^T\mathbf{B}^T$
 - Forward $\dot{\mathbf{C}} = \dot{\mathbf{A}}^T\mathbf{B}^T + \mathbf{A}^T\dot{\mathbf{B}}^T$
 - Reverse $\bar{\mathbf{A}} = \mathbf{B}^T\bar{\mathbf{C}}^T, \bar{\mathbf{B}} = \bar{\mathbf{C}}^T\mathbf{A}^T$
14. Linear Lagrange-Green strain $\mathbf{E} = \frac{1}{2}(\mathbf{U} + \mathbf{U}^T)$
 - Forward $\dot{\mathbf{E}} = \frac{1}{2}(\dot{\mathbf{U}} + \dot{\mathbf{U}}^T)$
 - Reverse $\bar{\mathbf{U}} = \bar{\mathbf{E}}$

15. Lagrange-Green strain $\mathbf{E} = \frac{1}{2} (\mathbf{U} + \mathbf{U}^T + \mathbf{U}^T \mathbf{U})$

- Forward $\dot{\mathbf{E}} = \frac{1}{2} (\dot{\mathbf{U}} + \dot{\mathbf{U}}^T + \dot{\mathbf{U}}^T \mathbf{U} + \mathbf{U}^T \dot{\mathbf{U}})$
- Reverse $\bar{\mathbf{U}} = (\mathbf{I} + \mathbf{U})\bar{\mathbf{E}}$

16. Isotropic constitutive $\mathbf{S} = 2\mu\mathbf{E} + \lambda\text{tr}(\mathbf{E})\mathbf{I}$

- Forward $\dot{\mathbf{S}} = 2\mu\dot{\mathbf{E}} + \lambda\text{tr}(\dot{\mathbf{E}})\mathbf{I}$
- Reverse $\bar{\mathbf{E}} = 2\mu\bar{\mathbf{S}} + \lambda\text{tr}(\bar{\mathbf{S}})\mathbf{I}$