

# **Notes for MATH 6627: Statistical Consulting Practicum**

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# Preface

- These notes are to be used for MATH 6627: Practicum in Statistical Consulting
- Some references used are:
- Lastly, see below:

```
print("Make sure you install R!")
```

```
[1] "Make sure you install R!"
```

# 1 Mixed Models

## 1.1 Clustered data

### 1.1.1 Review of multiple linear regression

Recall the multiple linear regression model:

For the model  $Y|X = X\beta + \epsilon$ , we have

- $Y \in \mathbb{R}^{n \times 1}$  is the response variable (a continuous random vector)
- $X \in \mathbb{R}^{n \times p}$  is the covariate matrix (Note that the first column is often  $1_n$  – the column vector of ones)
- $X_i \in \mathbb{R}^{p \times 1}$  is the  $i^{th}$  observed explanatory variable ( $i = 1, \dots, n$ ) (not a random variable, in the sense that we condition on it)
- $\beta \in \mathbb{R}^{p \times 1}$  is the coefficient vector
- $\epsilon \in \mathbb{R}^n$  is the random error (continuous random variable)

The key assumptions of the (normal) MLR are that

- $\epsilon$  is multivariate normally distributed
- $E(\epsilon) = 0$
- $Cov(\epsilon) = \sigma^2 I_n$
- $Y|X = X\beta + \epsilon$

As such, it is critical that when applying MLR models, the observations are *independent*. However, there are many, many problems where the data contains dependent observations. If we have data that can be split into mutually independent clusters, then we call this *clustered data*.

### 1.1.2 An example

Consider the following simple example. Suppose a study wishes to prove/disprove the following:  
**Does Ozempic cause sustained weight loss over time?**

What type of data would we need to answer this question? We might start with the question: Can we collect data that would allow us to answer this question with a MLR model?

Could we:

- Take a sample of individuals on Ozempic and measure their weight? – **No.** How do we determine if their weight has decreased since starting it?
- Take a sample of individuals both on and not on Ozempic at a point in time, and compare their weights? – **No.** How can we rule out the fact that these are different populations?

It seems that this question could not be reliably answered using the above suggested methods. We would **need** to be able to follow individuals, starting when they begin Ozempic, recording their weights, and continue following them for a period of time. We might have data that looks like:

Month 1	Month 2	Month 3	Month 4	...
360	355	350	340	...
225	222	224	225	...
288	270	253	260	...

We could simply compare the weights in month 1 to the last month measured, and apply a one-sample t-test. What if the patients lose weight in the first 6 months and then gain it back? This would not be captured by such a model. We could run one t-test for each month, but of course then the type-1 error would be very large.

It is better to model the weights of patients on Ozempic over time. Inspired by the Normal MLR model, we might posit that patient  $i$ 's weight at time  $j$  is governed by the following equation:

$$Y_{ij} = \beta_0 + \beta_1 t_j + \epsilon_{ij}.$$

with  $\epsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$ . Now, if we want to apply the Normal MLR model, we would need to assume  $\epsilon_{ij} \perp \epsilon_{\ell k}$  when  $ij \neq \ell k$ . Is this reasonable? This would imply that  $Y_{ij} \perp Y_{ij+1}$ , i.e., a patient's weight in month  $j$  is independent of their weight in month  $j+1$ . This is, of course unreasonable. However, it would be reasonable to assume that  $Y_{ij} \perp Y_{\ell k}$  when  $\ell \neq i$ . This is an example of **clustered data**. Here the clusters are the patients.

- It is safe to assume that a patient's weight at a given time is unrelated to another patient's weight at any given time

- However, a patient's weight at a given time is related to their past and future weights; the within patient weights are dependent.

Therefore, a better assumption might be that the

$$Y_{ij} = \beta_{0i} + \beta_1 t_j + \epsilon_{ij}.$$

where  $\beta_{0i}$  are now **random variables**, where there exists one per patient. The coefficients  $\beta_{0i}$  contain the dependence of the weight within individuals, and allow us to model the random errors as independent:  $Cov(\epsilon) = \sigma^2 I$ . This is only one way to model the within-patient dependence between patients.

**Example 1.1.** Testing this theory...

Simplify the correlation between  $Y_{ik}$  and  $Y_{ij}$  in this model, and in the Normal MLR (the Normal MLR is the MLR, with the added assumption that the errors are normally distributed. Compare the results.

Solution:

Done in lecture.

The previous example is a longitudinal study, which is a sub-type of the more general clustered data. A longitudinal study is a research study in which subjects are followed over time. Typically this involves repeated measurements of the same variables. Longitudinal studies differ from cross-sectional studies and time series studies. Cross-sectional studies have no clusters, and time series follow one or more variates over time. The random errors in a time series may be correlated.

Longitudinal studies are useful for:

- detecting changes in outcomes, both at the population and individual level,
- assessing **Longitudinal effects**, as compared to cohort effects/cross sectional effects,
- understanding different sources of variation, e.g., between- and within-subject variation.
- detecting **time effects**, both directly and as interactions with other relevant factors.

One example of longitudinal data, which we will see later is the TLC trial data:

```
#####
TLC <- read.csv("data/TLC.csv", stringsAsFactors = T)
head(TLC)
```

ID	Treatment	W0	W1	W4	W6
1	1	P	30.8	26.9	25.8
2	2	A	26.5	14.8	19.5
3	3	A	25.8	23.0	19.1
4	4	P	24.7	24.5	22.0
5	5	A	20.4	2.8	3.2
6	6	A	20.4	5.4	4.5
					11.9

As mentioned, longitudinal data is one example of clustered data. Clustered data refers to data that can be divided into clusters, such that data within a given cluster are correlated. For longitudinal observations, observations taken from the same subject at different time points are correlated because they belong to the same subject. In general, real world data have a complex dependence structure - can often be fit into this clustered framework.

Another example of clustered data is hierarchical data. These data have clusters within clusters. You could make a super dated inception joke here. For instance, if we wanted to assess how a new way of teaching p-values affects statistical literacy, we could sample universities, then professors, then classes. Here, assuming professors teach multiple sections, we could safely assume that the effect of this new teaching method may differ by university, by professor, and by class. In other words, observations within these groups would be correlated.

### 1.1.3 Additional challenges with clustered data

Often, clustered data are accompanied by other, additional challenges.

- Missing data or dropouts
- Measurement errors
- Censoring
- Outliers

The below examples are adapted from Wu (2019) .

#### Example 1.2. Blood pressure

A researcher wishes to evaluate a treatment for reducing high blood pressure. Blood pressures of each subject in the study are measured before and after the treatment. The researcher is also interested in how blood pressures of the subjects change over time after the treatment, so blood pressure is also measured after treatment once a month for 5 months. What are some potential challenges associated with this data? One answer: The data may contain missing values, e.g., drop out. Blood pressure has measurement error – often repeatedly measured.

### Example 1.3. Mental distress

Investigate changes in subjects' mental distress over time in a treatment group and a control group. Mental distress in 239 subjects were measured at baseline, 4, 12, 24, and 60 months, based on their answers to questionnaires. Subjects randomly assigned into two groups: a treatment and a control group. The Global Severity Index (GSI) is used to measure subjects' distress levels. Other variables such as education, annual income, depression, anxiety, etc. were collected

Variable	Mean	Standard Deviation
GSI score of subjects (0 – 10)	1.13	0.72
Education of subjects (in years)	13.70	2.36
Income of subjects (in \$10,000)	4.68	1.90
Depression score of subjects (0 – 10)	1.55	0.99
Anxiety score of subjects (0 – 10)	1.23	0.92

Table 1.2 Missing data rate

Variable	baseline	3 months
GSI	0.04	0.14
Depression	0.03	0.13
Anxiety	0.03	0.13

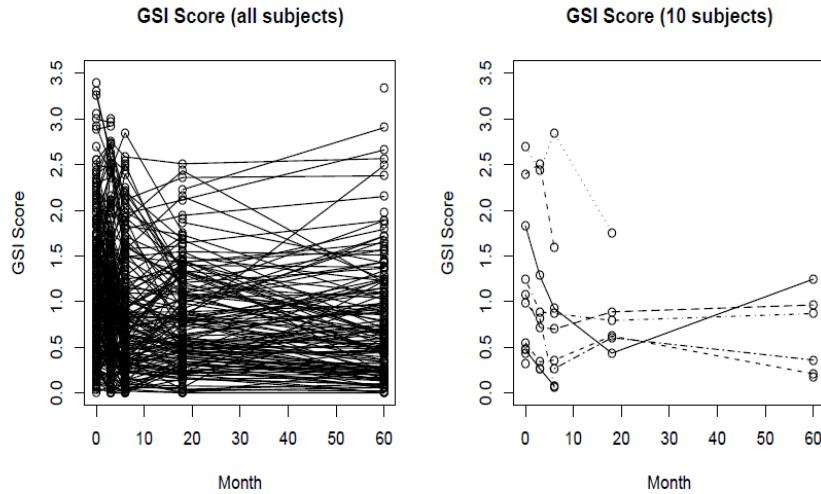


Figure 1.1 GSI scores over time. Left figure: GSI scores for all subjects. Right figure: GSI scores for 10 randomly selected subjects. The open dots are the observed values.

What are some potential issues for analysis?

Substantial individual variability, Missing data, Outliers, Measurement error? GSI influenced by short-term emotional state

### Example 1.4. AIDS Study

The following is an AIDS study designed to evaluate an anti-HIV treatment. 53 HIV infected patients were treated with an antiviral regimen. Viral load (RNA) was repeatedly quantified on days 0, 2, 7, 10, 14, 21, and 28, and weeks 8, 12, 24, and 48 after initiation of the treatment. Immunologic markers known as CD4 and CD8 cell counts were also measured along with viral load, as well as some other variables. Viral load has a lower detection limit of 100, i.e., viral loads below 100 are not quantifiable.

Table 1.3 *Summary statistics for viral load (RNA), CD4, and CD8 at five selected measurement times*

Variable	Day 2		Day 7		Day 14		Day 28		Day 56	
	Mean	S.D. <sup>b</sup>	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
RNA <sup>a</sup>	5.00	0.59	4.06	0.81	3.23	0.64	3.02	0.61	2.52	0.74
CD4	203	74	231	89	274	108	284	89	300	94
CD8	961	506	1026	643	1037	545	1086	627	1033	329

Note: a) RNA (viral load) is in  $\log_{10}$  scale; b) S.D.: standard deviation

Figure 1.1: AD1

Table 1.4 *Missing data rates for some variables at baseline*

Covariate	Definition	Missing Rate
AGE	age of the patient	0
WEIGHT	weight of the patient	0
LU20	NK activity	37.5%
TNF	plasma tumor necrosis factor	16.7%
APOP	% of cells that are apoptotic	0
CH50	complement CH50	18.75%
BIGG	gp120-binding IgG levels	22.92%
BIGC3	C3 binding to HIV-infected cells	27.08%

Figure 1.2: AD2

Other information about this data is given by:

- “HIV viral dynamic models model viral load trajectories during an anti-HIV treatment”
- “In an HIV viral dynamic model, the relationship between viral load and viral dynamic parameters is often nonlinear, and the viral dynamic parameters often vary substantially across patients.”
- Thus, nonlinear mixed effect models

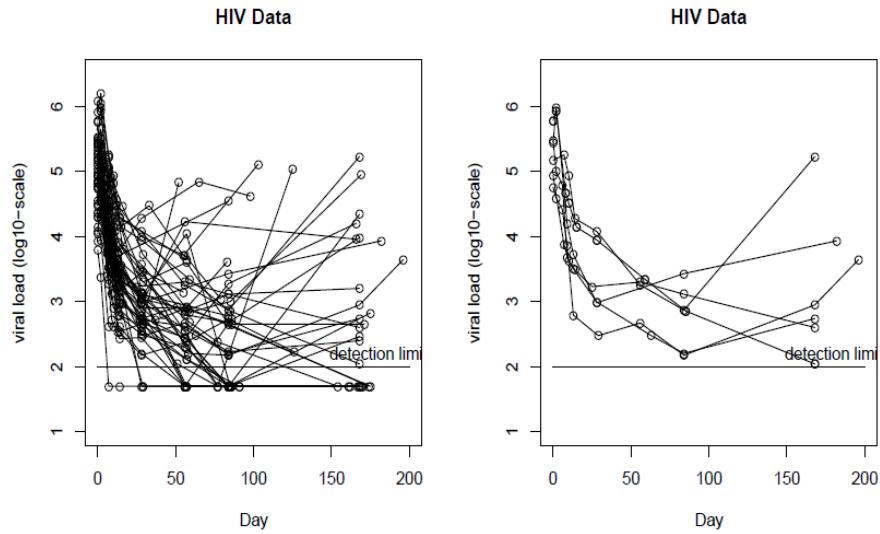


Figure 1.3 *Viral loads trajectories (in  $\log_{10}$  scale). The open circles are observed values. The viral load detection limit in this study is  $\log_{10}(100) = 2$ . Viral loads below the detection limit are substituted by half the limit.*

Figure 1.3: AD3

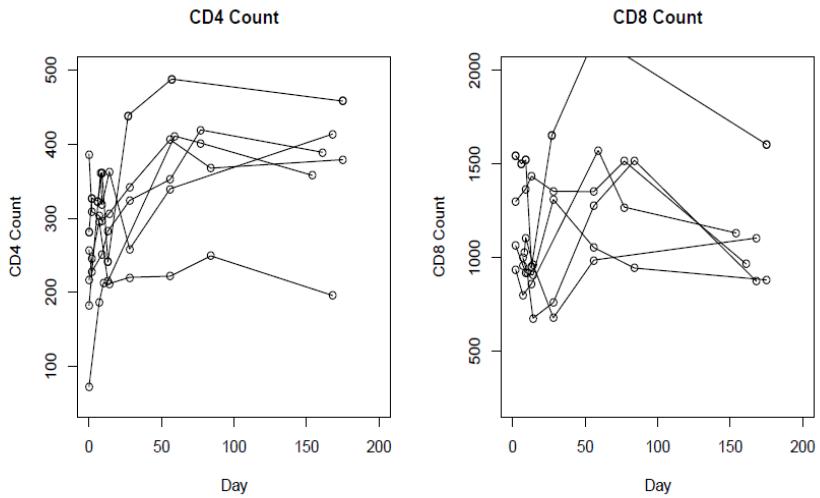


Figure 1.4 *CD4 Counts (left) and CD8 counts (right) of six randomly selected patients.*

Figure 1.4: AD4

- AIDS researchers are also interested in the relationship between viral loads and CD4 counts over time
- “CD4 counts are known to be measured with substantial errors, and patients often drop out because of drug side effects or other problems.”

What are some potential issues with this dataset?

- different measurement times across patients
- different numbers of within-individual measurements across patients
- large variation between patients
- large variation in the data within each patient
- some patients dropping out of the study
- some viral loads being censored (i.e., below the limit of detection)
- substantial measurement errors in the data
- complex long-term trajectories
- data being missing at measurement times

**Multilevel models:** Multilevel models/Hierarchical linear models/Nested data models: Statistical models for “nested clusters”. They contain parameters that vary at more than one level. An example could be a model of student performance, where the data are collected from students from multiple classes from multiple schools.

**Other model classes:** Marginal models/GEE models – Mean and the correlation (covariance) structure are modeled separately. Does not require distributional assumptions (see Chapter 10 Wu (2019)) Transitional models – Within-individual correlation is modeled via Markov structures.

#### 1.1.4 Homework questions

- Write down why clustered data are challenging to analyse?

## 1.2 Analysing clustered data with mixed models

A **mixed model** is a convenient modelling framework which can be used to model complex dependency structure within a data set. They are an extension of the familiar Normal MLR model, where the independent errors assumption is relaxed. In order to relax that assumption, a new concept is introduced: the **random effect**.

### 1.2.1 Random effects

In a mixed effect model, the effects of each of the covariates can be split into two categories: fixed and random effects. Deciding on what is a fixed effect and what is a random effect can be difficult, and there is no agreed upon definition: see [this post by Andrew Gelman](#). I will provide some guidance below, but ultimately, there is no binary rule for determining whether one should model an effect as fixed or random. Your model should reflect the assumptions that are reasonable to make about the data at hand, and answer the research questions adequately.

When we model a fixed effect, we only model the average across the whole population. On the other hand, when we model a covariate as a random effect, we are modelling the average effect of that covariate as well as how that effect might vary between clusters. So, if we want a measure of how an effect varies between clusters, one would use a random effect. Random effects can also be used to implicitly introduce a dependency structure within clusters. For instance, in the Ozempic example in Section [Section 1.1.2](#), we saw that the random effect introduced a correlation between the observations coming from the same individual. Furthermore, modelling the intercept as a random effect allows us to estimate the average “regression line” for the population taking Ozempic, as well as how that “regression line” varies from person to person. (This will be made precise later if you missed the lecture.)

**Example 1.5.** A first example:

A clinical trial is set up to compare a new drug with a standard drug. The drug effect is of interest in the trial. We propose a Normal MLR (or fixed-effects) model with “drug” and “gender” as the two-fixed effects factors. Each has finite levels: “drug” – “new drug” and “standard drug”; “gender” – “female”, “male”, “non-binary”. Is there a cluster variable? Should we introduce any random effects?

**Example 1.6.** Clinical trial:

In a clinical trial, several hospitals in Canada are sampled. In each of the selected hospitals, a new treatment is compared with an existing treatment. Is there a cluster variable? Should we introduce any random effects? What definition(s) do any of these random effects fit?

One way to decide on whether an effect is a fixed or random effect is to ask if the observations for that covariate contain the complete set of levels we are interested in for a given covariate. In Example [1.6](#), the data can be clustered by hospital. The treatment is a fixed effect, as we have observed the complete set of levels we are interested in for it. On the other hand, we would like our analysis to generalize beyond the selected hospitals, and so we would like to assess how the regression line varies between hospitals. This rule does not work well for continuous covariates, such as height, which may not require a random effect, but also, we cannot observe all levels of this factor.

**Example 1.7.** Antibiotics:

The efficacy an antibiotic maintains after it has been stored for two years is of scientific interest. Eight batches of the drug are selected at random from a population of available batches. From each batch, we take a sample of size two. The goal of the analysis: Estimate the overall mean concentration. Does the random batch have a significant effect on the variability of the responses?

batch	r1	r2
1	40.00	42.00
2	33.00	34.00
3	46.00	47.00
4	55.00	52.00
5	63.00	59.00
6	35.00	38.00
7	56.00	56.00
8	34.00	29.00

Since the batches are drawn randomly from a larger population, we could model the batch effect as a random effect. Obviously, the within batch observations will be correlated. The data are clustered by batch. Suppose instead that only eight batches exist in the whole world, and we are interested in knowing whether the batch number has an effect on the response. Then, the batch becomes a fixed effect.

**Example 1.8.** A Tale of Two Thieves (Cabrera and McDougall (2002)):

Recall that the client wanted us to assess the level of active ingredient in their tablets, as well as assess the variability in that can be attributed to the sampling technique.

	METHOD	LOCATION	REPLICATE	ASSAY
1	Intm	1	1	34.38
2	Intm	1	2	34.87
3	Intm	1	3	35.71
4	Intm	2	1	35.31
5	Intm	2	2	37.59
6	Intm	2	3	38.02

Number	methdb	drum	tablet	yb
1	Tablet	1	1	35.77
2	Tablet	1	2	39.44

Number	methdb	drum	tablet	yb
3	Tablet	1	3	36.43
4	Tablet	5	1	35.71
5	Tablet	5	2	37.08
6	Tablet	5	3	36.54

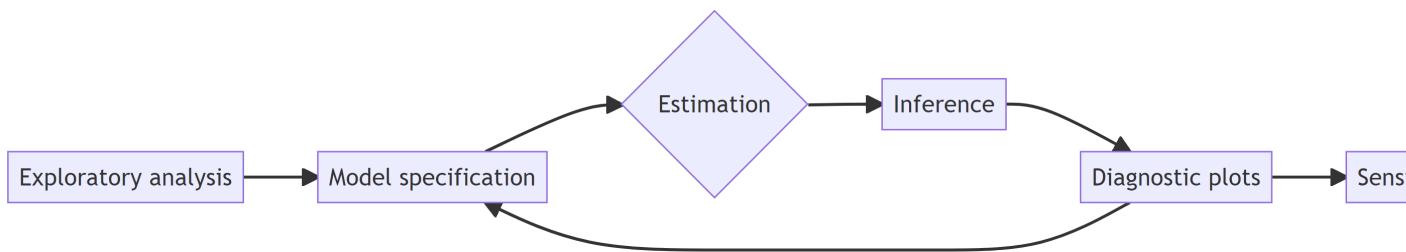
What are the clusters? What might be a random effect?

### 1.2.2 Homework questions

- Give an example of a study where a mixed effect model would apply, which effects are random and which are fixed?
- Describe the potential differences between a random and fixed effect. Compare and contrast the different definitions of random effects. Which one do you prefer and why?
- How would you determine which definition for random effects a client is using?

### 1.2.3 Regression and general data modelling review

By the end of this section, we will have covered all steps involved in analyzing data using mixed models:



**Example 1.9.** How do you do each of these steps in a simple linear regression model?

Let's review. Suppose  $Y_i$  are continuous and we want to model  $E[Y_i|X_i]$ . A linear regression model takes

$$E[Y_i|X_i] = X_i'\beta.$$

We take  $\hat{\beta} = (X'X)^{-1}X'Y$ , and call these ordinary least squares (OLS) estimators. If  $Y_i|X_i \sim N(X_i'\beta, \sigma^2)$ , then the OLS estimators are the maximum likelihood estimators.

If we take  $Y_i = X'_i\beta + \epsilon_i$ , where  $X_i$  is non-normal, then the OLS estimators minimize the MSE of any predictor:

$$\phi(\beta) = \frac{1}{n} \sum_{i=1}^n \|\beta - E[Y_i|X_i]\|^2$$

is minimized at  $\hat{\beta}$ .

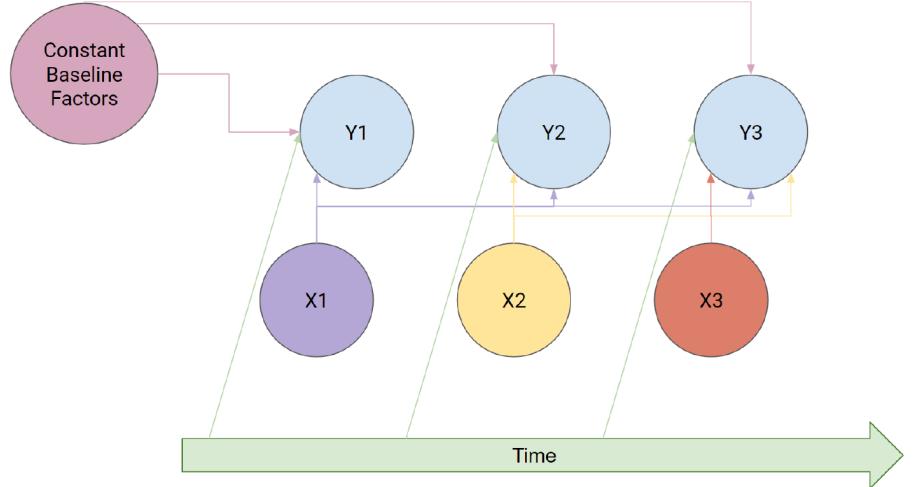
In this case, as discussed in Section 1.1.1, we assume that:

1. The conditional mean is linear (in parameters).
2. All values of  $Y_i$  have constant variance, denoted  $\sigma^2$  (conditionally).
3. The  $Y_i$  are independent.

Then, one can show that  $\hat{\beta}$  is asymptotically normal with  $Var(\hat{\beta}) = \sigma^2(X'X)^{-1}$ . We then use this fact to construct confidence intervals, hypothesis tests etc. We later analyze the residuals to diagnose any problems with the fit. Overall, linear regression allows us to estimate a functional form for the conditional mean of a continuous outcome. The ordinary least-squares estimators are valid MLE-type estimators when normality is assumed, and are least-squares estimators otherwise. The asymptotic analysis is valid in large samples, regardless of distributional assumptions. We now present equivalents of these results for the mixed model.

#### 1.2.4 Defining the linear mixed model

Back to analyzing clustered data. Let's start with longitudinal data, which could be represented by the following diagram:



sented by the following diagram:

In (goal?), we see that there are time-varying and constant covariates, as well as the time-varying response. To formalize this, we can write:

- $Y_{ij}$  response of subject  $i$  at  $j$ th time point for  $i \in [n]$  and  $j \in [J_i]$ .

- $X_{ijk}$  covariate  $k$  of subject  $i$  at  $j$ th time point for  $k \in [K]$ ,  $i \in [n]$  and  $j \in [J_i]$ .
- $X_{ij}$  covariate vector for subject  $i$  at  $j$ th time point for  $i \in [n]$  and  $j \in [J_i]$ .
- $t_{ij}$  actual time for subject  $i$  at time point  $j$  for  $i \in [n]$  and  $j \in [J_i]$ .

We can split the covariate matrix into time-varying covariates  $Z$  and constant covariates  $W$ . We have that  $X = [W|Z]$ . Each subject has  $J_i$  rows in  $X$  associated with it. Let  $X_i$  be the  $J_i \times p$  submatrix corresponding to the covariates for subject  $i$  and let  $Z_i$  be the  $J_i \times m$  submatrix corresponding to the time-varying covariates for subject  $i$ .

Now, the goal is to fit a model for  $E[Y_{ij}|X_{ij}, t_{ij}]$  with interpretable parameters.

To account for the correlation between subjects, we model the response as a vector  $Y_i$ , where

$$Y_i = X_i\alpha + Z_i\beta_i + \epsilon_i,$$

where

- $\alpha$ : Population level effects – constant between subjects ( $p \times 1$ )
- $\beta_i$ : Patient-level heterogeneity – varies between subjects ( $m \times 1$ )
- $\epsilon_{ij}$ : Individual measurement variation – varies between measurements (scalars)

Note that in the mixed model, we assume:  $\alpha$  is a fixed vector,  $\beta_i$  is randomly drawn for each individual,  $\epsilon_{ij}$  are also randomly drawn.

We can then assume that  $\beta_i \sim N(0, \Sigma_\beta)$ ,  $\epsilon_{ij} \sim N(0, \sigma^2)$  with  $\epsilon_{ij} \perp \beta_i$ .

Now, let's look at some properties of the model. Conditional on the random effects

$$E[Y_i|\beta_i, X_i] = X_i\alpha + Z_i\beta_i$$

and

$$\text{Cov}(Y_i|\beta_i, X_i) = \text{Cov}(\epsilon_i|\beta_i, X_i) = \sigma^2 I.$$

**Derive these.** If we consider the marginal distribution of  $Y_i$  we find:

$$E[Y_i|X_i] = X_i\alpha \quad \text{and} \quad \text{Cov}(Y_i|X_i) = Z_i\Sigma_\beta Z'_i + \sigma^2 I$$

**Derive this.** Combining these results we find that, under this assumed model,

$$Y_i|X_i \sim N(X_i\alpha, Z_i\Sigma_\beta Z'_i + \sigma^2 I).$$

### 1.2.5 Special cases of single layer models

We will now cover some special cases of the above model. The most basic mixed model is the random intercept model. Let  $\tilde{W}_i$  be the covariate matrix of fixed effects with the intercept column removed. The resulting model is

$$Y_i = \alpha_1 \mathbb{1}_{J_i} + \tilde{W}_i \alpha + \beta_i \mathbb{1}_{J_i} + \epsilon_i,$$

where  $\beta_i \sim N(0, \sigma_\beta^2)$  and  $\epsilon_i \sim N(0, \sigma^2 I_{J_i})$ . For  $\ell \neq j$ , it follows that

$$\text{Corr}(Y_{ij}, Y_{i\ell}) = \frac{\sigma_\beta^2}{\sigma_\beta^2 + \sigma^2}.$$

The variance is constant across time or clusters:  $\text{Cov}(Y_i | X_i) = (\sigma_\beta^2 + \sigma^2)I$ . **Derive these.** Observe that in this model, all subject level regression lines are parallel. This can be used when we suspect that the correlation is constant over time, and only the mean response is thought to vary between clusters.

If instead, we would like the regression lines to vary in general between subjects, we can introduce slopes are random effects. This is the **random intercept and slope model**. Here,

$$Y_i = \alpha_0 \mathbb{1}_{J_i} + \tilde{W}_i \alpha + \beta_{0i} \mathbb{1}_{J_i} + \alpha_1 t_i + \beta_{1i} t_i + \epsilon_i.$$

**What is  $Z_i$  here?** The within-subject correlation will be time dependent in this model automatically, in this model, we assume that  $\beta_i = (\beta_{0i}, \beta_{1i})' \sim N(0, \Sigma_\beta)$ , where

$$\Sigma_\beta = \begin{pmatrix} \sigma_{\beta_0}^2 & \sigma_{\beta_0, \beta_1} \\ \sigma_{\beta_0, \beta_1} & \sigma_{\beta_1}^2 \end{pmatrix}.$$

Now, let's understand some of the features of this model. For any  $i \in [n]$ , we have that

$$\text{Cov}(Y_i | X_i) = Z_i \Sigma_\beta Z_i' + \sigma^2 I = (1_{J_i} t_i) \Sigma_\beta (1_{J_i} t_i)' + \sigma^2 I.$$

We see that the variance of the response is not constant across time. Further, for  $\ell \neq j$ :

$$\text{Cov}(Y_{ij}, Y_{i\ell}) = \sigma_{\beta_0}^2 + \sigma_{\beta_0, \beta_1} (t_{ij} + t_{i\ell}) + \sigma_{\beta_1}^2 t_{ij} t_{i\ell}.$$

In this model, the correlation between subject responses at different time points is varying.

Use the following code to explore how the correlation between time points changes as the distance between the time points grows.

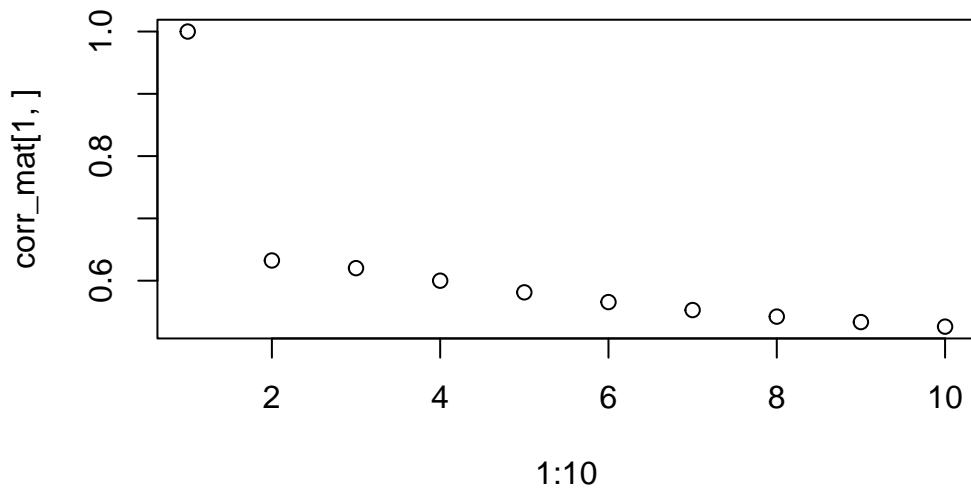
```

plot_cor=function(matt){

Zi=cbind(rep(1,10),1:10)
# err=s
cov_mat=Zi%*%matt%*%t(Zi)+diag(rep(1,10))
cov_mat
vars=sqrt(diag(cov_mat))
vars
corr_mat=apply(cov_mat,1,"*",1/vars)
corr_mat=apply(corr_mat,1,"*",1/vars)
corr_mat
plot(1:10,corr_mat[1,])
}

 matt=matrix(c(1,0.4,0.4,1),nrow=2)
 matt=matrix(c(1,0.4,0.4,1),nrow=2)
 matt=matrix(c(1,0,0,2),nrow=2)
 matt=matrix(c(1,0,0,6),nrow=2)
 matt=matrix(c(1,-0.9,-0.9,4),nrow=2)
 matt=matrix(c(1,0,0,0.5),nrow=2)
 plot_cor(matt)

```



### 1.2.6 Multi-level models

So far we have discussed single level mixed models, which do not admit a nested structure. In this way, there is a nested structure of clusters. For example, if we wanted to assess how a new way of teaching p-values affects statistical literacy, we could sample universities, then professors, then classes. Here, assuming professors teach multiple sections, we could assume that effects differ by university, by professor, and by class. We could write a mixed effect model with 3 levels as

$$Y_{ijk} = X_{ijk}\alpha + Z_{i,jk}\beta_i + Z_{ij,k}\beta_{ij} + Z_{ijk}\beta_{ijk} + \epsilon_{ijk}$$

$$i \in [n], j \in [J_i], k \in [m_{ij}],$$

$$\beta_i \sim N(0, \Sigma_1), \beta_{ij} \sim N(0, \Sigma_2), \beta_{ijk} \sim N(0, \Sigma_3), \epsilon_{ijk} \sim N(0, \sigma^2 I).$$

Note that the “,” tells us which columns of the covariate matrix we are concerned with:  $Z_{i,jk}$  denotes the covariates nested in the highest level,  $Z_{ij,k}$  the second highest and  $Z_{ijk}$  the innermost level. For instance, with respect to the above example,  $Z_{i,jk}$  would be the university level covariates. (Note that some books count the number of levels by the number of sources of random variation, which is 1+ the level definition used here.) In addition, Pinheiro and Bates (2009) represents  $\Sigma_1$  in form  $\Sigma_1^{-1}/\sigma^2 = \Delta'\Delta$ , where  $\Delta$  is a non-unique relative precision factor. Specifications of the covariance matrices depend on the context. For more details, see Pinheiro and Bates (2009), Gelman and Hill (2006), Wu (2019).

### 1.2.7 Parameter estimation and inference

Generally, parameters are estimated with either maximum likelihood or restricted maximum likelihood (REML). Recall that this model is parametric, we have assumed normality. Thus, we can write down the likelihood. Let  $V_i = Cov(Y_i) = Z_i \Sigma_\beta Z_i' + \sigma^2 I$ . Then, the (familiar) asymptotic result for both the MLE and the REML estimates:

$$\hat{\alpha} \stackrel{\text{asymptotic}}{\sim} N \left( \alpha, \left[ \sum_{i=1}^n X_i' V_i^{-1} X_i \right]^{-1} \right),$$

where  $\stackrel{\text{asymptotic}}{\sim}$  denotes asymptotically distributed as. For more details on how to derive the estimates, see Pinheiro and Bates (2009).

Restricted maximum likelihood is used because the MLE biases the variance estimates downward. In REML, we maximize

$$\mathcal{L}(\Sigma_\beta, \sigma^2 | y) = \int \mathcal{L}(\alpha, \Sigma_\beta, \sigma^2 | y) d\alpha.$$

This constitutes a uniform prior on  $\alpha$ . Note that REML estimates are not invariant under reparameterizations of the fixed effects – changing the units of the covariates  $X_i$  units changes the estimates. As a result, LRT are not valid for testing the significance of fixed effects – the restricted likelihoods cannot be compared to determine significance.

**Testing – Fixed effects – MLE:** When using the MLEs, we can use likelihood ratio tests to test significance of various parameters. The parameters for the covariances, denoted  $\sigma_{\beta_k, \beta_\ell}$ , will have some regularity concerns. Suppose we want to test whether a subset of the parameters are 0. Let  $k = \# \text{ df in alt} - \# \text{ df in null}$ . Recall that Wilks' Theorem gives  $-2(\ell_1(\hat{\theta}) - \ell_0(\hat{\theta})) \sim \chi_k^2$ , which can be used to conduct the test.

**Testing – Random effects:** However, this does not apply to random effects. The variance parameters lie on the boundary of the parameter space, and so Wilks' Theorem does not apply! Instead, we can simulate the distribution of the LRT statistic under the null and use the simulated distribution to obtain our critical value. If you are using a software where this is not feasible, then you can use  $\frac{1}{2}\chi_{\# \text{ RE Null}}^2 + \frac{1}{2}\chi_{\# \text{ RE ALT}}^2$ .

**Testing – Fixed effects – REML:** REML estimates are not invariant under reparameterizations of the fixed effects – changing the units for the covariates  $X_i$  changes the REML estimates. As a result, LRT are not valid for testing the significance of fixed effects – the restricted likelihoods cannot be compared to determine significance. To test the fixed effects, we can use tests conditional on the variance parameters/RE parameters. In this case, we can perform either marginal  $t$ -tests – tests which consider adding the parameter to the model with all other covariates, or sequential  $F$ -tests – a test that adds the variables sequentially in the order they enter the model.

**Confidence intervals – Fixed effects – REML:** Both REML and MLE give asymptotic normality of both  $\hat{\sigma}$  and fixed effect estimates. This can be used to obtain confidence intervals. For the parameters contained in  $\Sigma_\beta$ , constructing confidence intervals can be more difficult because  $\Sigma_\beta$  must be positive definite, which restricts the parameter space. In this case, we transform the parameters so that they are unconstrained, compute the confidence interval, and transform the interval back. See Section 2.4 in “Mixed Models in S and S-plus” for more details Pinheiro and Bates (2009).

### 1.2.8 Individual effects

One thing we may want to do is produce an estimate of  $\beta_i$  for observation  $i$ . One may notice that  $E[\beta_i | Y_i] = \Sigma_\beta Z'_i V_i^{-1} (Y_i - X_i \alpha)$ . Wait, we either know or have estimates of all of the values on the right-hand side. BLUP:  $\hat{\beta}_i = \hat{\Sigma}_\beta Z'_i \hat{V}_i^{-1} (Y_i - X_i \hat{\alpha})$ . Fitted values:

$$\hat{Y}_i = X_i \hat{\alpha} + Z_i \hat{\beta}_i.$$

Let's analyze  $V_i$ :

$$\begin{aligned}
V_i &= Z_i \Sigma_\beta Z'_i + \sigma^2 I \\
\implies V_i V_i^{-1} &= Z_i \Sigma_\beta Z'_i V_i^{-1} + \sigma^2 I V_i^{-1} \\
\implies I &= Z_i \Sigma_\beta Z'_i V_i^{-1} + \sigma^2 V_i^{-1}.
\end{aligned}$$

Now, the same logic gives that

$$I = Z_i \hat{\Sigma}_\beta Z'_i V_i^{-1} + \hat{\sigma}^2 V_i^{-1}.$$

We have

$$\begin{aligned}
\hat{Y}_i &= X_i \hat{\alpha} + Z_i \hat{\beta}_i \\
&= X_i \hat{\alpha} + Z_i (\hat{\Sigma}_\beta Z'_i \hat{V}_i^{-1} (Y_i - X_i \hat{\alpha})) \\
&= (I - \hat{\Sigma}_\beta Z'_i \hat{V}_i^{-1}) X_i \hat{\alpha} + Z_i \hat{\Sigma}_\beta Z'_i \hat{V}_i^{-1} Y_i \\
&= \hat{\sigma}^2 \hat{V}_i^{-1} X_i \hat{\alpha} + (I - \hat{\sigma}^2 \hat{V}_i^{-1}) Y_i \\
&= \hat{\sigma}^2 \hat{V}_i^{-1} X_i \hat{\alpha} + Z_i \hat{\Sigma}_\beta Z'_i V_i^{-1} Y_i.
\end{aligned}$$

Thus,

$$\hat{Y}_i = \hat{\sigma}^2 \hat{V}_i^{-1} X_i \hat{\alpha} + Z_i \hat{\Sigma}_\beta Z'_i V_i^{-1} Y_i.$$

We have that:

- $\hat{\sigma}^2 I$  within subject variation
- $Z_i \hat{\Sigma}_\beta Z'_i$  between subject variation
- Higher within subject variation – more weight to the population average

### 1.2.9 Exploratory analysis, checking assumptions and sensitivity testing

**Exploratory analysis:** Prior to setting up our model, we would ideally conduct exploratory analysis. Here, we look for outliers, inconsistencies in the data, and try to ascertain the relationships between the provided variables. This will help inform the model we will choose. It is also helpful to check that the model results approximately mirror what we saw in the EDA, as a sanity check. Some tools you can use in EDA are:

- Descriptive statistics
- xy plots – may have to subsample
- Box plots by cluster variable
- Cross-sectional plots

**Diagnostic plots:** These are used to check the fit of the model and check the assumptions. In a mixed model, we have independence and normality, and structure assumptions. This involves using graphics we are likely familiar with, such as qqplots. We may use:

- Checking independence between residuals across time – acf (may not be appropriate), variogram
- We have independence and normality, and structure assumptions
- Residuals vs. fitted values
- qqplot of residuals/random effects for normality
- Observed vs. fitted values

**Sensitivity testing:** In reality, there may be several models/frameworks with assumptions that could fit your data. For example, we may use a nonparametric method, a robust method, inclusion of different effects, use of different statistical tests. One thing you can do after performing a data analysis is to do the analysis under other models that may have been applied, and see if your results change. This helps support the conclusions made, and can reveal additional insights about your dataset. Be careful not to apply models whose assumptions are not reasonable for your data.

#### 1.2.10 Homework questions

- How would you analyse the TLC data discussed last class? What are some statistical tests you might conduct?
- Write down the likelihood function under the random intercept model.

### 1.3 Case study: Batches of antibiotic and quality control

```
library(nlme)
library(lme4)
```

```
Loading required package: Matrix
```

```
Warning: package 'Matrix' was built under R version 4.2.3
```

```
Attaching package: 'lme4'
```

```
The following object is masked from 'package:nlme':
```

```
lmList
```

### 1.3.1 Case information:

- After an antibiotic has been stored for two years, it is of scientific interest to know what concentration of active ingredient is.
- Eight batches of the drug are selected at random from a population of available batches.
- From each batch, we take a sample of size two.
- The goal of the analysis: Determine (to estimate) the overall mean concentration. A further question is whether or not the random batch has a significant effect on the variability of the responses.

From 8 batches of antibiotics, 2 samples are drawn.

Batch	1	2	3	4	5	6	7	8
Sample 1	40	33	46	55	63	35	56	34
Sample 2	42	34	47	52	59	38	56	29

```
batch=as.matrix(read.csv('data/batch.csv'))
```

```
Warning in read.table(file = file, header = header, sep = sep, quote = quote, :
incomplete final line found by readTableHeader on 'data/batch.csv'
```

```
batch=t(batch)
batch=unname(batch)
batch=data.frame(cbind(1:8,batch))
names(batch)=c("batch","r1","r2")
batch$r1=as.double(batch$r1)
batch$r2=as.double(batch$r2)
batch
```

```
batch r1 r2
1     1 40 42
2     2 33 34
3     3 46 47
4     4 55 52
5     5 63 59
6     6 35 38
7     7 56 56
8     8 34 29
```

Overall mean: You can just take the sample mean here – the batches have an equal number of samples in each of the batches.

```
mean(c(batch$r1,batch$r2))
```

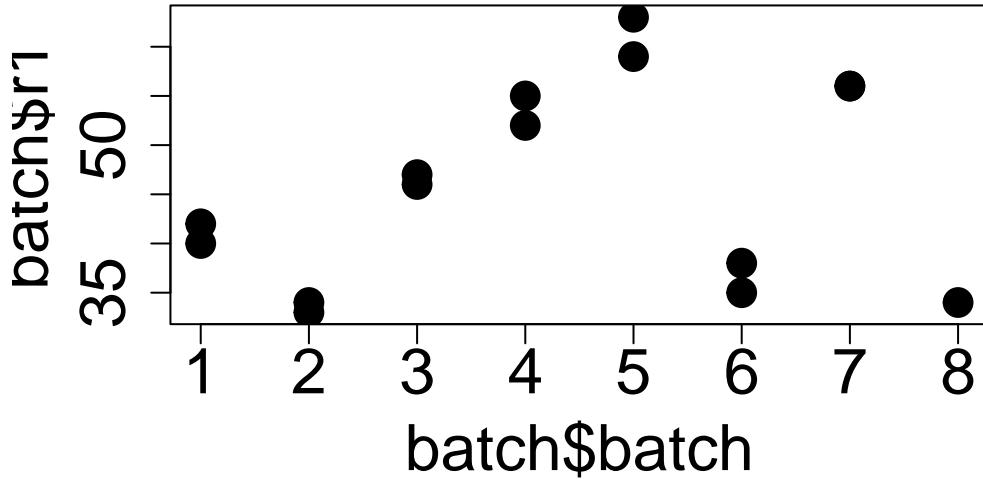
```
[1] 44.9375
```

```
summary(batch)
```

batch	r1	r2
Min. :1.00	Min. :33.00	Min. :29.00
1st Qu.:2.75	1st Qu.:34.75	1st Qu.:37.00
Median :4.50	Median :43.00	Median :44.50
Mean :4.50	Mean :45.25	Mean :44.62
3rd Qu.:6.25	3rd Qu.:55.25	3rd Qu.:53.00
Max. :8.00	Max. :63.00	Max. :59.00

Graphically, the within batch variability is low relative to the between batch.

```
par(cex.lab=2,cex.axis=2,mfrow=c(1,1))
plot(batch$batch,batch$r1,pch=21, bg=1, cex=2)
points(batch$batch,batch$r2,pch=21, bg=1, cex=2)
```



```
cor(batch$r1, batch$r2)
```

```
[1] 0.9672002
```

Okay what about a confidence intervals for the mean? What about whether or not the random batch has a significant effect on the variability of the responses? We need a model for this.

It appears that the within batch mean is not constant.

$$Y_{ij} = \mu + \beta_i + \epsilon_{ij},$$

where for  $i \in [8]$  and  $j \in [2]$ , we have

- $Y_{ij}$ : concentration
- $\mu$ : overall mean
- $\beta_i$ : effect of batch  $i$ , this effect is random!
- $\epsilon_{ij}$  random error

The assumptions are

- $\beta_i \sim N(0, \sigma_b^2)$  iid
- $\epsilon_{ij} \sim N(0, \sigma^2)$  iid
- $\beta_i$  is independent of  $\epsilon_{ij}$

Under these assumptions  $E(Y_{ij}) = \mu$  and  $Var(Y_{ij}) = \sigma^2 + \sigma_b^2$ .

In addition, we see that we capture the dependence structure: One can check that

- $Cov(Y_{i1}, Y_{i2}) = \sigma_b^2$
- $Cov(Y_{i1}, Y_{i'1}) = 0$

Recall we are interested in whether or not the random batch has a significant effect on the variability of the responses. This means we would like to estimate  $\sigma_b$  and test if it is negligible.

Let's estimate the parameters of this model. The relevant R package for (generalised) linear mixed models in R are `nlme`, `lme4` and `lmerTest`. Let's use REML to estimate our parameters.

```
#We need to reshape this data into long format!
batch_long=reshape(batch,
                    varying=c('r1','r2'),
                    timevar = 'replicate',
                    idvar = 'batch',
                    times=c(1,2),
                    direction = "long",sep = "")

head(batch_long)

  batch replicate   r
1.1      1       1 40
2.1      2       1 33
3.1      3       1 46
4.1      4       1 55
5.1      5       1 63
6.1      6       1 35

#using other package
# fit.lme<-lme4::lmer(r ~ 1 | batch, data=batch_long)
# summary(fit.lme)

rownames(batch_long) <- NULL

#defaults to REML
model_1=lme(
  fixed= r~1,
```

```

random= ~ 1 | batch, data=batch_long )
summary(model_1)

Linear mixed-effects model fit by REML
  Data: batch_long
    AIC      BIC      logLik
  101.0371 103.1613 -47.51855

Random effects:
Formula: ~1 | batch
  (Intercept) Residual
StdDev:     10.95445 2.015565

Fixed effects: r ~ 1
Value Std.Error DF t-value p-value
(Intercept) 44.9375 3.905623 8 11.50585      0

Standardized Within-Group Residuals:
Min       Q1       Med       Q3       Max
-1.35131912 -0.56486600  0.09135863  0.51634786  1.12937443

Number of Observations: 16
Number of Groups: 8

```

Okay, between batch variance is huge. Let's test if its non-zero anyways. Recall that for REML estimates, the asymptotic distribution for the LRT is not the same as usual. In this case, under the null hypothesis,  $Y_{ij} \sim N(\mu, \sigma^2)$ .

Therefore, in order to construct a hypothesis test for  $\sigma_\beta^2$ , we can do the following:

1. Compute the LRT statistic from the sample, call it  $\hat{T}$ .
2. Simulate many, say  $n_{sim}$ , new samples of the same size from the model  $Y_{ij} \sim N(\mu, \sigma^2)$ .
3. For each of the  $n_{sim}$  samples, compute the LRT statistic:  $\tilde{T}_1, \dots, \tilde{T}_{n_{sim}}$ .
4. The (empirical) p-value is then the proportion of  $\tilde{T}_1, \dots, \tilde{T}_{n_{sim}}$  larger than  $\hat{T}$ .

Thus,

```

#Step 1. Computing T-hat
fit_null<-lm(r ~ 1 , data=batch_long)

observed=lmtest::lrtest(fit_null,model_1)$Chisq[2]; observed

```

```
Warning in modelUpdate(objects[[i - 1]], objects[[i]]): original model was of  
class "lm", updated model is of class "lme"
```

```
[1] 25.40237
```

```
#Step 2.
```

```
#Get the fixed effects  
fe=nlme::fixed.effects(model_1); fe
```

```
(Intercept)
```

```
44.9375
```

```
#Get the estimated variance of the RE
```

```
sigma_batch_est= nlme::getVarCov(model_1); sigma_batch_est
```

```
Random effects variance covariance matrix
```

```
    (Intercept)
```

```
    (Intercept) 120
```

```
    Standard Deviations: 10.954
```

```
#Get the estimate of sigma
```

```
sigma_est=model_1$sigma; sigma_est
```

```
[1] 2.015565
```

```
n=nrow(batch_long)  
n_sim=100
```

```
#Step 2
```

```
simulated=t(replicate(n_sim,rnorm(n,fe[1],sigma_est)))
```

```
# n_sim x 16
```

```
dim(simulated)
```

```
[1] 100 16
```

```

#Step 3
#takes a simulated sample y and computes the LRT for Y
compute_lrt=function(y){

  #create a copy of the dataset
  batch_copy=batch_long

  #replace response with new sample
  batch_copy$r=y

  #replace response with new sampl
  alt=lme(
    fixed= r~1,
    random= r ~ 1 | batch, data=batch_copy )

  null<-lm(r ~ 1 , data=batch_copy)

  test=lmtest::lrtest(null,alt)$Chisq[2]

  return(test)
}

#compute LRT for each simulated sample
ts=suppressWarnings(apply(simulated, 1, compute_lrt))

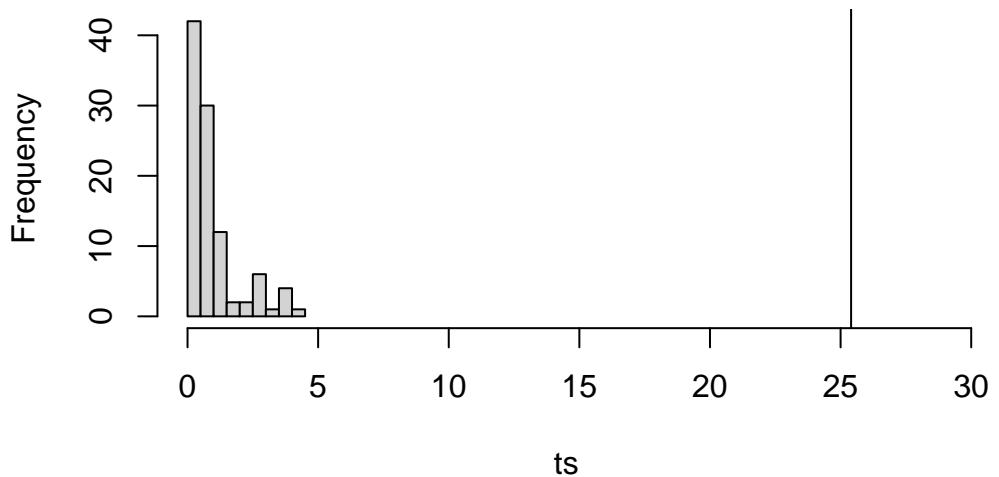
pvalue=mean(observed<=ts); pvalue

[1] 0

hist(ts,xlim=c(min(ts),29))
abline(v=observed)

```

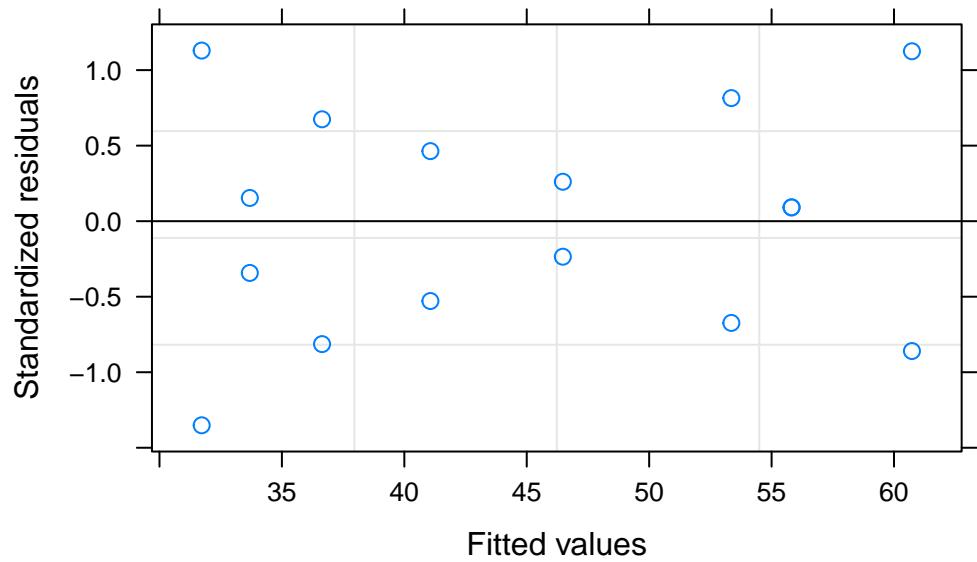
### Histogram of ts



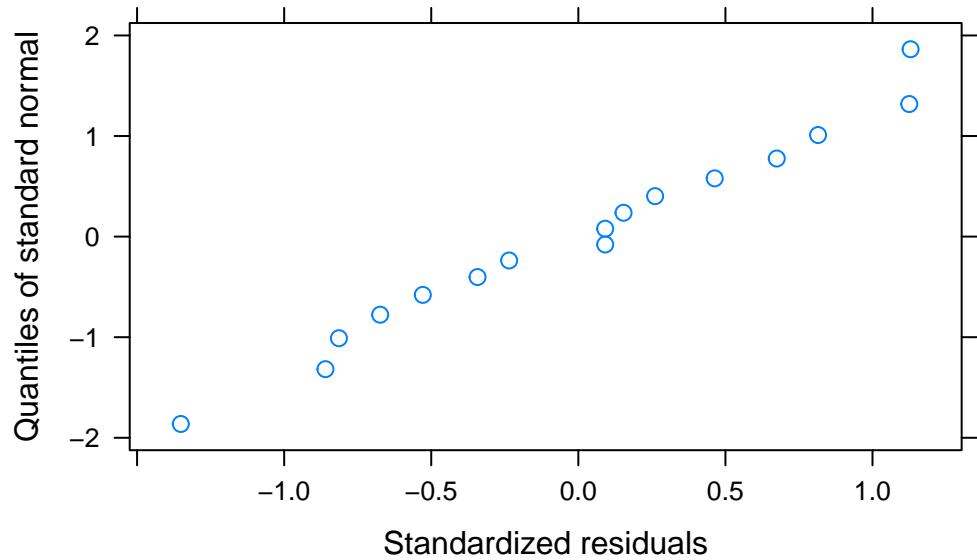
```
observed
```

```
[1] 25.40237
```

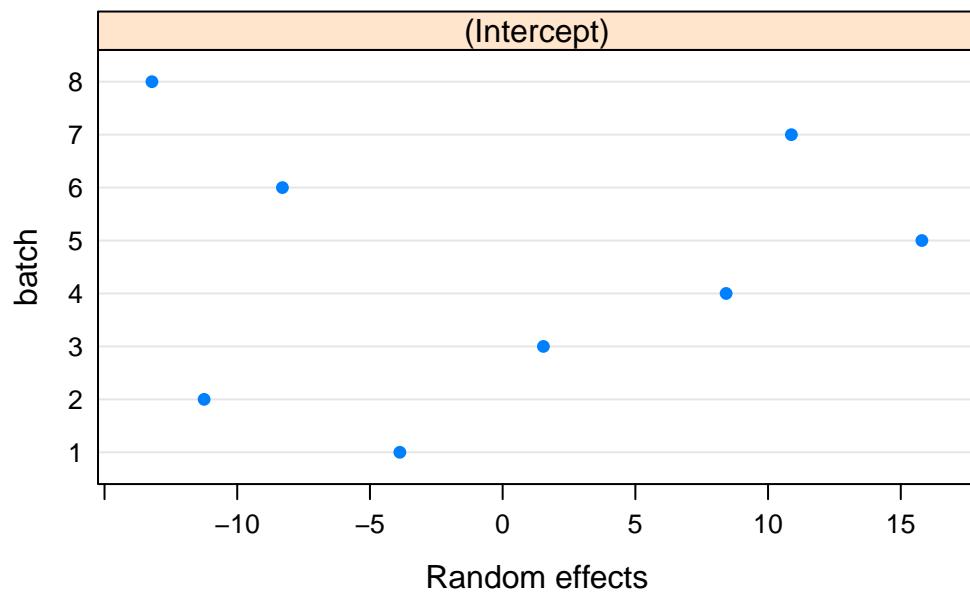
```
plot(model_1)
```



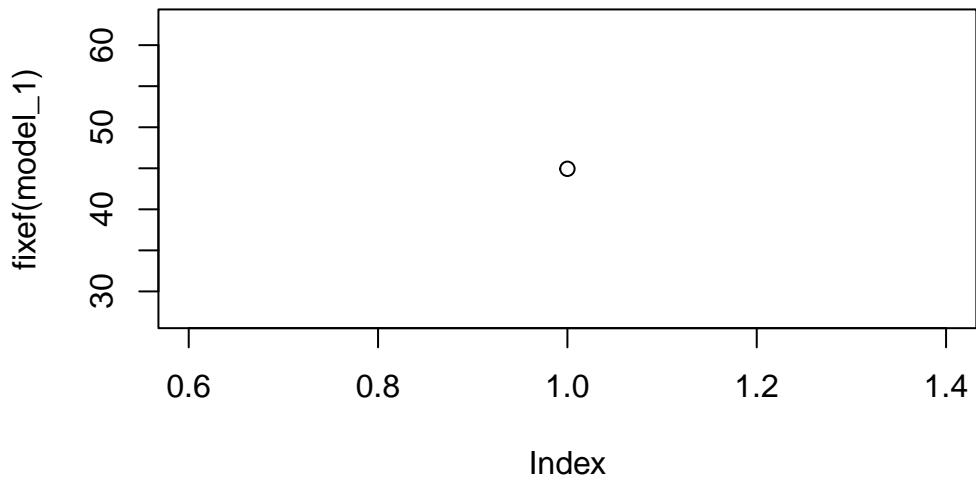
```
qqnorm(model_1, ~ residuals(., type="pearson"))
```



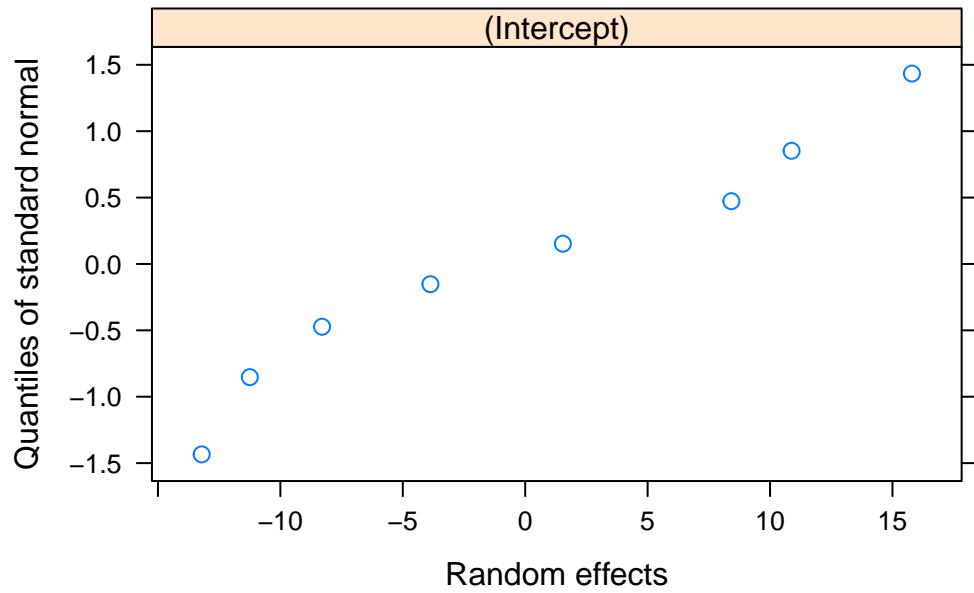
```
plot(ranef(model_1))
```



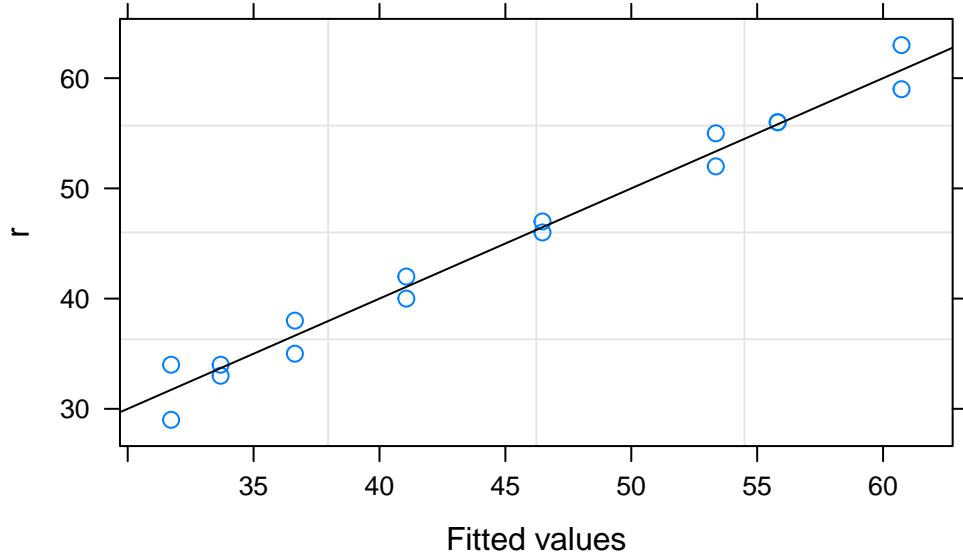
```
plot(fixef(model_1))
```



```
qqnorm(model_1, ~ ranef(.))
```



```
plot(model_1, r ~ fitted(.), abline=c(0,1))
```



```
intervals(model_1)
```

Approximate 95% confidence intervals

Fixed effects:

	lower	est.	upper
(Intercept)	35.93112	44.9375	53.94388

Random Effects:

Level: batch

	lower	est.	upper
sd((Intercept))	6.430117	10.95445	18.66216

Within-group standard error:

	lower	est.	upper
	1.234790	2.015565	3.290036

Results summary:

- Quick summary: the mean is estimated to be 45, and we saw significant variation between batches. The batch mean concentration has standard deviation 11.
- More on the mean: The mean concentration is estimated to be 45, at least, we estimate that the mean concentration is not below 36 and does not exceed 54.
- Batch variability: The concentration varies significantly between batches. The standard deviation of the mean batch concentration is estimated to be 12, ranging from (6,19). This means we estimate that roughly 68% of the batches have a mean concentration within 11 units of the overall mean (estimated to be 45) and 95% are within 22 units of the overall mean.

## 1.4 Case study: Air Pollution

```
library(nlme)
library(lme4)
```

### 1.4.1 Case information:

- Six Cities Air Pollution Data – Data on lung growth along with assorted patient information.
- How much of lung size do age and height explain?

Data Columns:

- id: Patient ID
- ht: Patient height at the corresponding visit
- age: Patient age
- baseht: Patient height at the first visit
- baseage: Patient age at the first visit
- logfev1: The log of FEV1 measurement (outcome based on lung function)

Data Info:

- <https://content.sph.harvard.edu/fitzmaur/ala2e/>
- Applied LDA: Garrett Fitzmaurice, Nan Laird & James Ware
- Dockery, D.W., Berkey, C.S., Ware, J.H., Speizer, F.E. and Ferris, B.G. (1983). Distribution of FVC and FEV1 in children 6 to 11 years old. American Review of Respiratory Disease, 128, 405-412.

### 1.4.2 EDA:

```
air_pollution <- read.csv("data/air_pollution.csv")
```

Let's explore the data.

```
par(cex.lab=2,cex.axis=2,mfrow=c(1,1))
```

```
head(air_pollution)
```

```
  id      ht      age baseht baseage logfev1
1 1 1.20  9.3415     1.2  9.3415 0.21511
2 1 1.28 10.3929     1.2  9.3415 0.37156
3 1 1.33 11.4524     1.2  9.3415 0.48858
4 1 1.42 12.4600     1.2  9.3415 0.75142
5 1 1.48 13.4182     1.2  9.3415 0.83291
6 1 1.50 15.4743     1.2  9.3415 0.89200
```

```
summary(air_pollution)
```

```
      id          ht          age        baseht
Min.   : 1.0   Min.   :1.110   Min.   : 6.434   Min.   :1.110
1st Qu.: 69.0  1st Qu.:1.370   1st Qu.: 9.719   1st Qu.:1.220
Median :129.0  Median :1.540   Median :12.597   Median :1.260
Mean   :135.7  Mean   :1.498   Mean   :12.568   Mean   :1.276
3rd Qu.:199.0  3rd Qu.:1.620   3rd Qu.:15.368   3rd Qu.:1.320
Max.   :300.0   Max.   :1.790   Max.   :18.691   Max.   :1.720
      baseage      logfev1
Min.   : 6.434   Min.   :-0.04082
1st Qu.: 7.135   1st Qu.: 0.54812
Median : 7.781   Median : 0.86710
Mean   : 8.030   Mean   : 0.81600
3rd Qu.: 8.449   3rd Qu.: 1.09861
Max.   :14.067   Max.   : 1.59534
```

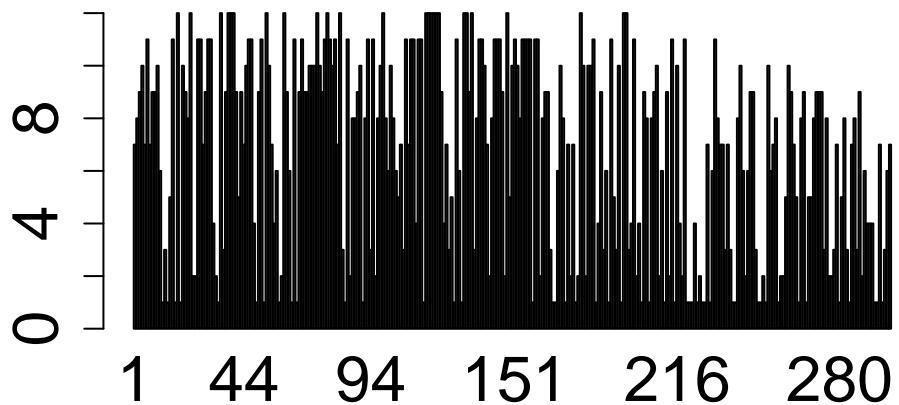
```
#Check for missing values
colSums(is.na(air_pollution))
```

```
  id      ht      age baseht baseage logfev1
0       0       0       0       0       0       0
```

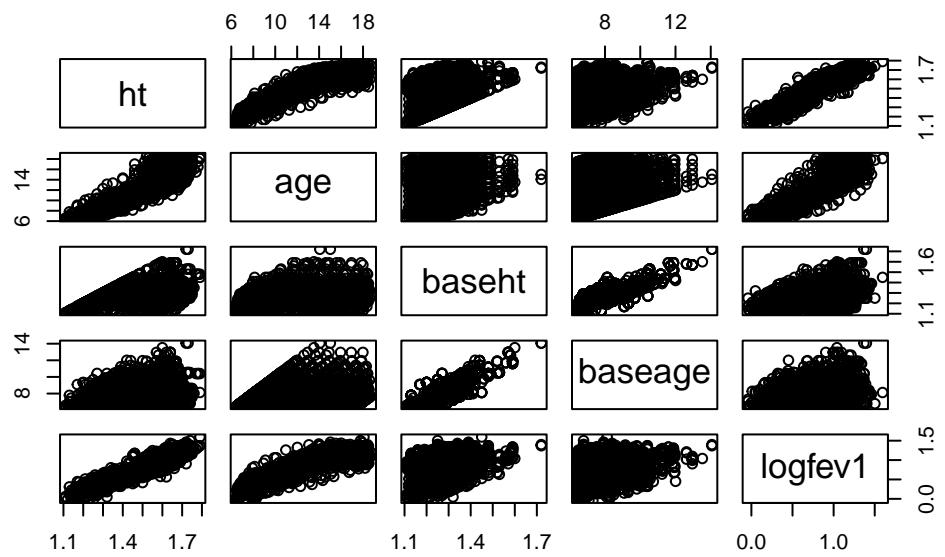
```
unique(air_pollution$baseht)
```

```
[1] 1.20 1.13 1.18 1.15 1.11 1.24 1.27 1.17 1.32 1.26 1.25 1.19 1.21 1.23 1.22  
[16] 1.30 1.37 1.41 1.14 1.29 1.31 1.28 1.36 1.33 1.38 1.12 1.35 1.34 1.45 1.39  
[31] 1.16 1.58 1.60 1.40 1.42 1.72 1.46 1.48 1.52 1.49 1.56 1.53 1.43 1.44 1.59  
[46] 1.57
```

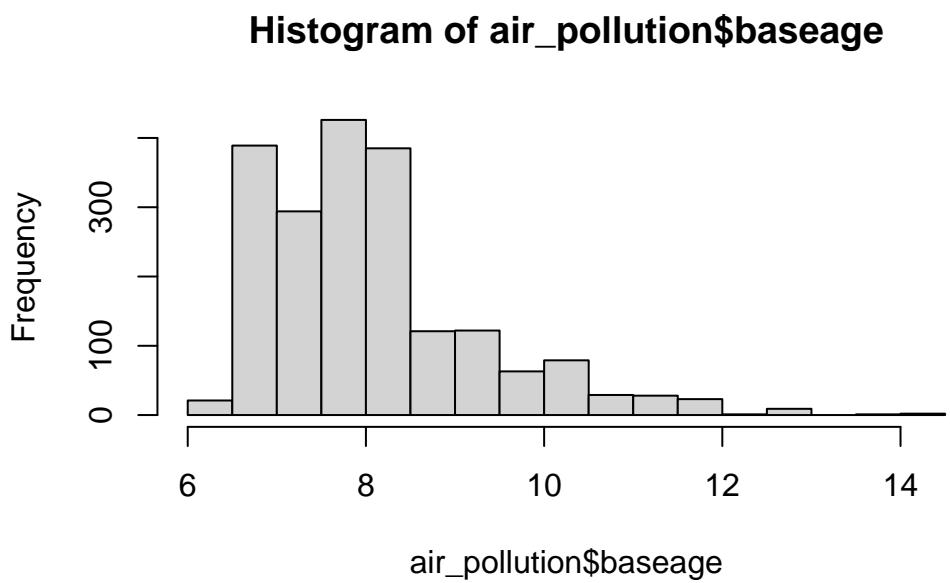
```
# Is it balanced?  
n=length(unique(air_pollution$id))  
# typeof(air_pollution$id)  
barplot(table(as.factor(air_pollution$id)))
```



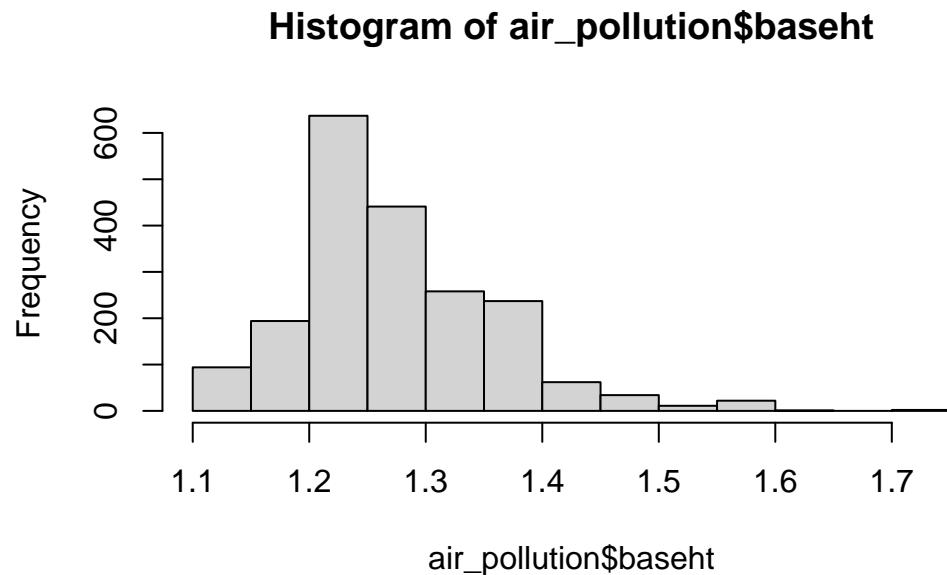
```
plot(air_pollution[,-1])
```



```
hist(air_pollution$baseage)
```

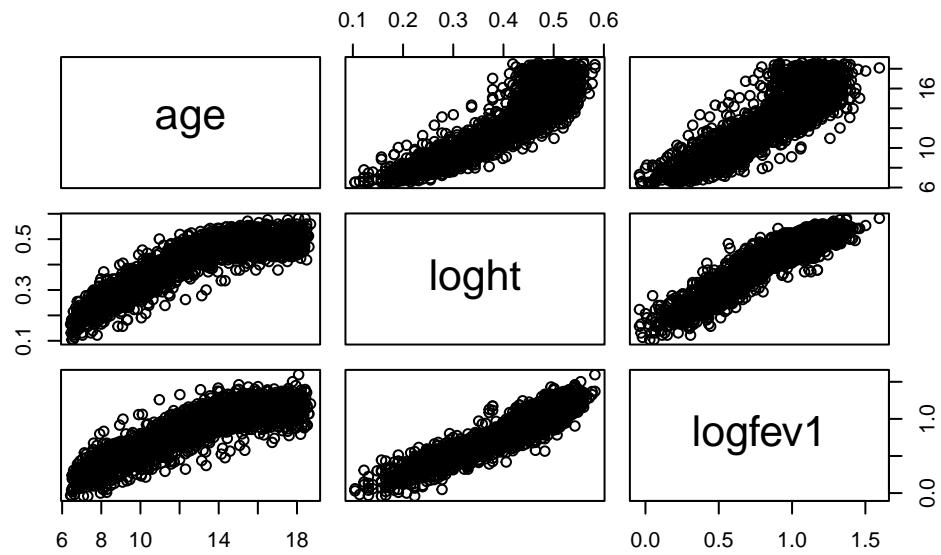


```
hist(air_pollution$baseht)
```

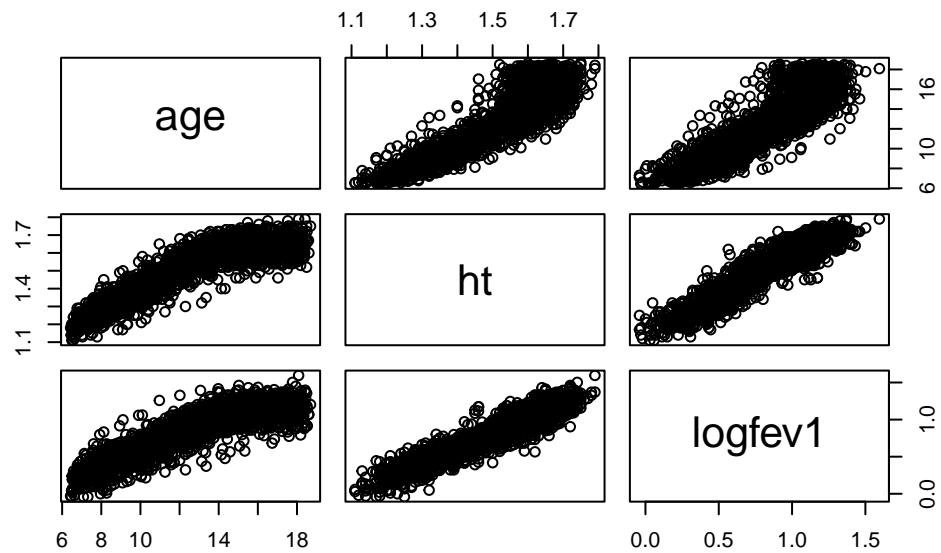


```
# hist(air_pollution$age)
# hist(air_pollution$ht)

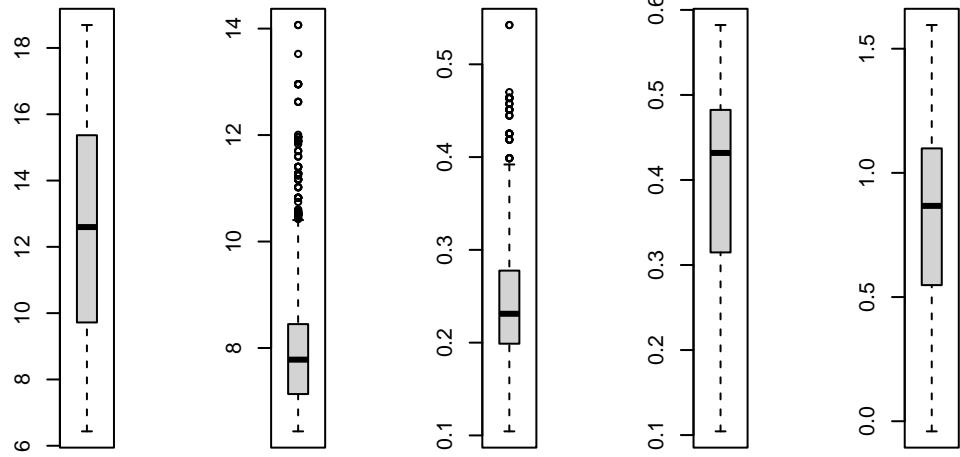
#Hint: height has been shown to be linearly associated with logfev1 on log scale
air_pollution$loght=log(air_pollution$ht)
air_pollution$logbht=log(air_pollution$baseht)
plot(air_pollution[,c('age','loght','logfev1')])
```



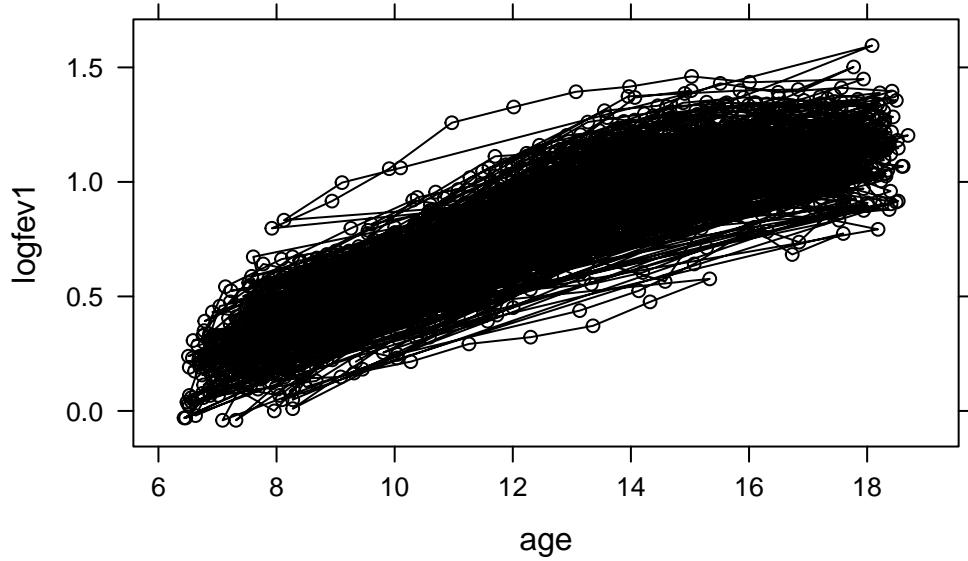
```
plot(air_pollution[,c('age','ht','logfev1')])
```



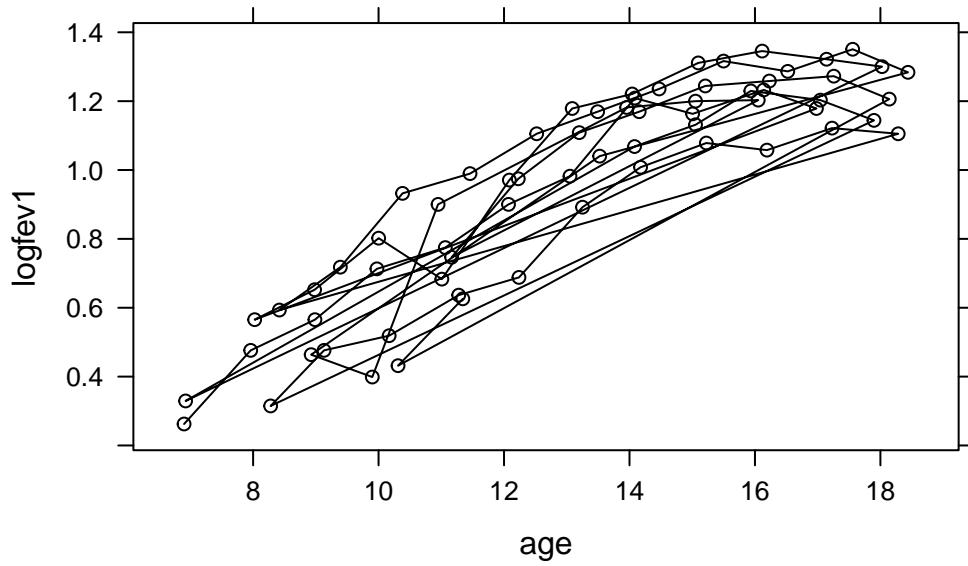
```
par(mfrow=c(1,5))
bx=apply(air_pollution[,c('age','baseage','logbht','loght','logfev1')],2,boxplot)
```



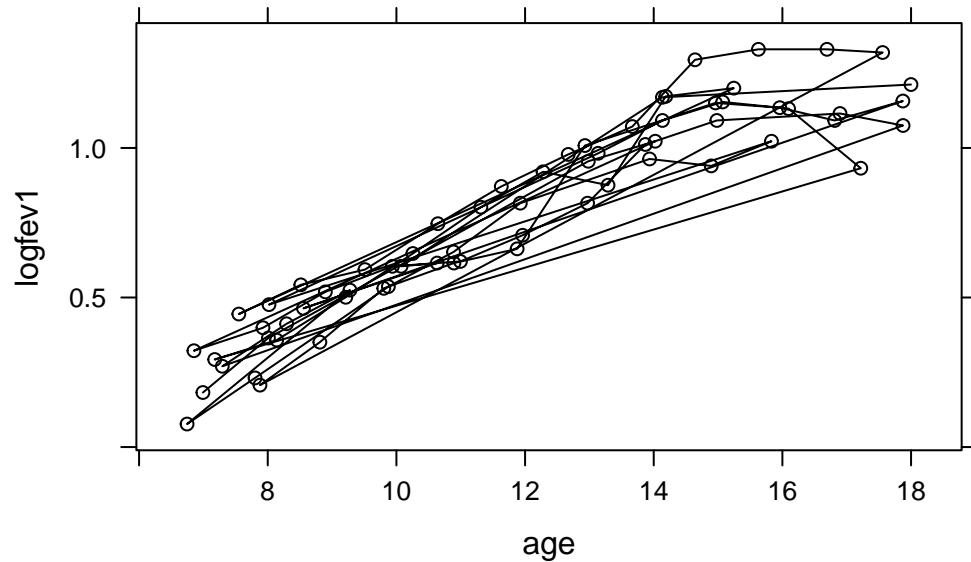
```
lattice::xyplot(logfev1~age, air_pollution, col = 'black', type = c('l', 'p'))
```



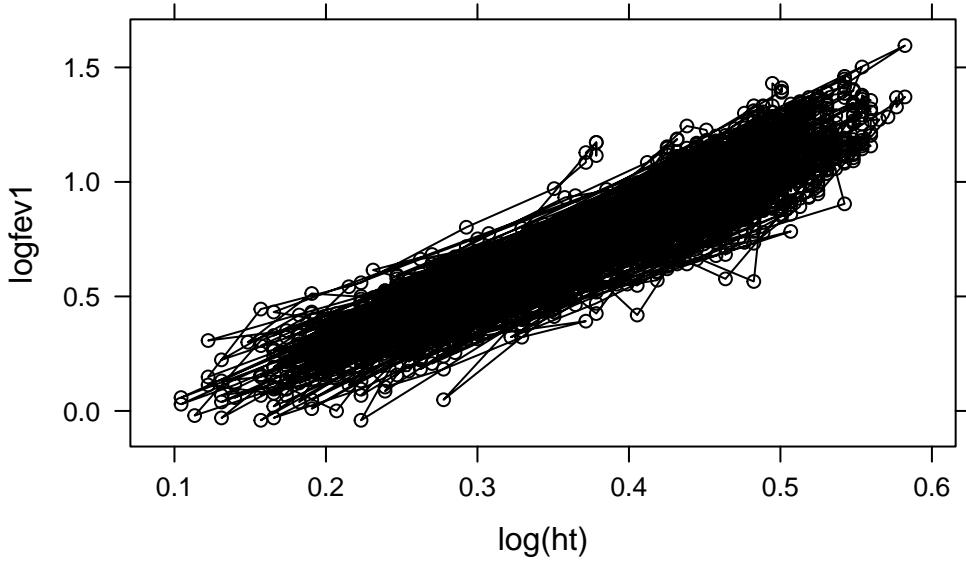
```
lattice::xyplot(logfev1~age, air_pollution[ air_pollution$id %in% sample(1:n,10),], col =
```



```
lattice::xyplot(logfev1~age, air_pollution[ air_pollution$id %in% sample(1:n,10),], col =
```



```
lattice::xyplot(logfev1~log(ht), air_pollution, col = 'black', type = c('l', 'p'))
```



What can we conclude from this exploratory analysis?

- 300 participants
- No missing values
- Not balanced
- Starting age ranges widely
- Somewhat linear relationships with the response, but not quite.
- Log scale might be better for height
- Heavy tail in base age, and base height.
- Parallel regression line between response and age

### 1.4.3 Specifying

- Let  $i$  index the individuals and  $j$  index the  $j$ th age recorded for a given individual.

$$Y_{ij} = X_{ij}^\top \alpha + Z_{ij}^\top \beta_i + \epsilon_{ij},$$

where for  $i \in [299]$  and  $j \in [J_i]$ .

Here,

- $Z_{ij}^\top = (1, age)$ ,  $X_{ij}^\top = (1, age, loght...)$
- Each observation has a random intercept and slope, which depends on their age.

#### 1.4.4 Estimation

Let's estimate the parameters of this model. Let's use REML to estimate our parameters.

```
#defaults to REML
model <- nlme::lme(
  fixed = logfev1 ~ age + log(ht) + baseage + log(baseht) ,
  random =~ age|id,
  correlation = NULL, # Defaults to sigma^2 I
  method = 'REML',
  data = air_pollution
)
summary(model)
```

```
Linear mixed-effects model fit by REML
Data: air_pollution
      AIC      BIC    logLik
-4549.882 -4499.528 2283.941

Random effects:
Formula: ~age | id
Structure: General positive-definite, Log-Cholesky parametrization
          StdDev     Corr
(Intercept) 0.110485541 (Intr)
age         0.007078381 -0.553
Residual    0.060237881

Fixed effects: logfev1 ~ age + log(ht) + baseage + log(baseht)
                Value Std.Error DF t-value p-value
(Intercept) -0.2883233 0.03871675 1692 -7.44699 0.0000
age          0.0235286 0.00139534 1692 16.86231 0.0000
log(ht)       2.2371984 0.04353724 1692 51.38585 0.0000
baseage      -0.0165088 0.00745785  296 -2.21362 0.0276
log(baseht)   0.2182148 0.14552087  296  1.49954 0.1348

Correlation:
          (Intr) age   lg(ht) baseag
age        0.023
log(ht)    -0.077 -0.875
baseage    -0.822 -0.184  0.180
log(baseht) 0.370  0.239 -0.275 -0.815

Standardized Within-Group Residuals:
```

Min	Q1	Med	Q3	Max
-6.45672792	-0.52534885	0.05351814	0.60114614	2.76671671

Number of Observations: 1993  
 Number of Groups: 299

### 1.4.5 Testing

Do we need the baseline fixed effects? Did the height and age they entered the study at effect the outcome?

```
library(nlme)
#marginal tests:
summary(model)
```

```
Linear mixed-effects model fit by REML
  Data: air_pollution
        AIC      BIC    logLik
 -4549.882 -4499.528 2283.941

Random effects:
Formula: ~age | id
Structure: General positive-definite, Log-Cholesky parametrization
          StdDev     Corr
(Intercept) 0.110485541 (Intr)
age         0.007078381 -0.553
Residual    0.060237881

Fixed effects: logfev1 ~ age + log(ht) + baseage + log(baseht)
                Value Std.Error DF t-value p-value
(Intercept) -0.2883233 0.03871675 1692 -7.44699 0.0000
age          0.0235286 0.00139534 1692 16.86231 0.0000
log(ht)       2.2371984 0.04353724 1692 51.38585 0.0000
baseage      -0.0165088 0.00745785  296 -2.21362 0.0276
log(baseht)   0.2182148 0.14552087  296  1.49954 0.1348

Correlation:
          (Intr) age    lg(ht) baseag
age        0.023
log(ht)    -0.077 -0.875
baseage    -0.822 -0.184  0.180
log(baseht) 0.370  0.239 -0.275 -0.815
```

```
Standardized Within-Group Residuals:  
      Min       Q1       Med       Q3       Max  
-6.45672792 -0.52534885  0.05351814  0.60114614  2.76671671
```

```
Number of Observations: 1993  
Number of Groups: 299
```

```
#sequential tests:  
anova(model)
```

	numDF	denDF	F-value	p-value
(Intercept)	1	1692	11406.811	<.0001
age	1	1692	16605.209	<.0001
log(ht)	1	1692	2905.336	<.0001
baseage	1	296	2.929	0.0881
log(baseht)	1	296	2.249	0.1348

```
#based on the above, lets remove the base effects  
model=update(model,fixed= logfev1 ~ age+log(ht))  
summary(model)
```

```
Linear mixed-effects model fit by REML
```

```
Data: air_pollution  
      AIC      BIC      logLik  
-4559.789 -4520.618 2286.895
```

```
Random effects:
```

```
Formula: ~age | id  
Structure: General positive-definite, Log-Cholesky parametrization  
          StdDev     Corr  
(Intercept) 0.11107952 (Intr)  
age         0.00706045 -0.552  
Residual    0.06025488
```

```
Fixed effects: logfev1 ~ age + log(ht)
```

	Value	Std.Error	DF	t-value	p-value
(Intercept)	-0.3693653	0.00916684	1692	-40.29366	0
age	0.0230800	0.00135469	1692	17.03715	0
log(ht)	2.2493934	0.04176179	1692	53.86247	0

```

Correlation:
  (Intr) age
age    -0.208
log(ht) -0.190 -0.869

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-6.44851637 -0.51754969  0.05387027  0.59382767  2.78517863

Number of Observations: 1993
Number of Groups: 299

```

Can the random slope be dropped?

```

model_null=update(model,random=logfev1 ~ 1|id)
summary(model_null)

```

```

Linear mixed-effects model fit by REML
Data: air_pollution
      AIC      BIC   logLik
-4489.806 -4461.827 2249.903

Random effects:
Formula: logfev1 ~ 1 | id
  (Intercept) Residual
StdDev:  0.09608834 0.06429074

Fixed effects: logfev1 ~ age + log(ht)
      Value Std.Error DF t-value p-value
(Intercept) -0.3645554 0.00833614 1692 -43.73192     0
age          0.0237689 0.00126198 1692  18.83462     0
log(ht)      2.2162876 0.04195216 1692  52.82893     0

Correlation:
  (Intr) age
age    -0.048
log(ht) -0.218 -0.928

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-6.00485051 -0.54072380  0.06768753  0.61714951  2.86626992

Number of Observations: 1993

```

```
Number of Groups: 299
```

```
LRT=anova(model_null,model)$L[2]; LRT  
  
[1] 73.98299  
  
simmed=nlme::simulate.lme(model_null,nsim=100,m2=model)  
sim_LRT=-2*(simmed>null$REML[,2]-simmed$alt$REML[,2])  
  
pval=mean(sim_LRT>=LRT); print(pval)
```

```
[1] 0
```

```
hist(sim_LRT)  
abline(v=LRT)
```



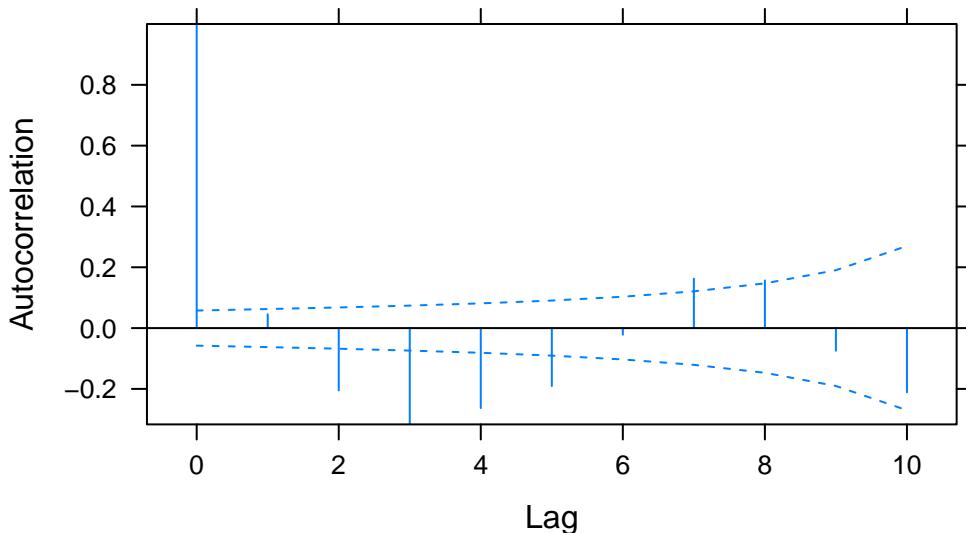
```
anova(model_null,model)
```

	Model	df	AIC	BIC	logLik	Test	L.Ratio	p-value
model_null	1	5	-4489.806	-4461.827	2249.903			
model	2	7	-4559.789	-4520.618	2286.894	1 vs 2	73.98299	<.0001

#### 1.4.6 Diagnostics

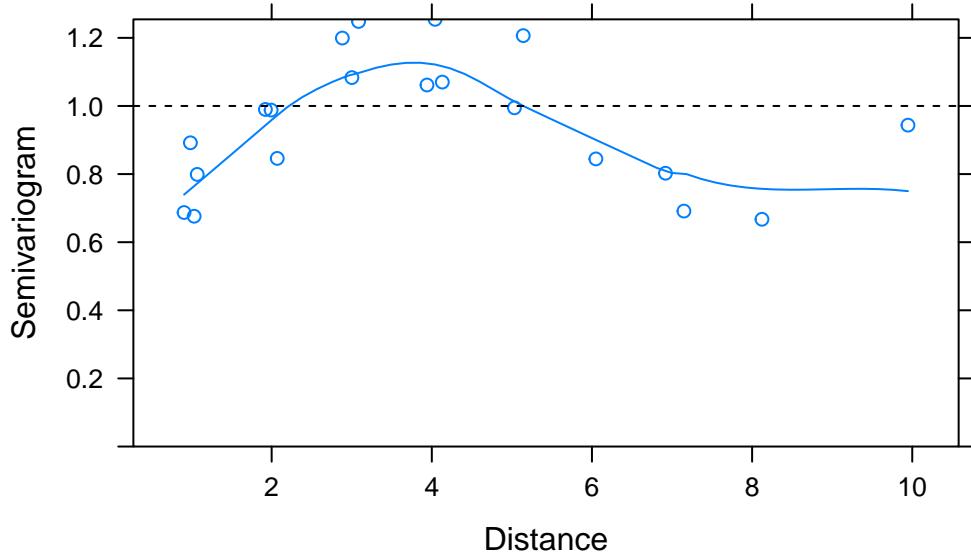
```
## ACF - Checks that errors are independent
plot(ACF(model), alpha = 0.01, main = "ACF plot for independent errors.") # This looks pro
```

**ACF plot for independent errors.**



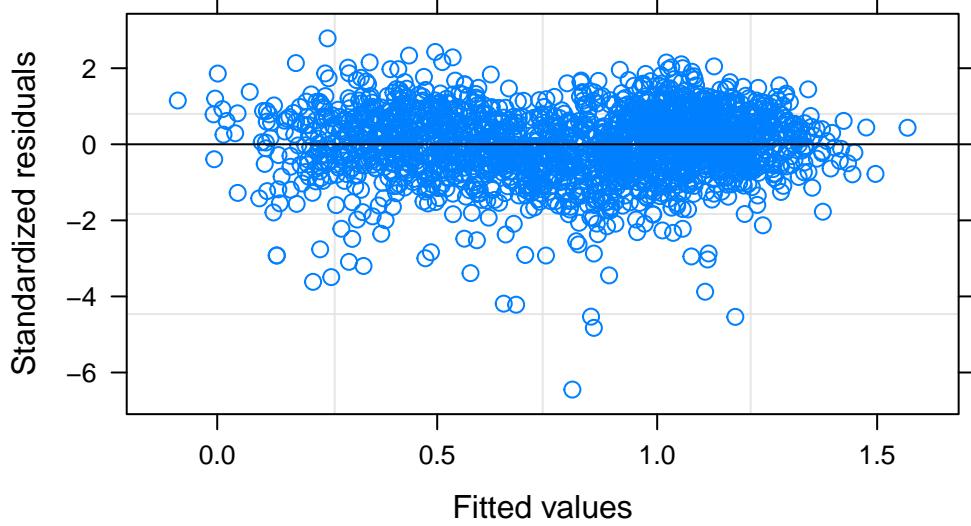
```
# This may not be the best way to check the model fit though, since there are not evenly s

# Instead we can use a 'semi-Variogram'.
# This should fluctuate randomly around 1
vg <- Variogram(model, form = ~age|id, resType = "pearson")
plot(vg, sigma=1) ## Looks okay, honestly, not the best.
```

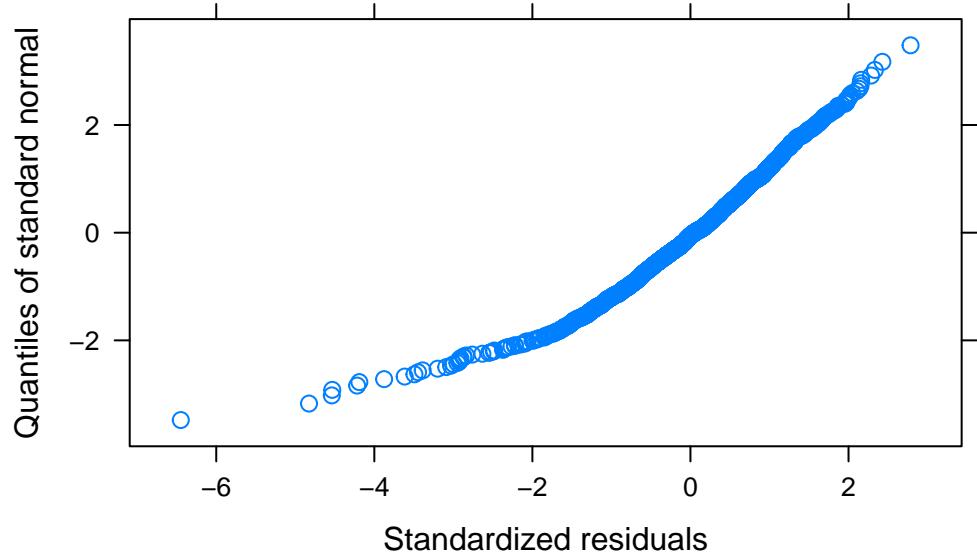


```
# Residuals vs. Fitted (no patterns)
plot(model, main = "Plot of residuals vs. fitted.")
```

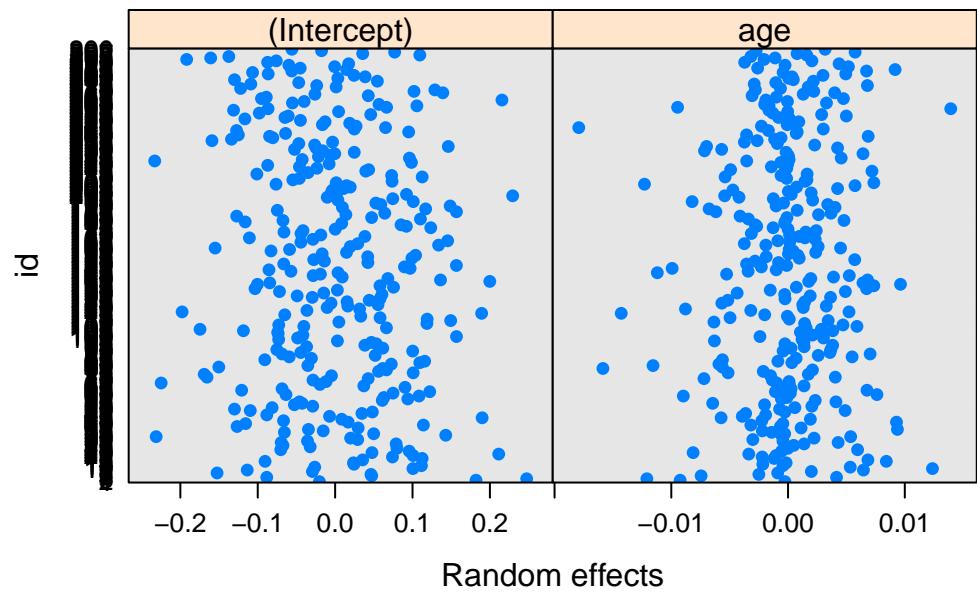
### Plot of residuals vs. fitted.



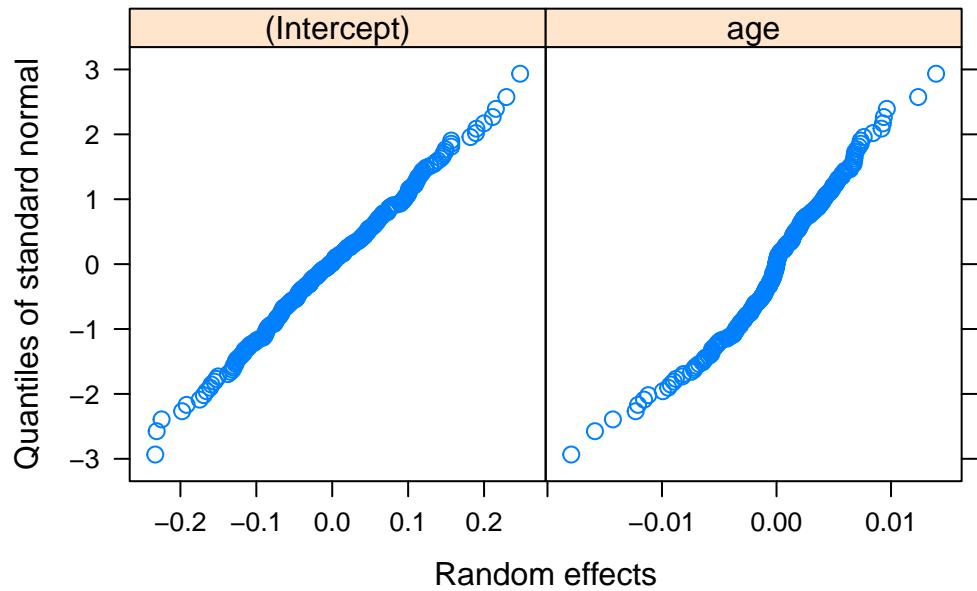
```
# QQPlot for normality of errors  
qqnorm(model, ~ residuals(., type="pearson")) # Some issues... probably
```



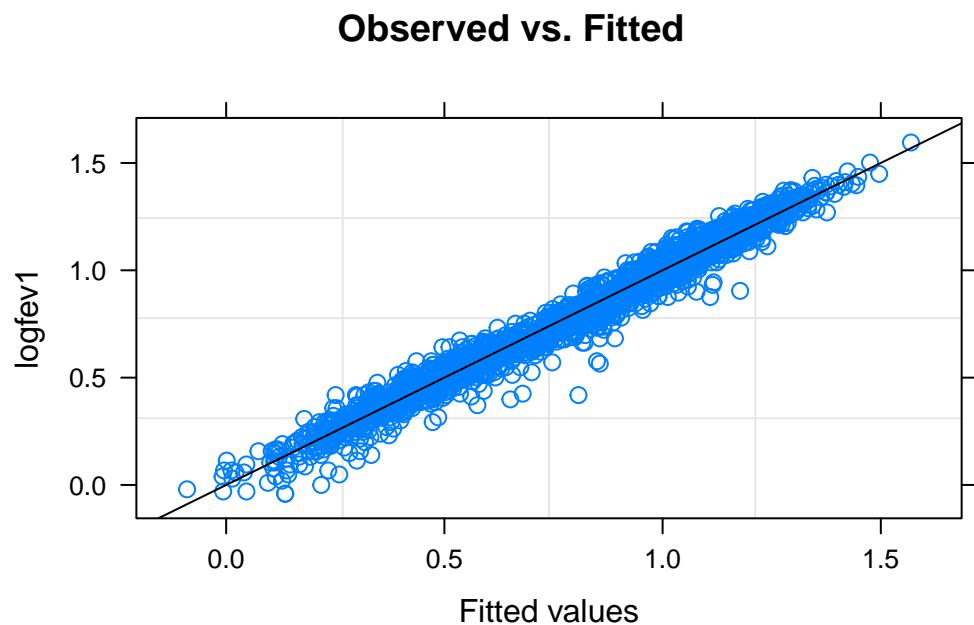
```
# Plots for the Predicted (BLUPs)  
plot(ranef(model))
```



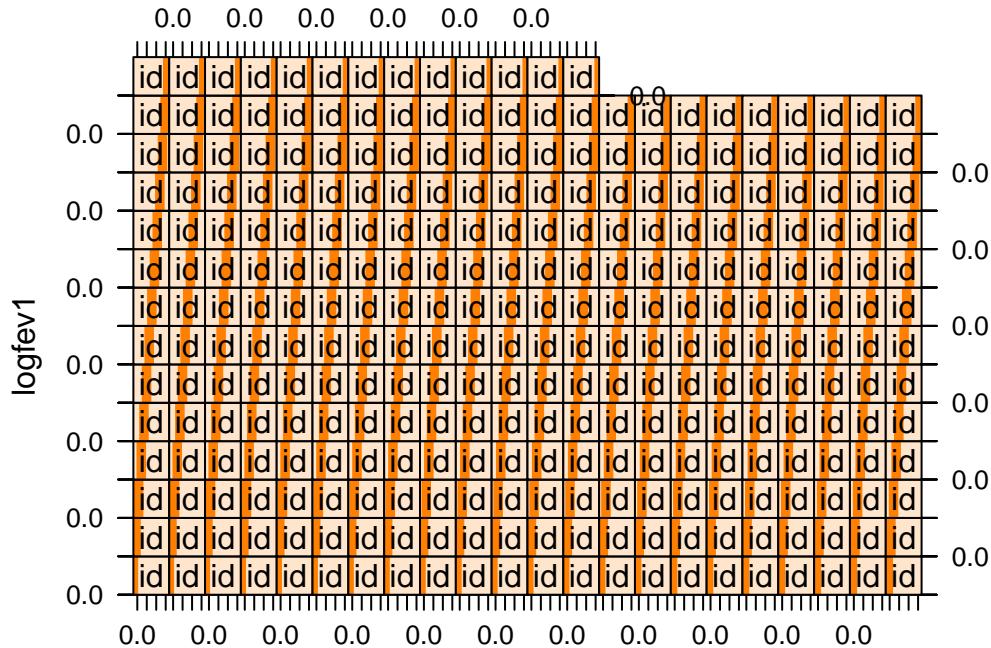
```
qqnorm(model, ~ranef(.)) # These look okay!
```



```
# Observed vs. Fitted  
plot(model, logfev1 ~ fitted(.), abline = c(0,1), main = "Observed vs. Fitted")
```



```
plot(model, logfev1 ~ fitted(.)|id, abline = c(0,1), main = "Observed vs. Fitted (By Subject)
```



```
# Could also look (e.g.) by treatment, if it existed!
```

```
### Intervals
intervals(model)
```

Approximate 95% confidence intervals

Fixed effects:

	lower	est.	upper
(Intercept)	-0.3873448	-0.36936532	-0.35138579
age	0.0204230	0.02308004	0.02573708
log(ht)	2.1674832	2.24939340	2.33130360

Random Effects:

Level: id

	lower	est.	upper
sd((Intercept))	0.095315000	0.11107952	0.129451390
sd(age)	0.005828914	0.00706045	0.008552184
cor((Intercept),age)	-0.685485568	-0.55220425	-0.383114275

Within-group standard error:

lower	est.	upper

```
0.05812314 0.06025488 0.06246480
```

```
intervals(model)
```

```
Approximate 95% confidence intervals
```

```
Fixed effects:
```

	lower	est.	upper
(Intercept)	-0.3873448	-0.36936532	-0.35138579
age	0.0204230	0.02308004	0.02573708
log(ht)	2.1674832	2.24939340	2.33130360

```
Random Effects:
```

```
Level: id
```

	lower	est.	upper
sd((Intercept))	0.095315000	0.11107952	0.129451390
sd(age)	0.005828914	0.00706045	0.008552184
cor((Intercept),age)	-0.685485568	-0.55220425	-0.383114275

```
Within-group standard error:
```

	lower	est.	upper
	0.05812314	0.06025488	0.06246480

```
### Predictions
```

```
new_data <- data.frame(id = c(1, 25, 25, 25),
                        age = c(18, 18, 18, 18),
                        ht = c(1.54, 1.85, 1.7, 2),
                        baseht = c(1.2, 1.32, 1.32, 1.32),
                        baseage = c(9.3415, 8.0274, 8.0274, 8.0274),
                        loght=log(c(1.54, 1.85, 1.7, 2)))
```

```
# level specifies whether at the population [0] or subject [1] level
predict(model, newdata = new_data, level = c(0,1))
```

	id	predict.fixed	predict.id
1	1	1.017324	0.9922821
2	25	1.429870	1.3972195
3	25	1.239667	1.2070167
4	25	1.605236	1.5725857

### 1.4.7 Results summary:

- Age explains a significant amount of the variability of lung size - a one year increase in age is roughly equivalent to a  $\exp(0.023)$  in lung size (fev1)
- Height also explains lung size, we see that for every 10 cm, we have that lungs are  $\exp(2.25 \cdot \log(0.1))$  (fev1) bigger
- Population average lung size is  $\exp(-0.308090040)$
- There is some evidence that the time at which the subject entered the study was predictive of their lung size. Investigate!
- Seems like lungs stop growing at 16 - may be no need to study after that age?

## 1.5 Case study: Tale of two thieves, see Cabrera and McDougall (2002)

```
library(nlme)
library(lme4)
library(ggplot2)
```

Warning: package 'ggplot2' was built under R version 4.2.3

### 1.5.1 Case information:

- Concentration Data – Data on concentration of active ingredient in tablets and samples from blender
- Recall: “Our main concern is that the amount of active ingredient is consistent in the X tablets. We would like you to analyse both samples to determine how much the active ingredient differs from tablet to tablet. We also want to compare the quality of the samples retrieved by the thieves to determine Which one is better?”

Suppose we receive the following documentation:

Prescription and over-the-counter drugs contain a mixture of both active and inactive ingredients, with the dosage determined by the amount of active ingredient in each tablet. Making sure the tablets contain the correct dosage is an important problem in the drug manufacturing industry and in this case study, we consider an experiment conducted by a pharmaceutical company to investigate sampling variability and bias associated with the manufacture of a certain type of tablet.

### **1.5.2 Outline of the Problem**

**Tablet Manufacture** The tablets were manufactured by mixing the active and inactive ingredients in a “V-blender,” so-named because it looks like a large V. (See Figure 8.1.) Mixing was achieved by rotating the V-blender in the vertical direction. After the mixture was thoroughly blended, the powder was discharged from the bottom of the V-blender and compressed into tablet form.

**Uniform Content:** The most important requirement of this manufacturing process was that the tablets have uniform content. That is, the correct amount of active ingredient must be present in each tablet. The content uniformity of the mixture within the V-blender will need to be assessed. **Thief Sampling** A “thief” instrument was used to obtain samples from different locations within the V-blender. This was essentially a long pole with a closed scoop at one end, which was plunged into the powder mixture by a mechanical device. At the appropriate depth for a given location, the scoop was opened and a sample collected. Considerable force was needed to insert a thief into the powder mixture and it was of interest to compare two types of thieves.

- The Unit Dose thief collects three individual unit dose samples at each location.
- The Intermediate Dose thief collects one large sample which is itself sampled to give three unit dose samples.

### **1.5.3 Experiment Procedure**

The objective of this experiment was to study bias and variability differences between the two thieves and to compare the thief-sampled results with those of the tablets. The experiment was implemented as follows.

1. Blend the mixture in the V-blender for 20 minutes.
2. Tie the thieves together and use them to obtain samples from six locations within the V-blender. A schematic of the V-blender and sampling locations was shown previously.
3. Discharge the powder from the V-blender and compress it to form tablets. Load tablets into 30 drums.
4. Select 10 drums and sample three tablets from each of these drums.
5. Assay all samples to determine the amount of active ingredient in each sample. The specified assay value is: 35 mg/100 mg.

The locations shown in the blender represented the “desired” sampling positions for the thieves. In the actual experiment, these “fixed” positions were subject to a certain amount of variability. The samples collected by the thieves can be regarded as random within each location.

In the Tablet experiment, the order in which the drums were filled was recorded and this information was incorporated into the random selection procedure. Specifically, one drum was

randomly selected from each triple sequence: {1, 2, 3} {4, 5, 6} . . . {28, 29, 30}. The factor DRUM could therefore be used to test for a “time” effect in the Tablet data.

Data Columns:

- method
- location
- replicate
- assay/yb
- drum

Data Info: see 8.1 in Cabrera and McDougall (2002)

```
thief=read.csv('data/thief.csv')
tablet=read.csv('data/tablet.csv')
```

#### 1.5.4 EDA

Let's explore the data

```
par(cex.lab=2,cex.axis=2,mfrow=c(1,1))

head(tablet)

methdb   drum  tablet    yb
1 Tablet     1      1 35.77
2 Tablet     1      2 39.44
3 Tablet     1      3 36.43
4 Tablet     5      1 35.71
5 Tablet     5      2 37.08
6 Tablet     5      3 36.54

summary(tablet)

  methdb          drum          tablet         yb
Length:30      Min.   : 1.0   Min.   :1   Min.   :33.09
Class :character 1st Qu.: 7.0   1st Qu.:1   1st Qu.:35.10
Mode  :character Median :15.5   Median :2   Median :35.69
                  Mean   :14.9   Mean   :2   Mean   :35.79
                  3rd Qu.:22.0   3rd Qu.:3   3rd Qu.:36.52
                  Max.   :28.0   Max.   :3   Max.   :39.44
```

```
head(thief)
```

	METHOD	LOCATION	REPLICATE	ASSAY
1	Intm	1	1	34.38
2	Intm	1	2	34.87
3	Intm	1	3	35.71
4	Intm	2	1	35.31
5	Intm	2	2	37.59
6	Intm	2	3	38.02

```
summary(thief)
```

	METHOD	LOCATION	REPLICATE	ASSAY
Length:	36	Min. :1.0	Min. :1	Min. :32.77
Class :	character	1st Qu.:2.0	1st Qu.:1	1st Qu.:35.39
Mode :	character	Median :3.5	Median :2	Median :36.67
		Mean :3.5	Mean :2	Mean :36.65
		3rd Qu.:5.0	3rd Qu.:3	3rd Qu.:37.88
		Max. :6.0	Max. :3	Max. :39.80

```
#Check for missing values  
colSums(is.na(thief))
```

METHOD	LOCATION	REPLICATE	ASSAY
0	0	0	0

```
colSums(is.na(tablet))
```

methdb	drum	tablet	yb
0	0	0	0

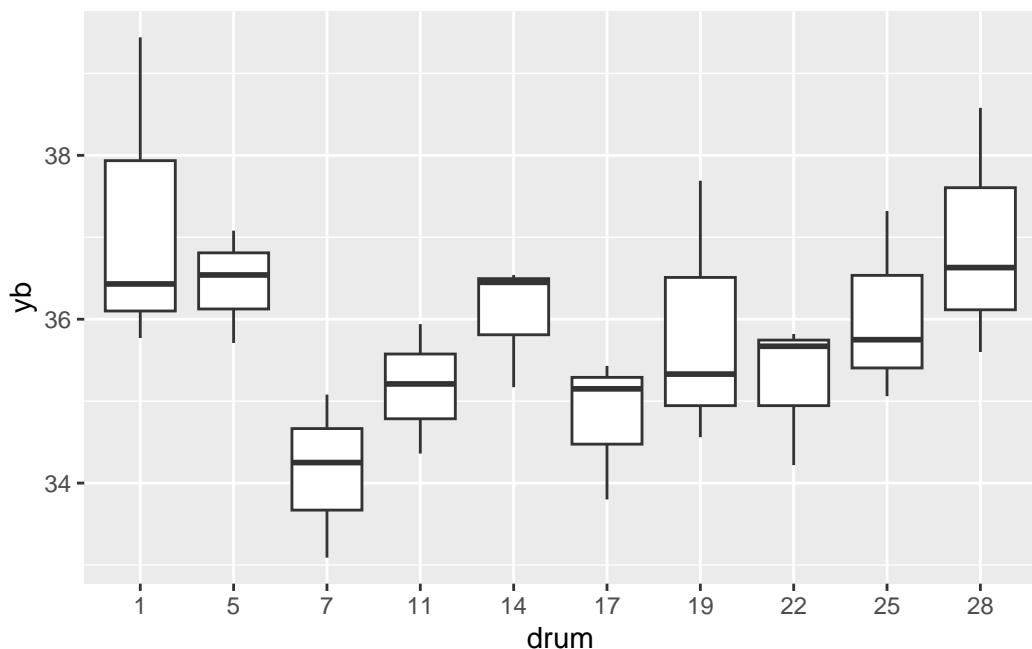
```
unique(tablet$methdb)
```

```
[1] "Tablet"
```

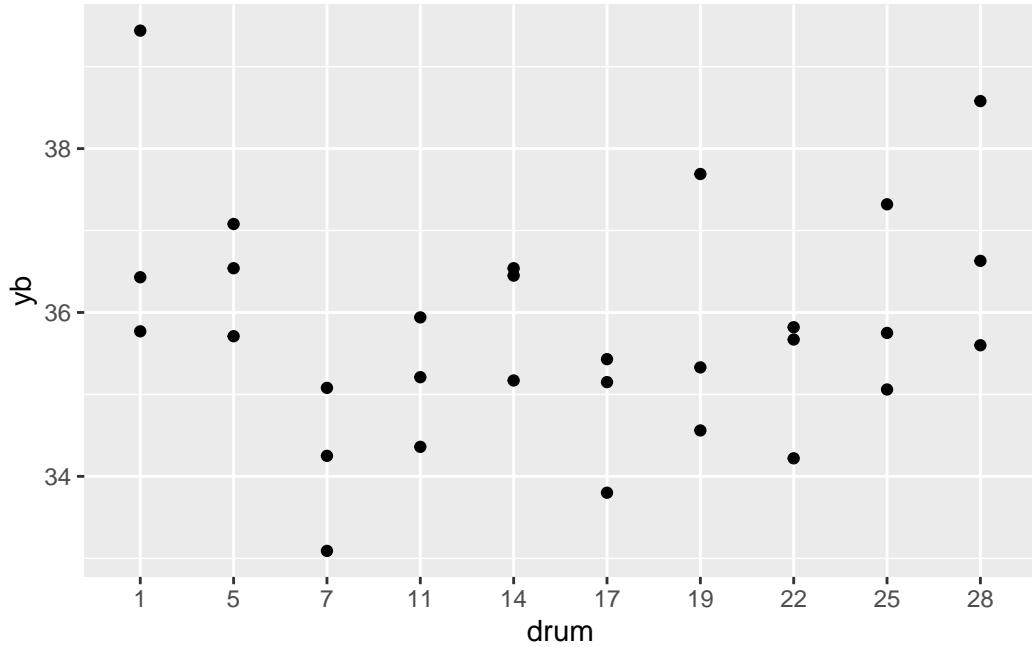
```
tablet$drum=as.factor(tablet$drum); tablet$drum
```

```
[1] 1 1 1 5 5 5 7 7 7 11 11 11 14 14 14 17 17 17 19 19 19 22 22 22 25  
[26] 25 25 28 28 28  
Levels: 1 5 7 11 14 17 19 22 25 28
```

```
e <- ggplot2::ggplot(tablet, ggplot2::aes(x = drum, y=yb)) + ggplot2::geom_boxplot()  
e
```

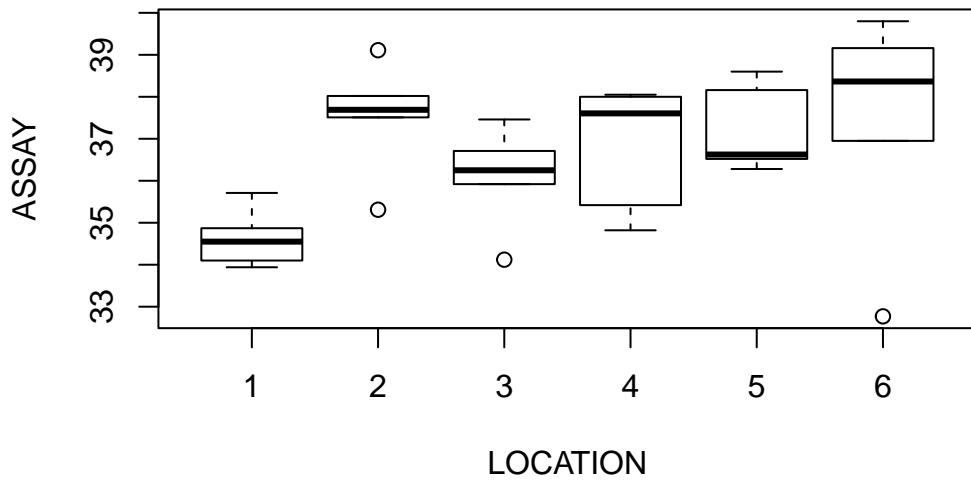


```
e <- ggplot2::ggplot(tablet, ggplot2::aes(x = drum, y=yb)) + ggplot2::geom_point()  
e
```

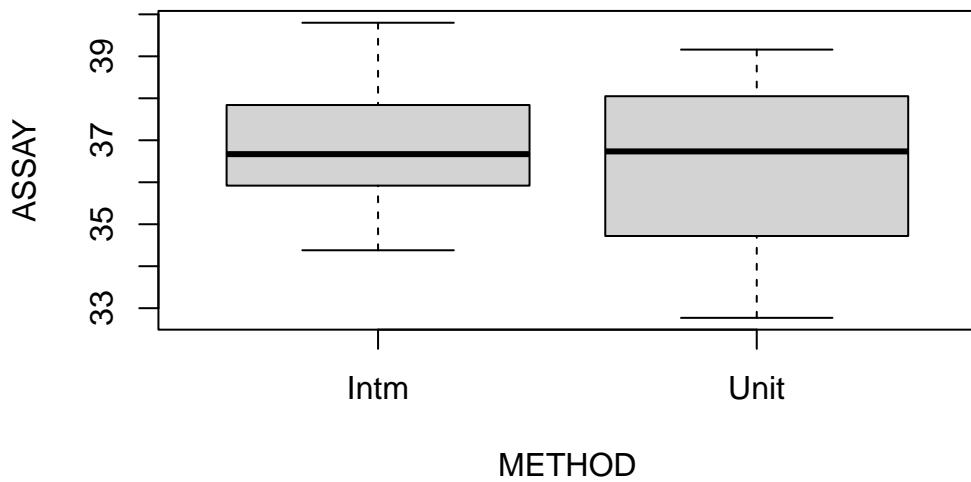


```
boxplot(ASSAY ~ LOCATION, col=as.numeric(thief$METHOD), data=thief)
```

```
Warning in boxplot.default(split(mf[[response]], mf[-response], drop = drop, :
NAs introduced by coercion
```



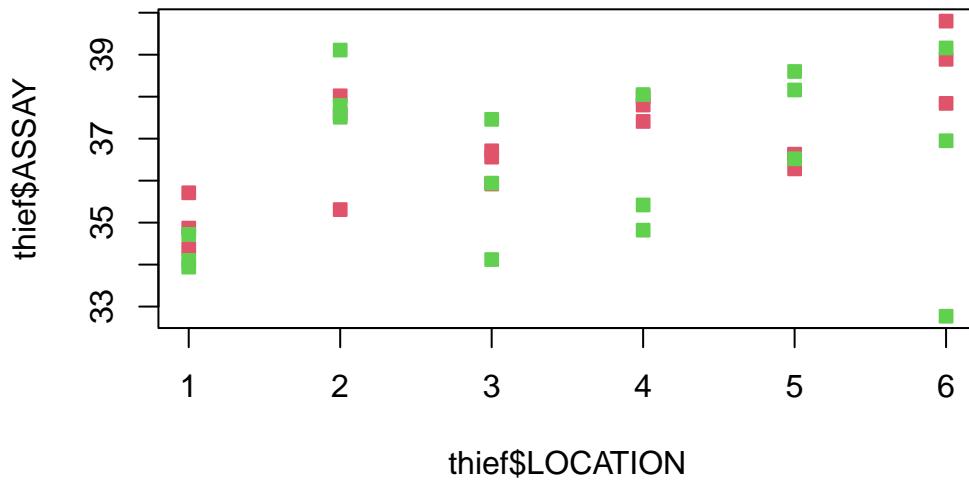
```
boxplot(ASSAY ~ METHOD,data=thief)
```



```

color=as.integer(as.factor(thief$METHOD))+1
plot(thief$LOCATION ,thief$ASSAY,col=color,pch=22,bg=color)

```



What can we conclude from this exploratory analysis?

### 1.5.5 Specification

Let's tackle the first question: how much does the active ingredient in the tablets vary? Write down the model fit below.

```

names(tablet)[4]="con"
model=lme(
  fixed= con ~1,
  random= con ~ 1 | drum, data=tablet )
summary(model)

```

```

Linear mixed-effects model fit by REML
Data: tablet
      AIC      BIC logLik
108.092 112.1939 -51.046

```

```

Random effects:
Formula: con ~ 1 | drum
            (Intercept) Residual
StdDev:    0.6673375 1.197821

Fixed effects: con ~ 1
              Value Std.Error DF t-value p-value
(Intercept) 35.789 0.3039075 20 117.7628      0

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-1.58945875 -0.62071916 -0.08544433  0.37232202  2.47467411

Number of Observations: 30
Number of Groups: 10

```

Okay, the “between drum standard deviation” is half the residual standard deviation. Let’s test if its non-zero. Recall that for REML estimates, the asymptotic distribution for the LRT is not the same as usual. In this case, under the null hypothesis,  $Y_{ij} \sim N(\mu, \sigma^2)$ . Thus,

```

fit_null<-lm(con ~ 1 , data=tablet)
observed=lmtest::lrtest(fit_null,model)$Chisq[2]

```

```

Warning in modelUpdate(objects[[i - 1]], objects[[i]]): original model was of
class "lm", updated model is of class "lme"

```

```

fe=nlme::fixed.effects(model); fe

(Intercept)
35.789

```

```

sigma_drum_est= nlme::getVarCov(model); sigma_drum_est

```

```

Random effects variance covariance matrix
            (Intercept)
(Intercept) 0.44534
Standard Deviations: 0.66734

```

```

sigma_est=model$sigma; sigma_est

[1] 1.197821

n=nrow(tablet)
n_sim=100

simulated=replicate(n,rnorm(n_sim,fe[1],sigma_est))

# n_sim x n
# dim(simulated)

compute_lrt=function(y){
  tablet_copy=tablet
  tablet_copy$con=y

  alt=lme(
    fixed= con~1,
    random= con ~ 1 | drum, data=tablet_copy )

  null<-lm(con ~ 1 , data=tablet_copy)

  test=lmtest::lrtest(null,alt)$Chisq[2]
  return(test)
}

ts=suppressWarnings(apply(simulated, 1, compute_lrt))

# ts

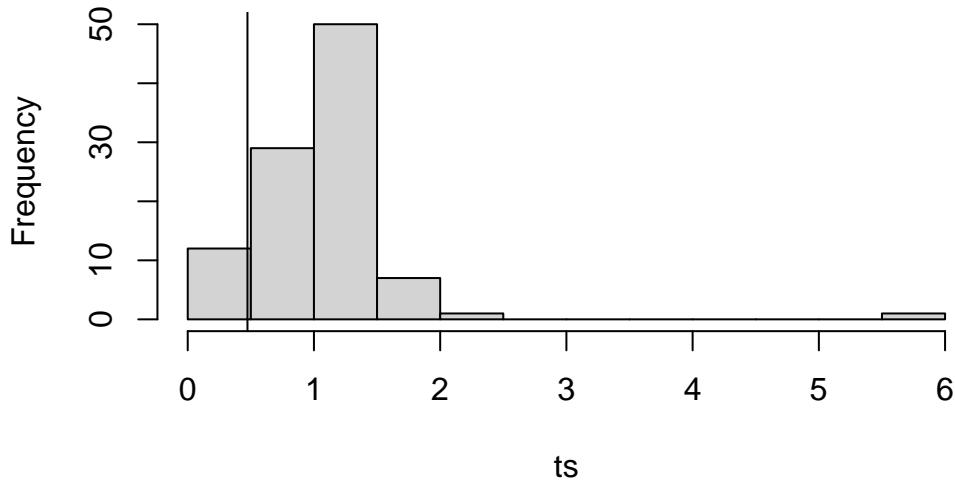
pvalue=mean(observed<=ts); pvalue

[1] 0.9

hist(ts)
abline(v=observed); observed

```

### Histogram of ts



```
[1] 0.4731465
```

```
1-pchisq(observed,1)/2-(observed>0)/2
```

```
[1] 0.2457716
```

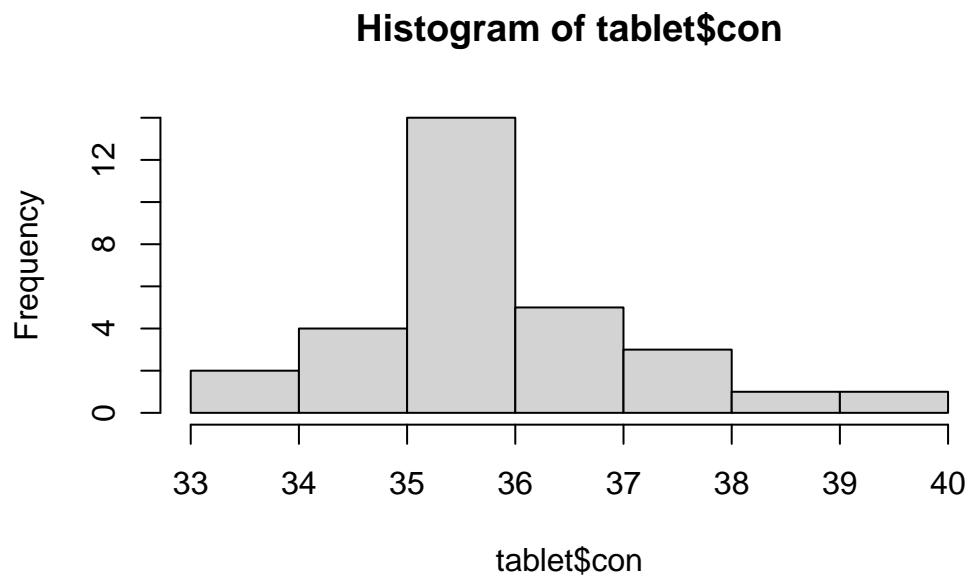
```
lmtest::lrtest(fit_null,model)
```

```
Warning in modelUpdate(objects[[i - 1]], objects[[i]]): original model was of
class "lm", updated model is of class "lme"
```

Likelihood ratio test

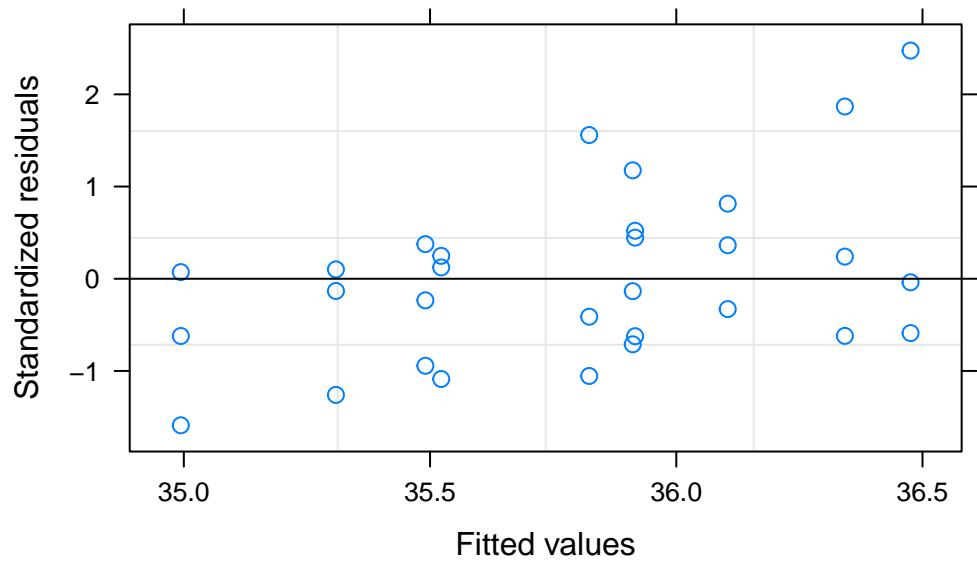
```
Model 1: con ~ 1
Model 2: con ~ 1
#Df LogLik Df Chisq Pr(>Chisq)
1    2 -51.283
2    3 -51.046  1 0.4731     0.4915
```

```
hist(tablet$con)
```

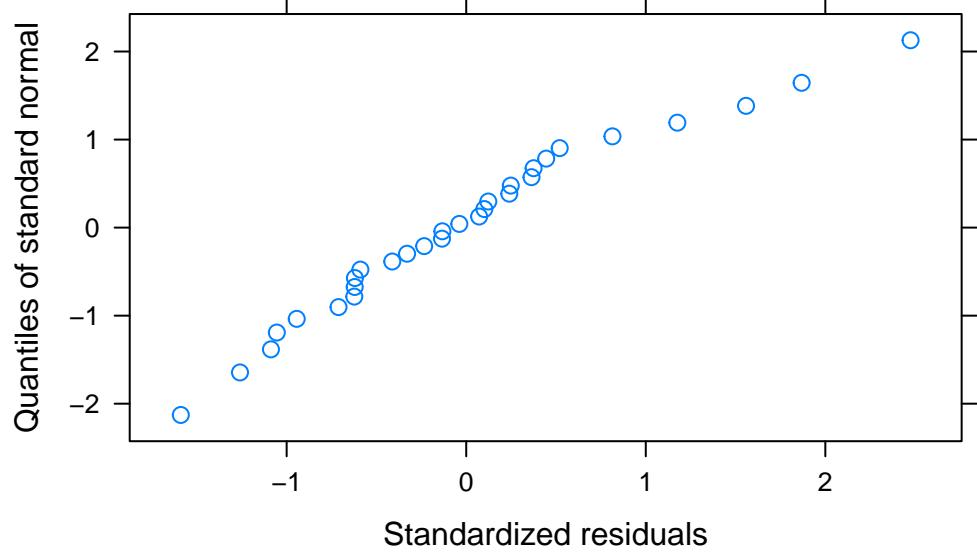


### 1.5.6 Diagnostics

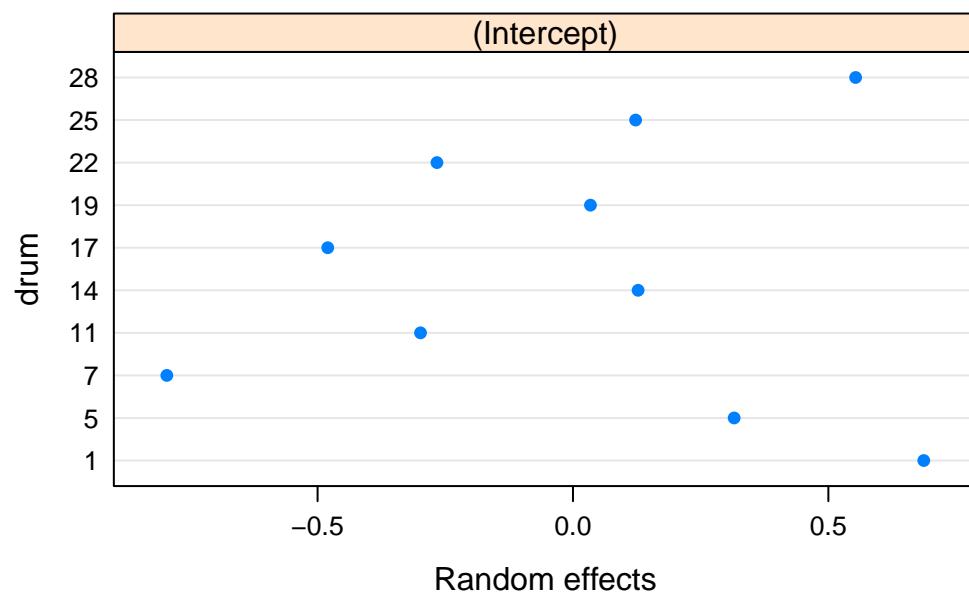
```
plot(model)
```



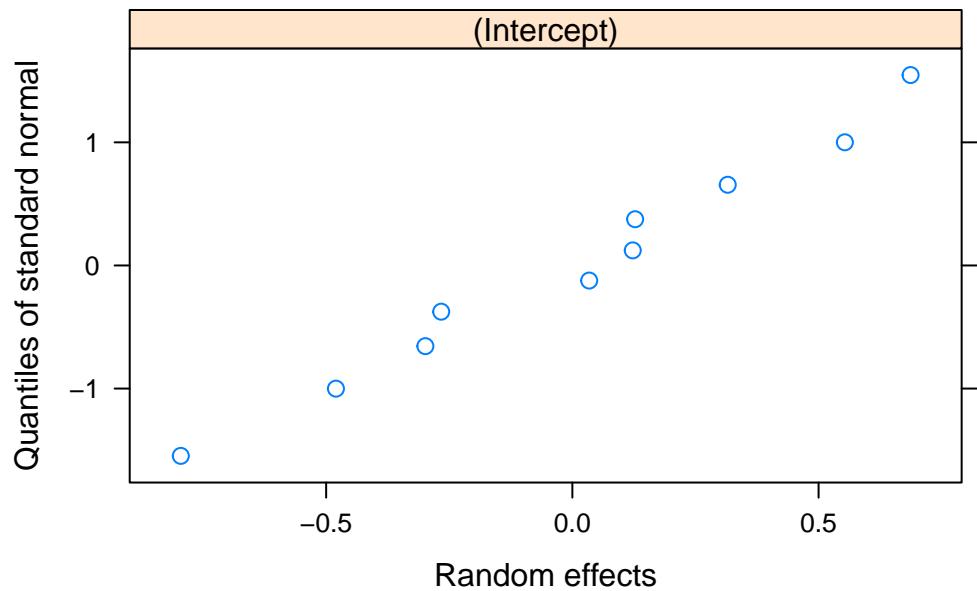
```
qqnorm(model, ~ residuals(.,type="pearson"))
```



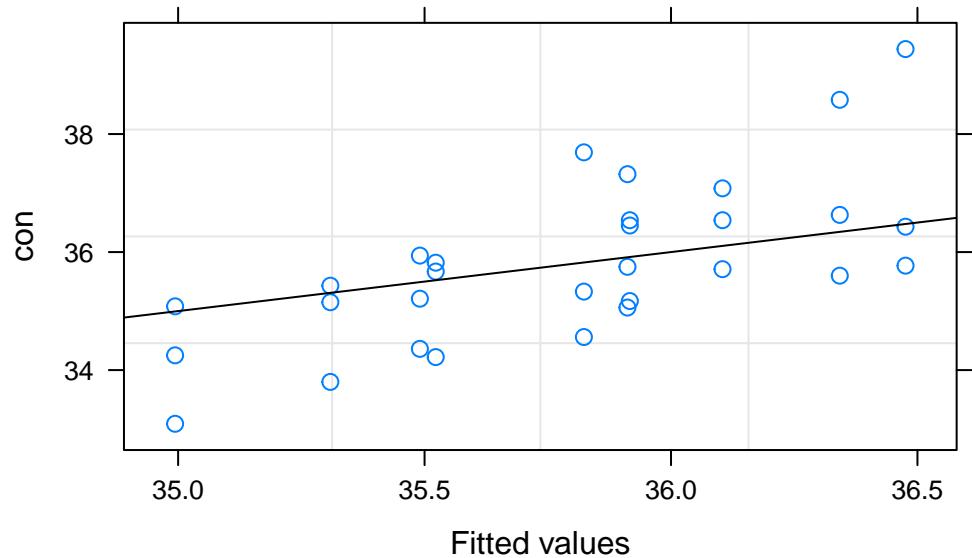
```
plot(ranef(model))
```



```
qqnorm(model, ~ ranef(.))
```



```
plot(model, con ~ fitted(.), abline=c(0,1))
```



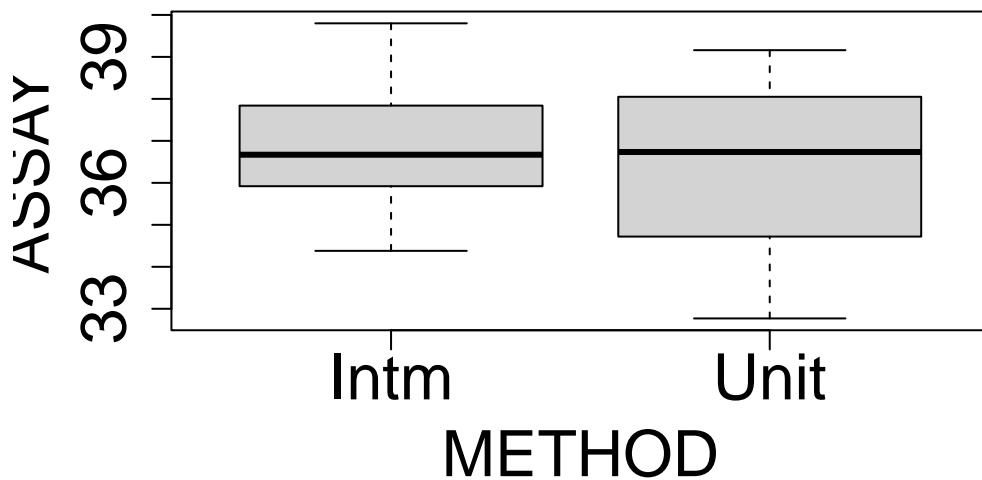
```
fit_null<-lm(con ~ 1 , data=tablet)
```

```
confint(fit_null)
```

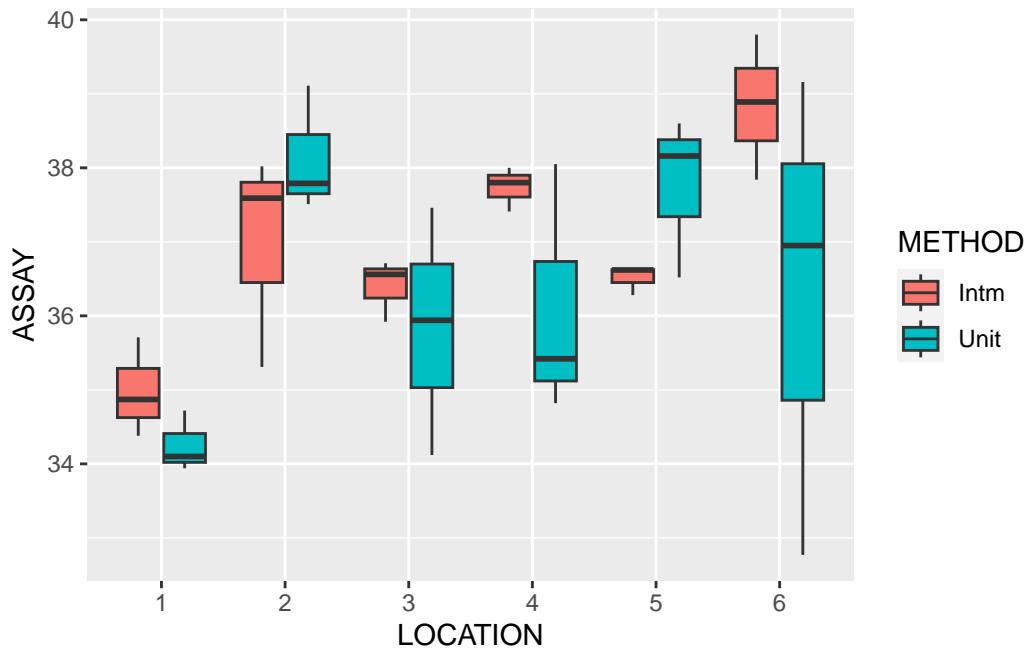
```
2.5 % 97.5 %  
(Intercept) 35.28119 36.29681
```

Let's tackle the next question: which sampling method is better? – which sampling method has a lower variability?

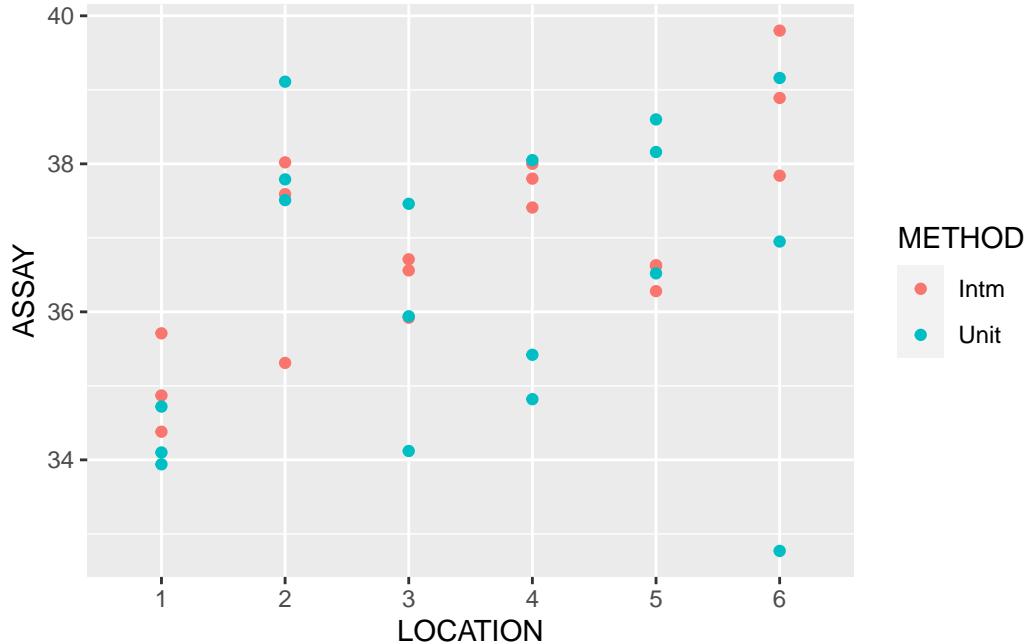
```
library(ggplot2)  
par(cex.lab=2,cex.axis=2,mfrow=c(1,1))  
boxplot(ASSAY~METHOD,data=thief)
```



```
thief$LOCATION=as.factor(thief$LOCATION)  
e <- ggplot(thief, aes(x = LOCATION, y=ASSAY,fill=METHOD)) + geom_boxplot()  
e
```



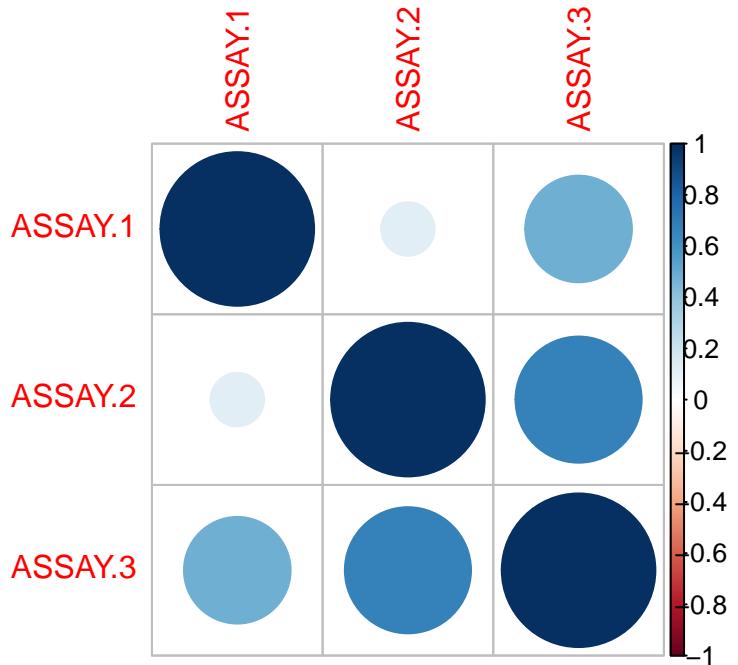
```
thief$LOCATION=as.factor(thief$LOCATION)
thief$METHOD=as.factor(thief$METHOD)
e <- ggplot(thief, aes(x = LOCATION, y=ASSAY, color=METHOD)) + geom_point()
e
```



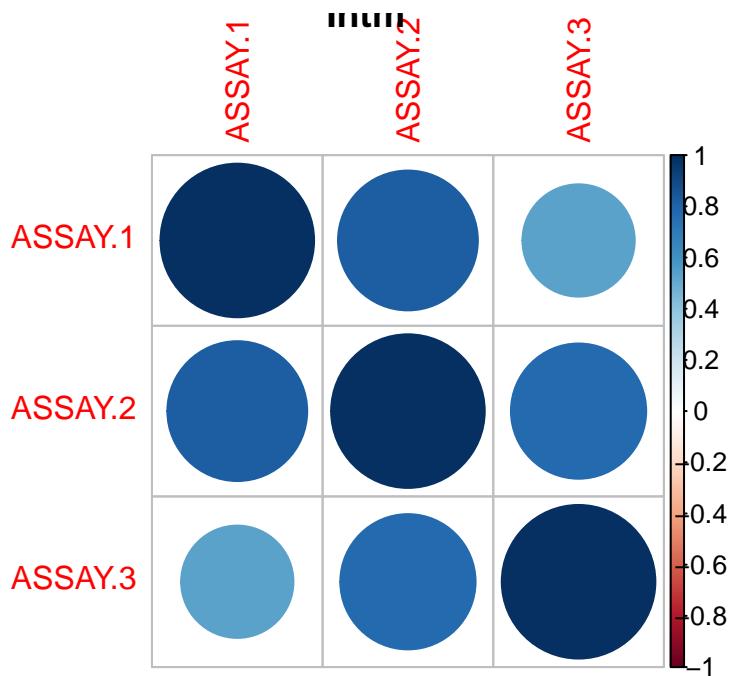
```
# boxplot(ASSAY ~ LOCATION, col=as.numeric(thief$METHOD), data=thief)
thief_wide=reshape(thief[,1:4], timevar='REPLICATE', idvar=c('LOCATION','METHOD'),
head(thief_wide)
```

	METHOD	LOCATION	ASSAY.1	ASSAY.2	ASSAY.3
1	Intm	1	34.38	34.87	35.71
4	Intm	2	35.31	37.59	38.02
7	Intm	3	36.71	36.56	35.92
10	Intm	4	37.80	37.41	38.00
13	Intm	5	36.28	36.63	36.62
16	Intm	6	38.89	39.80	37.84

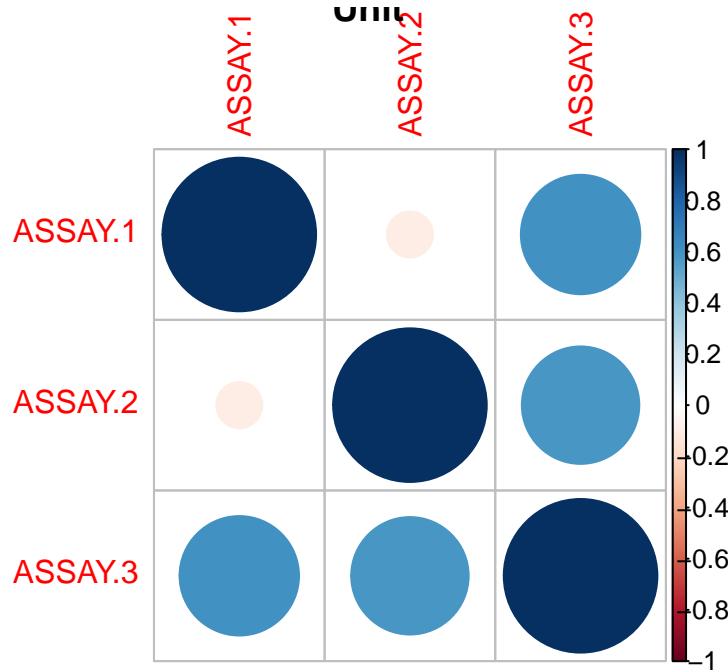
```
corrplot::corrplot(cor(thief_wide[,3:5]))
```



```
corrplot::corrplot(cor(thief_wide[1:6,3:5]),main="Intm")
```



```
corrplot::corrplot(cor(thief_wide[7:12,3:5]),main="Unit")
```



- We have 6 observations per location, three for each sampling type
- Each location could have its own mean concentration – see EDA  $\mu + \beta_i$
- We expect that the assays within each sampling type will vary around their location means -  $\beta_i$
- Nested within the locations is the replicates , 3 per sampling method
- We can capture the variability of sampling at each location via the random effect location.
- Let  $i$  be the location number,  $j$  be the sampling type and  $k$  be the replicate number

$$Y_{ijk} = \mu + \beta_i + \alpha_j + \epsilon_{ijk}.$$

We can also write this as follows:

- Let  $i$  be the location number and  $j$  be the sampling type
- Let  $W_{ij} = (1, j-1)$
- Let  $Z_{ij} = 1$  with  $\beta_i \sim N(0, \Sigma_\beta)$ .

$$Y_{ij} = W_{ij}\alpha + Z_{ij}\beta_i + \epsilon_{ij}.$$

```

thief=thief[,-5]
thief$REPLICATE=as.factor(thief$REPLICATE)
thief$LOCATION=as.factor(thief$LOCATION)
thief$METHOD=as.factor(thief$METHOD)
print(thief$METHOD)

[1] Intm Intm
[16] Intm Intm Intm Unit Unit
[31] Unit Unit Unit Unit Unit
Levels: Intm Unit

#Unit is True
thief$ASSAY=as.numeric(thief$ASSAY)
thief

      METHOD LOCATION REPLICATE ASSAY
1     Intm         1          1 34.38
2     Intm         1          2 34.87
3     Intm         1          3 35.71
4     Intm         2          1 35.31
5     Intm         2          2 37.59
6     Intm         2          3 38.02
7     Intm         3          1 36.71
8     Intm         3          2 36.56
9     Intm         3          3 35.92
10    Intm         4          1 37.80
11    Intm         4          2 37.41
12    Intm         4          3 38.00
13    Intm         5          1 36.28
14    Intm         5          2 36.63
15    Intm         5          3 36.62
16    Intm         6          1 38.89
17    Intm         6          2 39.80
18    Intm         6          3 37.84
19    Unit          1          1 33.94
20    Unit          1          2 34.72
21    Unit          1          3 34.10
22    Unit          2          1 39.11
23    Unit          2          2 37.51
24    Unit          2          3 37.79

```

```

25   Unit      3       1 37.46
26   Unit      3       2 34.12
27   Unit      3       3 35.94
28   Unit      4       1 38.05
29   Unit      4       2 34.82
30   Unit      4       3 35.42
31   Unit      5       1 36.52
32   Unit      5       2 38.60
33   Unit      5       3 38.16
34   Unit      6       1 39.16
35   Unit      6       2 32.77
36   Unit      6       3 36.95

model=lme(
  fixed= ASSAY ~ METHOD ,
  random= ~ 1|LOCATION, data=thief)
summary(model)

Linear mixed-effects model fit by REML
Data: thief
      AIC      BIC      logLik
142.2532 148.3586 -67.1266

Random effects:
Formula: ~1 | LOCATION
          (Intercept) Residual
StdDev:    0.9596085 1.456579

Fixed effects: ASSAY ~ METHOD
      Value Std.Error DF  t-value p-value
(Intercept) 36.90778 0.5209056 29 70.85310 0.0000
METHODUnit -0.51111 0.4855263 29 -1.05269 0.3012
Correlation:
          (Intr)
METHODUnit -0.466

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-2.94429462 -0.46995636  0.02331003  0.53623214  1.53118448

Number of Observations: 36
Number of Groups: 6

```

```

print(ranef(model))

(Intercept)
1 -1.4683710
2  0.6522971
3 -0.3857585
4  0.1910729
5  0.3488284
6  0.6619311

#Step 1. Computing T-hat
fit_null<-lm(ASSAY ~ 1 , data=thief)

observed=lmtest::lrtest(fit_null,model)$Chisq[2]; observed

Warning in modelUpdate(objects[[i - 1]], objects[[i]]): original model was of
class "lm", updated model is of class "lme"

[1] 5.442094

#Step 2.

#Get the fixed effects
fe=nlme::fixed.effects(model); fe

(Intercept) METHODUnit
36.9077778 -0.5111111

#Get the estimated variance of the RE
sigma_b_est= nlme::getVarCov(model); sigma_b_est

Random effects variance covariance matrix
(Intercept)
(Intercept) 0.92085
Standard Deviations: 0.95961

```

```

#Get the estimate of sigma
sigma_est=model$sigma; sigma_est

[1] 1.456579

n=nrow(thief)
n_sim=100

#Step 2
simulated=t(replicate(n_sim,rnorm(n,fe[1],sigma_est))); dim(simulated)

[1] 100 36

#Step 3
#takes a simulated sample y and computes the LRT for Y
compute_lrt=function(y){

  #create a copy of the dataset
  t_copy=thief

  #replace response with new sample
  t_copy$ASSAY=y

  #replace response with new sampl
  alt=lme(
    fixed= ASSAY~1,
    random=ASSAY ~ 1 | LOCATION, data=t_copy )

  null<-lm(ASSAY ~ 1 , data=t_copy)

  test=lmtest::lrtest(null,alt)$Chisq[2]

  return(test)
}

#compute LRT for each simulated sample
ts=suppressWarnings(apply(simulated, 1, compute_lrt))

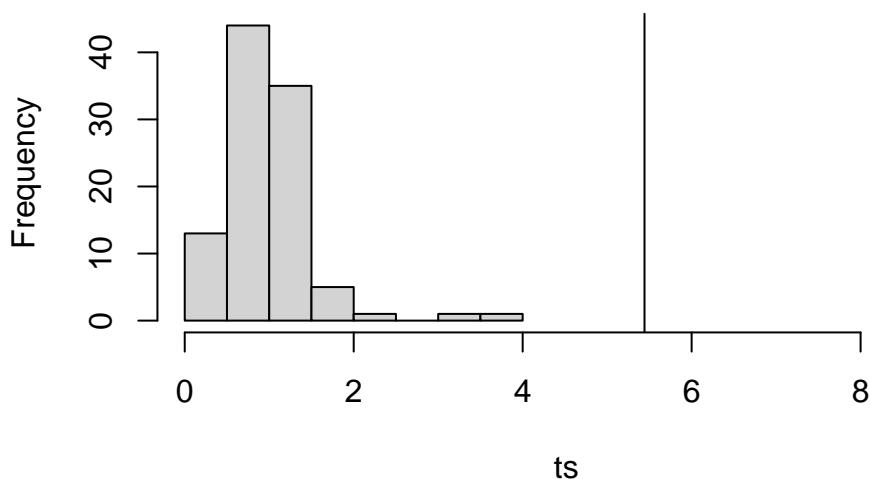
```

```
pvalue=mean(observed<=ts); pvalue
```

```
[1] 0
```

```
hist(ts,xlim=c(min(ts),9))
abline(v=observed)
```

**Histogram of ts**



```
observed
```

```
[1] 5.442094
```

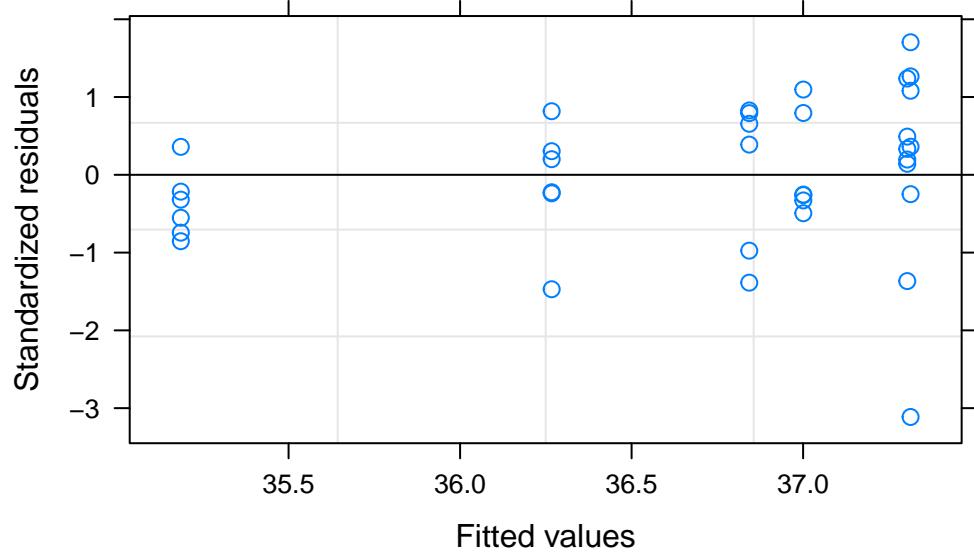
So then, the location effect introduces significant variability. We should recommend that they continue to sample multiple locations. The unit dose thief seems to estimate lower values of concentration, though this is erased by the standard error. Now, in general, we can see that this method is more variable, and potentially tends to underestimate the active ingredient. This fact leads me to recommend the intermediate sampling method.

- Are the assay values generally well behaved?

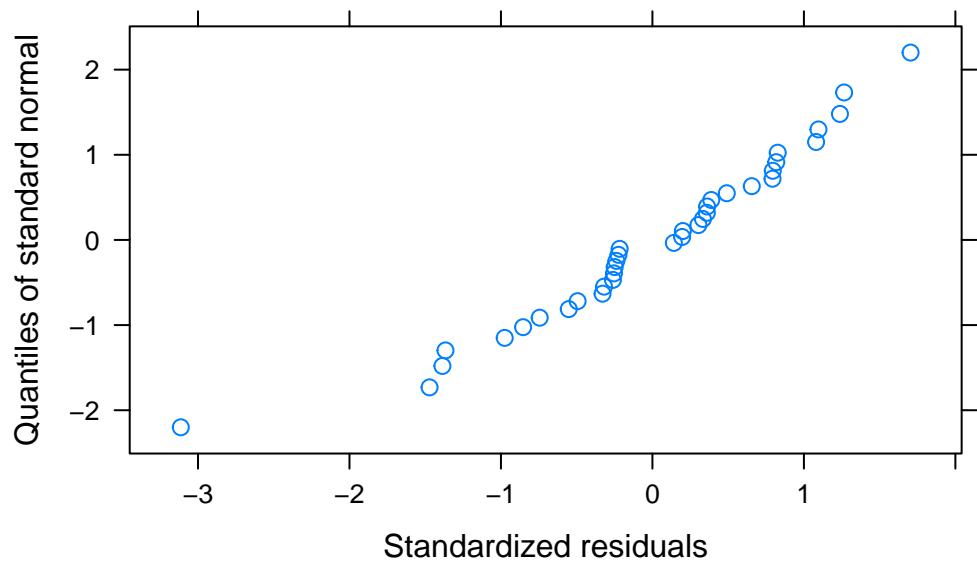
- Is there any evidence of a location effect
- Are the thief-sampled values comparable to the tablet values?
- Do the tablet data show any drum or time effect?

```
model=lme(
  fixed= ASSAY~1,
  random= ASSAY ~ 1 | LOCATION , data=thief)

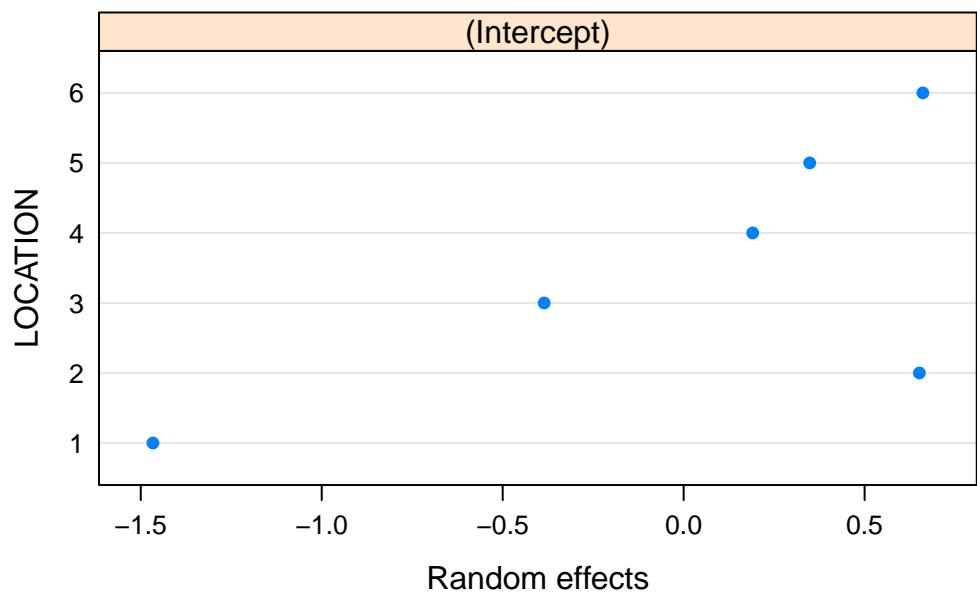
plot(model)
```



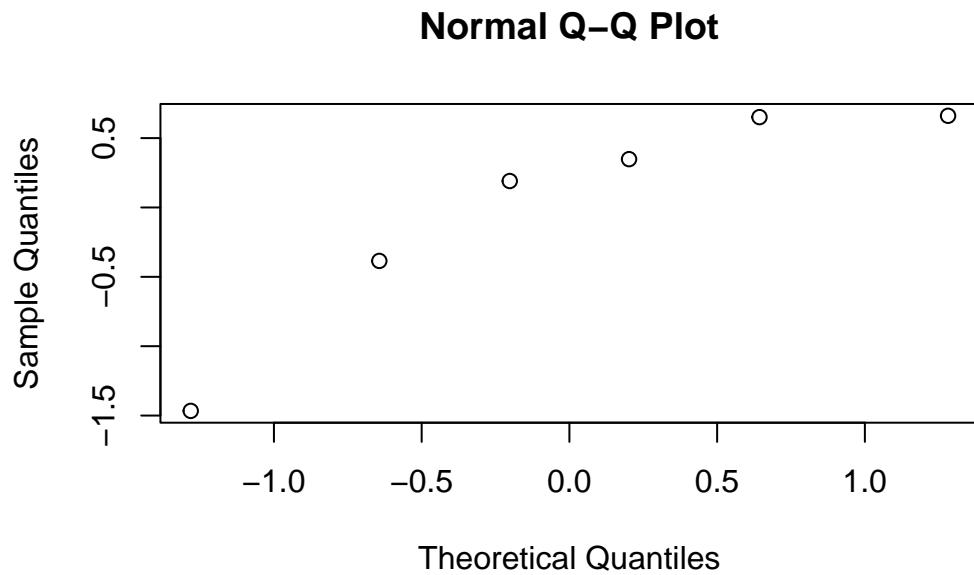
```
qqnorm(model, ~ residuals(.,type="pearson"))
```



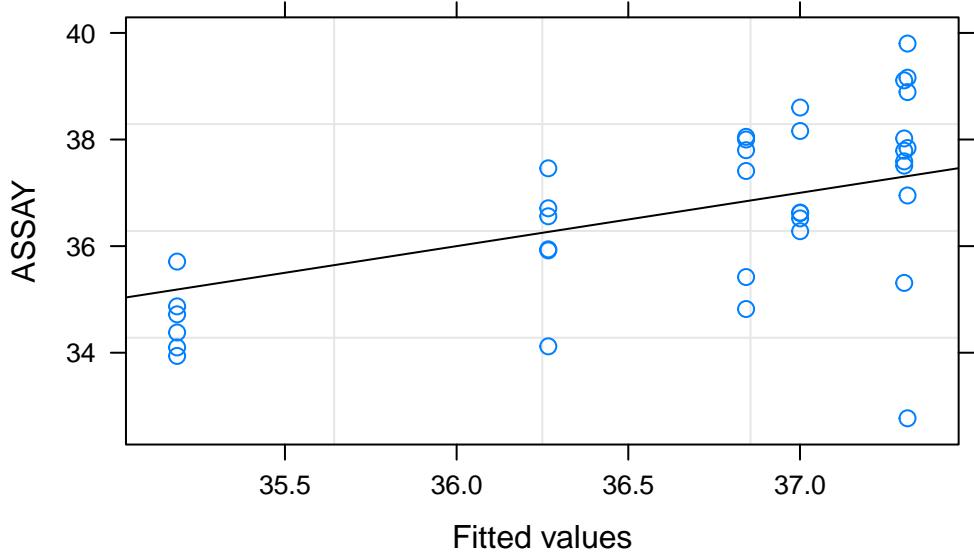
```
plot(ranef(model))
```



```
qqnorm(ranef(model) [,1])
```



```
plot(model, ASSAY ~ fitted(.), abline=c(0,1))
```



```
# fit_null<-lm(ASSAY ~ 1 , data=thief)
# confint(fit_null)
# fit_null
```

We can also fit a random effect for each of the methods and locations:

```
model=lme(
  fixed= ASSAY~METHOD,
  random= ASSAY ~ METHOD | LOCATION , data=thief)
summary(model)
```

```
Linear mixed-effects model fit by REML
Data: thief
      AIC      BIC      logLik
 145.3404 154.4986 -66.67022

Random effects:
Formula: ASSAY ~ METHOD | LOCATION
Structure: General positive-definite, Log-Cholesky parametrization
          StdDev   Corr
(Intercept) 1.045122 (Intr)
METHODUnit  1.021304 -0.363
```

```

Residual      1.360833

Fixed effects: ASSAY ~ METHOD
    Value Std.Error DF  t-value p-value
(Intercept) 36.90778 0.5337867 29 69.14331 0.0000
METHODUnit -0.51111 0.6161222 29 -0.82956 0.4136
Correlation:
          (Intr)
METHODUnit -0.509

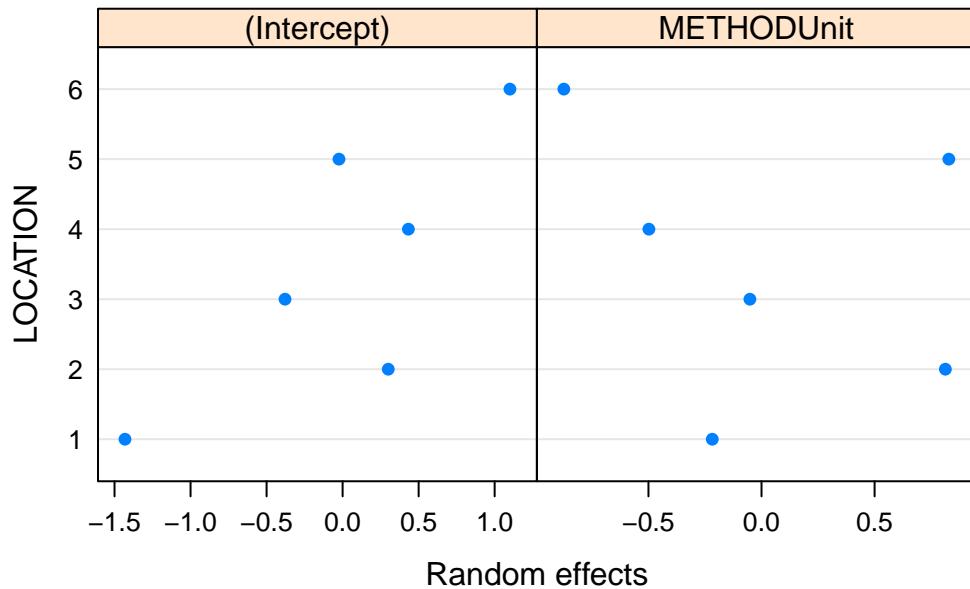
Standardized Within-Group Residuals:
    Min     Q1     Med     Q3     Max
-2.8314642 -0.4546280  0.0113704  0.5126063  1.8641882

```

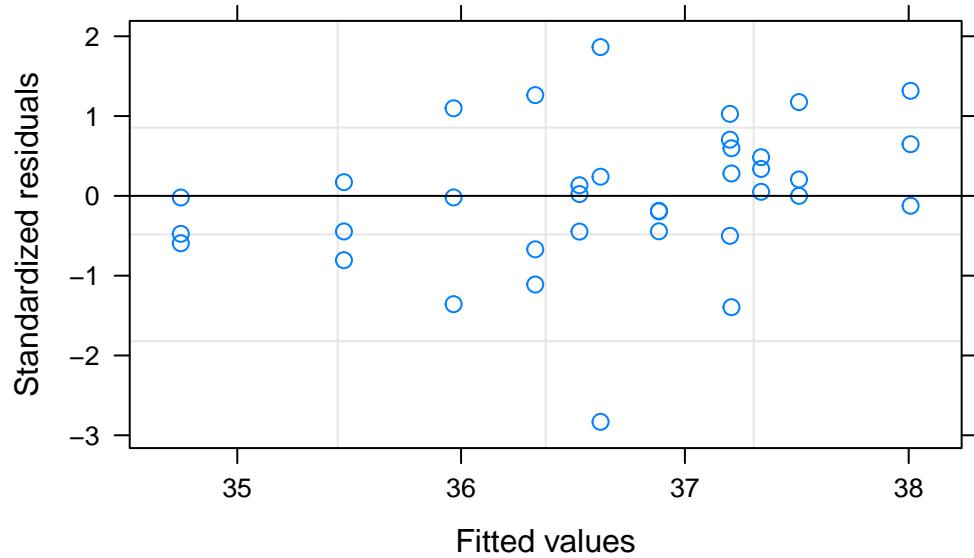
Number of Observations: 36  
 Number of Groups: 6

# We can see that the variability coming from each source is somewhat large.

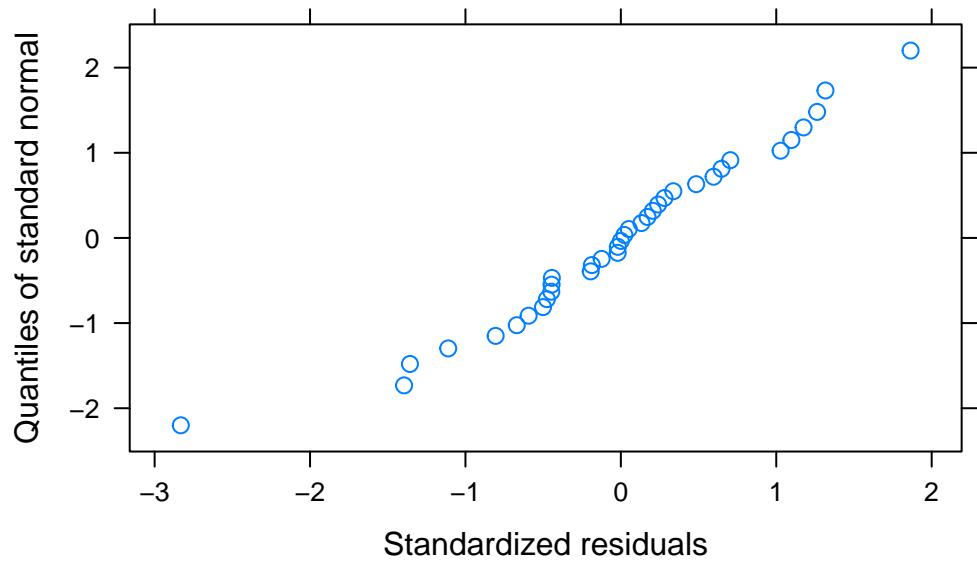
```
plot(ranef(model))
```



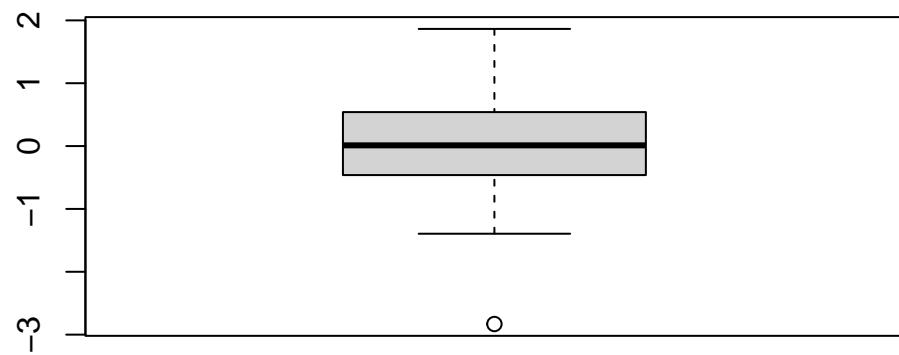
```
plot(model)
```



```
qqnorm(model, ~residuals(., type = "normalized"))
```



```
boxplot(residuals(model, type = "normalized"))
```



```

large=which.max(abs(residuals(model, type = "normalized"))); large

6
35

model=lme(
  fixed= ASSAY~METHOD,
  random= ASSAY ~ METHOD | LOCATION , data=thief[-large,])
summary(model)

Linear mixed-effects model fit by REML
Data: thief[-large, ]
    AIC      BIC    logLik
128.7654 137.7445 -58.3827

Random effects:
Formula: ASSAY ~ METHOD | LOCATION
Structure: General positive-definite, Log-Cholesky parametrization
          StdDev   Corr
(Intercept) 1.1562644 (Intr)
METHODUnit  0.7965863 0.066
Residual     1.0572951

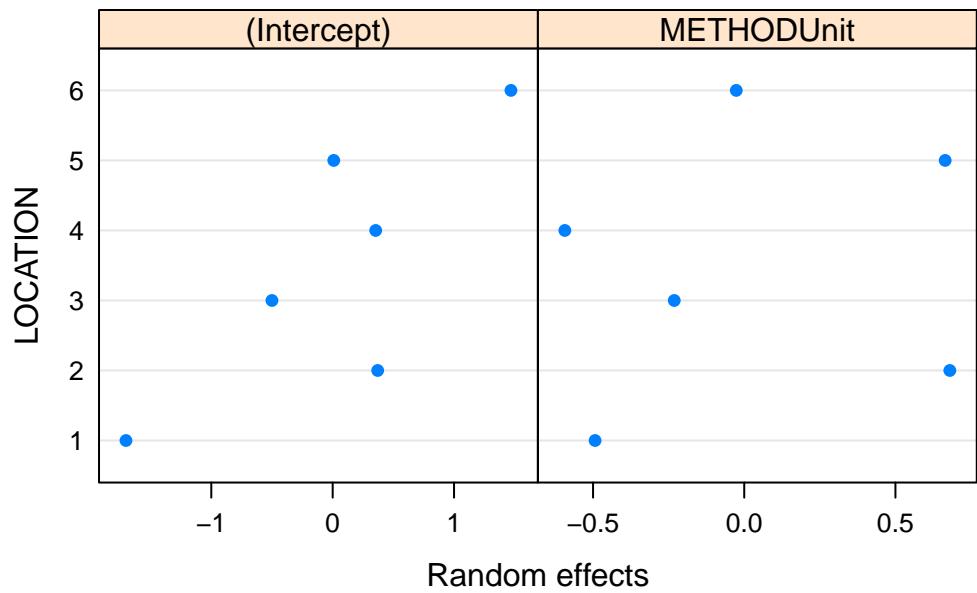
Fixed effects: ASSAY ~ METHOD
              Value Std.Error DF t-value p-value
(Intercept) 36.90778 0.5337871 28 69.14326 0.0000
METHODUnit -0.21293 0.4843533 28 -0.43961 0.6636
Correlation:
          (Intr)
METHODUnit -0.201

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-1.8620371 -0.5179546  0.0419608  0.6032329  1.5075857

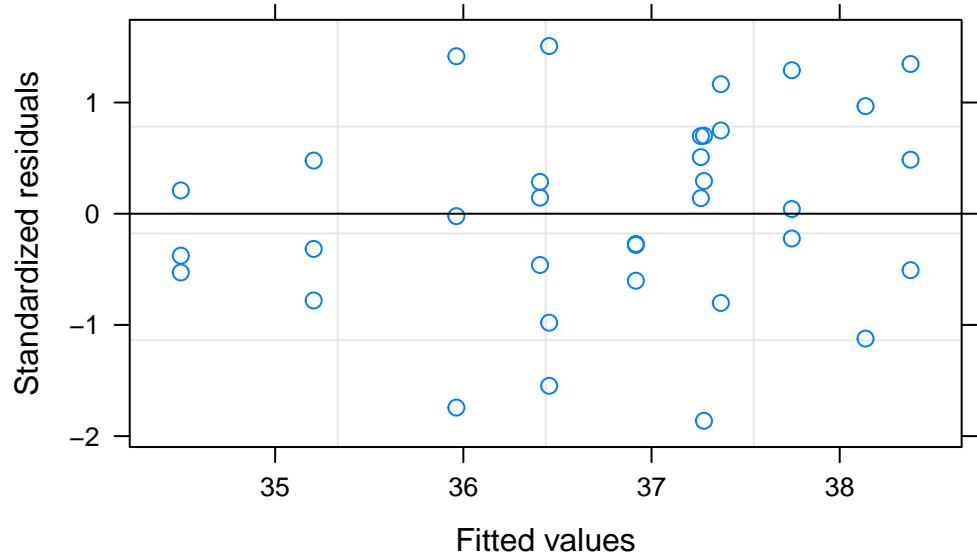
Number of Observations: 35
Number of Groups: 6

plot(ranef(model))

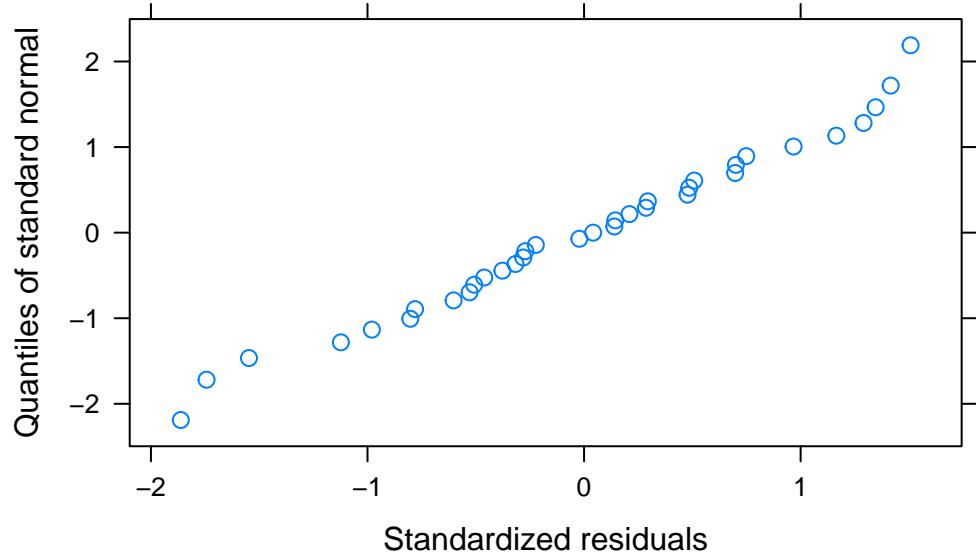
```



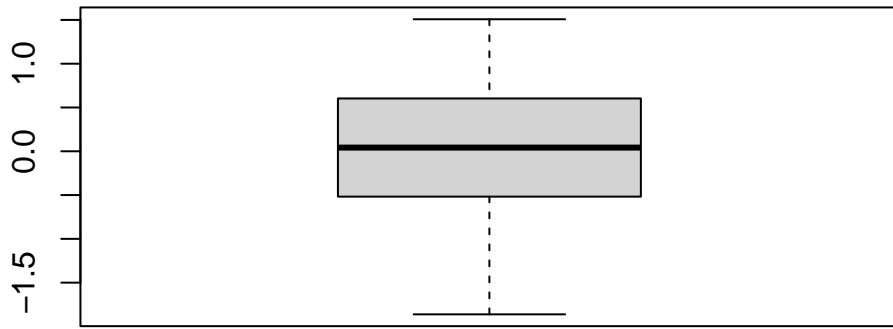
```
plot(model)
```



```
qqnorm(model, ~residuals(., type = "normalized"))
```



```
boxplot(residuals(model, type = "normalized"))
```



```

large=which.max(residuals(model, type = "normalized"))

model=lme(
  fixed= sqrt(ASSAY)~METHOD,
  random= sqrt(ASSAY) ~ METHOD | LOCATION , data=thief[-large,])
summary(model)

Linear mixed-effects model fit by REML
Data: thief[-large, ]
  AIC      BIC    logLik
-23.94854 -14.96949 17.97427

Random effects:
Formula: ASSAY ~ METHOD | LOCATION
Structure: General positive-definite, Log-Cholesky parametrization
          StdDev     Corr
(Intercept) 0.08767712 (Intr)
METHODUnit  0.10625944 -0.405
Residual    0.10789056

Fixed effects: sqrt(ASSAY) ~ METHOD

```

```

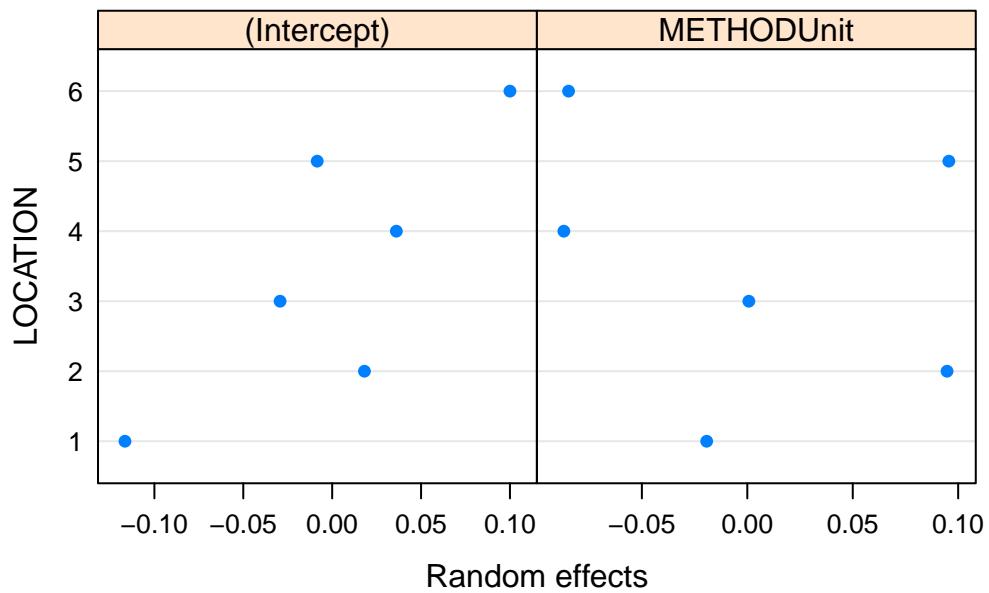
Value Std.Error DF t-value p-value
(Intercept) 6.074146 0.04390786 28 138.33847 0.0000
METHODUnit -0.054387 0.05677613 28 -0.95792 0.3463
Correlation:
          (Intr)
METHODUnit -0.511

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-2.87783191 -0.44462638  0.03546945  0.45330842  2.06501422

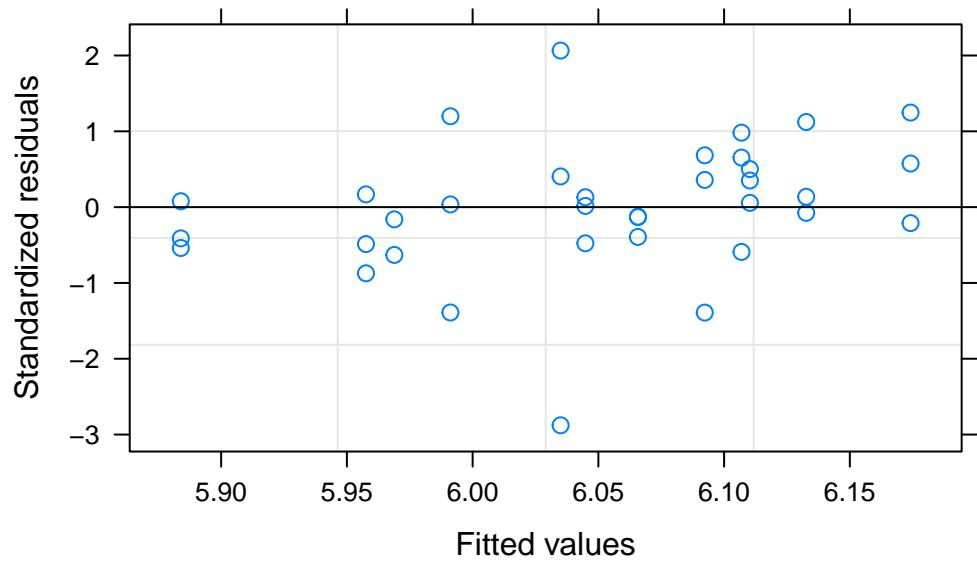
```

Number of Observations: 35  
 Number of Groups: 6

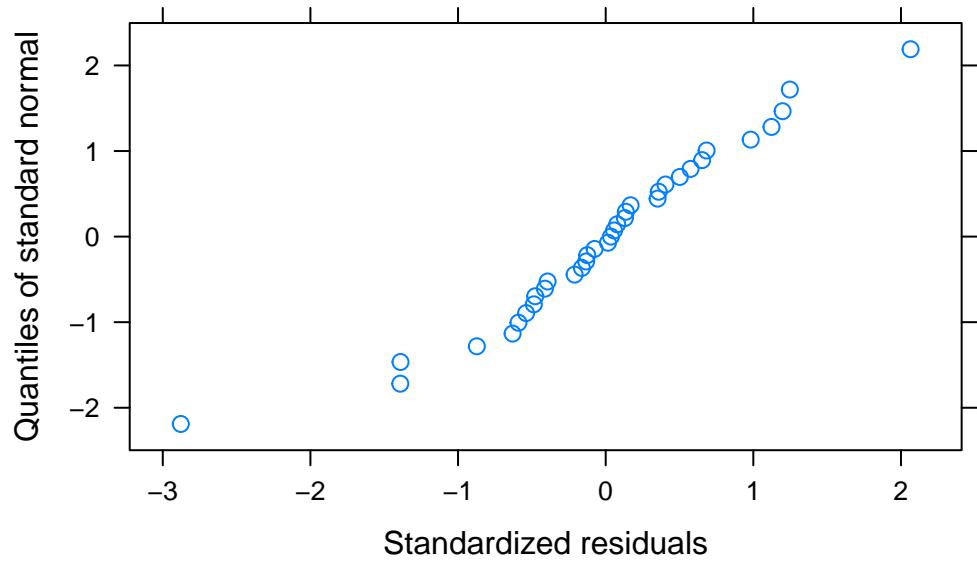
```
plot(ranef(model))
```



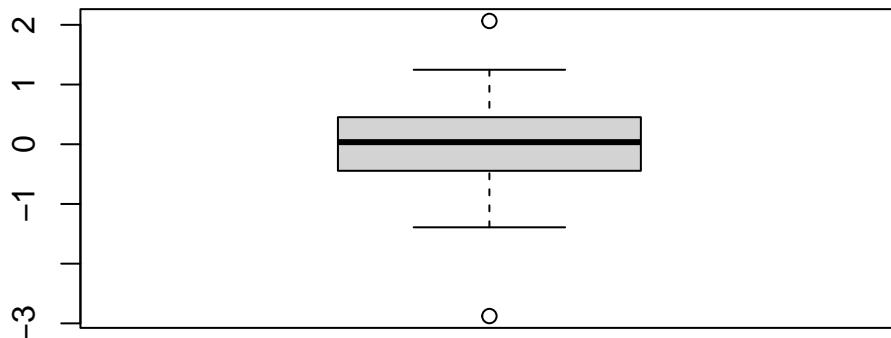
```
plot(model)
```



```
qqnorm(model, ~residuals(., type = "normalized"))
```



```
boxplot(residuals(model, type = "normalized"))
```



```
large=which.max(residuals(model, type = "normalized"))
```

```
thief$CEN=abs(thief$ASSAY-mean(thief$ASSAY))
model=lme(
  fixed= CEN ~ METHOD ,
  random= ~ 1|LOCATION, data=thief)
summary(model)
```

```
Linear mixed-effects model fit by REML
Data: thief
      AIC      BIC    logLik
99.44747 105.5529 -45.72373
```

```
Random effects:
Formula: ~1 | LOCATION
          (Intercept) Residual
StdDev:   0.5372663 0.7715252
```

```
Fixed effects: CEN ~ METHOD
              Value Std.Error DF  t-value p-value
(Intercept) 1.0988889 0.2849187 29 3.856850 0.0006
METHODUnit  0.5922222 0.2571751 29 2.302798 0.0287
Correlation:
          (Intr)
METHODUnit -0.451

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-2.59227946 -0.67605167 -0.04425782  0.47231782  2.05364082

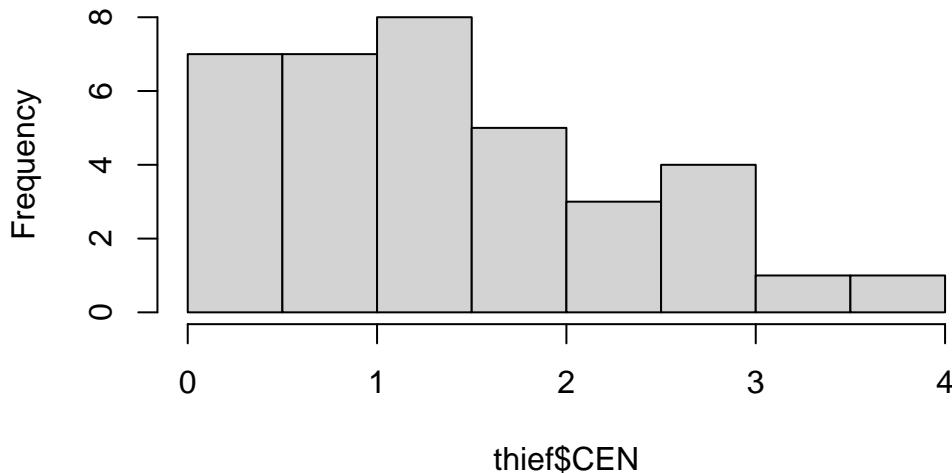
Number of Observations: 36
Number of Groups: 6
```

```
print(ranef(model))
```

```
(Intercept)
1 0.47423228
2 -0.03335199
3 -0.42613374
4 -0.08117489
5 -0.54024717
6 0.60667552
```

```
hist(thief$CEN)
```

### Histogram of thief\$CEN



```
# Non-normal (of course... maybe use simulation)
```

Results summary:

- Tablet mean concentration was estimated to be 35.8 with CI (35.3, 36.3)
- Active ingredient was uniform across blender and sampling types
- Tablets in the drums on the end points seem to have a higher active ingredient
- The blender has higher concentrations of active ingredient 36.65 with CI (36.0743, 37.23015)
- The location has significant variance ( $\sim 1$  mg)
- Sampling methods are equivalent in mean, but UNIT seems to have higher variance. Possibly produce smaller estimates of the concentration, but this is not statistically significant

## 1.6 Case study: Treatment of Lead-Exposed Children

*Does treatment A (chelation treatment with succimer) affect the levels of lead in the blood of lead-exposed children?*

[Data source](#) or [Data source](#)

Description:

The Treatment of Lead-Exposed Children (TLC) trial was a placebo-controlled, randomized study of succimer (a chelating agent) in children with blood lead levels of 20-44 micrograms/dL. These data consist of four repeated measurements of blood lead levels obtained at baseline (or week 0), week 1, week 4, and week 6 on 100 children who were randomly assigned to chelation treatment with succimer or placebo.

Data Column Descriptions:

- ID: Subject ID Number
- Treatment: Which treatment group (P=Placebo; A=Succimern)
- W0, W1, W4, W6: Blood-lead levels in micrograms per deciliter at Weeks 0, 1, 4, and 6

```
#####
TLC <- read.csv("data/TLC.csv",stringsAsFactors = T)
head(TLC)
```

	ID	Treatment	W0	W1	W4	W6
1	1	P	30.8	26.9	25.8	23.8
2	2	A	26.5	14.8	19.5	21.0
3	3	A	25.8	23.0	19.1	23.2
4	4	P	24.7	24.5	22.0	22.5
5	5	A	20.4	2.8	3.2	9.4
6	6	A	20.4	5.4	4.5	11.9

```
dim(TLC)
```

```
[1] 100    6
```

```
TLC$ID=as.factor(TLC$ID)
TLC$ID=as.factor(TLC$ID)
```

### 1.6.1 Modelling:

- Recall that the goal is to answer: “Does treatment  $A$  affect the levels of lead in the blood of lead-exposed children?”
- How are treatment group and time related to lead levels?
- Given treatment and time, what do we expect the blood levels to be  $E(W|Treatment, Time)$ ?

Putting this data into our notation gives:

- $n = 100$ ,  $J = 4$ ,  $K = 1$

- $Y_{ij}$  is the lead level of individual  $i$  at time  $j$
- $X_{ij}$  is treatment indicator of individual  $i$  at time  $j$
- $t_j \in \{0, 1, 4, 6\}$

Let's explore this data a bit. Notice that the data is in "wide format". To convert to a between long and wide format use `reshape()`

```
head(TLC)
```

ID	Treatment	W0	W1	W4	W6
1	P	30.8	26.9	25.8	23.8
2	A	26.5	14.8	19.5	21.0
3	A	25.8	23.0	19.1	23.2
4	P	24.7	24.5	22.0	22.5
5	A	20.4	2.8	3.2	9.4
6	A	20.4	5.4	4.5	11.9

```
# Convert from "wide" to "long" and back again, using reshape.
# If you're interested, you can also use `pivot_wider` and `pivot_longer` from the tidyverse
#   (If that doesn't mean anything to you, feel free to ignore it!)
TLC_long <- reshape(data = TLC,
                      varying = c("W0", "W1", "W4", "W6"),
                      timevar = "week",
                      idvar = "ID",
                      times = c(0, 1, 4, 6),
                      direction = "long",
                      sep = "")

head(TLC_long)
```

ID	Treatment	week	W
1.0	P	0	30.8
2.0	A	0	26.5
3.0	A	0	25.8
4.0	P	0	24.7
5.0	A	0	20.4
6.0	A	0	20.4

```
TLC_wide <- reshape(data = TLC_long,
                      timevar = "week",
                      v.names = "W",
```

```

    idvar = "ID",
    times = c(0, 1, 4, 6),
    direction = "wide",
    sep = "")

```

`head(TLC_wide)`

	ID	Treatment	W0	W1	W4	W6
1.0	1	P	30.8	26.9	25.8	23.8
2.0	2	A	26.5	14.8	19.5	21.0
3.0	3	A	25.8	23.0	19.1	23.2
4.0	4	P	24.7	24.5	22.0	22.5
5.0	5	A	20.4	2.8	3.2	9.4
6.0	6	A	20.4	5.4	4.5	11.9

```

#balanced
table(TLC_long$ID)

```

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4

```

#Check for missing values
colSums(is.na(TLC_long))

```

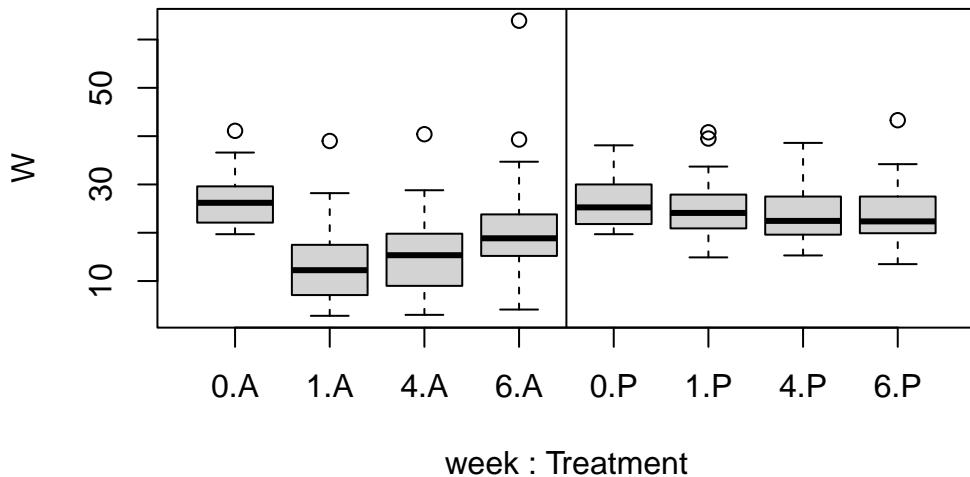
ID	Treatment	week	W
0	0	0	0

```

# Create a Basic Boxplot to get a Sense of the Data
boxplot(W ~ week + Treatment, data = TLC_long)

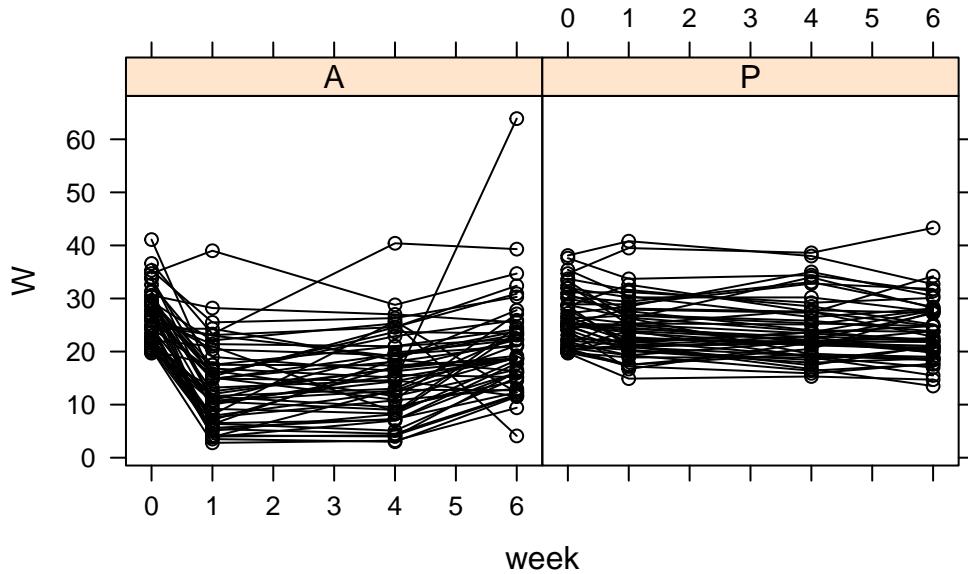
```

```
abline(v=4.5) # Abline v=... draws a vertical line at 4.5
```



```
# Start with an xyplot
# This requires the package 'lattice'
# You can install using: install.packages("lattice")

lattice::xyplot(W ~ week | Treatment,
                 data = TLC_long,
                 groups = ID,
                 col = 'black',
                 type = c('l', 'p'))
```



```

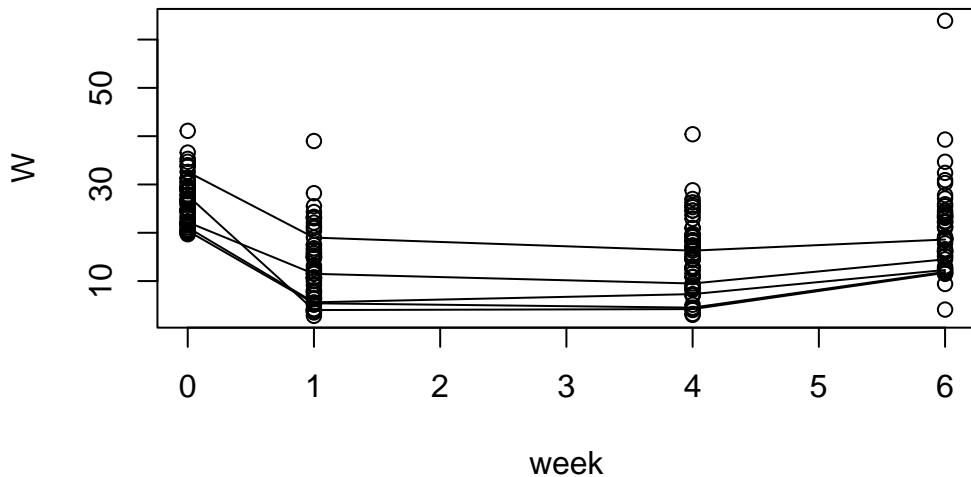
# The plot is a mess, as-is, so instead we can subset!
plot_num <- 5 # Select a fixed number

# This is Just Randomly Sampling from Each Group
random_samples_P <- sample(unique(TLC_long$ID[which(TLC_long$Treatment == 'P')]),
                           size = plot_num,
                           replace = FALSE)
random_samples_A <- sample(unique(TLC_long$ID[which(TLC_long$Treatment == 'A')]),
                           size = plot_num,
                           replace = FALSE)

## Actually Draw the Plots
plot(W ~ week, data = TLC_long, subset = (Treatment == 'A'))
for (rid in random_samples_A){
  lines(W ~ week,
        data = TLC_long,
        subset = (ID==rid),
        type = 'l')
}

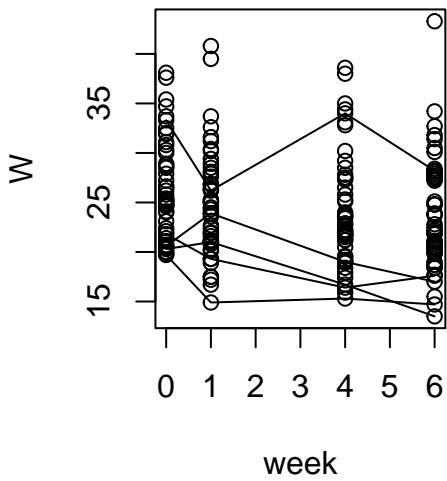
```

```
}
```



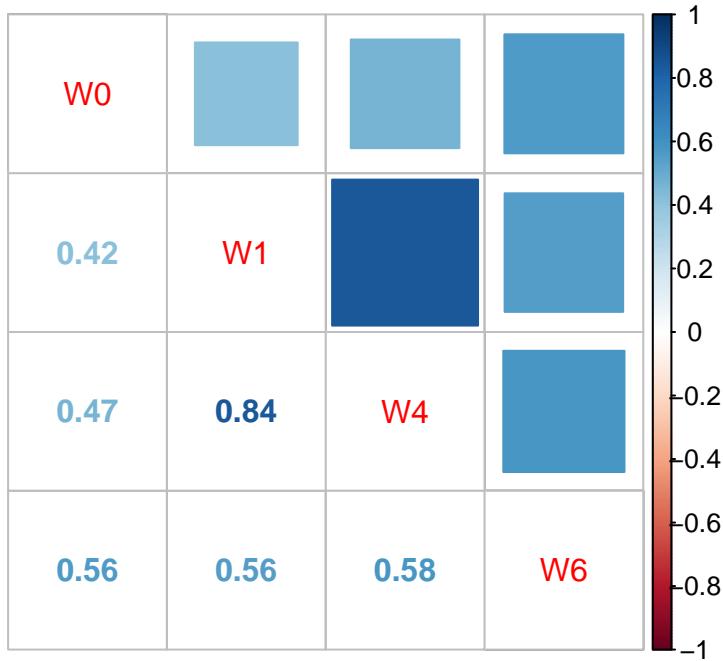
```
# Repeat it for Placebo

par(mfrow=c(1,2))
plot(W ~ week, data = TLC_long, subset = (Treatment == 'P'))
for (rid in random_samples_P){
  # Loop through the Random Points and Draw the Corresponding Lines
  lines(W ~ week,
        data = TLC_long,
        subset = (ID==rid),
        type = 'l')
}
```



Correlation plot

```
# This is a basic correlation plot
# It requires the 'corrplot' library, which can be installed with
# install.packages("corrplot")
corrplot::corrplot.mixed(cor(TLC_wide[c("W0", "W1", "W4", "W6")]),
                        lower = 'number',
                        upper = 'square')
```

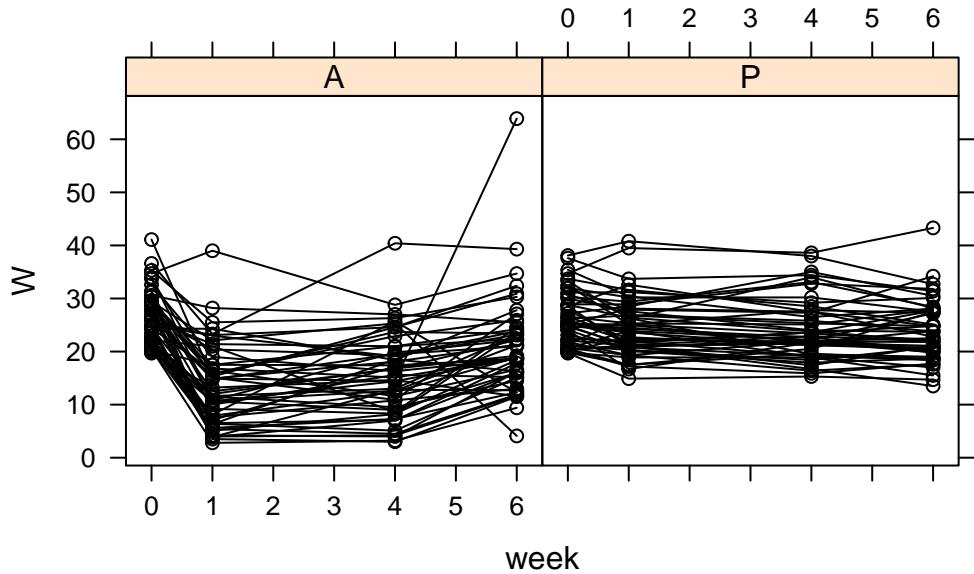


What can we conclude from the EDA? What model could we propose in this case?

### 1.6.2 Longitudinal Data as a mixed effects model

- Looking at the XY plots, we see that the individual means seem to vary.
- This means that each individual is likely to have a different mean
- The treatment effect should be modelled as a fixed effect
- How to model time effect?

```
lattice::xyplot(W ~ week | Treatment,
                 data = TLC_long,
                 groups = ID,
                 col = 'black',
                 type = c('l', 'p'))
```



What are some ways we can model the time effect here?

Let's fit a linear mixed effects model.

```
# head(TLC_long
# head(TLC_long[order(TLC_long$ID),])
typeof(TLC_long)

[1] "list"

#REML
#treat the time points as factors
TLC_long$week=as.factor(TLC_long$week)
# head(TLC_long)
model <- nlme::lme(fixed= W ~ 1 +Treatment+week+Treatment*week,
random= ~1| ID, data = TLC_long) #to run the model

summary(model)
```

```
Linear mixed-effects model fit by REML
Data: TLC_long
```

```
AIC      BIC      logLik  
2480.621 2520.334 -1230.311
```

Random effects:

```
Formula: ~1 | ID  
          (Intercept) Residual  
StdDev:    5.112717 4.214287
```

Fixed effects: W ~ 1 + Treatment + week + Treatment \* week

	Value	Std.Error	DF	t-value	p-value
(Intercept)	26.540	0.9370175	294	28.323912	0.0000
TreatmentP	-0.268	1.3251428	98	-0.202242	0.8401
week1	-13.018	0.8428574	294	-15.445080	0.0000
week4	-11.026	0.8428574	294	-13.081691	0.0000
week6	-5.778	0.8428574	294	-6.855252	0.0000
TreatmentP:week1	11.406	1.1919804	294	9.568950	0.0000
TreatmentP:week4	8.824	1.1919804	294	7.402807	0.0000
TreatmentP:week6	3.152	1.1919804	294	2.644339	0.0086

Correlation:

	(Intr)	TrtmnP	week1	week4	week6	TrtP:1	TrtP:4
TreatmentP	-0.707						
week1	-0.450	0.318					
week4	-0.450	0.318	0.500				
week6	-0.450	0.318	0.500	0.500			
TreatmentP:week1	0.318	-0.450	-0.707	-0.354	-0.354		
TreatmentP:week4	0.318	-0.450	-0.354	-0.707	-0.354	0.500	
TreatmentP:week6	0.318	-0.450	-0.354	-0.354	-0.707	0.500	0.500

Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-4.18502705	-0.46501691	-0.04732964	0.36499878	7.66712566

Number of Observations: 400

Number of Groups: 100

```
nlme::intervals(model)
```

Approximate 95% confidence intervals

Fixed effects:

lower	est.	upper
-------	------	-------

```

(Intercept)      24.6958881  26.540  28.384112
TreatmentP       -2.8977028  -0.268   2.361703
week1            -14.6767987 -13.018  -11.359201
week4            -12.6847987 -11.026  -9.367201
week6             -7.4367987  -5.778  -4.119201
TreatmentP:week1  9.0601043  11.406  13.751896
TreatmentP:week4  6.4781043  8.824  11.169896
TreatmentP:week6  0.8061043  3.152  5.497896

```

Random Effects:

Level: ID

	lower	est.	upper
sd((Intercept))	4.33785	5.112717	6.025997

Within-group standard error:

	lower	est.	upper
3.887059	4.214287	4.569062	

What do we notice here?

- We see that the treatment effect is not significant in this model, but the interaction terms are.

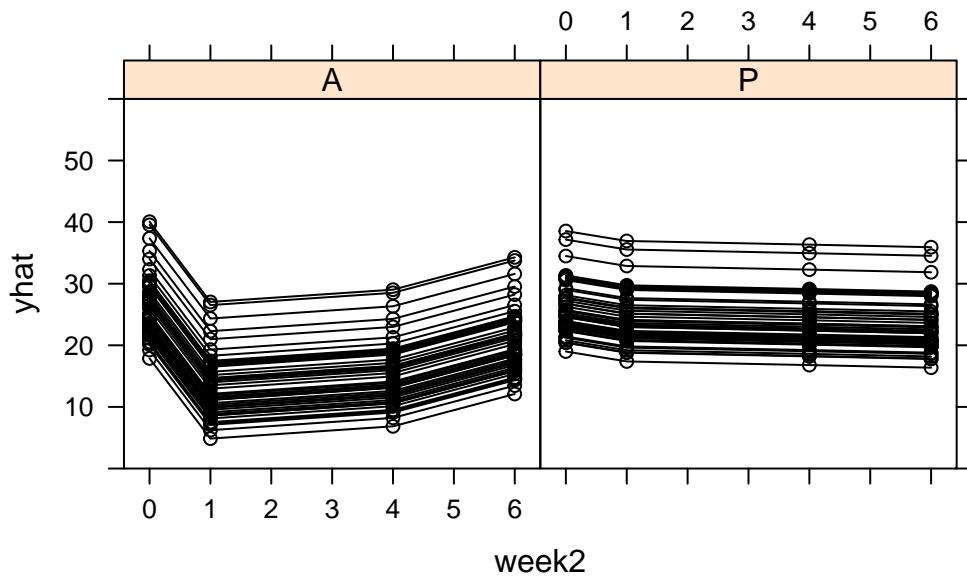
```
#we can plot the xy plot of the fitted values
```

```

yhat=predict(model,newdata = TLC_long[,-4],level=0:1)
TLC_long_2=TLC_long
TLC_long_2$yhat=yhat[,3]
TLC_long_2$week2=as.numeric(as.character(TLC_long_2$week))

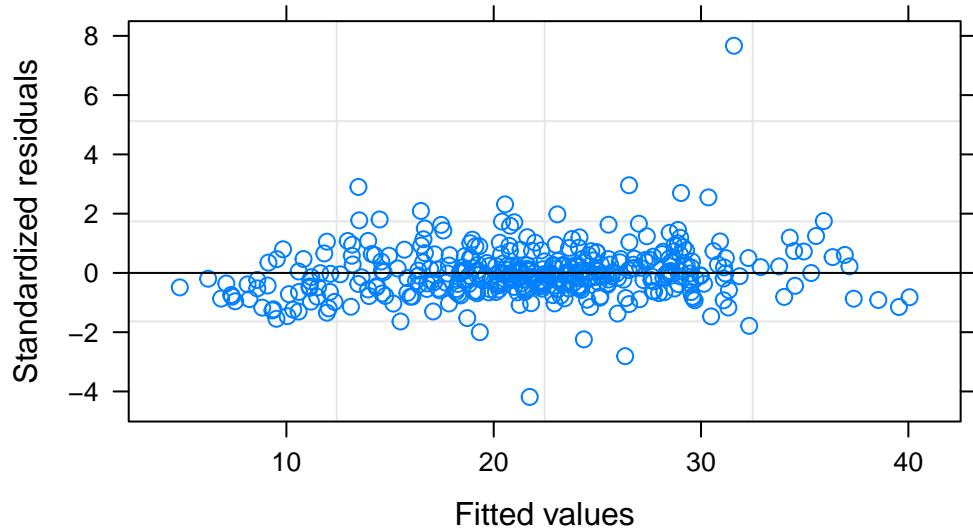
lattice::xyplot(yhat ~ week2|Treatment,data=TLC_long_2 , groups = ID,
                 col = 'black',
                 type = c('l', 'p'),ylim=c(0,60))

```

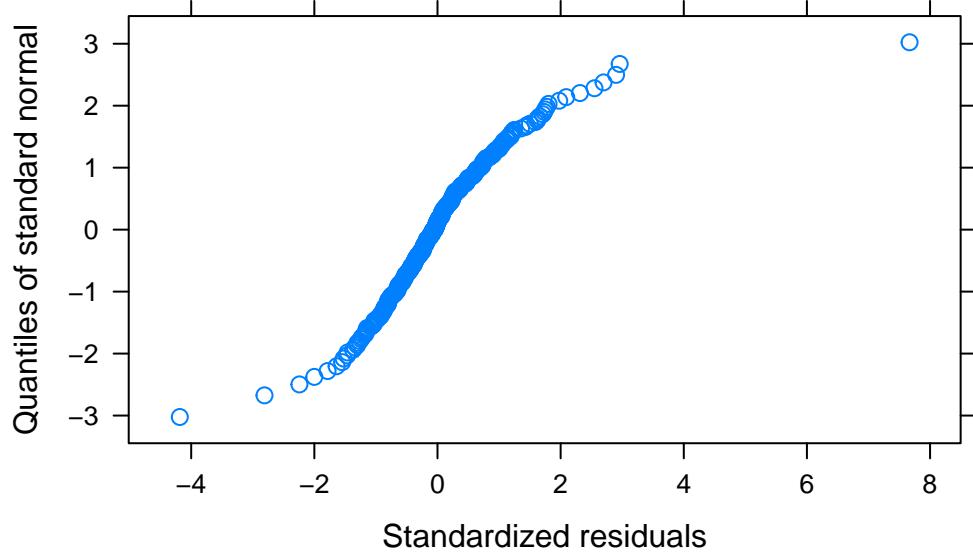


```
# residuals over time?  
  
# Residuals vs. Fitted (no patterns)  
plot(model, main = "Plot of residuals vs. fitted.")
```

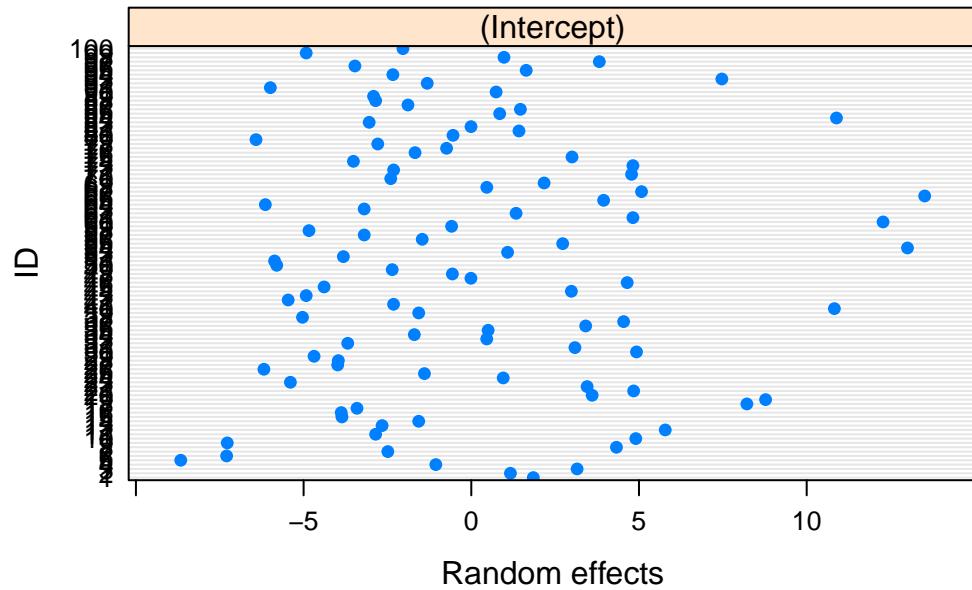
### Plot of residuals vs. fitted.



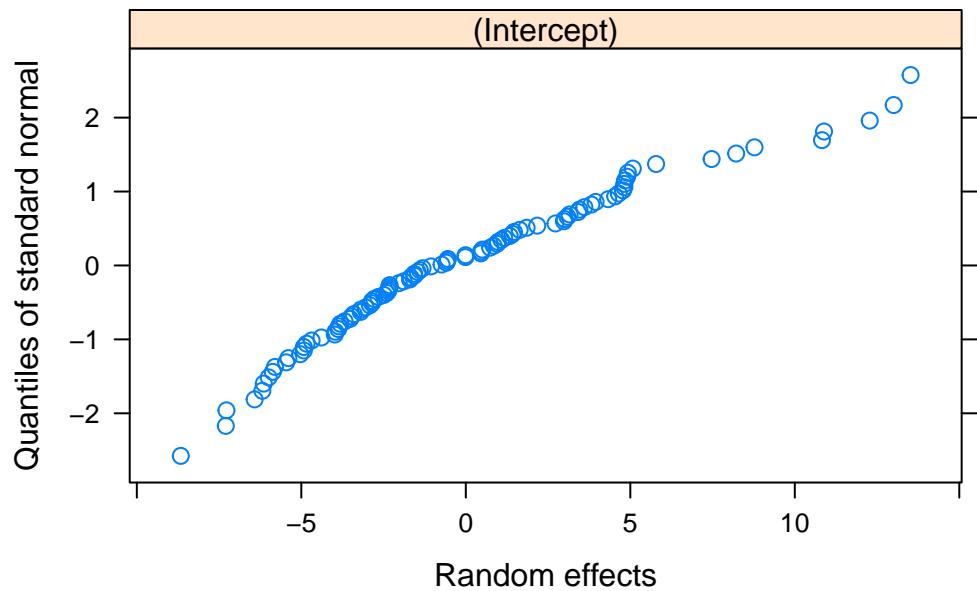
```
# QQPlot for normality of errors  
qqnorm(model, ~ residuals(., type="pearson")) # Some issues... probably
```



```
# Plots for the Predicted (BLUPs)
plot(nlme:::ranef(model))
```

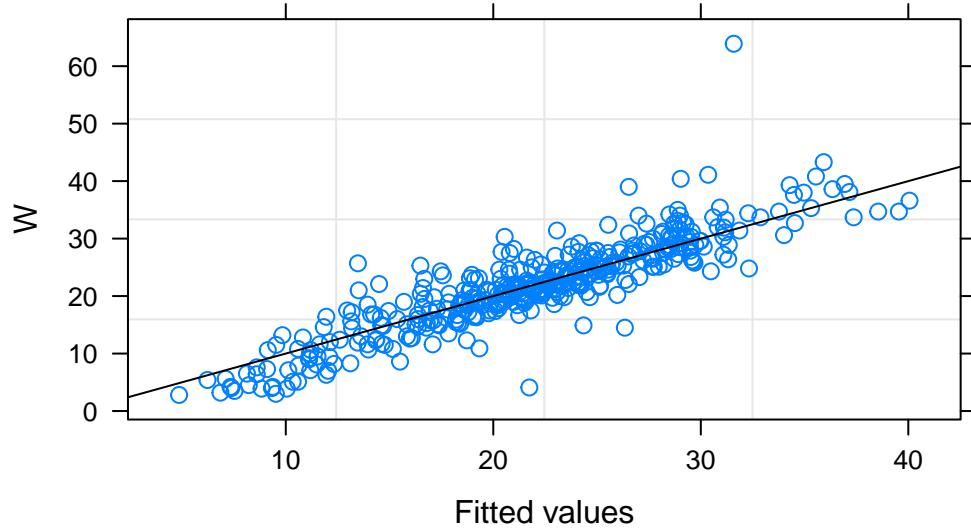


```
qqnorm(model, ~ranef(.)) # These look okay!
```



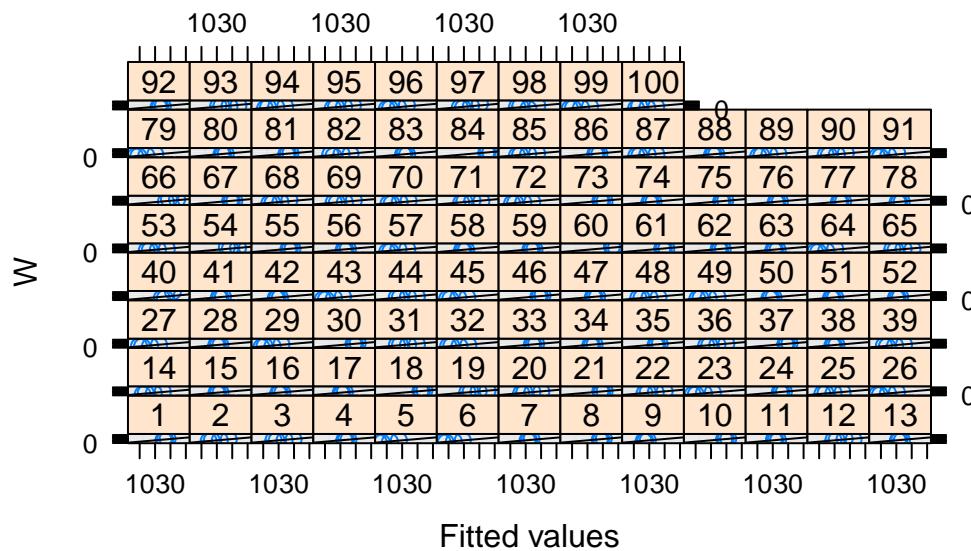
```
# model$residuals  
  
# Observed vs. Fitted  
plot(model, W ~ fitted(.), abline = c(0,1), main = "Observed vs. Fitted")
```

### Observed vs. Fitted



```
plot(model, W~ fitted(.)|ID, abline = c(0,1), main = "Observed vs. Fitted (By Subject)")
```

### Observed vs. Fitted (By Subject)



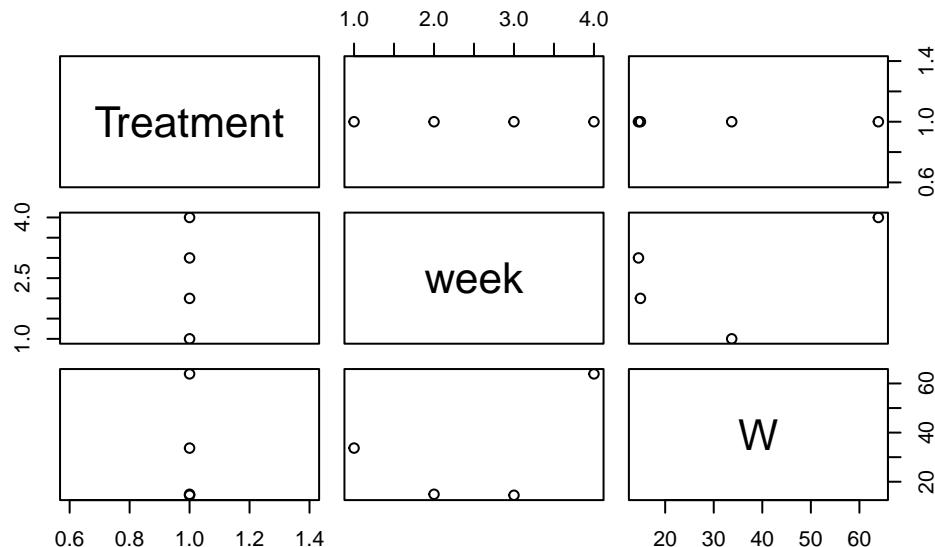
```
# Could also look (e.g.) by treatment, if it existed!
```

Some thoughts

- Maybe we should investigate this residual... If this patient is outlying for idiosyncratic reasons we may want to remove them and redo the analysis
- The fitted xy plots look like the empirical ones - good sign
- Everything else looks pretty good

```
id= TLC_long$ID[which(residuals(model, type="pearson")>5)]
```

```
plot(TLC_long[TLC_long$ID==id,-1])
```

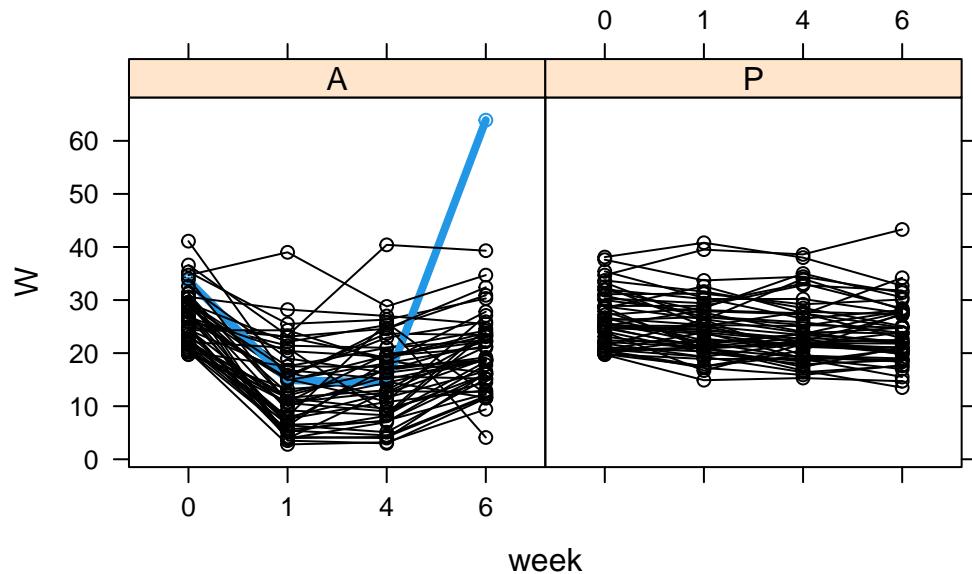


```
col=rep(1,400)
col[TLC_long$ID==id]=4

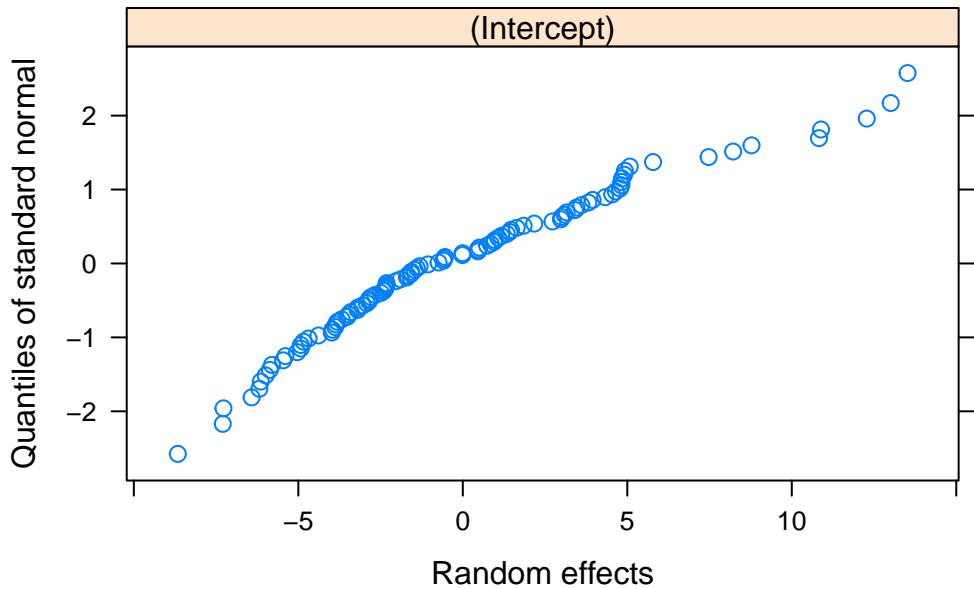
lwd=rep(1,400)
lwd[TLC_long$ID==id]=4

lattice::xyplot(W ~ week | Treatment,
                 data = TLC_long,
```

```
groups = ID,  
col = col,  
type = c('l', 'p'), lwd=lwd)
```



```
qqnorm(model, ~ranef(.))
```



```

TLC_long_3=TLC_long[TLC_long$ID!=id,]

model2 <- nlme::lme(fixed= W ~ 1 +Treatment+week+Treatment*week,random= ~1| ID, data = TLC_3)

summary(model2)

Linear mixed-effects model fit by REML
  Data: TLC_long
    AIC      BIC      logLik
 2480.621 2520.334 -1230.311

Random effects:
Formula: ~1 | ID
        (Intercept) Residual
StdDev:   5.112717 4.214287

Fixed effects: W ~ 1 + Treatment + week + Treatment * week
                Value Std.Error DF t-value p-value
(Intercept)     26.540 0.9370175 294 28.323912 0.0000
TreatmentP      -0.268 1.3251428  98 -0.202242 0.8401
week1           -13.018 0.8428574 294 -15.445080 0.0000
week4           -11.026 0.8428574 294 -13.081691 0.0000

```

```

week6          -5.778 0.8428574 294  -6.855252  0.0000
TreatmentP:week1 11.406 1.1919804 294   9.568950  0.0000
TreatmentP:week4  8.824 1.1919804 294   7.402807  0.0000
TreatmentP:week6  3.152 1.1919804 294   2.644339  0.0086
Correlation:
              (Intr) TrtmnP week1 week4 week6 TrtP:1 TrtP:4
TreatmentP     -0.707
week1          -0.450  0.318
week4          -0.450  0.318  0.500
week6          -0.450  0.318  0.500  0.500
TreatmentP:week1 0.318 -0.450 -0.707 -0.354 -0.354
TreatmentP:week4 0.318 -0.450 -0.354 -0.707 -0.354  0.500
TreatmentP:week6 0.318 -0.450 -0.354 -0.354 -0.707  0.500  0.500

```

#### Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-4.18502705	-0.46501691	-0.04732964	0.36499878	7.66712566

Number of Observations: 400

Number of Groups: 100

```
summary(model2)
```

Linear mixed-effects model fit by REML

Data: TLC\_long\_3  
 AIC BIC logLik  
 2371.01 2410.62 -1175.505

Random effects:

Formula: ~1 | ID  
 (Intercept) Residual  
 StdDev: 5.083126 3.671238

Fixed effects: W ~ 1 + Treatment + week + Treatment \* week

	Value	Std.Error	DF	t-value	p-value
(Intercept)	26.393878	0.8957514	291	29.465627	0.0000
TreatmentP	-0.121878	1.2604340	97	-0.096695	0.9232
week1	-12.900000	0.7417021	291	-17.392428	0.0000
week4	-10.859184	0.7417021	291	-14.640897	0.0000
week6	-6.512245	0.7417021	291	-8.780135	0.0000
TreatmentP:week1	11.288000	1.0436674	291	10.815707	0.0000

```

TreatmentP:week4    8.657184 1.0436674 291    8.294964  0.0000
TreatmentP:week6    3.886245 1.0436674 291    3.723643  0.0002
Correlation:
              (Intr) TrtmnP week1 week4 week6 TrtP:1 TrtP:4
TreatmentP      -0.711
week1          -0.414  0.294
week4          -0.414  0.294  0.500
week6          -0.414  0.294  0.500  0.500
TreatmentP:week1  0.294 -0.414 -0.711 -0.355 -0.355
TreatmentP:week4  0.294 -0.414 -0.355 -0.711 -0.355  0.500
TreatmentP:week6  0.294 -0.414 -0.355 -0.355 -0.711  0.500  0.500

```

#### Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-4.63582605	-0.49846350	-0.05679961	0.40026479	3.28119867

Number of Observations: 396

Number of Groups: 99

```
nlme::intervals(model)
```

Approximate 95% confidence intervals

#### Fixed effects:

	lower	est.	upper
(Intercept)	24.6958881	26.540	28.384112
TreatmentP	-2.8977028	-0.268	2.361703
week1	-14.6767987	-13.018	-11.359201
week4	-12.6847987	-11.026	-9.367201
week6	-7.4367987	-5.778	-4.119201
TreatmentP:week1	9.0601043	11.406	13.751896
TreatmentP:week4	6.4781043	8.824	11.169896
TreatmentP:week6	0.8061043	3.152	5.497896

#### Random Effects:

Level: ID	lower	est.	upper
sd((Intercept))	4.33785	5.112717	6.025997

#### Within-group standard error:

lower	est.	upper
3.887059	4.214287	4.569062

```
nlme::intervals(model2)
```

Approximate 95% confidence intervals

Fixed effects:

	lower	est.	upper
(Intercept)	24.630905	26.3938776	28.156850
TreatmentP	-2.623490	-0.1218776	2.379735
week1	-14.359781	-12.9000000	-11.440219
week4	-12.318964	-10.8591837	-9.399403
week6	-7.972026	-6.5122449	-5.052464
TreatmentP:week1	9.233907	11.2880000	13.342093
TreatmentP:week4	6.603090	8.6571837	10.711277
TreatmentP:week6	1.832151	3.8862449	5.940338

Random Effects:

Level: ID

	lower	est.	upper
sd((Intercept))	4.334069	5.083126	5.961643

Within-group standard error:

lower	est.	upper
3.384769	3.671238	3.981952

```
nlme::fixef(model)-nlme::fixef(model2)
```

(Intercept)	TreatmentP	week1	week4
0.1461224	-0.1461224	-0.1180000	-0.1668163
week6	TreatmentP:week1	TreatmentP:week4	TreatmentP:week6
0.7342449	0.1180000	0.1668163	-0.7342449

```
anova(model2)
```

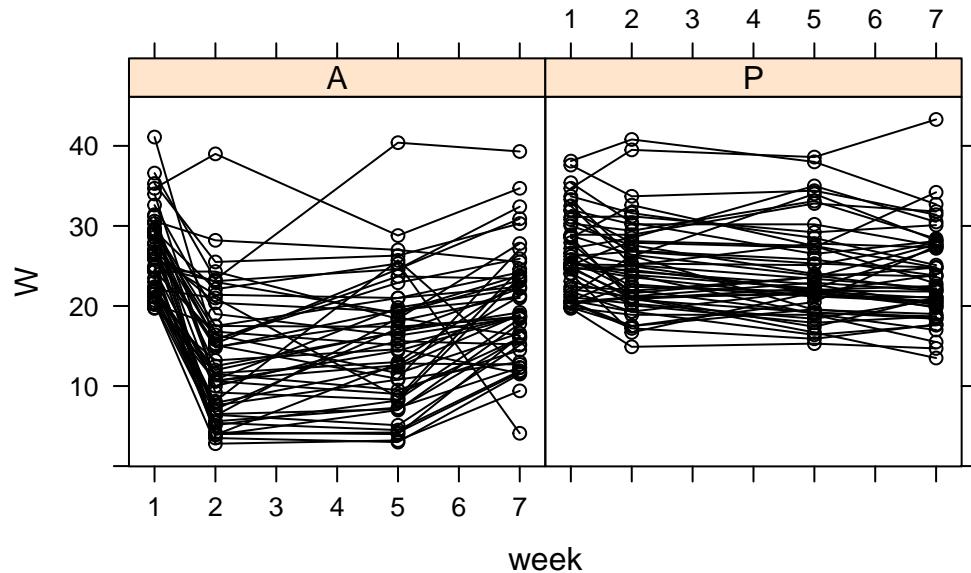
	numDF	denDF	F-value	p-value
(Intercept)	1	291	1606.9197	<.0001
Treatment	1	97	28.8577	<.0001
week	3	291	77.0542	<.0001
Treatment:week	3	291	46.2000	<.0001

### 1.6.3 Sensitivity analysis - We could have fit a quadratic or piece-wise linear model to the data.

```
TLC_long_pl=TLC_long_3
TLC_long_pl$week=as.numeric(as.character(TLC_long_3$week))+1
TLC_long_pl$time1=(TLC_long_pl$week<2)*TLC_long_pl$week
# TLC_long_pl$time2=(TLC_long_pl$week>=3)*(TLC_long_pl$week-TLC_long_pl$week)
head(TLC_long_pl)
```

	ID	Treatment	week	W	time1
1.0	1	P	1	30.8	1
2.0	2	A	1	26.5	1
3.0	3	A	1	25.8	1
4.0	4	P	1	24.7	1
5.0	5	A	1	20.4	1
6.0	6	A	1	20.4	1

```
lattice::xyplot(W ~ week | Treatment, data=TLC_long_pl, groups = ID,
                 col = 'black',
                 type = c('l', 'p'))
```



```

model_pl <- nlme::lme(fixed= W ~ week+time1+Treatment+Treatment*week+Treatment*time1,random=~1|ID,method="REML")
summary(model_pl)

Linear mixed-effects model fit by REML
Data: TLC_long_pl
      AIC      BIC    logLik
2382.985 2414.714 -1183.492

Random effects:
Formula: ~1 | ID
          (Intercept) Residual
StdDev:     5.076697 3.706653

Fixed effects: W ~ week + time1 + Treatment + Treatment * week + Treatment * time1
                Value Std.Error DF t-value p-value
(Intercept) 10.561547 1.0495337 293 10.063085 0
week         1.230397 0.1487828 293  8.269756 0
time1        14.601933 0.8194318 293 17.819584 0
TreatmentP   14.507927 1.4768248  97  9.823729 0
week:TreatmentP -1.432713 0.2093560 293 -6.843432 0
time1:TreatmentP -13.197091 1.1530427 293 -11.445449 0

Correlation:
            (Intr) week   time1 TrtmnP wk:TrP
week       -0.662
time1      -0.549  0.666
TreatmentP -0.711  0.470  0.390
week:TreatmentP  0.470 -0.711 -0.473 -0.662
time1:TreatmentP  0.390 -0.473 -0.711 -0.549  0.666

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-4.39986915 -0.47078045 -0.04895001  0.39625544  3.32467517

Number of Observations: 396
Number of Groups: 99

summary(model2)

```

```

Linear mixed-effects model fit by REML
Data: TLC_long_3

```

```
AIC      BIC      logLik  
2371.01 2410.62 -1175.505
```

Random effects:

```
Formula: ~1 | ID  
          (Intercept) Residual  
StdDev:    5.083126 3.671238
```

Fixed effects: W ~ 1 + Treatment + week + Treatment \* week

	Value	Std.Error	DF	t-value	p-value
(Intercept)	26.393878	0.8957514	291	29.465627	0.0000
TreatmentP	-0.121878	1.2604340	97	-0.096695	0.9232
week1	-12.900000	0.7417021	291	-17.392428	0.0000
week4	-10.859184	0.7417021	291	-14.640897	0.0000
week6	-6.512245	0.7417021	291	-8.780135	0.0000
TreatmentP:week1	11.288000	1.0436674	291	10.815707	0.0000
TreatmentP:week4	8.657184	1.0436674	291	8.294964	0.0000
TreatmentP:week6	3.886245	1.0436674	291	3.723643	0.0002

Correlation:

	(Intr)	TrtmnP	week1	week4	week6	TrtP:1	TrtP:4
TreatmentP	-0.711						
week1	-0.414	0.294					
week4	-0.414	0.294	0.500				
week6	-0.414	0.294	0.500	0.500			
TreatmentP:week1	0.294	-0.414	-0.711	-0.355	-0.355		
TreatmentP:week4	0.294	-0.414	-0.355	-0.711	-0.355	0.500	
TreatmentP:week6	0.294	-0.414	-0.355	-0.355	-0.711	0.500	0.500

Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-4.63582605	-0.49846350	-0.05679961	0.40026479	3.28119867

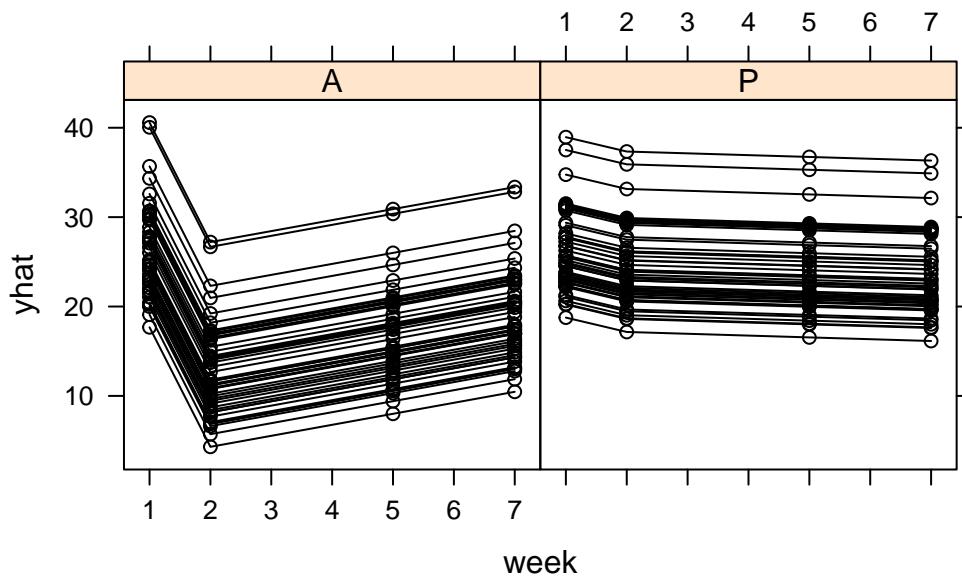
Number of Observations: 396

Number of Groups: 99

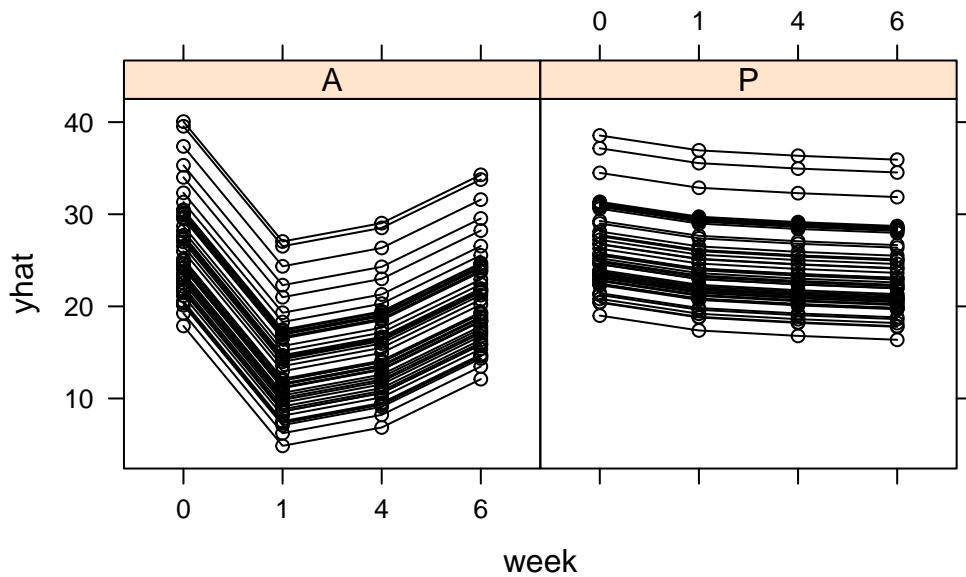
```
#we can plot the xy plot of the fitted values
```

```
yhat=predict(model_pl,newdata = TLC_long_pl[,-4],level=0:1)  
TLC_long_4=TLC_long_pl  
TLC_long_4$yhat=yhat[,3]
```

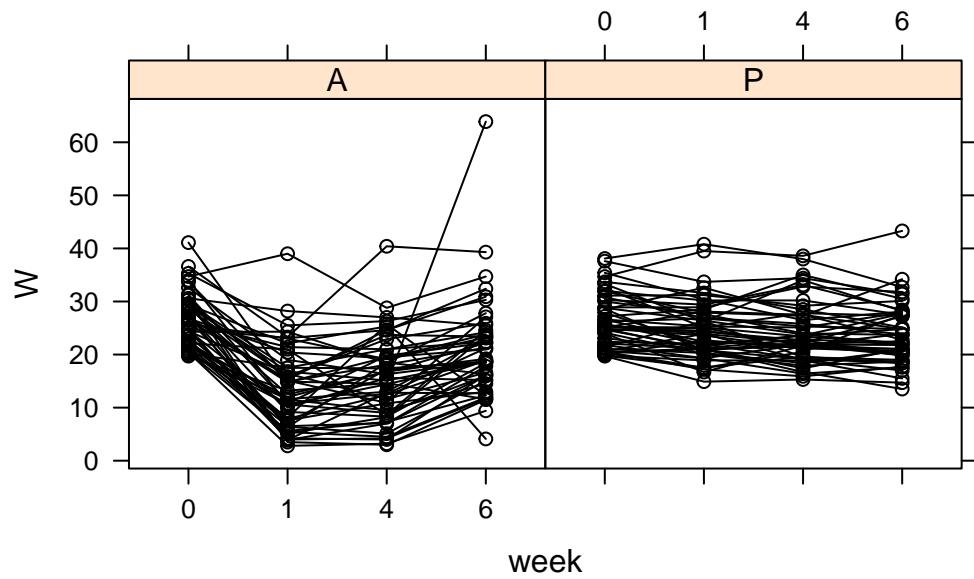
```
lattice::xyplot(yhat ~ week|Treatment,data=TLC_long_4 , groups = ID,
                 col = 'black',
                 type = c('l', 'p'))
```



```
lattice::xyplot(yhat ~ week|Treatment,data=TLC_long_2 , groups = ID,
                 col = 'black',
                 type = c('l', 'p'))
```

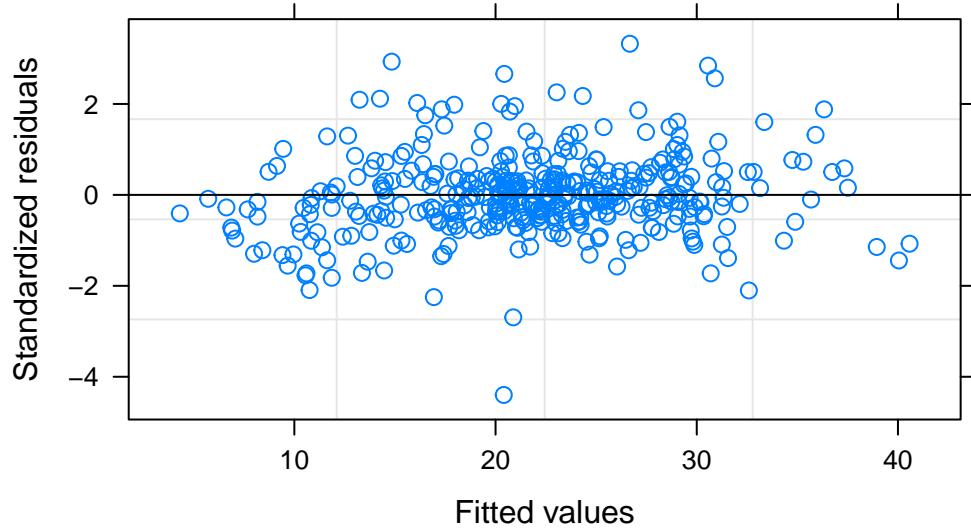


```
lattice::xyplot(W ~ week | Treatment, data= TLC_long_2 , groups = ID,
                 col = 'black',
                 type = c('l', 'p'))
```

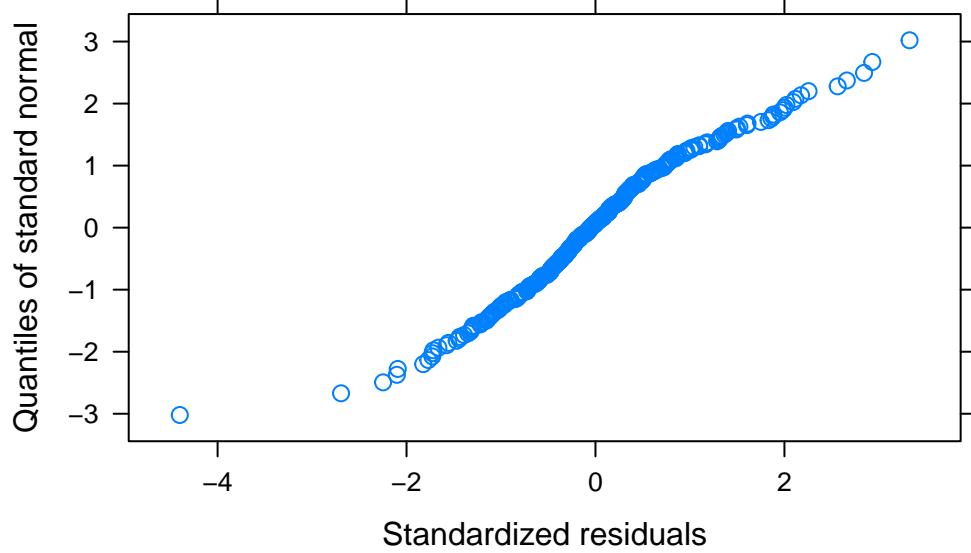


```
# residuals over time?  
  
# Residuals vs. Fitted (no patterns)  
plot(model_pl, main = "Plot of residuals vs. fitted.")
```

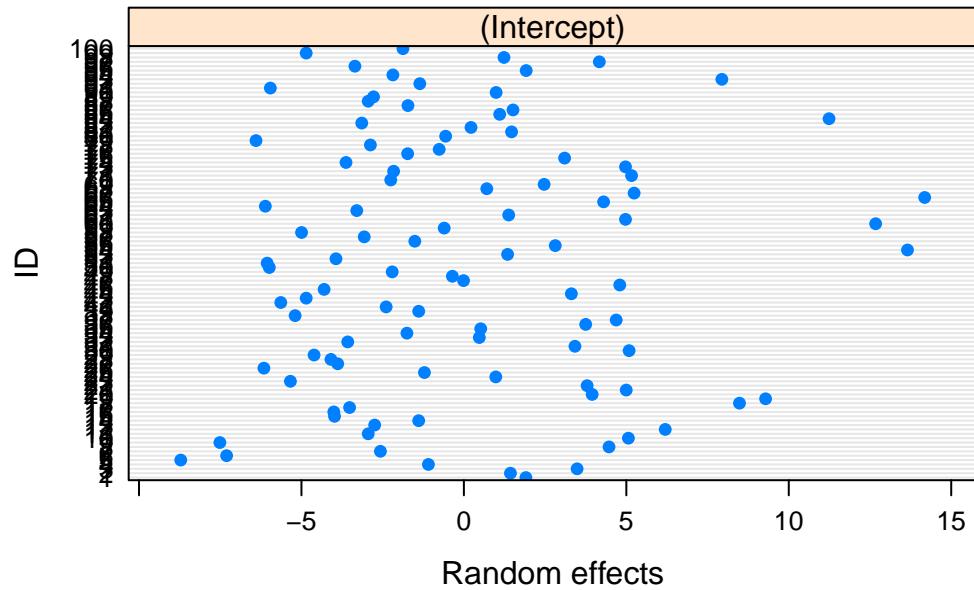
### Plot of residuals vs. fitted.



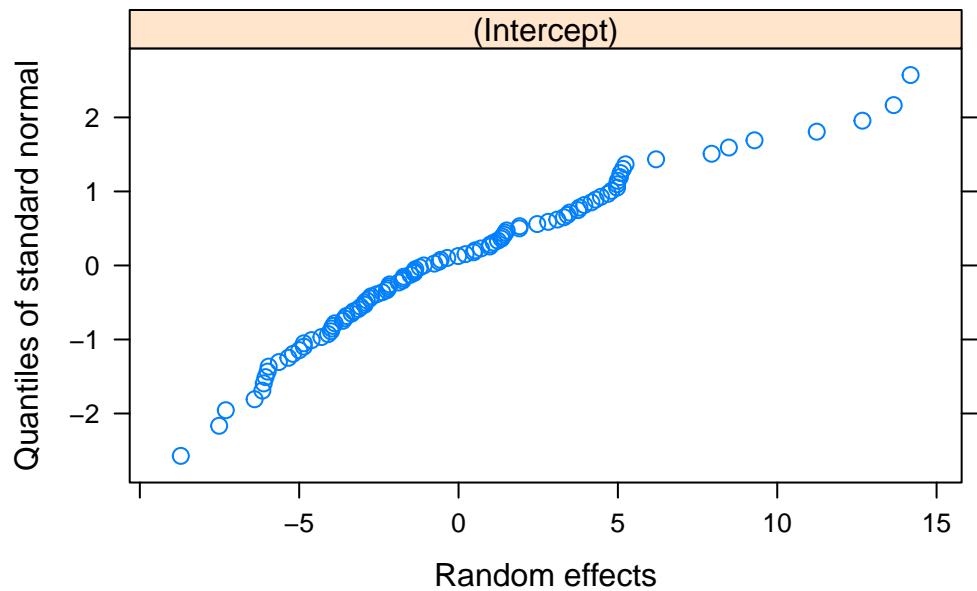
```
# QQPlot for normality of errors  
qqnorm(model_pl, ~ residuals(., type="pearson")) # Some issues... probably
```



```
# Plots for the Predicted (BLUPs)
plot(nlme::ranef(model_pl))
```

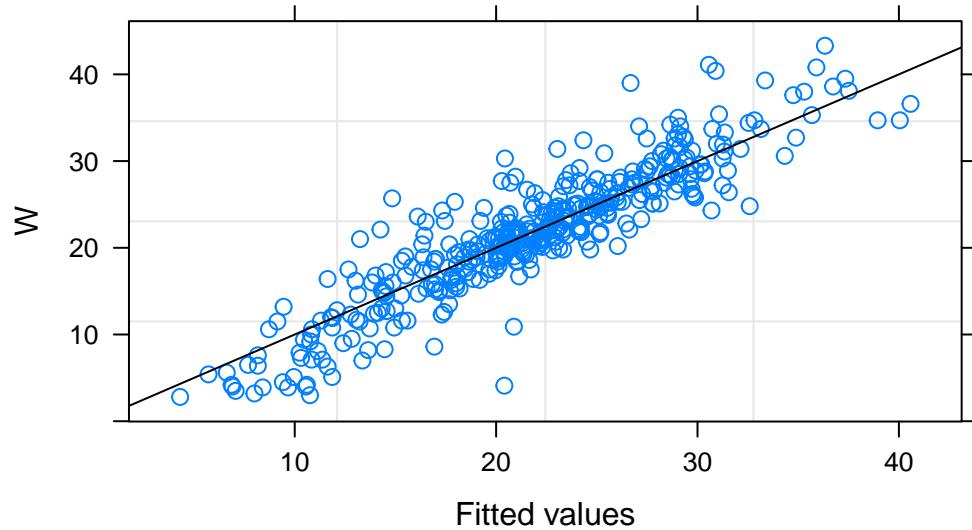


```
qqnorm(model_pl, ~ranef(.)) # These look okay!
```



```
# model$residuals  
  
# Observed vs. Fitted  
plot(model_pl, W ~ fitted(.), abline = c(0,1), main = "Observed vs. Fitted")
```

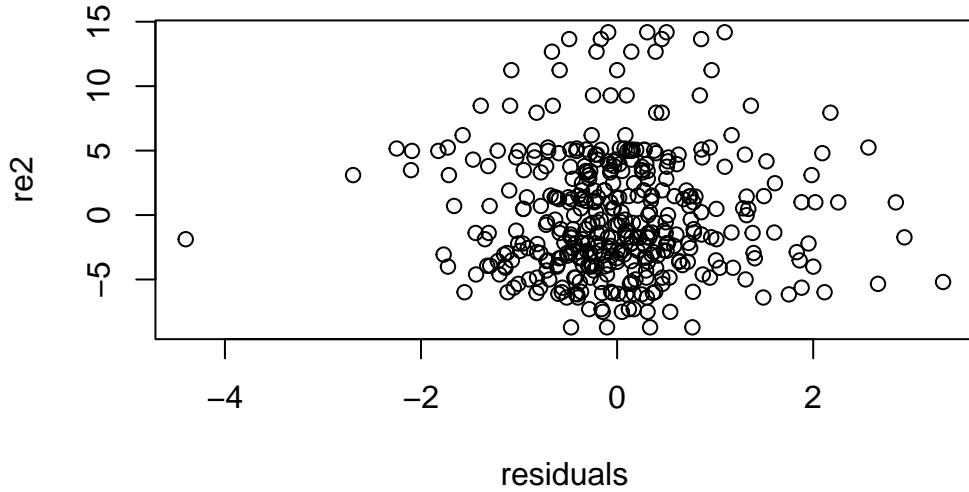
## Observed vs. Fitted



```
# Could also look (e.g.) by treatment, if it existed!

residuals=residuals(model_pl, type="pearson")
re=nlme::ranef(model_pl)
re2=rep(re$`Intercept` ,each=4)

plot(residuals,re2)
```



```
length(residuals)
```

```
[1] 396
```

```
length(re)
```

```
[1] 1
```

```
TLC_long_pl$weeksq= TLC_long_pl$week^2
model_q <- nlme::lme(fixed= W ~ week+weeksq+Treatment+week*Treatment+weeksq*Treatment, rand
summary(model_q)
```

Linear mixed-effects model fit by REML

Data: TLC\_long\_pl  
 AIC      BIC      logLik  
 2485.78 2517.509 -1234.89

Random effects:  
 Formula: ~1 | ID

```

(Intercept) Residual
StdDev:      4.948122 4.346842

Fixed effects: W ~ week + weeksq + Treatment + week * Treatment + weeksq * Treatment
                Value Std.Error DF     t-value p-value
(Intercept)    32.08886 1.2914204 293  24.847722 0.000
week          -9.43993 0.7337490 293 -12.865333 0.000
weeksq         1.12085 0.0908875 293 12.332330 0.000
TreatmentP    -5.11030 1.8171896  97 -2.812198 0.006
week:TreatmentP 8.33921 1.0324764 293  8.076897 0.000
weeksq:TreatmentP -1.02915 0.1278901 293 -8.047157 0.000
Correlation:
              (Intr) week   weeksq TrtmnP wk:TrP
week        -0.763
weeksq       0.707 -0.984
TreatmentP  -0.711  0.542 -0.502
week:TreatmentP 0.542 -0.711  0.699 -0.763
weeksq:TreatmentP -0.502  0.699 -0.711  0.707 -0.984

Standardized Within-Group Residuals:
      Min        Q1        Med        Q3        Max
-4.14185550 -0.45450738 -0.04543433  0.47296981  3.35222137

Number of Observations: 396
Number of Groups: 99

```

```
# model_q <- nlme::lme(fixed= W ~ week+weeksq+Treatment+weeksq*Treatment,random= ~1|ID, data=TLC_long_pl)
summary(model_q)
```

```

Linear mixed-effects model fit by REML
Data: TLC_long_pl
      AIC      BIC   logLik
2485.78 2517.509 -1234.89

Random effects:
Formula: ~1 | ID
(Intercept) Residual
StdDev:      4.948122 4.346842

```

```

Fixed effects: W ~ week + weeksq + Treatment + week * Treatment + weeksq * Treatment
                Value Std.Error DF     t-value p-value

```

(Intercept)	32.08886	1.2914204	293	24.847722	0.000
week	-9.43993	0.7337490	293	-12.865333	0.000
weeksq	1.12085	0.0908875	293	12.332330	0.000
TreatmentP	-5.11030	1.8171896	97	-2.812198	0.006
week:TreatmentP	8.33921	1.0324764	293	8.076897	0.000
weeksq:TreatmentP	-1.02915	0.1278901	293	-8.047157	0.000

Correlation:

	(Intr)	week	weeksq	TrtmnP	wk:TrP
week	-0.763				
weeksq	0.707	-0.984			
TreatmentP	-0.711	0.542	-0.502		
week:TreatmentP	0.542	-0.711	0.699	-0.763	
weeksq:TreatmentP	-0.502	0.699	-0.711	0.707	-0.984

Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-4.14185550	-0.45450738	-0.04543433	0.47296981	3.35222137

Number of Observations: 396

Number of Groups: 99

```
summary(model2)
```

Linear mixed-effects model fit by REML

Data: TLC\_long\_3  
 AIC      BIC      logLik  
 2371.01  2410.62 -1175.505

Random effects:

Formula: ~1 | ID  
 (Intercept) Residual  
 StdDev:    5.083126 3.671238

Fixed effects: W ~ 1 + Treatment + week + Treatment \* week

	Value	Std.Error	DF	t-value	p-value
(Intercept)	26.393878	0.8957514	291	29.465627	0.0000
TreatmentP	-0.121878	1.2604340	97	-0.096695	0.9232
week1	-12.900000	0.7417021	291	-17.392428	0.0000
week4	-10.859184	0.7417021	291	-14.640897	0.0000
week6	-6.512245	0.7417021	291	-8.780135	0.0000
TreatmentP:week1	11.288000	1.0436674	291	10.815707	0.0000

```

TreatmentP:week4    8.657184 1.0436674 291    8.294964  0.0000
TreatmentP:week6    3.886245 1.0436674 291    3.723643  0.0002
Correlation:
              (Intr) TrtmnP week1  week4  week6  TrtP:1 TrtP:4
TreatmentP      -0.711
week1          -0.414  0.294
week4          -0.414  0.294  0.500
week6          -0.414  0.294  0.500  0.500
TreatmentP:week1  0.294 -0.414 -0.711 -0.355 -0.355
TreatmentP:week4  0.294 -0.414 -0.355 -0.711 -0.355  0.500
TreatmentP:week6  0.294 -0.414 -0.355 -0.355 -0.711  0.500  0.500

```

Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-4.63582605	-0.49846350	-0.05679961	0.40026479	3.28119867

Number of Observations: 396

Number of Groups: 99

#we can plot the xy plot of the fitted values

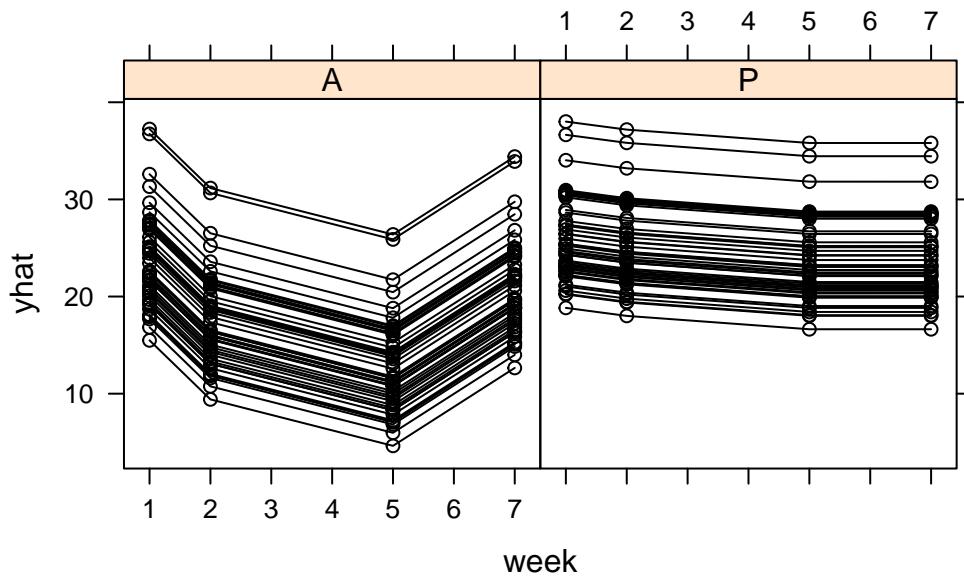
```

yhat=predict(model_q,newdata = TLC_long_pl[,-4],level=0:1)
TLC_long_5=TLC_long_pl
TLC_long_5$yhat=yhat[,3]

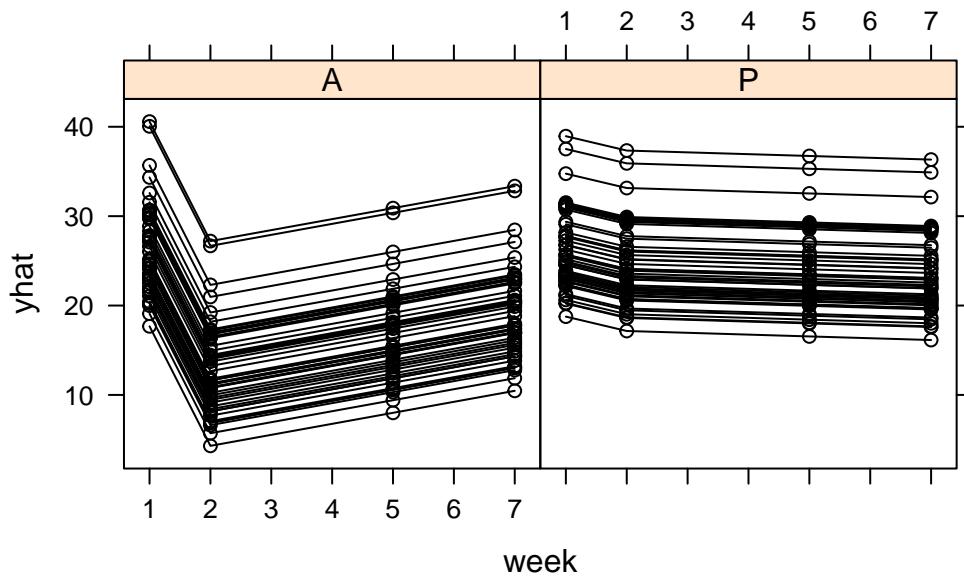
par(mfrow=c(2,2))

lattice::xyplot(yhat ~ week|Treatment,data=TLC_long_5 , groups = ID,
                 col = 'black',
                 type = c('l', 'p'))

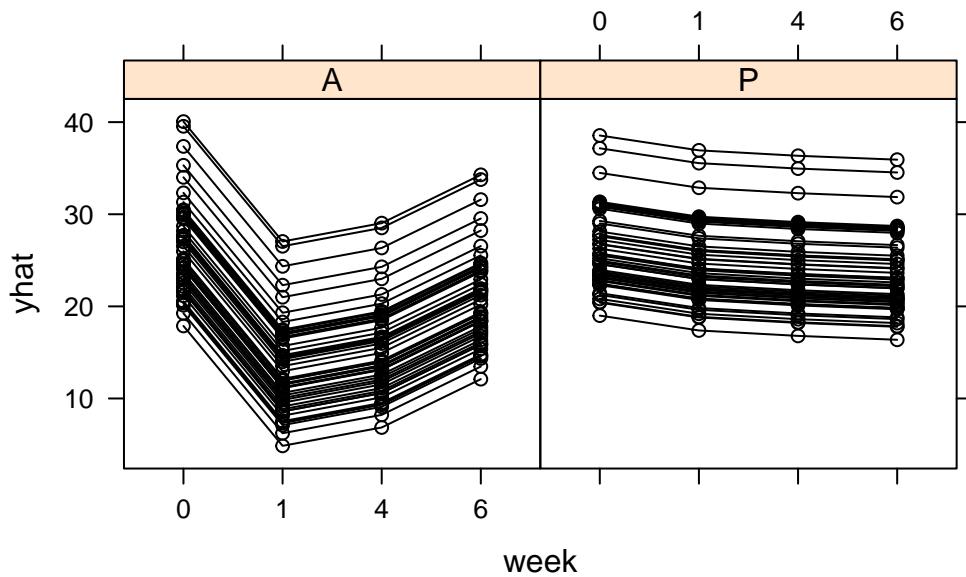
```



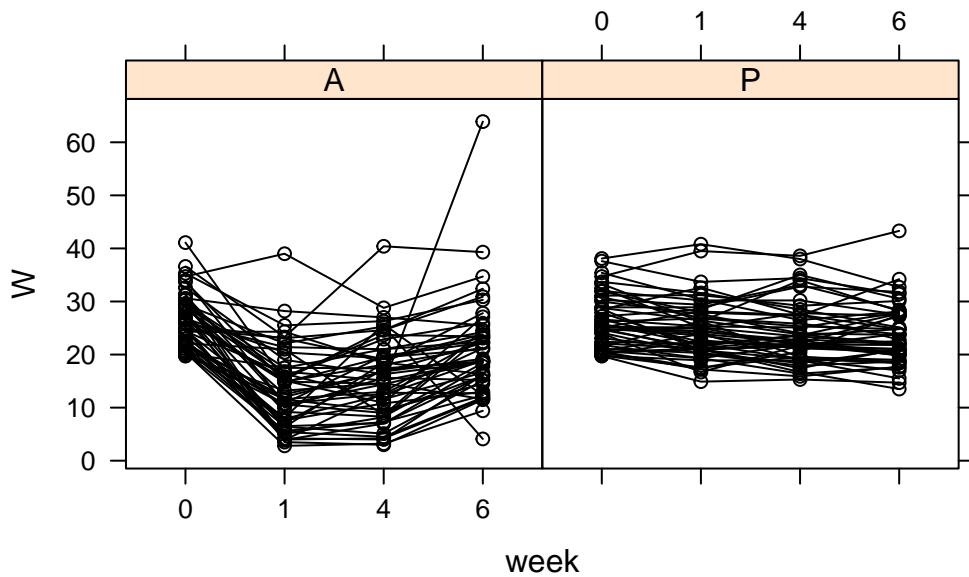
```
lattice::xyplot(yhat ~ week | Treatment, data= TLC_long_4 , groups = ID,  
                 col = 'black',  
                 type = c('l', 'p'))
```



```
lattice::xyplot(yhat ~ week | Treatment, data= TLC_long_2 , groups = ID,  
                 col = 'black',  
                 type = c('l', 'p'))
```



```
lattice::xyplot(W ~ week | Treatment, data= TLC_long_2 , groups = ID,
                 col = 'black',
                 type = c('l', 'p'))
```



Conclusions:

- All models indicate a significant effect of treatment, with the largest drop being a time point
  - 1.
- The lead levels seem to be returning to baseline over time
- The treatment certainly reduces lead levels for a few weeks
- Investigate subject data with ID 40.

We can get more specific too

- At time point 1, individual lead levels seem to drop by 11 points over the placebo.
- After that, the gap starts closing over time - valued at 11, 8, and then 3

## References

- Cabrera, Javier, and Andrew McDougall. 2002. *Statistical Consulting*. Springer New York. <https://doi.org/10.1007/978-1-4757-3663-2>.
- Gelman, Andrew, and Jennifer Hill. 2006. *Data Analysis Using Regression and Multi-level/Hierarchical Models*. Cambridge University Press. <https://doi.org/10.1017/cbo9780511790942>.
- Pinheiro, J. C., and D. Bates. 2009. *Mixed-Effects Models in s and s-PLUS*. Statistics and Computing. Springer. <https://books.google.ca/books?id=y54QDUTmvDcC>.
- Wu, Lang. 2019. *Mixed Effects Models for Complex Data*. Chapman & Hall/CRC Monographs on Statistics & Applied Probability. Philadelphia, PA: Chapman & Hall/CRC.