DIFFERENTIATION

2.1 DERIVATIVES

We are now ready to explain what is meant by the slope of a curve or the velocity of a moving point. Consider a real function f and a real number a in the domain of f. When x has value a, $f(x)$ has value $f(a)$. Now suppose the value of x is changed from a to a hyperreal number $a + \Delta x$ which is infinitely close to but not equal to a. Then the new value of $f(x)$ will be $f(a + \Delta x)$. In this process the value of x will be changed by a nonzero infinitesimal amount Δx, while the value of $f(x)$ will be changed by the amount

$$f(a + \Delta x) - f(a).$$

The ratio of the change in the value of $f(x)$ to the change in the value of x is

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

This ratio is used in the definition of the slope of f which we now give.

DEFINITION

S is said to be the **slope** of f at a if

$$S = st\left(\frac{f(a + \Delta x) - f(a)}{\Delta x}\right)$$

for every nonzero infinitesimal Δx.

The slope, when it exists, is infinitely close to the ratio of the change in $f(x)$ to an infinitely small change in x. Given a curve $y = f(x)$, the slope of f at a is also called the slope of the curve $y = f(x)$ at $x = a$. Figure 2.1.1 shows a nonzero infinitesi-

mal Δx and a hyperreal straight line through the two points on the curve at a and $a + \Delta x$. The quantity

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

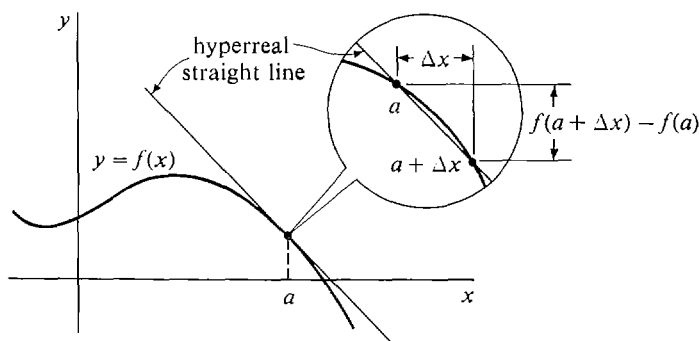is the slope of this line, and its standard part is the slope of the curve.

Figure 2.1.1

The slope of f at a does not always exist. Here is a list of all the possibilities.

(1) The slope of f at a exists if the ratio

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

is finite and has the same standard part for all infinitesimal $\Delta x \neq 0$. It has the value

$$S = st\left(\frac{f(a + \Delta x) - f(a)}{\Delta x}\right).$$

(2) The slope of f at a can fail to exist in any of four ways:
   (a) $f(a)$ is undefined.
   (b) $f(a + \Delta x)$ is undefined for some infinitesimal $\Delta x \neq 0$.
   (c) The term $\dfrac{f(a + \Delta x) - f(a)}{\Delta x}$ is infinite for some infinitesimal $\Delta x \neq 0$.
   (d) The term $\dfrac{f(a + \Delta x) - f(a)}{\Delta x}$ has different standard parts for different infinitesimals $\Delta x \neq 0$.

We can consider the slope of f at any point x, which gives us a new function of x.

DEFINITION

*Let f be a real function of one variable. The **derivative** of f is the new function f′ whose value at x is the slope of f at x. In symbols,*

$$f'(x) = st\left(\frac{f(x + \Delta x) - f(x)}{\Delta x}\right)$$

whenever the slope exists.

The derivative $f'(x)$ is undefined if the slope of f does not exist at x.

For a given point a, the slope of f at a and the derivative of f at a are the same thing. We usually use the word "slope" to emphasize the geometric picture and "derivative" to emphasize the fact that f' is a function.

The process of finding the derivative of f is called *differentiation*. We say that f is *differentiable* at a if $f'(a)$ is defined; i.e., the slope of f at a exists.

Independent and dependent variables are useful in the study of derivatives. Let us briefly review what they are. A *system of formulas* is a finite set of equations and inequalities. If we are given a system of formulas which has the same graph as a simple equation $y = f(x)$, we say that y *is a function of* x, or that y *depends on* x, and we call x the *independent variable* and y the *dependent variable*.

When $y = f(x)$, we introduce a new independent variable Δx and a new dependent variable Δy, with the equation

(1) $\Delta y = f(x + \Delta x) - f(x).$

This equation determines Δy as a real function of the two variables x and Δx, when x and Δx vary over the real numbers. We shall usually want to use the Equation 1 for Δy when x is a real number and Δx is a nonzero infinitesimal. The Transfer Principle implies that Equation 1 also determines Δy as a hyperreal function of two variables when x and Δx are allowed to vary over the hyperreal numbers.

Δy is called the *increment* of y. Geometrically, the increment Δy is the change in y along the curve corresponding to the change Δx in x. The symbol y' is sometimes used for the derivative, $y' = f'(x)$. Thus the hyperreal equation

$$f'(x) = st\left(\frac{f(x + \Delta x) - f(x)}{\Delta x}\right)$$

now takes the short form

$$y' = st\left(\frac{\Delta y}{\Delta x}\right).$$

The infinitesimal Δx may be either positive or negative, but not zero. The various possibilities are illustrated in Figure 2.1.2 using an infinitesimal microscope. The signs of Δx and Δy are indicated in the captions.

Our rules for standard parts can be used in many cases to find the derivative of a function. There are two parts to the problem of finding the derivative f' of a function f:

(1)   Find the domain of f'.
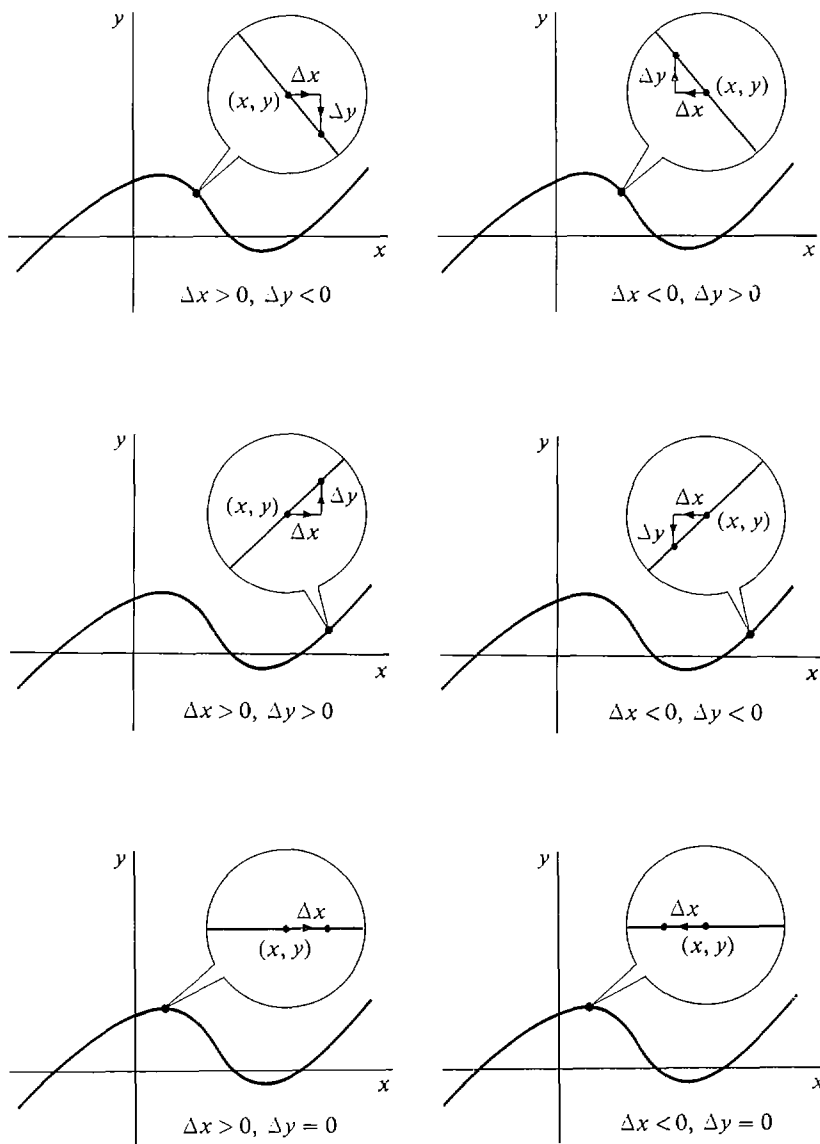(2)   Find the value of $f'(x)$ when it is defined.

Figure 2.1.2

EXAMPLE 1   Find the derivative of the function

$$f(x) = x^3.$$

In this and the following examples we let x vary over the real numbers and Δx vary over the nonzero infinitesimals. Let us introduce the new variable y with the equation $y = x^3$. We first find $\Delta y/\Delta x$.

$$y = x^3,$$
$$y + \Delta y = (x + \Delta x)^3,$$
$$\Delta y = (x + \Delta x)^3 - x^3,$$
$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^3 - x^3}{\Delta x}.$$

Next we simplify the expression for $\Delta y/\Delta x$.

$$\frac{\Delta y}{\Delta x} = \frac{(x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3) - x^3}{\Delta x}$$
$$= \frac{3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$
$$= 3x^2 + 3x \Delta x + (\Delta x)^2.$$

Then we take the standard part,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st(3x^2 + 3x \Delta x + (\Delta x)^2)$$
$$= st(3x^2) + st(3x \Delta x) + st((\Delta x)^2)$$
$$= 3x^2 + 0 + 0 = 3x^2.$$

Therefore, $f'(x) = st\left(\dfrac{\Delta y}{\Delta x}\right) = 3x^2.$

We have shown that the derivative of the function

$$f(x) = x^3$$

is the function $f'(x) = 3x^2$

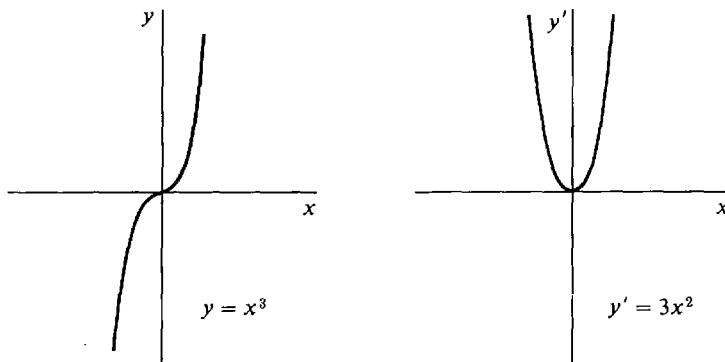with the whole real line as domain. $f(x)$ and $f'(x)$ are shown in Figure 2.1.3.

Figure 2.1.3

EXAMPLE 2   Find $f'(x)$ given $f(x) = \sqrt{x}$.

Case 1 $x < 0$. Since $\sqrt{x}$ is not defined, $f'(x)$ does not exist.

Case 2 $x = 0$. When Δx is a negative infinitesimal, the term

$$\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \frac{\sqrt{0 + \Delta x} - \sqrt{0}}{\Delta x}$$

is not defined because $\sqrt{\Delta x}$ is undefined. When Δx is a positive infinitesimal, the term

$$\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \frac{\sqrt{\Delta x}}{\Delta x} = \frac{1}{\sqrt{\Delta x}}$$

is defined but its value is infinite. Thus for two reasons, $f'(x)$ does not exist.

Case 3 $x > 0$. Let $y = \sqrt{x}$. Then

$$y + \Delta y = \sqrt{x + \Delta x},$$
$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x},$$
$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}.$$

We then make the computation

$$\frac{\Delta y}{\Delta x} = \frac{(\sqrt{x + \Delta x} - \sqrt{x})}{\Delta x} \cdot \frac{(\sqrt{x + \Delta x} + \sqrt{x})}{(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$= \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$= \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

Taking standard parts, $st\left(\frac{\Delta y}{\Delta x}\right) = st\left(\frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}\right)$

$$= \frac{1}{st(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$= \frac{1}{st(\sqrt{x + \Delta x}) + st(\sqrt{x})}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Therefore, when $x > 0$, $f'(x) = \dfrac{1}{2\sqrt{x}}.$

So the derivative of $f(x) = \sqrt{x}$

is the function $f'(x) = \dfrac{1}{2\sqrt{x}},$

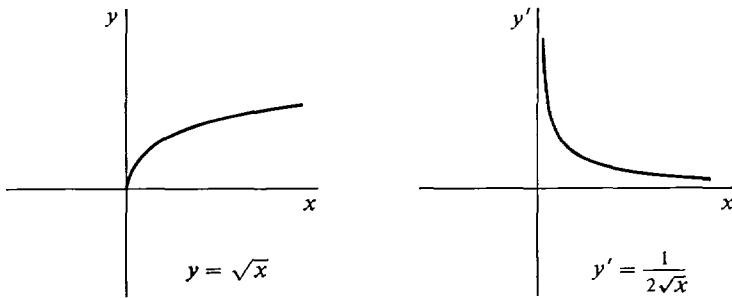and the set of all $x > 0$ is its domain (see Figure 2.1.4).

$y = \sqrt{x}$

$y' = \frac{1}{2\sqrt{x}}$

Figure 2.1.4

EXAMPLE 3 Find the derivative of $f(x) = 1/x$.

Case 1 $x = 0$. Then $1/x$ is undefined so $f'(x)$ is undefined.

Case 2 $x \neq 0$.

$$y = 1/x,$$

$$y + \Delta y = \frac{1}{x + \Delta x},$$

$$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x},$$

$$\frac{\Delta y}{\Delta x} = \frac{1/(x + \Delta x) - 1/x}{\Delta x}.$$

Simplifying,

$$\frac{1/(x + \Delta x) - 1/x}{\Delta x} = \frac{x - (x + \Delta x)}{x(x + \Delta x)\,\Delta x} = \frac{-\Delta x}{x(x + \Delta x)\,\Delta x}$$

$$= \frac{-1}{x(x + \Delta x)}.$$

Taking the standard part,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(-\frac{1}{x(x + \Delta x)}\right) = -\frac{1}{st(x(x + \Delta x))}$$

$$= -\frac{1}{st(x)st(x + \Delta x)} = -\frac{1}{x \cdot x} = -\frac{1}{x^2}.$$

Thus

$$f'(x) = -1/x^2.$$

The derivative of the function $f(x) = 1/x$ is the function $f'(x) = -1/x^2$ whose domain is the set of all $x \neq 0$. Both functions are graphed in Figure 2.1.5.
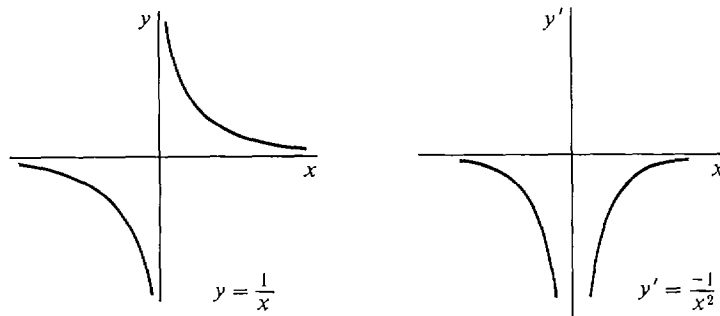
Figure 2.1.5

$$y = \frac{1}{x}$$

$$y' = \frac{-1}{x^2}$$

EXAMPLE 4   Find the derivative of $f(x) = |x|$.

Case 1 $x > 0$. In this case $|x| = x$, and we have

$$y = x,$$
$$y + \Delta y = x + \Delta x,$$
$$\Delta y = \Delta x,$$
$$\frac{\Delta y}{\Delta x} = 1, \quad f'(x) = 1.$$

Case 2 $x < 0$. Now $|x| = -x$, and

$$y = -x,$$
$$y + \Delta y = -(x + \Delta x),$$
$$\Delta y = -(x + \Delta x) - (-x) = -\Delta x,$$
$$\frac{\Delta y}{\Delta x} = -\frac{\Delta x}{\Delta x} = -1, \quad f'(x) = -1.$$

Case 3 $x = 0$. Then

$$y = 0,$$
$$y + \Delta y = |0 + \Delta x| = |\Delta x|,$$
$$\Delta y = |\Delta x|,$$

and
$$\frac{\Delta y}{\Delta x} = \frac{|\Delta x|}{\Delta x} = \begin{cases} 1 & \text{if } \Delta x > 0, \\ -1 & \text{if } \Delta x < 0. \end{cases}$$

The standard part of $\Delta y/\Delta x$ is then 1 for some values of Δx and -1 for others. Therefore $f'(x)$ does not exist when $x = 0$.

In summary,

$$f'(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

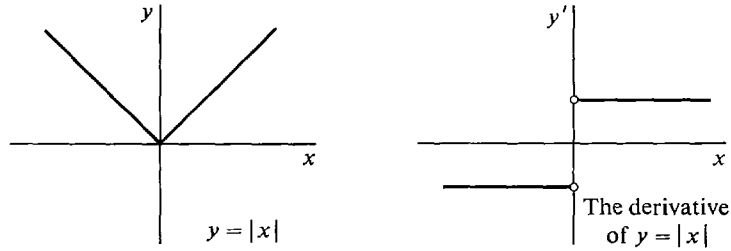Figure 2.1.6 shows $f(x)$ and $f'(x)$.

Figure 2.1.6 $y = |x|$ The derivative of $y = |x|$

The derivative has a variety of applications to the physical, life, and social sciences. It may come up in one of the following contexts.

Velocity: If an object moves according to the equation $s = f(t)$ where t is time and s is distance, the derivative $v = f'(t)$ is called the *velocity* of the object at time t.

Growth rates: A population y (of people, bacteria, molecules, etc.) grows according to the equation $y = f(t)$ where t is time. Then the derivative $y' = f'(t)$ is the *rate of growth* of the population y at time t.

Marginal values (economics): Suppose the total cost (or profit, etc.) of producing x items is $y = f(x)$ dollars. Then the cost of making one additional item is approximately the derivative $y' = f'(x)$ because y' is the change in y per unit change in x. This derivative is called the *marginal cost*.

EXAMPLE 5   A ball thrown upward with initial velocity b ft per sec will be at a height

$$y = bt - 16t^2$$

feet after t seconds. Find the velocity at time t. Let t be real and $\Delta t \neq 0$, infinitesimal.

$$y + \Delta y = b(t + \Delta t) - 16(t + \Delta t)^2,$$
$$\Delta y = [b(t + \Delta t) - 16(t + \Delta t)^2] - [bt - 16t^2],$$
$$\frac{\Delta y}{\Delta t} = \frac{[b(t + \Delta t) - 16(t + \Delta t)^2] - [bt - 16t^2]}{\Delta t}$$
$$= \frac{b\,\Delta t - 32t\,\Delta t - 16(\Delta t)^2}{\Delta t}$$
$$= b - 32t - 16\,\Delta t.$$
$$st\left(\frac{\Delta y}{\Delta t}\right) = st(b - 32t - 16\,\Delta t)$$
$$= st(b - 32t) - st(16\,\Delta t)$$
$$= b - 32t - 0 = b - 32t.$$

At time t sec, $v = y' = b - 32t$ ft/sec.

Both functions are graphed in Figure 2.1.7.

Figure 2.1.7 $y = bt - 16t^2$ $v = b - 32t$

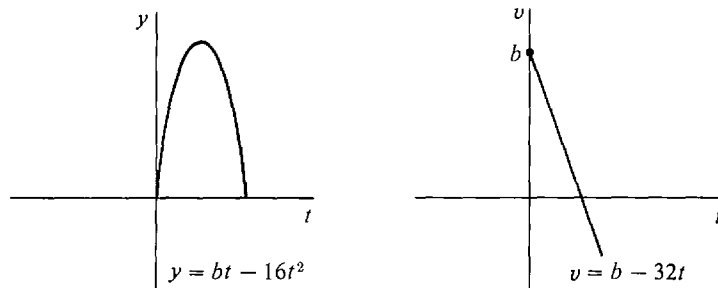**EXAMPLE 6**   Suppose a bacterial culture grows in such a way that at time t there are t^3 bacteria. Find the rate of growth at time $t = 1000$ sec.

$$y = t^3 \qquad y' = 3t^2 \quad \text{by Example 1.}$$

At $t = 1000$, $y' = 3{,}000{,}000$ bacteria/sec.

EXAMPLE 7   Suppose the cost of making x needles is $\sqrt{x}$ dollars. What is the marginal cost after 10,000 needles have been made?

$$y = \sqrt{x}, \qquad y' = \frac{1}{2\sqrt{x}} \qquad \text{by Example 2.}$$

At $x = 10{,}000$, $y' = \dfrac{1}{2\sqrt{10{,}000}} = \dfrac{1}{200}$ dollars per needle.

Thus the marginal cost is one half of a cent per needle.

PROBLEMS FOR SECTION 2.1

Find the derivative of the given function in Problems 1–21.

1	$f(x) = x^2$	**2**	$f(t) = t^2 + 3$
3	$f(x) = 1 - 2x^2$	**4**	$f(x) = 3x^2 + 2$
5	$f(t) = 4t$	**6**	$f(x) = 2 - 5x$
7	$f(t) = 4t^3$	**8**	$f(t) = -t^3$
9	$f(u) = 5\sqrt{u}$	**10**	$f(u) = \sqrt{u + 2}$
11	$g(x) = x\sqrt{x}$	**12**	$g(x) = 1/\sqrt{x}$
13	$g(t) = t^{-2}$	**14**	$g(t) = t^{-3}$
15	$f(y) = 3y^{-1} + 4y$	**16**	$f(y) = 2y^3 + 4y^2$
17	$f(x) = ax + b$	**18**	$f(x) = ax^2$
19	$f(x) = \sqrt{ax + b}$	**20**	$f(x) = 1/(x + 2)$
21	$f(x) = 1/(3 - 2x)$			

22   Find the derivative of $f(x) = 2x^2$ at the point $x = 3$.

23   Find the slope of the curve $f(x) = \sqrt{x - 1}$ at the point $x = 5$.

24   An object moves according to the equation $y = 1/(t + 2), t \geq 0$. Find the velocity as a function of t.

25   A particle moves according to the equation $y = t^4$. Find the velocity as a function of t.

26   Suppose the population of a town grows according to the equation $y = 100t + t^2$. Find the rate of growth at time $t = 100$ years.

27   Suppose a company makes a total profit of $1000x - x^2$ dollars on x items. Find the marginal profit in dollars per item when $x = 200$, $x = 500$, and $x = 1000$.

28   Find the derivative of the function $f(x) = |x + 1|$.

29   Find the derivative of the function $f(x) = |x^3|$.

30   Find the slope of the parabola $y = ax^2 + bx + c$ where a, b, c are constants.

2.2 DIFFERENTIALS AND TANGENT LINES

Suppose we are given a curve $y = f(x)$ and at a point (a, b) on the curve the slope $f'(a)$ is defined. Then the *tangent line* to the curve at the point (a, b), illustrated in Figure 2.2.1, is defined to be the straight line which passes through the point (a, b) and has the same slope as the curve at $x = a$. Thus the tangent line is given by the equation

$$l(x) - b = f'(a)(x - a),$$
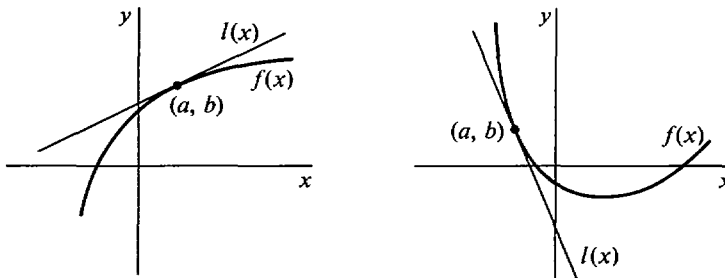
or

$$l(x) = f'(a)(x - a) + b.$$

Figure 2.2.1   Tangent lines.

EXAMPLE 1   For the curve $y = x^3$, find the tangent lines at the points $(0, 0)$, $(1, 1)$, and $(-\frac{1}{2}, -\frac{1}{8})$ (Figure 2.2.2).

The slope is given by $f'(x) = 3x^2$. At $x = 0$, $f'(0) = 3 \cdot 0^2 = 0$. The tangent line has the equation

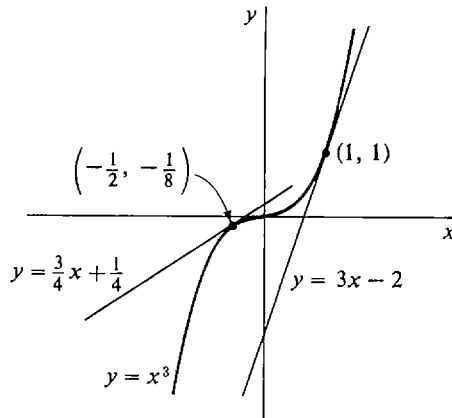$$y = 0(x - 0) + 0, \quad \text{or} \quad y = 0.$$

Figure 2.2.2

At $x = 1$, $f'(1) = 3$, whence the tangent line is

$$y = 3(x - 1) + 1, \quad \text{or} \quad y = 3x - 2.$$

At $x = -\frac{1}{2}$, $f'(-\frac{1}{2}) = 3 \cdot (-\frac{1}{2})^2 = \frac{3}{4}$, so the tangent line is

$$y = \tfrac{3}{4}(x - (-\tfrac{1}{2})) + (-\tfrac{1}{8}), \quad \text{or} \quad y = \tfrac{3}{4}x + \tfrac{1}{4}.$$

Given a curve $y = f(x)$, suppose that x starts out with the value a and then changes by an infinitesimal amount Δx. What happens to y? Along the curve, y will change by the amount

$$f(a + \Delta x) - f(a) = \Delta y.$$

But along the tangent line y will change by the amount

$$l(a + \Delta x) - l(a) = [f'(a)(a + \Delta x - a) + b] - [f'(a)(a - a) + b]$$
$$= f'(a)\,\Delta x.$$

When x changes from a to $a + \Delta x$, we see that:

$$\text{change in } y \text{ along curve} = f(a + \Delta x) - f(a),$$
$$\text{change in } y \text{ along tangent line} = f'(a)\,\Delta x.$$

In the last section we introduced the dependent variable Δy, the increment of y, with the equation

$$\Delta y = f(x + \Delta x) - f(x).$$

Δy is equal to the change in y along the curve as x changes to $x + \Delta x$.

The following theorem gives a simple but useful formula for the increment Δy.

INCREMENT THEOREM

Let $y = f(x)$. Suppose $f'(x)$ exists at a certain point x, and Δx is infinitesimal. Then Δy is infinitesimal, and

$$\Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x$$

for some infinitesimal ε, which depends on x and Δx.

PROOF

Case 1 $\Delta x = 0$. In this case, $\Delta y = f'(x)\,\Delta x = 0$, and we put $\varepsilon = 0$.

Case 2 $\Delta x \neq 0$. Then

$$\frac{\Delta y}{\Delta x} \approx f'(x);$$

so for some infinitesimal ε,

$$\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon.$$

Multiplying both sides by Δx,

$$\Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x.$$

EXAMPLE 2   Let $y = x^3$, so that $y' = 3x^2$. According to the Increment Theorem,

$$\Delta y = 3x^2\,\Delta x + \varepsilon\,\Delta x$$

for some infinitesimal ε. Find ε in terms of x and Δx when $\Delta x \neq 0$. We have

$$\Delta y = 3x^2\,\Delta x + \varepsilon\,\Delta x,$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + \varepsilon,$$

$$\varepsilon = \frac{\Delta y}{\Delta x} - 3x^2.$$

We must still eliminate Δy. From Example 1 in Section 2.1,

$$\Delta y = (x + \Delta x)^3 - x^3,$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\,\Delta x + (\Delta x)^2.$$

Substituting, $\varepsilon = (3x^2 + 3x\,\Delta x + (\Delta x)^2) - 3x^2.$

Since $3x^2$ cancels,

$$\varepsilon = 3x\,\Delta x + (\Delta x)^2.$$

We shall now introduce a new dependent variable dy, called the differential of y, with the equation

$$dy = f'(x)\,\Delta x.$$

dy is equal to the change in y along the tangent line as x changes to $x + \Delta x$. In Figure 2.2.3 we see dy and Δy under the microscope.

$$\Delta y = \text{change in } y \text{ along curve}$$
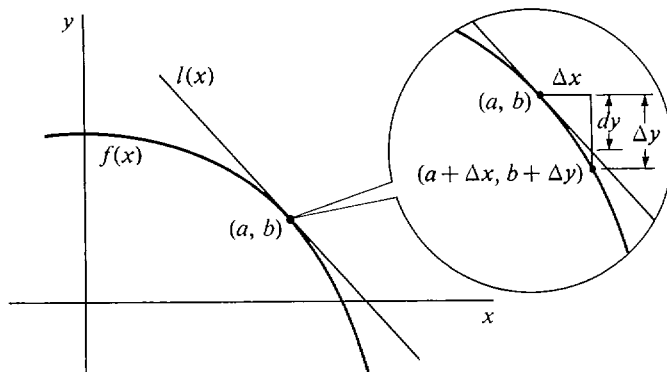$$dy = \text{change in } y \text{ along tangent line}$$

Figure 2.2.3

To keep our notation uniform we also introduce the symbol dx as another name for Δx. For an independent variable x, Δx and dx are the same, but for a dependent variable y, Δy and dy are different.

DEFINITION

Suppose y depends on x, $y = f(x)$.

(i)   *The **differential of** x is the independent variable $dx = \Delta x$.*
(ii)   *The **differential of** y is the dependent variable dy given by*

$$dy = f'(x)\, dx.$$

When $dx \neq 0$, the equation above may be rewritten as

$$\frac{dy}{dx} = f'(x).$$

Compare this equation with

$$\frac{\Delta y}{\Delta x} \approx f'(x).$$

The quotient dy/dx is a very convenient alternative symbol for the derivative $f'(x)$. In fact we shall write the derivative in the form dy/dx most of the time.

The differential dy depends on two independent variables x and dx. In functional notation,

$$dy = df(x, dx)$$

where df is the real function of two variables defined by

$$df(x, dx) := f'(x)\, dx.$$

When dx is substituted for Δx and dy for $f'(x)\,dx$, the Increment Theorem takes the short form

$$\Delta y = dy + \varepsilon\,dx.$$

The Increment Theorem can be explained graphically using an infinitesimal microscope. Under an infinitesimal microscope, a line of length Δx is magnified to a line of unit length, but a line of length $\varepsilon\,\Delta x$ is only magnified to an infinitesimal length ε. Thus the Increment Theorem shows that when $f'(x)$ exists:

(1) The differential dy and the increment $\Delta y = dy + \varepsilon\,dx$ are so close to each other that they cannot be distinguished under an infinitesimal microscope.

(2) The curve $y = f(x)$ and the tangent line at (x, y) are so close to each other that they cannot be distinguished under an infinitesimal microscope; both look like a straight line of slope $f'(x)$.

Figure 2.2.3 is not really accurate. The curvature had to be exaggerated in order to distinguish the curve and tangent line under the microscope. To give an accurate picture, we need a more complicated figure like Figure 2.2.4, which has a second infinitesimal microscope trained on the point $(a + \Delta x, b + \Delta y)$ in the field of view of the original microscope. This second microscope magnifies $\varepsilon\,dx$ to a unit length and magnifies Δx to an infinite length.
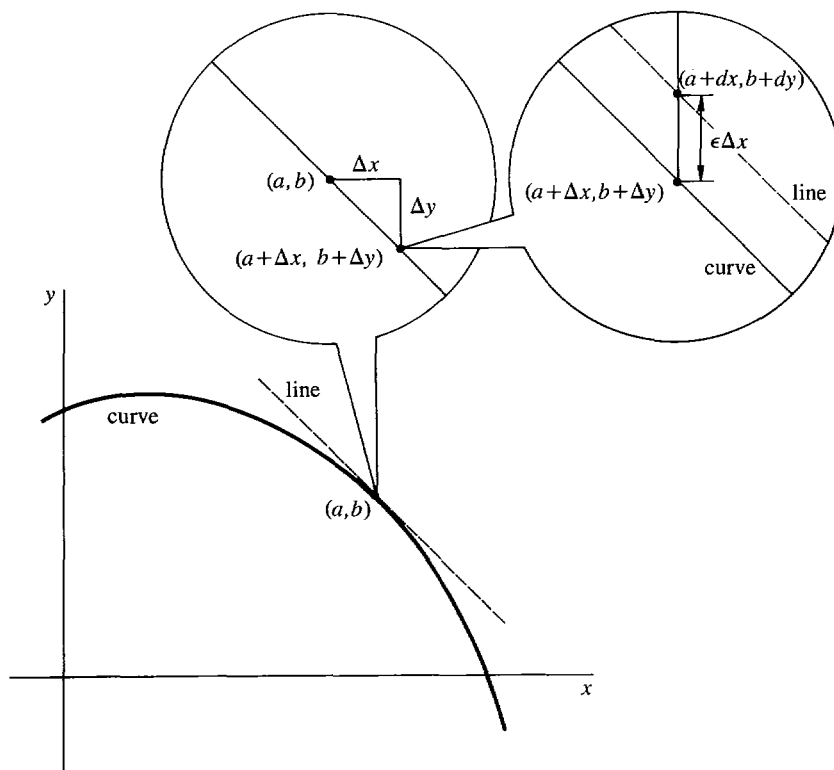

Figure 2.2.4

EXAMPLE 3   Whenever a derivative $f'(x)$ is known, we can find the differential dy at once by simply multiplying the derivative by dx, using the formula $dy = f'(x)\, dx$. The examples in the last section give the following differentials.

(a) $\qquad\quad y = x^3, \qquad\qquad dy = 3x^2\, dx.$

(b) $\qquad\quad y = \sqrt{x}, \qquad\qquad dy = \dfrac{dx}{2\sqrt{x}}$ where $x > 0$.

(c) $\qquad\quad y = 1/x, \qquad\qquad dy = -dx/x^2$ when $x \neq 0$.

(d) $\qquad\quad y = |x|, \qquad\qquad dy = \begin{cases} dx & \text{when } x > 0, \\ -dx & \text{when } x < 0, \\ \text{undefined} & \text{when } x = 0. \end{cases}$

(e) $\qquad\quad y = bt - 16t^2, \quad dy = (b - 32t)\, dt.$

The differential notation may also be used when we are given a system of formulas in which two or more dependent variables depend on an independent variable. For example if y and z are functions of x,

$$y = f(x), \qquad z = g(x),$$

then $\Delta y, \Delta z, dy, dz$ are determined by

$$\Delta y = f(x + \Delta x) - f(x), \qquad \Delta z = g(x + \Delta x) - g(x),$$
$$dy = f'(x)\, dx, \qquad\qquad\quad dz = g'(x)\, dx.$$

EXAMPLE 4   Given $y = \tfrac{1}{2}x, z = x^3$, with x as the independent variable, then

$$\Delta y = \tfrac{1}{2}(x + \Delta x) - \tfrac{1}{2}x = \tfrac{1}{2}\Delta x,$$
$$\Delta z = 3x^2\, \Delta x + 3x(\Delta x)^2 + (\Delta x)^3,$$
$$dy = \tfrac{1}{2}\, dx, \qquad dz = 3x^2\, dx.$$

The meaning of the symbols for increment and differential in this example will be different if we take y as the independent variable. Then x and z are functions of y.

$$x = 2y, \qquad z = 8y^3.$$

Now $\Delta y = dy$ is just an independent variable, while

$$\Delta x = 2(y + \Delta y) - 2y = 2\,\Delta y,$$
$$\Delta z = 8(y + \Delta y)^3 - 8y^3$$
$$= 8[3y^2\, \Delta y + 3y(\Delta y)^2 + (\Delta y)^3]$$
$$= 24y^2\, \Delta y + 24y(\Delta y)^2 + 8(\Delta y)^3.$$

Moreover, $\qquad dx = 2\, dy, \qquad dz = 24y^2\, dy.$

We may also apply the differential notation to terms. If $\tau(x)$ is a term with the variable x, then $\tau(x)$ determines a function f,

$$\tau(x) = f(x),$$

and the differential $d(\tau(x))$ has the meaning

$$d(\tau(x)) = f'(x)\,dx.$$

EXAMPLE 5

(a) $$d(x^3) = 3x^2\,dx.$$

(b) $$d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}, \qquad x > 0.$$

(c) $$d(1/x) = -\frac{dx}{x^2}, \qquad x \neq 0.$$

(d) $$d(|x|) = \begin{cases} dx & \text{when } x > 0, \\ -dx & \text{when } x < 0, \\ \text{undefined} & \text{when } x = 0. \end{cases}$$

(e)   Let $u = bt$ and $w = -16t^2$. Then

$$u + w = bt - 16t^2, \qquad d(u + w) = (b - 32t)\,dt.$$

PROBLEMS FOR SECTION 2.2

In Problems 1–8, express Δy and dy as functions of x and Δx, and for Δx infinitesimal find an infinitesimal ε such that $\Delta y = dy + \varepsilon\,\Delta x$.

1	$y = x^2$	2	$y = -5x^2$
3	$y = 2\sqrt{x}$	4	$y = x^4$
5	$y = 1/x$	6	$y = x^{-2}$
7	$y = x - 1/x$	8	$y = 4x + x^3$

9   If $y = 2x^2$ and $z = x^3$, find Δy, Δz, dy, and dz.

10   If $y = 1/(x + 1)$ and $z = 1/(x + 2)$, find Δy, Δz, dy, and dz.

11	Find $d(2x + 1)$	12	Find $d(x^2 - 3x)$
13	Find $d(\sqrt{x + 1})$	14	Find $d(\sqrt{2x + 1})$
15	Find $d(ax + b)$	16	Find $d(ax^2)$
17	Find $d(3 + 2/x)$	18	Find $d(x\sqrt{x})$
19	Find $d(1/\sqrt{x})$	20	Find $d(x^3 - x^2)$

21   Let $y = \sqrt{x}, z = 3x$. Find $d(y + z)$ and $d(y/z)$.

22   Let $y = x^{-1}$ and $z = x^3$. Find $d(y + z)$ and $d(yz)$.

In Problems 23–30 below, find the equation of the line tangent to the given curve at the given point.

23	$y = x^2$; (2, 4)	24	$y = 2x^2$; $(-1, 2)$
25	$y = -x^2$; (0, 0)	26	$y = \sqrt{x}$; (1, 1)

27 $y = 3x - 4;$ $(1, -1)$ **28** $y = \sqrt{t - 1};$ $(5, 2)$

29 $y = x^4;$ $(-2, 16)$ **30** $y = x^3 - x;$ $(0, 0)$

31   Find the equation of the line tangent to the parabola $y = x^2$ at the point (x_0, x_0^2).

32   Find all points $P(x_0, x_0^2)$ on the parabola $y = x^2$ such that the tangent line at P passes through the point $(0, -4)$.

☐ **33**   Prove that the line tangent to the parabola $y = x^2$ at $P(x_0, x_0^2)$ does not meet the parabola at any point except P.

2.3 DERIVATIVES OF RATIONAL FUNCTIONS

A term of the form

$$a_1 x + a_0$$

where a_1, a_0 are real numbers, is called a *linear term* in x; if $a_1 \neq 0$, it is also called *polynomial of degree one* in x. A term of the form

$$a_2 x^2 + a_1 x + a_0, \qquad a_2 \neq 0$$

is called a *polynomial of degree two* in x, and, in general, a term of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0$$

is called a *polynomial of degree n* in x.

A *rational term* in x is any term which is built up from the variable x and real numbers using the operations of addition, multiplication, subtraction, and division. For example every polynomial is a rational term and so are the terms

$$\frac{(3x^2 - 5)(x + 2)^3}{5x - 11}, \qquad \frac{(1 + 1/x)^9}{x^3 + 1/(2 - x)}.$$

A *linear function, polynomial function*, or *rational function* is a function which is given by a linear term, polynomial, or rational term, respectively. In this section we shall establish a set of rules which enable us to quickly differentiate any rational function. The rules will also be useful later on in differentiating other functions.

THEOREM 1

The derivative of a linear function is equal to the coefficient of x. That is,

$$\frac{d(bx + c)}{dx} = b, \qquad d(bx + c) = b\, dx.$$

PROOF Let $y = bx + c$, and let $\Delta x \neq 0$ be infinitesimal. Then

$$y + \Delta y = b(x + \Delta x) + c,$$
$$\Delta y = (b(x + \Delta x) + c) - (bx + c) = b\, \Delta x,$$
$$\frac{\Delta y}{\Delta x} = \frac{b\, \Delta x}{\Delta x} = b.$$

Therefore $\dfrac{dy}{dx} = st(b) = b.$

Multiplying through by dx, we obtain at once

$$dy = b\,dx.$$

If in Theorem 1 we put $b = 1, c = 0$, we see that the derivative of the identity function $f(x) = x$ is $f'(x) = 1$; i.e.,

$$\frac{dx}{dx} = 1, \qquad dx = dx.$$

On the other hand, if we put $b = 0$ in Theorem 1 then the term $bx + c$ is just the constant c, and we find that the derivative of the constant function $f(x) = c$ is $f'(x) = 0$; i.e.,

$$\frac{dc}{dx} = 0, \qquad dc = 0.$$

THEOREM 2 (Sum Rule)

Suppose u and v depend on the independent variable x. Then for any value of x where du/dx and dv/dx exist,

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \qquad d(u + v) = du + dv.$$

In other words, the derivative of the sum is the sum of the derivatives.

PROOF Let $y = u + v$, and let $\Delta x \neq 0$ be infinitesimal. Then

$$y + \Delta y = (u + \Delta u) + (v + \Delta v),$$

$$\Delta y = [(u + \Delta u) + (v + \Delta v)] - [u + v] = \Delta u + \Delta v,$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u + \Delta v}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

Taking standard parts,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right) = st\left(\frac{\Delta u}{\Delta x}\right) + st\left(\frac{\Delta v}{\Delta x}\right).$$

Thus
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

By using the Sum Rule $n - 1$ times, we see that

$$\frac{d(u_1 + \cdots + u_n)}{dx} = \frac{du_1}{dx} + \cdots + \frac{du_n}{dx}, \quad \text{or} \quad d(u_1 + \cdots + u_n) = du_1 + \cdots + du_n.$$

THEOREM 3 (Constant Rule)

Suppose u depends on x, and c is a real number. Then for any value of x where du/dx exists,

$$\frac{d(cu)}{dx} = c\frac{du}{dx}, \qquad d(cu) = c\,du.$$

PROOF Let $y = cu$, and let $\Delta x \neq 0$ be infinitesimal. Then

$$y + \Delta y = c(u + \Delta u),$$

$$\Delta y = c(u + \Delta u) - cu = c\,\Delta u,$$

$$\frac{\Delta y}{\Delta x} = \frac{c\,\Delta u}{\Delta x} = c\frac{\Delta u}{\Delta x}.$$

Taking standard parts,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(c\frac{\Delta u}{\Delta x}\right) = c\,st\left(\frac{\Delta u}{\Delta x}\right)$$

whence

$$\frac{dy}{dx} = c\frac{du}{dx}.$$

The Constant Rule shows that in computing derivatives, a constant factor may be moved "outside" the derivative. It can only be used when c is a constant. For products of two functions of x, we have:

THEOREM 4 (Product Rule)

Suppose u and v depend on x. Then for any value of x where du/dx and dv/dx exist,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad d(uv) = u\,dv + v\,du.$$

PROOF Let $y = uv$, and let $\Delta x \neq 0$ be infinitesimal.

$$y + \Delta y = (u + \Delta u)(v + \Delta v),$$

$$\Delta y = (u + \Delta u)(v + \Delta v) - uv = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v,$$

$$\frac{\Delta y}{\Delta x} = \frac{u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

Δu is infinitesimal by the Increment Theorem, whence

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right)$$

$$= u \cdot st\left(\frac{\Delta v}{\Delta x}\right) + v \cdot st\left(\frac{\Delta u}{\Delta x}\right) + 0 \cdot st\left(\frac{\Delta v}{\Delta x}\right).$$

So

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The Constant Rule is really the special case of the Product Rule where v is a constant function of x, $v = c$. To check this we let v be the constant c and see what the Product Rule gives us:

$$\frac{d(u \cdot c)}{dx} = u\frac{dc}{dx} + c\frac{du}{dx} = u \cdot 0 + c\frac{du}{dx} = c\frac{du}{dx}.$$

This is the Constant Rule.

The Product Rule can also be used to find the derivative of a power of u.

THEOREM 5 (Power Rule)

Let u depend on x and let n be a positive integer. For any value of x where du/dx exists,

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}, \qquad d(u^n) = nu^{n-1}\,du.$$

PROOF To see what is going on we first prove the Power Rule for $n = 1, 2, 3, 4$.

$n = 1$:   We have $u^n = u$ and $u^0 = 1$, whence

$$\frac{d(u^n)}{dx} = \frac{du}{dx} = 1 \cdot u^0 \cdot \frac{du}{dx}.$$

$n = 2$:   We use the Product Rule,

$$\frac{d(u^2)}{dx} = \frac{d(u \cdot u)}{dx} = u\frac{du}{dx} + u\frac{du}{dx} = 2 \cdot u^1 \cdot \frac{du}{dx}.$$

$n = 3$:   We write $u^3 = u \cdot u^2$, use the Product Rule again, and then use the result for $n = 2$.

$$\frac{d(u^3)}{dx} = \frac{d(u \cdot u^2)}{dx} = u\frac{d(u^2)}{dx} + u^2\frac{du}{dx}$$

$$= u \cdot 2u\frac{du}{dx} + u^2\frac{du}{dx} = 3u^2\frac{du}{dx}.$$

$n = 4$:   Using the Product Rule and then the result for $n = 3$,

$$\frac{d(u^4)}{dx} = \frac{d(u \cdot u^3)}{dx} = u\frac{d(u^3)}{dx} + u^3\frac{du}{dx}$$

$$= u \cdot 3u^2\frac{du}{dx} + u^3\frac{du}{dx} = 4u^3\frac{du}{dx}.$$

We can continue this process indefinitely and prove the theorem for every positive integer n. To see this, assume that we have proved the theorem for m. That is, assume that

(1)
$$\frac{d(u^m)}{dx} = mu^{m-1}\frac{du}{dx}.$$

We then show that it is also true for $m + 1$. Using the Product Rule and the Equation 1,

$$\frac{d(u^{m+1})}{dx} = \frac{d(u \cdot u^m)}{dx} = u\frac{d(u^m)}{dx} + u^m\frac{du}{dx}$$

$$= u \cdot mu^{m-1}\frac{du}{dx} + u^m\frac{du}{dx} = (m + 1)u^m\frac{du}{dx}.$$

Thus
$$\frac{d(u^{m+1})}{dx} = (m + 1)u^m\frac{du}{dx}.$$

This shows that the theorem holds for $m + 1$.

We have shown the theorem is true for $1, 2, 3, 4$. Set $m = 4$; then the theorem

holds for $m + 1 = 5$. Set $m = 5$; then it holds for $m + 1 = 6$. And so on. Hence the theorem is true for all positive integers n.

In the proof of the Power Rule, we used the following principle:

PRINCIPLE OF INDUCTION

Suppose a statement $P(n)$ about an arbitrary integer n is true when $n = 1$. Suppose further that for any positive integer m such that $P(m)$ is true, $P(m + 1)$ is also true. Then the statement $P(n)$ is true of every positive integer n.

In the previous proof, $P(n)$ was the Power Rule,

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

The Principle of Induction can be made plausible in the following way. Let a positive integer n be given. Set $m = 1$; since $P(1)$ is true, $P(2)$ is true. Now set $m = 2$; since $P(2)$ is true, $P(3)$ is true. We continue reasoning in this way for n steps and conclude that $P(n)$ is true.

The Power Rule also holds for $n = 0$ because when $u \neq 0, u^0 = 1$ and $d1/dx = 0$.

Using the Sum, Constant, and Power rules, we can compute the derivative of a polynomial function very easily. We have

$$\frac{d(x^n)}{dx} = nx^{n-1},$$

$$\frac{d(cx^n)}{dx} = cnx^{n-1},$$

and thus

$$\frac{d(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)}{dx} = a_n \cdot nx^{n-1} + a_{n-1}(n-1)x^{n-2} + \cdots + a_1.$$

EXAMPLE 1 $\dfrac{d(-3x^5)}{dx} = -3 \cdot 5x^4 = -15x^4.$

EXAMPLE 2 $\dfrac{d(6x^4 - 2x^3 + x - 1)}{dx} = 24x^3 - 6x^2 + 1.$

Two useful facts can be stated as corollaries.

COROLLARY 1

The derivative of a polynomial of degree $n > 0$ is a polynomial of degree $n - 1$. (A nonzero constant is counted as a polynomial of degree zero.)

COROLLARY 2

If u depends on x, then
$$\frac{d(u + c)}{dx} = \frac{du}{dx}$$

whenever du/dx exists. That is, adding a constant to a function does not change its derivative.

In Figure 2.3.1 we see that the effect of adding a constant is to move the curve up or down the y-axis without changing the slope.

For the last two rules in this section we need the formula for the derivative of $1/v$.
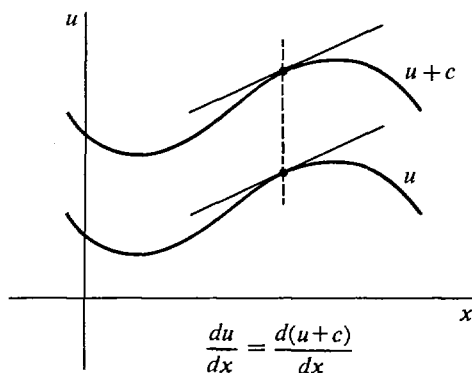

$$\frac{du}{dx} = \frac{d(u+c)}{dx}$$

Figure 2.3.1

LEMMA

Suppose v depends on x. Then for any value of x where $v \neq 0$ and dv/dx exists,
$$\frac{d(1/v)}{dx} = -\frac{1}{v^2}\frac{dv}{dx}, \qquad d\left(\frac{1}{v}\right) = -\frac{1}{v^2}dv.$$

PROOF Let $y = 1/v$ and let $\Delta x \neq 0$ be infinitesimal.

$$y + \Delta y = \frac{1}{v + \Delta v},$$

$$\Delta y = \frac{1}{v + \Delta v} - \frac{1}{v},$$

$$\frac{\Delta y}{\Delta x} = \frac{1/(v + \Delta v) - 1/v}{\Delta x}$$

$$= \frac{v - (v + \Delta v)}{\Delta x v(v + \Delta v)}$$

$$= -\frac{1}{v(v + \Delta v)}\frac{\Delta v}{\Delta x}.$$

Taking standard parts,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(-\frac{1}{v(v + \Delta v)}\frac{\Delta v}{\Delta x}\right)$$

$$= st\left(-\frac{1}{v(v + \Delta v)}\right) st\left(\frac{\Delta v}{\Delta x}\right)$$

$$= -\frac{1}{v^2} st\left(\frac{\Delta v}{\Delta x}\right).$$

Therefore

$$\frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}.$$

THEOREM 6 (Quotient Rule)

Suppose u, v depend on x. Then for any value of x where du/dx, dv/dx exist and v ≠ 0,

$$\frac{d(u/v)}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2}, \qquad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

PROOF We combine the Product Rule and the formula for $d(1/v)$. Let $y = u/v$. We write y in the form

$$y = \frac{1}{v}\cdot u.$$

Then

$$dy = d\left(\frac{1}{v}u\right) = \frac{1}{v}\,du + u\,d\left(\frac{1}{v}\right)$$

$$= \frac{1}{v}\,du + u(-v^{-2})\,dv$$

$$= \frac{v\,du - u\,dv}{v^2}.$$

THEOREM 7 (Power Rule for Negative Exponents)

Suppose u depends on x and n is a negative integer. Then for any value of x where du/dx exists and u ≠ 0, $d(u^n)/dx$ exists and

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}, \qquad d(u^n) = nu^{n-1}\,du.$$

PROOF Since n is negative, $n = -m$ where m is positive. Let $y = u^n = u^{-m}$. Then $y = 1/u^m$. By the Lemma and the Power Rule,

$$\frac{dy}{dx} = -\frac{1}{(u^m)^2}\cdot\frac{d(u^m)}{dx}$$

$$= -\frac{1}{u^{2m}}\cdot mu^{m-1}\frac{du}{dx}$$

$$= (-m)u^{-2m}u^{m-1}\frac{du}{dx}$$

$$= (-m)u^{-m-1}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

The Quotient Rule together with the Constant, Sum, Product, and Power Rules make it easy to differentiate any rational function.

EXAMPLE 3   Find dy when

$$y = \frac{1}{x^2 - 3x + 1}.$$

Introduce the new variable u with the equation

$$u = x^2 - 3x + 1.$$

Then $y = 1/u$, and $du = (2x - 3)\,dx$, so

$$dy = -\frac{1}{u^2}\,du = \frac{-(2x - 3)}{(x^2 - 3x + 1)^2}\,dx.$$

EXAMPLE 4   Let $y = \dfrac{(x^4 - 2)^3}{5x - 1}$ and find dy.

Let $\qquad u = (x^4 - 2)^3, \qquad v = 5x - 1.$

Then $\qquad y = \dfrac{u}{v}, \qquad dy = \dfrac{v\,du - u\,dv}{v^2}.$

Also, $\qquad du = 3\cdot(x^4 - 2)^2\cdot 4x^3\,dx = 12(x^4 - 2)^2\cdot x^3\,dx,$

$dv = 5\,dx.$

Therefore $\qquad dy = \dfrac{(5x - 1)12(x^4 - 2)^2 x^3\,dx - (x^4 - 2)^3 5\,dx}{(5x - 1)^2}$

$$= \frac{(x^4 - 2)^2[12(5x - 1)x^3 - 5(x^4 - 2)]}{(5x - 1)^2}\,dx.$$

EXAMPLE 5   Let $y = 1/x^3 + 3/x^2 + 4/x + 5$.

Then $\qquad dy = \left(-\dfrac{3}{x^4} - \dfrac{6}{x^3} - \dfrac{4}{x^2}\right)dx.$

EXAMPLE 6   Find dy where

$$y = \left(\frac{1}{x^2 + x} + 1\right)^2.$$

This problem can be worked by means of a double substitution. Let

$$u = x^2 + x, \qquad v = \frac{1}{u} + 1.$$

Then
$$y = v^2.$$

We find dy, dv, and du,

$$dy = 2v\,dv,$$
$$dv = -u^{-2}\,du,$$
$$du = (2x + 1)\,dx.$$

Substituting, we get dy in terms of x and dx,

$$
\begin{aligned}
dy &= 2v(-u^{-2}\,du) \\
&= -2vu^{-2}(2x + 1)\,dx \\
&= -2\left(\frac{1}{u} + 1\right)u^{-2}(2x + 1)\,dx \\
&= -2\left(\frac{1}{x^2 + x} + 1\right)(x^2 + x)^{-2}(2x + 1)\,dx.
\end{aligned}
$$

EXAMPLE 7   Assume that u and v depend on x. Given $y = (uv)^{-2}$, find dy/dx in terms of du/dx and dv/dx.

Let $s = uv$, whence $y = s^{-2}$. We have

$$dy = -2s^{-3}\,ds,$$
$$ds = u\,dv + v\,du.$$

Substituting, $dy = -2(uv)^{-3}(u\,dv + v\,du),$

and
$$\frac{dy}{dx} = -2(uv)^{-3}\left(u\frac{dv}{dx} + v\frac{du}{dx}\right).$$

The six rules for differentiation which we have proved in this section are so useful that they should be memorized. We list them all together.

Table 2.3.1   Rules for Differentiation

(1)	$\dfrac{d(bx + c)}{dx} = b.$	$d(bx + c) = b\,dx.$
(2)	$\dfrac{d(u + v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}.$	$d(u + v) = du + dv.$
(3)	$\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}.$	$d(cu) = c\,du.$
(4)	$\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}.$	$d(uv) = u\,dv + v\,du.$
(5)	$\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}.$	$d(u^n) = nu^{n-1}\,du$ (n is any integer).
(6)	$\dfrac{d(u/v)}{dx} = \dfrac{v\,du/dx - u\,dv/dx}{v^2}.$	$d(u/v) = \dfrac{v\,du - u\,dv}{v^2}.$

An easy way to remember the way the signs are in the Quotient Rule 6 is to put $u = 1$ and use the Power Rule 5 with $n = -1$,

$$d(1/v) = d(v^{-1}) = -1 \cdot v^{-2} \, dv = \frac{-1 \, dv}{v^2}.$$

PROBLEMS FOR SECTION 2.3

In Problems 1–42 below, find the derivative.

1    $f(x) = 3x^2 + 5x - 4$

2    $s = \frac{1}{3}t^3 + \frac{1}{2}t^2 + t$

3    $y = (x + 8)^5$

4    $z = (2 + 3x)^4$

5    $f(t) = (4 - t)^3$

6    $g(x) = 3(2 - 5x)^6$

7    $y = (x^2 + 5)^3$

8    $u = (6 + 2x^2)^3$

9    $u = (6 - 2x^2)^3$

10   $w = (1 + 4x^3)^{-2}$

11   $w = (1 - 4x^3)^{-2}$

12   $y = 1 + x^{-1} + x^{-2} + x^{-3}$

13   $f(x) = 5(x + 1 - 1/x)$

14   $u = (x^2 + 3x + 1)^4$

15   $v = 4(2x^2 - x + 3)^{-2}$

16   $y = -(2x + 3 + 4x^{-1})^{-1}$

17   $y = \dfrac{1}{1 + 1/t}$

18   $y = \dfrac{1}{2x^2 + 1}$

19   $s = \dfrac{-3}{4t^2 - 2t + 1}$

20   $s = (2t + 1)(3t - 2)$

21   $h(x) = \frac{1}{2}(x^2 + 1)(5 - 2x)$

22   $y = (2x^3 + 4)(x^2 - 3x + 1)$

23   $v = (3t^2 + 1)(2t - 4)^3$

24   $z = (-2x + 4 + 3x^{-1})(x + 1 - 5x^{-1})$

25   $y = \dfrac{x + 1}{x - 1}$

26   $w = \dfrac{2 - 3x}{1 + 2x}$

27   $y = \dfrac{x^2 - 1}{x^2 + 1}$

28   $u = \dfrac{x}{x^2 + 1}$

29   $x = \dfrac{(s - 1)(s - 2)}{s - 3}$

30   $y = \dfrac{t}{1 + 1/t}$

31   $y = \dfrac{2x^{-1} - x^{-2}}{3x^{-1} - 4x^{-2}}$

32   $y = 4x - 5$

33   $y = 6$

34   $y = 2x(3x - 1)(4 - 2x)$

35   $y = 3(x^2 + 1)(2x^2 - 1)(2x + 3)$

36   $y = (4x + 3)^{-1} + (x - 4)^{-2}$

37   $z = \dfrac{1}{(2x + 1)(x - 3)}$

38   $y = (x^2 + 1)^{-1}(3x - 1)^{-2}$

39   $y = [(2x + 1)^{-1} + 3]^{-1}$

40   $s = [(t^2 + 1)^3 + t]^{-1}$

41   $y = (2x + 1)^3(x^2 + 1)^2$

42   $y = \left(\dfrac{2}{x - 1} - x^{-3}\right)^4$

In Problems 43–48, assume u and v depend on x and find dy/dx in terms of du/dx and dv/dx.

43   $y = u - v$

44   $y = u^2 v$

45   $y = 4u + v^2$

46   $y = 1/(u + v)$

47   $y = 1/uv$

48   $y = (u + v)(2u - v)$

49   Find the line tangent to the curve $y = 1 + x + x^2 + x^3$ at the point $(1, 4)$.

50   Find the line tangent to the curve $y = 9x^{-2}$ at the point $(3, 1)$.

☐ **51**   Consider the parabola $y = x^2 + bx + c$. Find values of b and c such that the line $y = 2x$ is tangent to the parabola at the point $x = 2$, $y = 4$.

☐ **52**   Show that if u, v, and w are differentiable functions of x and $y = uvw$, then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.$$

☐ **53**   Use the principle of induction to show that if n is a positive integer, $u_1, \ldots, u_n$ are differentiable functions of x, and $y = u_1 + \cdots + u_n$, then

$$\frac{dy}{dx} = \frac{du_1}{dx} + \cdots + \frac{du_n}{dx}.$$

☐ **54**   Use the principle of induction to prove that for every positive integer n,

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}.$$

☐ **55**   Every rational function can be written as a quotient of two polynomials, $p(x)/q(x)$. Using this fact, show that the derivative of every rational function is a rational function.

2.4 INVERSE FUNCTIONS

Two real functions f and g are called *inverse functions* if the two equations

$$y = f(x), \qquad x = g(y)$$

have the same graphs in the (x, y) plane. That is, a point (x, y) is on the curve $y = f(x)$ if, and only if, it is on the curve $x = g(y)$. (In general, the graph of the equation $x = g(y)$ is different from the graph of $y = g(x)$, but is the same as the graph of $y = f(x)$; see Figure 2.4.1.)
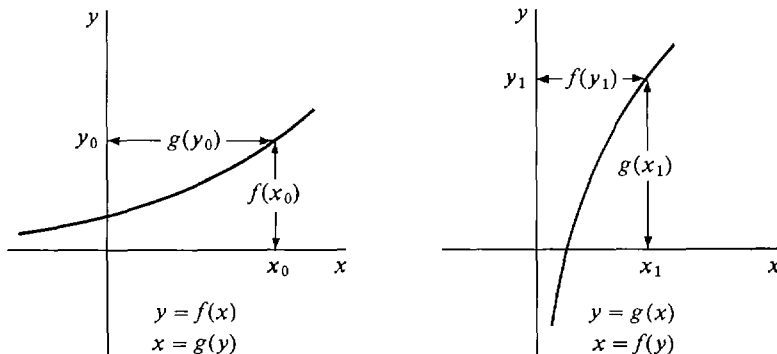

Figure 2.4.1   Inverse Functions

For example, the function $y = x^2$, $x \geq 0$, has the inverse function $x = \sqrt{y}$; the function $y = x^3$ has the inverse function $x = \sqrt[3]{y}$.

If we think of f as a black box operating on an input x to produce an output $f(x)$, the inverse function g is a black box operating on the output $f(x)$ to undo the work of f and produce the original input x (see Figure 2.4.2).
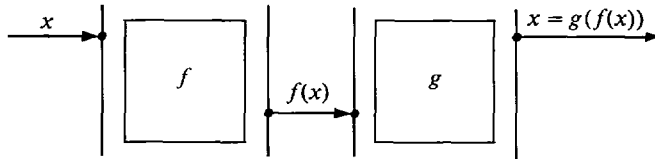

Figure 2.4.2

Many functions, such as $y = x^2$, do not have inverse functions. In Figure 2.4.3, we see that x is not a function of y because at $y = 1$, x has the two values $x = 1$ and $x = -1$.

Often one can tell whether a function f has an inverse by looking at its graph. If there is a horizontal line $y = c$ which cuts the graph at more than one point, the function f has no inverse. (See Figure 2.4.3.) If no horizontal line cuts the graph at more than one point, then f has an inverse function g. Using this rule, we can see in Figure 2.4.4 that the functions $y = |x|$ and $y = \sqrt{1 - x^2}$ do not have inverses.
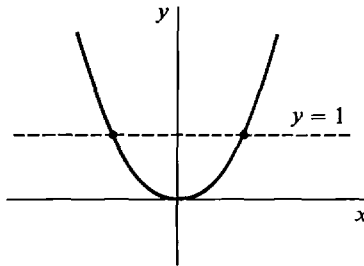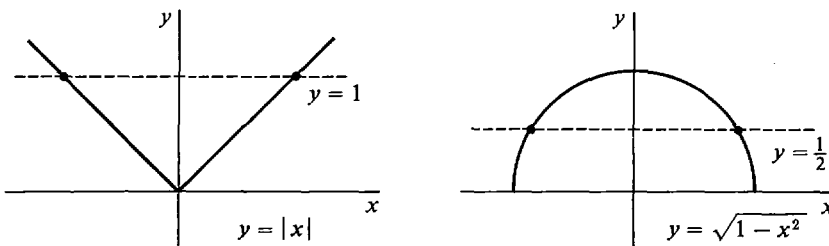

Figure 2.4.3

No inverse functions

Figure 2.4.4

Table 2.4.1 shows some familiar functions which do have inverses. Note that in each case, $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$.

Table 2.4.1

function $y = f(x)$	$\dfrac{dy}{dx}$	inverse function $x = g(y)$	$\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$
$y = x + c$	1	$x = y - c$	1
$y = kx$	k	$x = y/k$	$1/k$
$y = x^2, \quad x \geq 0$	$2x$	$x = \sqrt{y}$	$\dfrac{1}{2\sqrt{y}} = \dfrac{1}{2x}$
$y = x^2, \quad x \leq 0$	$2x$	$x = -\sqrt{y}$	$-\dfrac{1}{2\sqrt{y}} = \dfrac{1}{2x}$
$y = 1/x$	$-\dfrac{1}{x^2}$	$x = 1/y$	$-\dfrac{1}{y^2} = -x^2$

Suppose the (x, y) plane is flipped over about the diagonal line $y = x$. This will make the x- and y-axes change places, forming the (y, x) plane. If f has an inverse function g, the graph of the function $y = f(x)$ will become the graph of the inverse function $x = g(y)$ in the (y, x) plane, as shown in Figure 2.4.5.

The following rule shows that the derivatives of inverse functions are always reciprocals of each other.

INVERSE FUNCTION RULE

Suppose f and g are inverse functions, so that the two equations

$$y = f(x) \qquad \text{and} \qquad x = g(y)$$

have the same graphs. If both derivatives $f'(x)$ and $g'(y)$ exist and are nonzero, then

$$f'(x) = \frac{1}{g'(y)};$$

that is,

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

PROOF Let Δx be a nonzero infinitesimal and let Δy be the corresponding change in y. Then Δy is also infinitesimal because $f'(x)$ exists and is nonzero because $f(x)$ has an inverse function. By the rules for standard parts,

$$f'(x) \cdot g'(y) = st\left(\frac{\Delta y}{\Delta x}\right) \cdot st\left(\frac{\Delta x}{\Delta y}\right)$$

$$= st\left(\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y}\right) = st(1) = 1.$$

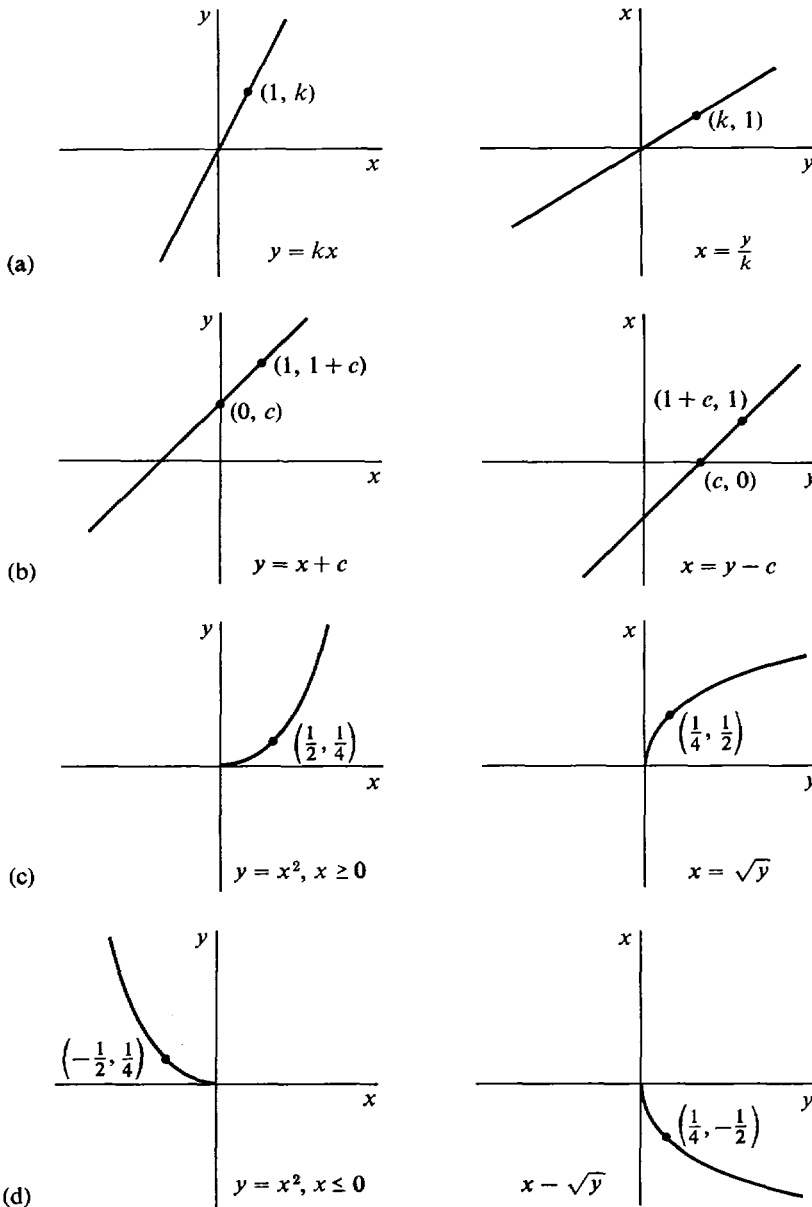Therefore

$$f'(x) = \frac{1}{g'(y)}.$$

Figure 2.4.5

The formula

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

in the Inverse Function Rule is not as trivial as it looks. A more complete statement is

$\dfrac{dy}{dx}$ computed with x the independent variable

$= \dfrac{1}{dx/dy}$ computed with y the independent variable.

Sometimes it is easier to compute dx/dy than dy/dx, and in such cases the Inverse Function Rule is a useful method.

EXAMPLE 1   Find dy/dx where $x = 1 + y^{-3}$.

Before solving the problem we note that

$$y = \frac{1}{\sqrt[3]{x-1}},$$

so x and y are inverse functions of each other. We want to find

$$\frac{dy}{dx} = \frac{d(1/\sqrt[3]{x-1})}{dx}$$

with x the independent variable. This looks hard, but it is easy to compute

$$\frac{dx}{dy} = \frac{d(1 + y^{-3})}{dy}$$

with y the independent variable.

SOLUTION $\dfrac{dx}{dy} = -3y^{-4}$,

$$\frac{dy}{dx} = \frac{1}{-3y^{-4}} = -\frac{1}{3}y^{4}.$$

We can write dy/dx in terms of x by substituting,

$$\frac{dy}{dx} = -\frac{1}{3}(x-1)^{-4/3}.$$

EXAMPLE 2   Find dy/dx where $x = y^5 + y^3 + y$. Compute dy/dx at the point $(3, 1)$.

Although we cannot solve the equation explicitly for y as a function of x, we can see from the graph in Figure 2.4.6 that there is an inverse function $y = f(x)$.
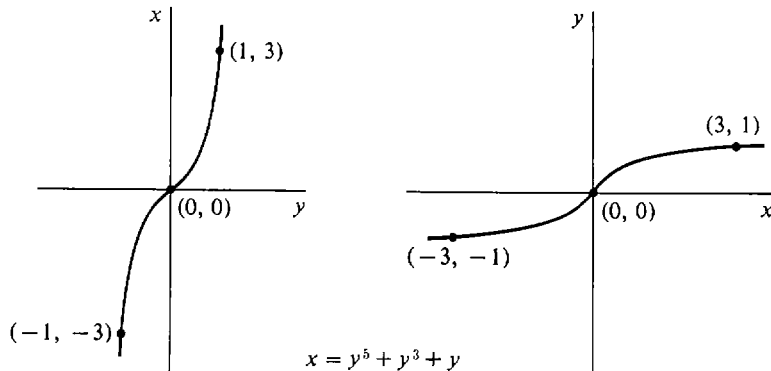

Figure 2.4.6

By the Inverse Function Rule,

$$\frac{dx}{dy} = 5y^4 + 3y^2 + 1,$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 3y^2 + 1}.$$

This time we must leave the answer in terms of y. At the point $(3,1)$, we substitute 1 for y and get $dy/dx = 1/9$.

For $y \geq 0$, the function $x = y^n$ has the inverse function $y = x^{1/n}$. In the next theorem, we use the Inverse Function Rule to find a new derivative, that of $y = x^{1/n}$.

THEOREM 1

If n is a positive integer and

$$y = x^{1/n},$$

then
$$\frac{dy}{dx} = \frac{1}{n}x^{(1/n)-1}.$$

Remember that $y = x^{1/n}$ is defined for all x if n is odd and for $x > 0$ if n is even. The derivative $\frac{1}{n}x^{(1/n)-1}$ is defined for $x \neq 0$ if n is odd and for $x > 0$ if n is even.

If we are willing to assume that dy/dx exists, then we can quickly find dy/dx by the Inverse Function Rule.

$$x = y^n,$$

$$\frac{dx}{dy} = ny^{n-1},$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{ny^{n-1}} = \frac{1}{n}y^{1-n}$$

$$= \frac{1}{n}(x^{1/n})^{1-n} = \frac{1}{n}x^{(1-n)/n} = \frac{1}{n}x^{(1/n)-1}.$$

Here is a longer but complete proof which shows that dy/dx exists and computes its value.

PROOF OF THEOREM 1 Let $x \neq 0$ and let Δx be nonzero infinitesimal. We first show that

$$\Delta y = (x + \Delta x)^{1/n} - x^{1/n}$$

is a nonzero infinitesimal. $\Delta y \neq 0$ because $x + \Delta x \neq x$. The standard part of Δy is

$$st(\Delta y) = st((x + \Delta x)^{1/n}) - st(x^{1/n})$$
$$= x^{1/n} - x^{1/n} = 0.$$

Therefore Δy is nonzero infinitesimal.

Now
$$x = y^n,$$

$$\frac{dx}{dy} = ny^{n-1},$$

$$\frac{\Delta x}{\Delta y} \approx ny^{n-1}.$$

Therefore
$$\frac{\Delta y}{\Delta x} \approx \frac{1}{ny^{n-1}} = \frac{1}{n}x^{(1/n)-1},$$

$$\frac{dy}{dx} = \frac{1}{n}x^{(1/n)-1}.$$
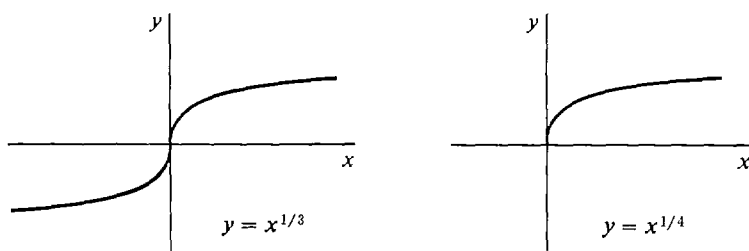

$$y = x^{1/3} \qquad\qquad y = x^{1/4}$$

Figure 2.4.7

Figure 2.4.7 shows the graphs of $y = x^{1/3}$ and $y = x^{1/4}$. At $x = 0$, the curves are vertical and have no slope.

EXAMPLE 3   Find the derivatives of $y = x^{1/n}$ for $n = 2, 3, 4$.

$$\frac{d(x^{1/2})}{dx} = \frac{1}{2}x^{-1/2}, \qquad x > 0.$$

$$\frac{d(x^{1/3})}{dx} = \frac{1}{3}x^{-2/3}, \qquad x \neq 0.$$

$$\frac{d(x^{1/4})}{dx} = \frac{1}{4}x^{-3/4}, \qquad x > 0.$$

Using Theorem 1 we can show that the Power Rule holds when the exponent is any rational number.

POWER RULE FOR RATIONAL EXPONENTS

Let $y = x^r$ where r is a rational number. Then whenever $x > 0$,

$$\frac{dy}{dx} = rx^{r-1}.$$

PROOF Let $r = m/n$ where m and n are integers, $n > 0$. Let

$$u = x^{1/n}, \qquad y = u^m.$$

Then

$$\frac{du}{dx} = \frac{1}{n}x^{(1/n)-1}$$

and

$$\frac{dy}{dx} = mu^{m-1}\frac{du}{dx}$$

$$= m(x^{1/n})^{m-1}\left(\frac{1}{n}x^{(1/n)-1}\right)$$

$$= \frac{m}{n}x^{(m/n)-1} = rx^{r-1}.$$

EXAMPLE 4   Find dy/dx where

$$y = x^{-3/7}.$$

$$\frac{dy}{dx} = -\frac{3}{7}x^{(-3/7)-1} = -\frac{3}{7}x^{-10/7}.$$

EXAMPLE 5   Find dy/dx where

$$y = \frac{1}{2 + x^{3/2}}.$$

Let

$$u = 2 + x^{3/2}, \qquad y = u^{-1}.$$

Then

$$\frac{du}{dx} = \frac{3}{2}x^{1/2},$$

$$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}$$

$$= -u^{-2}\left(\frac{3}{2}x^{1/2}\right) = -\frac{3}{2}\frac{x^{1/2}}{(2 + x^{3/2})^2}.$$

PROBLEMS FOR SECTION 2.4

In Problems 1–16, find dy/dx.

1	$x = 3y^3 + 2y$	**2**	$x = y^2 + 1, \quad y > 0$
3	$x = 1 - 2y^2, \quad y > 0$	**4**	$x = 2y^5 + y^3 + 4$
5	$x = (y^2 + 2)^{-1}, \quad y > 0$	**6**	$y = 1/\sqrt{x}$
7	$y = x^{4/3}$	**8**	$y = \sqrt{2x}$
9	$y = (\sqrt{x} + 1)/(\sqrt{x} - 1)$	**10**	$y = (2x^{1/3} + 1)^3$
11	$y = 1 + 2x^{1/3} + 4x^{2/3} + 6x$	**12**	$y = x^{-1/4} + 3x^{-3/4}$
13	$y = (x^{5/3} - x)^{-2}$	**14**	$x = y + 2\sqrt{y}$
15	$x = 3y^{1/3} + 2y, \quad y > 0$	**16**	$x = 1/(1 + \sqrt{y})$

In Problems 17–25, find the inverse function y and its derivative dy/dx as functions of x.

17 $x = ky + c, \quad k \neq 0$ **18** $x = y^3 + 1$

19 $x = 2y^2 + 1, \quad y \geq 0$ **20** $x = 2y^2 + 1, \quad y \leq 0$

21 $x = y^4 - 3, \quad y \geq 0$ **22** $x = y^2 + 3y - 1, \quad y \geq -\frac{3}{2}$

23 $x = y^4 + y^2 + 1, \quad y \geq 0$ **24** $x = 1/y^2 + 1/y - 1, \quad y > 0$

25 $x = \sqrt{y} + 2y, \quad y > 0$

☐ **26**   Show that no second degree polynomial $x = ay^2 + by + c$ has an inverse function.

☐ **27**   Show that $x = ay^2 + by + c, \ y \geq -b/2a$, has an inverse function. What does its graph look like?

☐ **28**   Prove that a function $y = f(x)$ has an inverse function if and only if whenever $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

2.5 TRANSCENDENTAL FUNCTIONS

The transcendental functions include the trigonometric functions $\sin x$, $\cos x$, $\tan x$, the exponential function e^x, and the natural logarithm function $\ln x$. These functions are developed in detail in Chapters 7 and 8. This section contains a brief discussion.

1 TRIGONOMETRIC FUNCTIONS

The Greek letters θ (theta) and ϕ (phi) are often used for angles. In the calculus it is convenient to measure angles in *radians* instead of degrees. An angle θ in radians is defined as the length of the arc of the angle on a circle of radius one (Figure 2.5.1). Since a circle of radius one has circumference 2π,

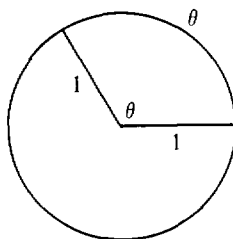$$360 \text{ degrees} = 2\pi \text{ radians.}$$

Figure 2.5.1

Thus a right angle is

$$90 \text{ degrees} = \pi/2 \text{ radians.}$$

To define the sine and cosine functions, we consider a point $P(x, y)$ on the unit circle $x^2 + y^2 = 1$. Let θ be the angle measured counterclockwise in radians from the point $(1, 0)$ to the point $P(x, y)$ as shown in Figure 2.5.2. Both coordinates

x and y depend on θ. The value of x is called the *cosine* of θ, and the value of y is the *sine* of θ. In symbols,

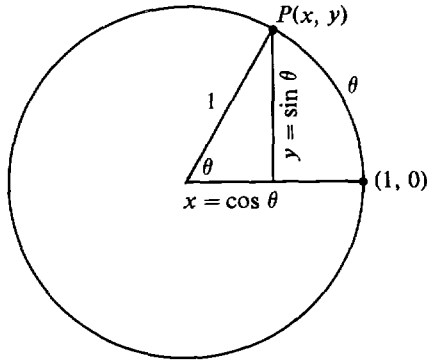$$x = \cos \theta, \qquad y = \sin \theta .$$

Figure 2.5.2

The *tangent* of θ is defined by

$$\tan \theta = \sin \theta / \cos \theta.$$

Negative angles and angles greater than 2π radians are also allowed.

The trigonometric functions can also be defined using the sides of a right triangle, but this method only works for θ between 0 and $\pi/2$. Let θ be one of the acute angles of a right triangle as shown in Figure 2.5.3.

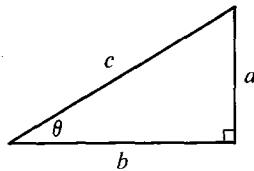**Figure 2.5.3**

Then $\sin \theta = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{a}{c},$

$\cos \theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{b}{c},$

$\tan \theta = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{a}{b}.$

The two definitions, with circles and right triangles, can be seen to be equivalent using similar triangles.

Table 2.5.1 gives the values of sin θ and cos θ for some important values of θ.

Table 2.5.1

θ in degrees	0°	30°	45°	60°	90°	180°	270°	360°
θ in radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
sin θ	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	-1	0
cos θ	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	-1	0	1

A useful identity which follows from the unit circle equation $x^2 + y^2 = 1$ is

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Here $\sin^2 \theta$ means $(\sin \theta)^2$.

Figure 2.5.4 shows the graphs of sin θ and cos θ, which look like waves that oscillate between 1 and -1 and repeat every 2π radians.

The derivatives of the sine and cosine functions are:

$$\frac{d(\sin \theta)}{d\theta} = \cos \theta.$$

$$\frac{d(\cos \theta)}{d\theta} = -\sin \theta.$$

Figure 2.5.4

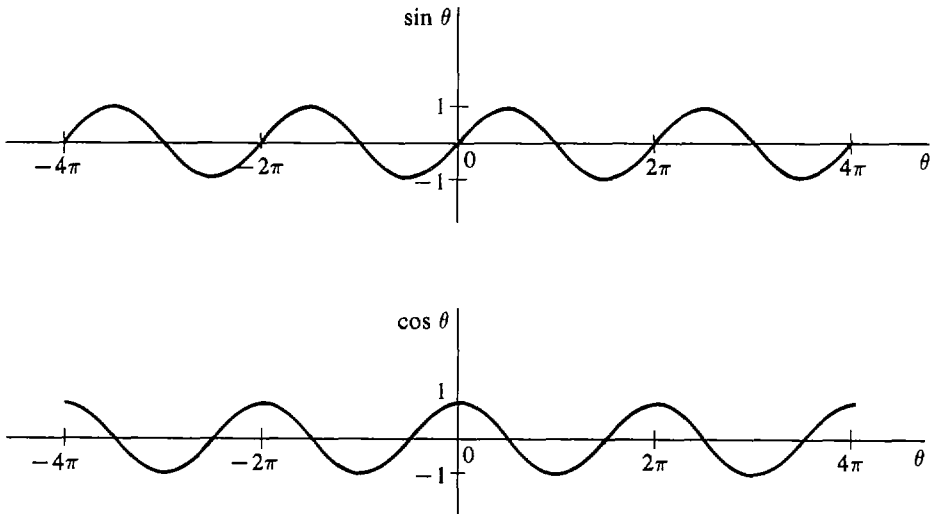In both formulas θ is measured in radians. We can see intuitively why these are the derivatives in Figure 2.5.5.

In the triangle under the infinitesimal microscope,

$$\frac{\Delta(\sin \theta)}{\Delta\theta} \approx \frac{\text{adjacent side}}{\text{hypotenuse}} = \cos \theta;$$

$$\frac{\Delta(\cos \theta)}{\Delta\theta} \approx \frac{-\text{opposite side}}{\text{hypotenuse}} = -\sin \theta.$$
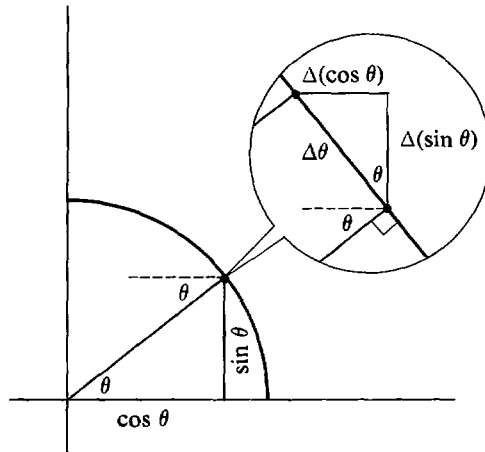
Figure 2.5.5

Notice that cos θ decreases, and $\Delta(\cos \theta)$ is negative in the figure, so the derivative of cos θ is $-\sin \theta$ instead of just sin θ.

Using the rules of differentiation we can find other derivatives.

EXAMPLE 1 Differentiate $y = \sin^2 \theta$. Let $u = \sin \theta$, $y = u^2$. Then

$$\frac{dy}{d\theta} = 2u \frac{du}{d\theta} = 2 \sin \theta \cos \theta.$$

EXAMPLE 2 Differentiate $y = \sin \theta(1 - \cos \theta)$. Let $u = \sin \theta$, $v = 1 - \cos \theta$. Then $y = u \cdot v$, and

$$\frac{dy}{d\theta} = u \frac{dv}{d\theta} + v \frac{du}{d\theta} = \sin \theta(-(-\sin \theta)) + (1 - \cos \theta) \cos \theta$$

$$= \sin^2 \theta + \cos \theta - \cos^2 \theta.$$

The other trigonometric functions (the secant, cosecant, and cotangent functions) and the inverse trigonometric functions are discussed in Chapter 7.

2 EXPONENTIAL FUNCTIONS

Given a positive real number b and a rational number m/n, the rational power $b^{m/n}$ is defined as

$$b^{m/n} = \sqrt[n]{b^m},$$

the positive nth root of b^m. The negative power $b^{-m/n}$ is

$$b^{-m/n} = \frac{1}{b^{m/n}}.$$

As an example consider $b = 10$. Several values of $10^{m/n}$ are shown in Table 2.5.2.

Table 2.5.2

10^{-3}	$10^{-3/2}$	10^{-1}	$10^{-2/3}$	$10^{-1/3}$	10^{0}	$10^{1/3}$	$10^{2/3}$	10^{1}	$10^{3/2}$	10^{3}
$\dfrac{1}{1000}$	$\dfrac{1}{10\sqrt{10}}$	$\dfrac{1}{10}$	$\dfrac{1}{\sqrt[3]{100}}$	$\dfrac{1}{\sqrt[3]{10}}$	1	$\sqrt[3]{10}$	$\sqrt[3]{100}$	10	$10\sqrt{10}$	1000

If we plot all the rational powers $10^{m/n}$, we get a dotted line, with one value for each rational number m/n, as in Figure 2.5.6.
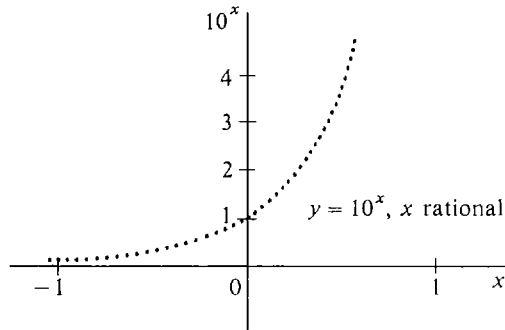

Figure 2.5.6

By connecting the dots with a smooth curve, we obtain a function $y = 10^x$, where x varies over all real numbers instead of just the rationals. 10^x is called the *exponential function with base* 10. It is positive for all x and follows the rules

$$10^{a+b} = 10^a \cdot 10^b, \qquad 10^{a \cdot b} = (10^a)^b.$$

The derivative of 10^x is a constant times 10^x, approximately

$$\frac{d(10^x)}{dx} \sim (2.303)10^x.$$

To see this let Δx be a nonzero infinitesimal. Then

$$\frac{d(10^x)}{dx} = st\left[\frac{10^{x+\Delta x} - 10^x}{\Delta x}\right] = st\left[\frac{(10^{\Delta x} - 1)10^x}{\Delta x}\right] = st\left[\frac{10^{\Delta x} - 1}{\Delta x}\right]10^x.$$

The number $st[(10^{\Delta x} - 1)/\Delta x]$ is a constant which does not depend on x and can be shown to be approximately 2.303.

If we start with a given positive real number b instead of 10, we obtain the *exponential function with base* b, $y = b^x$. The derivative of b^x is equal to the constant $st[(b^{\Delta x} - 1)/\Delta x]$ times b^x. This constant depends on b. The derivative is computed as follows:

$$\frac{d(b^x)}{dx} = st\left[\frac{b^{x+\Delta x} - b^x}{\Delta x}\right] = st\left[\frac{(b^{\Delta x} - 1)b^x}{\Delta x}\right] = st\left[\frac{b^{\Delta x} - 1}{\Delta x}\right]b^x.$$

The most useful base for the calculus is the number e. e is defined as the real number such that the derivative of e^x is e^x itself,

$$\frac{d(e^x)}{dx} = e^x.$$

In other words, e is the real number such that the constant

$$st\left[\frac{e^{\Delta x} - 1}{\Delta x}\right] = 1$$

(where Δx is a nonzero infinitesimal). It will be shown in Section 8.3 that there is such a number e and that e has the approximate value

$$e \sim 2.71828.$$

The function $y = e^x$ is called the *exponential function*. e^x is always positive and follows the rules

$$e^{a+b} = e^a \cdot e^b, \qquad e^{a \cdot b} = (e^a)^b, \qquad e^0 = 1.$$

Figure 2.5.7 shows the graph of $y = e^x$.
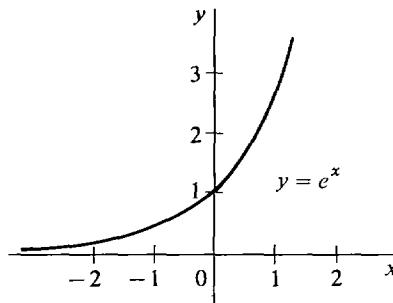
Figure 2.5.7

EXAMPLE 3   Find the derivative of $y = x^2 e^x$. By the Product Rule,

$$\frac{dy}{dx} = x^2 \frac{d(e^x)}{dx} + e^x \frac{d(x^2)}{dx} = x^2 e^x + 2xe^x.$$

3 THE NATURAL LOGARITHM

The inverse of the exponential function $x = e^y$ is the *natural logarithm function*, written

$$y = \ln x.$$

Verbally, $\ln x$ is the number y such that $e^y = x$. Since $y = \ln x$ is the inverse function of $x = e^y$, we have

$$e^{\ln a} = a, \qquad \ln(e^a) = a.$$

The simplest values of $y = \ln x$ are

$$\ln(1/e) = -1, \qquad \ln(1) = 0, \qquad \ln e = 1.$$

Figure 2.5.8 shows the graph of $y = \ln x$. It is defined only for $x > 0$.
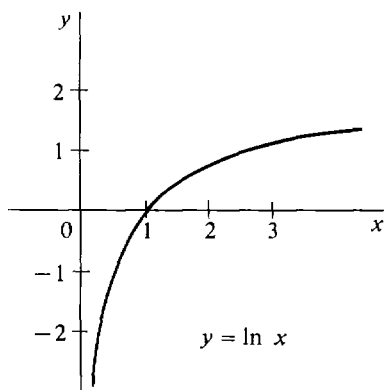
Figure 2.5.8

The most important rules for logarithms are

$$\ln (ab) = \ln a + \ln b,$$

$$\ln (a^b) = b \cdot \ln a.$$

The natural logarithm function is important in calculus because its derivative is simply $1/x$,

$$\frac{d(\ln x)}{dx} = \frac{1}{x}, \qquad (x > 0).$$

This can be derived from the Inverse Function Rule.

If $y = \ln x,$

then $x = e^y,$

$$\frac{dx}{dy} = e^y,$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}.$$

The natural logarithm is also called the logarithm to the base e and is sometimes written $\log_e x$. Logarithms to other bases are discussed in Chapter 8.

EXAMPLE 4   Differentiate $y = \dfrac{1}{\ln x}$.

$$\frac{dy}{dx} = \frac{-1}{(\ln x)^2} \frac{d(\ln x)}{dx} = -\frac{1}{x(\ln x)^2}.$$

4 SUMMARY

Here is a list of the new derivatives given in this section.

$$\frac{d(\sin x)}{dx} = \cos x.$$

$$\frac{d(\cos x)}{dx} = -\sin x.$$

$$\frac{d(e^x)}{dx} = e^x.$$

$$\frac{d(\ln x)}{dx} = \frac{1}{x} \quad (x > 0).$$

Tables of values for $\sin x$, $\cos x$, e^x, and $\ln x$ can be found at the end of the book.

PROBLEMS FOR SECTION 2.5

In Problems 1–20, find the derivative.

1	$y = \cos^2 \theta$	**2**	$s = \tan^2 t$
3	$y = 2 \sin x + 3 \cos x$	**4**	$y = \sin x \cdot \cos x$
5	$w = \dfrac{1}{\cos z}$	**6**	$w = \dfrac{1}{\sin z}$
7	$y = \sin^n \theta$	**8**	$y = \tan^n \theta$
9	$s = t \sin t$	**10**	$s = \dfrac{\cos t}{t - 1}$
11	$y = xe^x$	**12**	$y = 1/(1 + e^x)$
13	$y = (\ln x)^2$	**14**	$y = x \ln x$
15	$y = e^x \cdot \ln x$	**16**	$y = e^x \cdot \sin x$
17	$u = \sqrt{v}(1 - e^v)$	**18**	$u = (1 + e^v)(1 - e^v)$
19	$y = x^n \ln x$	**20**	$y = (\ln x)^n$

In Problems 21–24, find the equation of the tangent line at the given point.

21	$y = \sin x$ at $(\pi/6, \frac{1}{2})$	**22**	$y = \cos x$ at $(\pi/4, \sqrt{2}/2)$
23	$y = x - \ln x$ at $(e, e - 1)$	**24**	$y = e^{-x}$ at $(0, 1)$

2.6 CHAIN RULE

The Chain Rule is more general than the Inverse Function Rule and deals with the case where x and y are both functions of a third variable t.

Suppose $\qquad\qquad x = f(t), \qquad y = G(x).$

Thus x depends on t, and y depends on x. But y is also a function of t,

$$y = g(t),$$

where g is defined by the rule

$$g(t) = G(f(t)).$$

The function g is sometimes called the *composition* of G and f (sometimes written $g = G \circ f$).

The composition of G and f may be described in terms of black boxes. The function $g = G \circ f$ is a large black box operating on the input t to produce $g(t) = G(f(t))$. If we look inside this black box (pictured in Figure 2.6.1), we see two smaller black boxes, f and G. First f operates on the input t to produce $f(t)$, and then G operates on $f(t)$ to produce the final output $g(t) = G(f(t))$.

The Chain Rule expresses the derivative of g in terms of the derivatives of f and G. It leads to the powerful method of "change of variables" in computing derivatives and, later on, integrals.
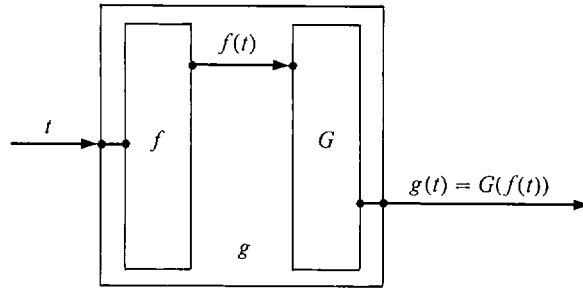

Figure 2.6.1   Composition $g = G \circ f$

CHAIN RULE

Let f, G be two real functions and define the new function g by the rule

$$g(t) = G(f(t)).$$

At any value of t where the derivatives $f'(t)$ and $G'(f(t))$ exist, $g'(t)$ also exists and has the value

$$g'(t) = G'(f(t))f'(t).$$

PROOF Let $\qquad x = f(t), \qquad y = g(t), \qquad y = G(x).$

Take t as the independent variable, and let $\Delta t \neq 0$ be infinitesimal. Form the corresponding increments Δx and Δy. By the Increment Theorem for $x = f(t)$, Δx is infinitesimal. Using the Increment Theorem again but this time for $y = G(x)$, we have

$$\Delta y = G'(x)\,\Delta x + \varepsilon\,\Delta x$$

for some infinitesimal ε. Dividing by Δt,

$$\frac{\Delta y}{\Delta t} = G'(x)\frac{\Delta x}{\Delta t} + \varepsilon\frac{\Delta x}{\Delta t}.$$

Then taking standard parts,

$$st\!\left(\frac{\Delta y}{\Delta t}\right) = G'(x)st\!\left(\frac{\Delta x}{\Delta t}\right) + 0,$$

or $\qquad\qquad g'(t) = G'(x)f'(t) = G'(f(t))f'(t).$

EXAMPLE 1   Find the derivative of $g(t) = \ln(\sin t)$. $g(t)$ is the natural logarithm of the sine of t. It can be written in the form

$$g(t) = G(f(t))$$

where $\qquad f(t) = \sin t, \qquad G(x) = \ln x.$

We have $\qquad f'(t) = \cos t, \qquad G'(x) = \dfrac{1}{x}.$

By the Chain Rule,

$$g'(t) = G'(f(t))f'(t)$$

$$= \frac{1}{\sin t} \cdot \cos t = \frac{\cos t}{\sin t}.$$

EXAMPLE 2 Find the derivative of $g(t) = \sqrt{3t + 1}$. $g(t)$ has the form

$$g(t) = G(f(t)),$$

where $\qquad f(t) = 3t + 1, \qquad G(x) = \sqrt{x}.$

We have $\qquad f'(t) = 3, \qquad G'(x) = \dfrac{1}{2}x^{-1/2}.$

Then $\qquad g'(t) = G'(f(t))f'(t)$

$$= \frac{1}{2}(3t + 1)^{-1/2}3 = \frac{3}{2\sqrt{3t + 1}}.$$

In practice it is more convenient to use the Chain Rule with dependent variables x and y instead of functions f and g.

CHAIN RULE WITH DEPENDENT VARIABLES

Let

$$x = f(t), \qquad y = g(t) = G(x).$$

Assume $g'(t)$ and $G'(x)$ exist. Then

$$\text{(i)} \quad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \qquad \text{(ii)} \quad dy = \frac{dy}{dx}dx$$

where dx/dt, dy/dt are computed with t as the independent variable, and dy/dx is computed with x as the independent variable.

Let us work Examples 1 and 2 again with dependent variables.

EXAMPLE 1 (Continued) Let $x = \sin t, \qquad y = \ln x.$

Find dy/dt using Chain Rule (i) and dy by using Chain Rule (ii).

(i) $\qquad \dfrac{dx}{dt} = \cos t, \qquad \dfrac{dy}{dx} = \dfrac{1}{x},$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{x} \cdot \cos t = \frac{\cos t}{\sin t}.$$

(ii)
$$dx = \cos t \, dt, \qquad \frac{dy}{dx} = \frac{1}{x},$$

$$dy = \frac{1}{x} dx = \frac{1}{x} \cos t \, dt = \frac{\cos t}{\sin t} dt.$$

EXAMPLE 2 (Continued)   Let $x = 3t + 1, \qquad y = \sqrt{x}.$

(i)
$$\frac{dx}{dt} = 3, \qquad \frac{dy}{dx} = \frac{1}{2} x^{-1/2},$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{3}{2} x^{-1/2} = \frac{3}{2}(3t + 1)^{-1/2}.$$

(ii)
$$dx = 3 \, dt, \qquad \frac{dy}{dx} = \frac{1}{2} x^{-1/2},$$

$$dy = \frac{1}{2} x^{-1/2} \, dx = \frac{1}{2}(3t + 1)^{-1/2} 3 \, dt = \frac{3}{2}(3t + 1)^{-1/2} \, dt.$$

The equation

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

with t as the independent variable is trivial. We simply cancel the dx's. But when dy/dx is computed with x as the independent variable while dx/dt is computed with t as the independent variable, the two dx's have different meanings, and the equation is not trivial.

Similarly, the equation

$$dy = \frac{dy}{dx} dx$$

is trivial with x as the independent variable but not when t is the independent variable in dy and dx, while x is independent in dy/dx.

The Chain Rule shows that when we change independent variables the equations

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}, \qquad dy = \frac{dy}{dx} dx$$

remain true.

The Inverse Function Rule can be proved from the Chain Rule as follows. Let

$$y = f(x), \qquad x = g(y)$$

be inverse functions whose derivatives exist. Then

$$\frac{dy}{dx}\frac{dx}{dy} = \frac{dy}{dy} = 1,$$

whence
$$\frac{dy}{dx} = \frac{1}{dx/dy}, \qquad f'(x) = \frac{1}{g'(y)}.$$

Using the Chain Rule we may write the Power Rule in a general form.

POWER RULE

Let r be a rational number, and let u depend on x. If u > 0 and du/dx exists, then

$$\frac{d(u^r)}{dx} = ru^{r-1}\frac{du}{dx}.$$

This is proved by letting $y = u^r$ and computing dy/dx by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}.$$

The Chain Rule has two types of applications.

(1) Given $x = f(t)$ and $y = G(x)$, find $\dfrac{dy}{dt}$. Use $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$.

(2) Given $x = f(t)$ and $y = g(t)$, find $\dfrac{dy}{dx}$. Use $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

Applications of type (1) often arise when a new dependent variable x is introduced to help compute $\dfrac{dy}{dt}$. Applications of type (2) arise when two variables x and y both depend on a third variable t, for example, when x and y are the co-ordinates of a moving particle and t is time.

We give three examples of type (1) and then two of type (2).

EXAMPLE 3   Suppose that by investing t dollars a company can produce

$$x = \frac{t}{10} - 100, \qquad t \geq 1000,$$

items, and that it can sell x items for a total profit of

$$y = 5x - \frac{x^2}{100}.$$

Find $\dfrac{dy}{dt}$, which is the marginal profit with respect to the amount invested.

We have $$\frac{dx}{dt} = \frac{1}{10}, \qquad \frac{dy}{dx} = 5 - \frac{x}{50}.$$

By the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \left(5 - \frac{x}{50}\right)\frac{1}{10}$$

$$= \left(5 - \frac{\frac{t}{10} - 100}{50}\right)\frac{1}{10}$$

$$= 0.7 - \frac{t}{5000}.$$

Thus after t dollars have been invested, an additional dollar invested will bring $0.7 - t/5000$ dollars of additional profit.

EXAMPLE 4 Find dy/dt where $y = (5t^2 - 2)^{1/4}$.

Let
$$x = 5t^2 - 2, \qquad y = x^{1/4}.$$

Then
$$\frac{dx}{dt} = 10t, \qquad \frac{dy}{dx} = \frac{1}{4}x^{-3/4},$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \left(\frac{1}{4}x^{-3/4}\right)(10t)$$

$$= \frac{10}{4}(5t^2 - 2)^{-3/4}t.$$

EXAMPLE 5 Find dy/dx where $y = \sqrt{\sin(4x + 1) + \cos(4x - 1)}$. This problem requires three uses of the Chain Rule.

Let
$$u = \sin(4x + 1) + \cos(4x - 1), \qquad y = \sqrt{u}.$$

Then by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}.$$

Now let $v = \sin(4x + 1)$, $w = \cos(4x - 1)$, $u = v + w$.

Then
$$\frac{du}{dx} = \frac{dv}{dx} + \frac{dw}{dx}.$$

We use the Chain Rule twice more to find dv/dx and dw/dx.

$$v = \sin(4x + 1).$$
$$\frac{dv}{dx} = \cos(4x + 1)\frac{d(4x + 1)}{dx} = 4\cos(4x + 1).$$
$$w = \cos(4x - 1).$$
$$\frac{dw}{dx} = -\sin(4x - 1)\frac{d(4x - 1)}{dx} = -4\sin(4x - 1).$$

Finally, we combine everything to get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}}\left(\frac{dv}{dx} + \frac{dw}{dx}\right)$$

$$= \frac{4\cos(4x + 1) - 4\sin(4x - 1)}{2\sqrt{\sin(4x + 1) + \cos(4x - 1)}}.$$

If a particle is moving in the plane, its position (x, y) at time t will be given by a pair of equations

$$x = f(t), \qquad y = g(t).$$

These are called *parametric equations*. The slope of the curve traced out by this particle can be found by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)},$$

whenever the derivatives exist and $f'(t) \neq 0$. This is a Chain Rule application of type (2).

EXAMPLE 6   A ball thrown horizontally from a 100 ft cliff at a velocity of 50 ft/sec will follow the parametric equations

$$x = 50t, \qquad y = 100 - 16t^2, \quad \text{in feet.}$$

Find the slope of its path at time t (Figure 2.6.2).

$$\frac{dx}{dt} = 50, \qquad \frac{dy}{dt} = -32t,$$

so

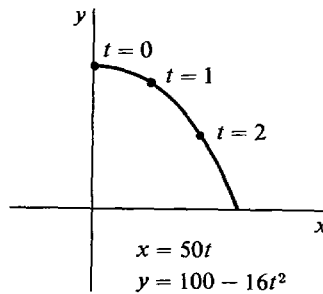$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{32t}{50}.$$

Figure 2.6.2

EXAMPLE 7   A particle moves according to the parametric equations

$$x = t^3 - t, \qquad y = t^2.$$

Find the slope of its path.

$$\frac{dx}{dt} = 3t^2 - 1, \qquad \frac{dy}{dt} = 2t,$$

so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 1}, \qquad t \neq \pm\sqrt{1/3}.$$

We see from Figure 2.6.3 that the path of this particle is not the graph of a function, and in fact contains a loop and crosses the point $(0, 1)$ twice, at $t = -1$ and $t = 1$. The path is vertical at the points $t = \pm\sqrt{1/3}$, where there is no slope. At the point $(0, 1)$, the two slopes of the path are $dy/dx = -1$ when $t = -1$, and $dy/dx = 1$ when $t = 1$.

EXAMPLE 8   A particle moving according to the parametric equations

$$x = \cos t, \qquad y = \sin t$$

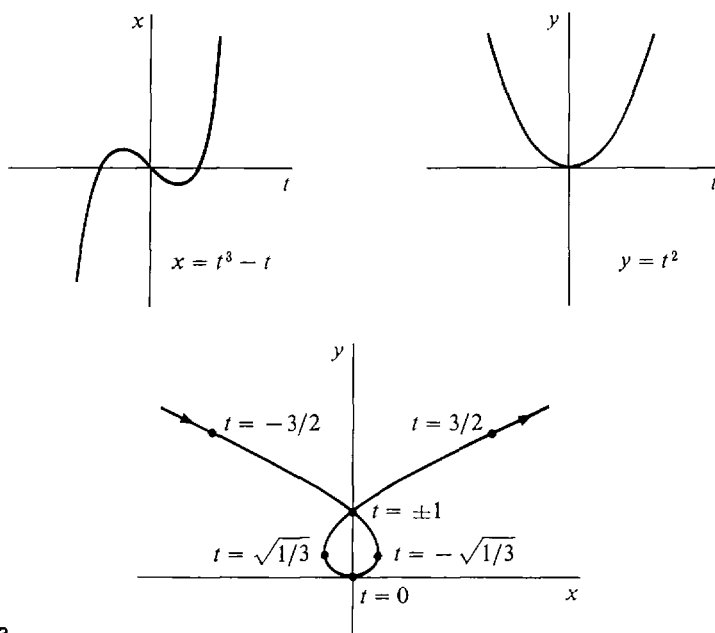will move counterclockwise around the unit circle at one radian per second

Figure 2.6.3

beginning at the point $(1, 0)$, as shown in Figure 2.6.4. Find the slope of its path at time t.

$$\frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t.$$

The slope is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{\cos t}{\sin t}.$$

In terms of x and y the slope is
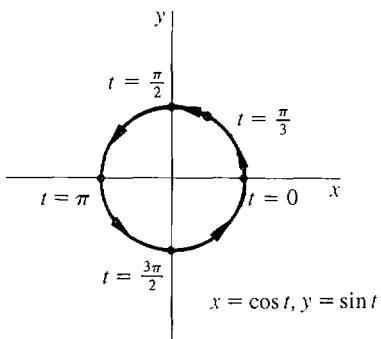
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Figure 2.6.4

PROBLEMS FOR SECTION 2.6

In Problems 1–44, find dy/dx.

1	$y = \sqrt{x + 2}$	**2**	$y = \sqrt{7 + 4x}$
3	$y = \sqrt{5 - x}$	**4**	$y = \sqrt{1 - 10x}$
5	$y = \dfrac{1}{\sqrt{2 + 3x}}$	**6**	$y = \dfrac{1}{\sqrt{4 - x}}$
7	$y = \sqrt[3]{6x + 1}$	**8**	$y = \sqrt[5]{2 - 3x}$
9	$y = \sqrt{x^2 + 1}$	**10**	$y = \sqrt{1 - x^2}$
11	$y = \sin(3x)$	**12**	$y = \cos(4 - 2x)$
13	$y = \sin(x^{-2})$	**14**	$y = \cos\sqrt{x}$
15	$y = e^{4x}$	**16**	$y = e^{-x^2}$
17	$y = e^{\cos x}$	**18**	$y = \ln(\ln x)$
19	$y = \cos u, \quad u = e^x$	**20**	$y = \tan u, u = \ln x$
21	$y = u^{10}, \quad u = 1 - 4x$	**22**	$y = u^{-10}, u = 1 - x^2$

23 $y = \sin u + \sin v, \quad u = 1 - x^2, \quad v = 2x - 1$

24 $y = e^u + e^v, \quad u = 1 - 3x, \quad v = 3 - 4x$

25 $y = e^u, \quad u = \sqrt{v}, \quad v = \sin x$

26 $y = \ln u, \quad u = \tan v, \quad v = 1/x$

27 $y = u^{-1/3}, \quad u = 1 + \sqrt{v}, \quad v = x^2 - 1$

28 $y = u^{-1}, \quad u = 3v + 4, \quad v = 1/(x + 1)$

29 $y = u^4, \quad u = 1 + 1/v, \quad v = x^3 + 1$

30 $y = u^2 + 1, \quad u = v^2 + 1, \quad v = x^2 + 1$

31	$y = (\sqrt{x^2 - 1} + \sqrt{x^2 + 1})^{1/3}$	**32**	$y = (x + \sqrt{3 - 4x})^{-1/2}$
33	$y = 3x \sin(2x - 1)$	**34**	$y = \sin(2x)\cos(3x)$
35	$x = \cos(3t), \quad y = \sin(3t)$	**36**	$x = e^t, \quad y = \ln t$
37	$x = \sin t, \quad y = \sin(2t)$	**38**	$x = \sin(e^t), \quad y = \cos(e^t)$
39	$x = \ln(t + 1), \quad y = t^2$	**40**	$x = e^{\cos t}, \quad y = e^{\sin t}$
41	$x = \sqrt{t^2 - 4}, \quad y = \sqrt{t^2 + 4}$	**42**	$x = 1 + \sqrt[3]{t}, \quad y = 2 + \sqrt[3]{t}$
43	$x = \sqrt[3]{t + 1}, \quad y = \sqrt[3]{t + 2}$	**44**	$x = \dfrac{2t + 1}{t + 2}, \quad y = \dfrac{2t + 3}{t + 2}$

45   A particle moves in the plane according to the parametric equations

$$x = t^2 + 1, \qquad y = 3t^3.$$

Find the slope of its path.

46   An ant moves in the plane according to the equations

$$x = (1 - t^2)^{-1}, \qquad y = \sqrt{t}.$$

Find the slope of its path.

☐ **47**   Let y depend on u, u depend on v, and v depend on x. Assume the derivatives dy/du, du/dv, and dv/dx exist. Prove that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

☐ **48**   Let the function $f(x)$ be differentiable for all x, and let $g(x) = f(f(x))$. Show that $g'(x) = f'(f(x))f'(x)$.

2.7 HIGHER DERIVATIVES

DEFINITION

*The **second derivative** of a real function f is the derivative of the derivative of f, and is denoted by f''. The third derivative of f is the derivative of the second derivative, and is denoted by f''', or $f^{(3)}$. In general, the nth derivative of f is denoted by $f^{(n)}$.*

*If y depends on x, $y = f(x)$, then the **second differential** of y is defined to be*

$$d^2y = f''(x)\, dx^2.$$

In general the nth differential of y is defined by

$$d^ny = f^{(n)}(x)\, dx^n.$$

Here dx^2 means $(dx)^2$ and dx^n means $(dx)^n$.

We thus have the alternative notations

$$\frac{d^2y}{dx^2} = f''(x), \qquad \frac{d^ny}{dx^n} = f^{(n)}(x)$$

for the second and nth derivatives. The notation

$$y'' = f''(x), \qquad y^{(n)} = f^{(n)}(x)$$

is also used.

The definition of the second differential can be remembered in the following way. By definition,

$$dy = f'(x)\, dx.$$

Now hold dx constant and formally apply the Constant Rule for differentiation, obtaining

$$d(dy) = f''(x)\, dx\, dx,$$

or
$$d^2y = f''(x)\, dx^2.$$

(This is not a correct use of the Constant Rule because the rule applies to a real constant c, and dx is not a real number. It is only a mnemonic device to remember the definition of d^2y, not a proof.)

The third and higher differentials can be motivated in the same way. If we hold dx constant and formally use the Constant Rule again and again, we obtain

$$dy = f'(x)\, dx,$$
$$d^2y = f''(x)\, dx\, dx = f''(x)\, dx^2,$$
$$d^3y = f'''(x)\, dx^2\, dx = f'''(x)\, dx^3,$$
$$d^4y = f^{(4)}(x)\, dx^3\, dx = f^{(4)}(x)\, dx^4,$$

and so on.

The *acceleration* of a moving particle is defined to be the derivative of the velocity with respect to time,

$$a = dv/dt.$$

Thus the velocity is the first derivative of the distance and the acceleration is the second derivative of the distance. If s is distance, we have

$$v = \frac{ds}{dt}, \qquad a = \frac{d^2s}{dt^2}.$$

EXAMPLE 1   A ball thrown up with initial velocity b moves according to the equation

$$y = bt - 16t^2$$

with y in feet, t in seconds. Then the velocity is

$$v = b - 32t \text{ ft/sec},$$

and the acceleration (due to gravity) is a constant,

$$a = -32 \text{ ft/sec}^2.$$

EXAMPLE 2   Find the second derivative of $y = \sin(2\theta)$.

First derivative   Put $u = 2\theta$. Then

$$y = \sin u, \qquad \frac{dy}{du} = \cos u, \qquad \frac{du}{d\theta} = 2.$$

By the Chain Rule,

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} = \cos(2\theta) \cdot 2,$$

$$\frac{dy}{d\theta} = 2\cos(2\theta).$$

Second derivative   Let $v = 2\cos(2\theta)$. We must find $dv/d\theta$. Put $u = 2\theta$. Then

$$v = 2\cos u, \qquad \frac{dv}{du} = -2\sin u, \qquad \frac{du}{d\theta} = 2.$$

Using the Chain Rule again,

$$\frac{d^2y}{d\theta^2} = \frac{dv}{d\theta} = \frac{dv}{du} \cdot \frac{du}{d\theta} = (-2\sin(2\theta)) \cdot 2.$$

This simplifies to

$$\frac{d^2y}{d\theta^2} = -4\sin(2\theta).$$

EXAMPLE 3   A particle moves so that at time t it has gone a distance s along a straight line, its velocity is v, and its acceleration is a. Show that

$$a = v\frac{dv}{ds}.$$

By definition we have

$$v = \frac{ds}{dt}, \qquad a = \frac{dv}{dt},$$

so by the Chain Rule,

$$a = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}.$$

EXAMPLE 4   If a polynomial of degree n is repeatedly differentiated, the kth derivative will be a polynomial of degree $n - k$ for $k \le n$, and the $(n + 1)$st derivative will be zero. For example,

$$y = 3x^5 - 10x^4 + x^2 - 7x + 4.$$
$$dy/dx = 15x^4 - 40x^3 + 2x - 7.$$
$$d^2y/dx^2 = 60x^3 - 120x^2 + 2.$$
$$d^3y/dx^3 = 180x^2 - 240x.$$
$$d^4y/dx^4 = 360x - 240.$$
$$d^5y/dx^5 = 360, \qquad d^6y/dx^6 = 0.$$

Geometrically, the second derivative $f''(x)$ is the slope of the curve $y' = f'(x)$ and is also the rate of change of the slope of the curve $y = f(x)$.

PROBLEMS FOR SECTION 2.7

In Problems 1–23, find the second derivative.

1 $\quad y = 1/x$	2 $\quad y = x^5$	3	$y = \dfrac{-5}{x + 1}$

4    $f(x) = 3x^{-2}$ 5 $f(x) = x^{1/2} + x^{-1/2}$ 6 $f(t) = t^3 - 4t^2$

7    $f(t) = t\sqrt{t}$ 8 $y = (3t - 1)^{10}$ 9 $y = \sin x$

10    $y = \cos x$ 11 $y = A \sin(Bx)$

12    $y = A \cos(Bx)$ 13 $y = e^{ax}$

14    $y = e^{-ax}$ 15 $y = \ln x$

16    $y = x \ln x$ 17 $y = \dfrac{1}{t^2 + 1}$

18    $y = \sqrt{3t + 2}$ 19 $z = \dfrac{x - 5}{x + 2}$ 20 $z = \dfrac{2x - 1}{3x - 2}$

21    $z = x\sqrt{x + 1}$ 22 $s = \left(\dfrac{t + 1}{t + 2}\right)^2$ 23 $s = \sqrt{\dfrac{t}{t + 3}}$

24   Find the third derivative of $y = x^2 - 2/x$.

25   A particle moves according to the equation $s = 1 - 1/t^2$, $t > 0$. Find its acceleration.

26   An object moves in such a way that when it has moved a distance s its velocity is $v = \sqrt{s}$. Find its acceleration. (Use Example 3.)

27   Suppose u depends on x and d^2u/dx^2 exists. If $y = 3u$, find d^2y/dx^2.

28   If d^2u/dx^2 and d^2v/dx^2 exist and $y = u + v$, find d^2y/dx^2.

29   If d^2u/dx^2 exists and $y = u^2$, find d^2y/dx^2.

30   If d^2u/dx^2 and d^2v/dx^2 exist and $y = uv$, find d^2y/dx^2.

31   Let $y = ax^2 + bx + c$ be a polynomial of degree two. Show that dy/dx is a linear function and d^2y/dx^2 is a constant function.

☐ 32   Prove that the nth derivative of a polynomial of degree n is constant. (Use the fact that the derivative of a polynomial of degree k is a polynomial of degree $k - 1$.)

2.8 IMPLICIT FUNCTIONS

We now turn to the topic of implicit differentiation. We say that y is an *implicit function* of x if we are given an equation

$$\sigma(x, y) = \tau(x, y)$$

which determines y as a function of x. An example is $x + xy = 2y$. Implicit differentiation is a way of finding the derivative of y without actually solving for y as a function of x. Assume that dy/dx exists. The method has two steps:

Step 1 Differentiate both sides of the equation $\sigma(x, y) = \tau(x, y)$ to get a new equation

(1)
$$\frac{d(\sigma(x, y))}{dx} = \frac{d(\tau(x, y))}{dx}.$$

The Chain Rule is often used in this step.

Step 2 Solve the new Equation 1 for dy/dx. The answer will usually involve both x and y.

In each of the examples below, we assume that dy/dx exists and use implicit differentiation to find the value of dy/dx.

EXAMPLE 1 Given the equation $x + xy = 2y$, find dy/dx.

Step 1 $\dfrac{d(x + xy)}{dx} = \dfrac{d(2y)}{dx}$. We find each side by the Sum and Product Rules,

$$\frac{d(x + xy)}{dx} = \frac{dx + d(xy)}{dx} = \frac{dx + x\,dy + y\,dx}{dx}$$

$$= 1 + x\frac{dy}{dx} + y.$$

$$\frac{d(2y)}{dx} = 2\frac{dy}{dx}.$$

Thus our new equation is

$$1 + x\frac{dy}{dx} + y = 2\frac{dy}{dx}.$$

Step 2 Solve for dy/dx.

$$2\frac{dy}{dx} - x\frac{dy}{dx} = 1 + y.$$

$$\frac{dy}{dx} = \frac{1 + y}{2 - x}.$$

We can check our answer by solving the original equation for y and using ordinary differentiation:

$$x + xy = 2y.$$
$$2y - xy = x.$$

$$y = \frac{x}{2 - x}.$$

By the Quotient Rule,

$$\frac{dy}{dx} = \frac{(2 - x) \cdot 1 - x(-1)}{(2 - x)^2} = \frac{2}{(2 - x)^2}.$$

A third way to find dy/dx is to solve the original equation for x, find dx/dy, and then use the Inverse Function Rule.

$$x + xy = 2y.$$

$$x = \frac{2y}{1 + y}.$$

$$\frac{dx}{dy} = \frac{(1 + y) \cdot 2 - 2y \cdot 1}{(1 + y)^2} = \frac{2}{(1 + y)^2}.$$

$$\frac{dy}{dx} = \frac{1}{2}(1 + y)^2.$$

To see that our three answers

$$\frac{dy}{dx} = \frac{1 + y}{2 - x}, \qquad \frac{dy}{dx} = \frac{2}{(2 - x)^2}, \qquad \frac{dy}{dx} = \frac{1}{2}(1 + y)^2$$

are all the same we substitute $\dfrac{x}{2 - x}$ for y:

$$\frac{dy}{dx} = \frac{1 + y}{2 - x} = \frac{1 + \dfrac{x}{2 - x}}{2 - x} = \frac{2}{(2 - x)^2}.$$

$$\frac{dy}{dx} = \frac{1}{2}(1 + y)^2 = \frac{1}{2}\left(1 + \frac{x}{2 - x}\right)^2 = \frac{2}{(2 - x)^2}.$$

In Example 1, we found dy/dx by three different methods.

(a) Implicit differentiation. We get dy/dx in terms of both x and y.

(b) Solve for y as a function of x and differentiate directly. This gives dy/dx in terms of x only.

(c) Solve for x as a function of y, find dx/dy directly, and use the Inverse Function Rule. This method gives dy/dx in terms of y only.

EXAMPLE 2   Given $y + \sqrt{y} = x^2$, find dy/dx.

$$\frac{d(y + \sqrt{y})}{dx} = \frac{d(x^2)}{dx}.$$

$$\frac{dy}{dx} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 2x.$$

$$\frac{dy}{dx} = \frac{2x}{1 + \frac{1}{2}y^{-1/2}}.$$

This answer can be used to find the slope at any point on the curve. For example, at the point $(\sqrt{2}, 1)$ the slope is

$$\frac{2\sqrt{2}}{1 + \frac{1}{2} \cdot 1^{-1/2}} = \frac{2\sqrt{2}}{3/2} = \frac{4\sqrt{2}}{3}$$

while at the point $(-\sqrt{2}, 1)$ the slope is

$$\frac{2(-\sqrt{2})}{1 + \frac{1}{2} \cdot 1^{-1/2}} = \frac{-4\sqrt{2}}{3}.$$

To get dy/dx in terms of x, we solve the original equation for y using the quadratic formula:

$$y + \sqrt{y} - x^2 = 0,$$

$$\sqrt{y} = \frac{-1 \pm \sqrt{1 + 4x^2}}{2}.$$

Since $\sqrt{y} \geq 0$, only one solution may occur,

$$\sqrt{y} = \frac{-1 + \sqrt{1 + 4x^2}}{2}.$$

Then

$$y = \left(\frac{-1 + \sqrt{1 + 4x^2}}{2}\right)^2$$

The graph of this function is shown in Figure 2.8.1. By substitution we get

$$\frac{dy}{dx} = \frac{2x}{1 + \frac{1}{2}y^{-1/2}} = \frac{2x}{1 + (-1 + \sqrt{1 + 4x^2})^{-1}}.$$
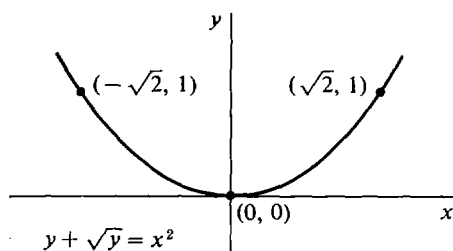

Figure 2.8.1

Often one side of an implicit function equation is constant and has derivative zero.

EXAMPLE 3   Given $x^2 - 2y^2 = 4$, $y \leq 0$, find dy/dx.

$$\frac{d(x^2 - 2y^2)}{dx} = \frac{d(4)}{dx}.$$

$$\frac{d(x^2 - 2y^2)}{dx} = 2x - 4y\frac{dy}{dx}.$$

$$\frac{d(4)}{dx} = 0.$$

$$2x - 4y\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = \frac{-2x}{-4y} = \frac{x}{2y}.$$

Solving the original equation for y, we get

$$-2y^2 = 4 - x^2, \qquad y \leq 0:$$

$$y^2 = \frac{x^2 - 4}{2}, \qquad y \leq 0:$$

$$y = -\sqrt{\frac{x^2 - 4}{2}}.$$

Thus dy/dx in terms of x is

$$\frac{dy}{dx} = \frac{x}{2y} = \frac{x}{-2\sqrt{\frac{x^2 - 4}{2}}} = -\frac{x}{\sqrt{2(x^2 - 4)}}.$$

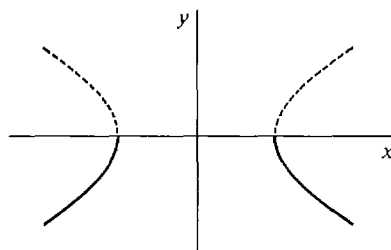The graph of this function is shown in Figure 2.8.2.

Figure 2.8.2 $x^2 - 2y^2 = 4,\ y \leq 0$

Implicit differentiation can even be applied to an equation that does not by itself determine y as a function x. Sometimes extra inequalities must be assumed in order to make y a function of x.

EXAMPLE 4   Given

(2)
$$x^2 + y^2 = 1,$$

find dy/dx. This equation does not determine y as a function of x; its graph is the unit circle. Nevertheless we differentiate both sides with respect to x and solve for dy/dx,

$$2x + 2y\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

We can conclude that for any system of formulas S which contains the Equation 2 and also determines y as a function of x, it is true that

(3)
$$\frac{dy}{dx} = -\frac{x}{y}.$$

We can use Equation 3 to find the slope of the line tangent to the unit circle at any point on the circle. The following examples are illustrated in Figure 2.8.3.
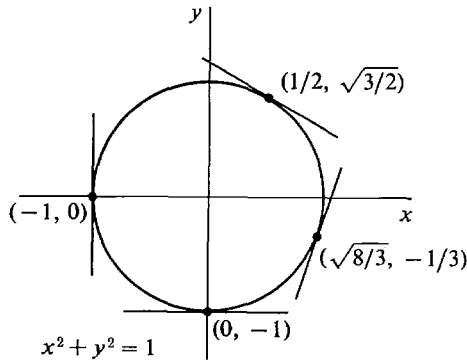
Figure 2.8.3 $x^2 + y^2 = 1$

$$\frac{dy}{dx} = 0 \text{ at } (0, -1), \qquad \frac{dy}{dx} = -\frac{1}{\sqrt{3}} \text{ at } \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

$$\frac{dy}{dx} = \sqrt{8} \text{ at } \left(\frac{\sqrt{8}}{3}, -\frac{1}{3}\right), \qquad \frac{dy}{dx} \text{ is undefined at } (-1, 0).$$

The system of formulas

$$x^2 + y^2 = 1, \qquad y \geq 0$$

gives us $\qquad y = \sqrt{1 - x^2}, \qquad \dfrac{dy}{dx} = -\dfrac{x}{y} = -\dfrac{x}{\sqrt{1 - x^2}}.$

On the other hand the system

$$x^2 + y^2 = 1, \qquad y \leq 0$$

gives us $\qquad y = -\sqrt{1 - x^2}, \qquad \dfrac{dy}{dx} = -\dfrac{x}{y} = \dfrac{x}{\sqrt{1 - x^2}}.$

EXAMPLE 5 Find the slope of the line tangent to the curve

(4) $$x^5 y^3 + xy^6 = y + 1$$

at the points $(1, 1)$, $(1, -1)$, and $(0, -1)$.

The three points are all on the curve, and the first two points have the same x coordinate, so Equation 4 does not by itself determine y as a function of x. We differentiate with respect to x,

$$\frac{d(x^5 y^3 + xy^6)}{dx} = \frac{d(y + 1)}{dx},$$

$$5x^4 y^3 + x^5 \cdot 3y^2 \frac{dy}{dx} + y^6 + 6xy^5 \frac{dy}{dx} = \frac{dy}{dx},$$

and then solve for dy/dx,

$$5x^4 y^3 + y^6 + (3x^5 y^2 + 6xy^5 - 1)\frac{dy}{dx} = 0,$$

(5) $$\frac{dy}{dx} = -\frac{5x^4 y^3 + y^6}{3x^5 y^2 + 6xy^5 - 1}.$$

Substituting, $\dfrac{dy}{dx} = -\dfrac{6}{8}$ at (1, 1).

$\dfrac{dy}{dx} = -1$ at (1, −1).

$\dfrac{dy}{dx} = +1$ at (0, −1).

Equation 5 for dy/dx is true of any system S of formulas which contains Equation 4 and determines y as a function of x.

Here is what generally happens in the method of implicit differentiation. Given an equation

(6) $$\tau(x, y) = \sigma(x, y)$$

between two terms which may involve the variables x and y, we differentiate both sides of the equation and obtain

(7) $$\frac{d(\tau(x, y))}{dx} = \frac{d(\sigma(x, y))}{dx}.$$

We then solve Equation 7 to get dy/dx equal to a term which typically involves both x and y. We can conclude that for any system of formulas which contains Equation 6 and determines y as a function of x, Equation 7 is true. Also, Equation 7 can be used to find the slope of the tangent line at any point on the curve $\tau(x, y) = \sigma(x, y)$.

PROBLEMS FOR SECTION 2.8

In Problems 1–26, find dy/dx by implicit differentiation. The answer may involve both x and y.

1	$xy = 1$	**2**	$2x^2 - 3y^2 = 4, \quad y \leq 0$
3	$x^3 + y^3 = 2$	**4**	$x^3 = y^5$
5	$y = 1/(x + y)$	**6**	$y^2 + 3y - 5 = x$
7	$x^{-2} + y^{-2} = 1$	**8**	$xy^3 = y + x$
9	$x^2 + 3xy + y^2 = 0$	**10**	$x/y + 3y = 2$
11	$x^5 = y^2 - y + 1$	**12**	$\sqrt{x} + \sqrt{y} = x + y$
13	$y = \sqrt{xy + 1}$	**14**	$x^4 + y^4 = 5$
15	$xy^2 - 3x^2y + x = 1$	**16**	$2xy^{-2} + x^{-2} = y$
17	$y = \sin(xy)$	**18**	$y = \cos(x + y)$
19	$x = \cos^2 y$	**20**	$x = \sin y + \cos y$
21	$y = e^{x+2y}$	**22**	$e^y = x^2 + y$
23	$e^x = \ln y$	**24**	$\ln y = \sin x$
25	$y^2 = \ln(2x + 3y)$	**26**	$\ln(\cos y) = 2x + 5$

In Problems 27–33, find the slope of the line tangent to the given curve at the given point or points.

27 $x^2 + xy + y^2 = 7$ at (1, 2) and (−1, 3)

28 $x + y^3 = y$ at (0, 0), (0, 1), (0, −1), (−6, 2)

29 $x^2 - y^2 = 3$ at (2, 1), (2, −1), ($\sqrt{3}$, 0)

30 $\tan y = x^2$ at ($\pi/4$, 1)

31 $2 \sin^2 x = 3 \cos y$ at $(\pi/3, \pi/3)$

32 $y + e^y = 1 + \ln x$ at $(1, 0)$

33 $e^{\sin x} = \ln y$ at $(0, e)$

34 Given the equation $x^2 + y^2 = 1$, find dy/dx and d^2y/dx^2.

35 Given the equation $2x^2 - y^2 = 1$, find dy/dx and d^2y/dx^2.

36 Differentiating the equation $x^2 = y^2$ implicitly, we get $dy/dx = x/y$. This is undefined at the point $(0, 0)$. Sketch the graph of the equation to see what happens at the point $(0, 0)$.

EXTRA PROBLEMS FOR CHAPTER 2

1 Find the derivative of $f(x) = 4x^3 - 2x + 1$.

2 Find the derivative of $f(t) = 1/\sqrt{2t - 3}$.

3 Find the slope of the curve $y = x(2x + 4)$ at the point $(1, 6)$.

4 A particle moves according to the equation $y = 1/(t^2 - 4)$. Find the velocity as a function of t.

5 Given $y = 1/x^3$, express Δy and dy as functions of x and Δx.

6 Given $y = 1/\sqrt{x}$, express Δy and dy as functions of x and Δx.

7 Find $d(x^2 + 1/x^2)$.

8 Find $d(x - 1/x)$.

9 Find the equation of the line tangent to the curve $y = 1/(x - 2)$ at the point $(1, -1)$.

10 Find the equation of the line tangent to the curve $y = 1 + x\sqrt{x}$ at the point $(1, 2)$.

11 Find dy/dx where $y = -3x^3 - 5x + 2$.

12 Find dy/dx where $y = (2x - 5)^{-2}$.

13 Find ds/dt where $s = (3t + 4)(t^2 - 5)$.

14 Find ds/dt where $s = (4t^2 - 6)^{-1} + (1 - 2t)^{-2}$.

15 Find du/dv where $u = (2v^2 - 5v + 1)/(v^3 - 4)$.

16 Find du/dv where $u = (v + (1/v))/(v - (1/v))$.

17 Find dy/dx where $y = x^{1/2} + 4x^{3/2}$.

18 Find dy/dx where $y = (1 + \sqrt{x})^2$.

19 Find dy/dx where $y = x^{1/3} - x^{-1/4}$.

20 Find dy/dx where $y = e^x \cos^2 x$.

21 Find dy/dx where $x = \sqrt{y} + y^2, y > 0$.

22 Find dy/dx where $x = y^{-1/2} + y^{-1}, y > 0$.

23 Find dy/dx where $y = \sqrt{1 - 3x}$.

24 Find dy/dx where $y = \sin(2 + \sqrt{x})$.

25 Find dy/dx where $y = u^{-1/2}, u = 5x + 4$.

26 Find dy/dx where $y = u^5, u = 2 - x^3$.

27 Find the slope dy/dx of the path of a particle moving so that $y = 3t + \sqrt{t}, x = (1/t) - t^2$.

28 Find the slope dy/dx of the path of a particle moving so that $y = \sqrt{4t - 5}, x = \sqrt{3t + 6}$.

29 Find d^2y/dx^2 where $y = \sqrt{4x - 1}$.

30 Find d^2y/dx^2 where $y = x/(x^2 + 2)$.

31 An object moves so that $s = t\sqrt{t + 3}$. Find the velocity $v = ds/dt$ and the acceleration $a = d^2s/dt^2$.

32 Find dy/dx by implicit differentiation when $x + y + 2x^2 + 3y^3 = 2$.

33   Find dy/dx by implicit differentiation when $3xy^3 + 2x^3y = 1$.

34   Find the slope of the line tangent to the curve $2x\sqrt{y} - y^2 = \sqrt{x}$ at $(1, 1)$.

☐ **35**   Find the derivative of $f(x) = |x^2 - 1|$.

☐ **36**   Find the derivative of the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

☐ **37**   Let $f(x) = (x - c)^{4/3}$. Show that $f'(x)$ exists for all real x but that $f''(c)$ does not exist.

☐ **38**   Let n be a positive integer and c a real number. Show that there is a function $g(x)$ which has an nth derivative at $x = c$ but does not have an $(n + 1)$st derivative at $x = c$. That is, $g^{(n)}(c)$ exists but $g^{(n+1)}(c)$ does not.

☐ **39**   (a)   Let $u = |x|$, $y = u^2$. Show that at $x = 0$, dy/dx exists even though du/dx does not.
        (b)   Let $u = x^4$, $y = |u|$. Show that at $x = 0$, dy/dx exists even though dy/du does not.

☐ **40**   Suppose $g(x)$ is differentiable at $x = c$ and $f(x) = |g(x)|$. Show that
        (a) $f'(c) = g'(c)$ if $g(c) > 0$,
        (b) $f'(c) = -g'(c)$ if $g(c) < 0$,
        (c) $f'(c) = 0$ if $g(c) = 0$ and $g'(c) = 0$,
        (d) $f'(c)$ does not exist if $g(c) = 0$ and $g'(c) \neq 0$.

☐ **41**   Prove by induction that for every positive integer n, $n < 2^n$.

☐ **42**   Prove by induction that the sum of the first n odd positive integers is equal to n^2,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$