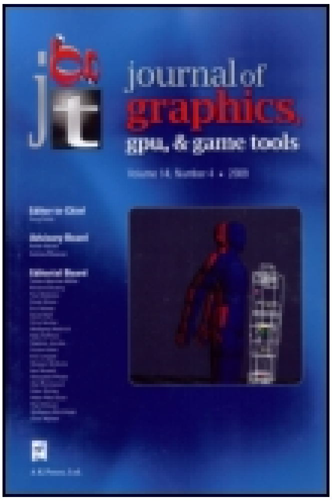

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Building an Orthonormal Basis from a Unit Vector

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Abstract. We show how to easily create a right-handed orthonormal basis, given a unit vector, in 2-, 3-, and 4-space.

1. Introduction

Often in graphics, we have a unit vector, $\mathbf{u}$, that we wish to extend to a basis (i.e., we want to enlarge the set $\{\mathbf{u}\}$ by adding new vectors to it until $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \dots\}$ is a basis, as in, e.g., [Hoffman, Kunze 71], Section 2.5, Theorem 5). For example, when we want to put a coordinate system (e.g., for texture-mapping) on a user-specified plane in 3-space, the natural specification of the plane is to give its normal, but this leaves the choice of plane-basis ambiguous up to a rotation in the plane. We describe the solution to this problem in two, three, and four dimensions.

2. Two Dimensions and Four Dimensions

Two dimensions and four dimensions are the easy cases: To extend $\mathbf{u} = (x, y)$ to an orthonormal basis of $\Re^2$, let

$$\mathbf{v} = (-y, x).$$

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This corresponds to taking the complex number $x + iy$ and mutiplying by i, which rotates 90 degrees clockwise. To extend $\mathbf{u} = (a, b, c, d)$ to an orthonormal basis of $\Re^4$, let

$$\begin{aligned} \mathbf{v} &= (-b, \quad a, -d, \quad c) \\ \mathbf{w} &= (-c, \quad d, \quad a, -b) \\ \mathbf{x} &= (-d, -c, \quad b, \ a). \end{aligned}$$

This corresponds to multiplying the quaternion $a + bi + cj + dk$ by i, j, and k, respectively.

3. Three Dimensions

Oddly, three dimensions are harder—there is no continuous solution to the problem. If there were, we could take each unit vector $\mathbf{u}$ and extend it to a basis $\mathbf{u}, \mathbf{v}(\mathbf{u}), \mathbf{w}(\mathbf{u})$, where $\mathbf{v}$ is a continuous function. By drawing the vector $\mathbf{v}(\mathbf{u})$ at the tip of the vector $\mathbf{u}$, we would create a continuous nonzero vector field on the sphere, which is impossible [Milnor 65].

Here is a numerically stable and simple way to solve the problem, although it is not continuous in the input: Take the smallest entry (in absolute value) of $\mathbf{u}$ and set it to zero; swap the other two entries and negate the first of them. The resulting vector $\bar{\mathbf{v}}$ is orthogonal to $\mathbf{u}$ and its length is at least $\sqrt{2/3} \approx .82$. Thus, given $\mathbf{u} = (x, y, z)$ let

$$\begin{aligned} \bar{\mathbf{v}} &= \begin{cases} (\ \ 0, -z, y), \text{ if } |x|<|y| \text{ and } |x|<|z| \\ (-z, \quad 0, x), \text{ if } |y|<|x| \text{ and } |y|<|z| \\ (-y, \quad x, 0), \text{ if } |z|<|x| \text{ and } |z|<|y| \end{cases} \\ \mathbf{v} &= \bar{\mathbf{v}}/||\bar{\mathbf{v}}|| \\ \mathbf{w} &= \mathbf{u} \times \mathbf{v}. \end{aligned}$$

Then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is an orthonormal basis. As a simple example, consider $\mathbf{u} = (-2/7, 6/7, 3/7)$. In this case, $\bar{\mathbf{v}} = (0, -3/7, 6/7)$, $v = \frac{1}{\sqrt{45}}(0, -3, 6)$, and $\mathbf{w} = \mathbf{u} \times \mathbf{v} = \frac{1}{7\sqrt{45}}(45, 12, 6)$.

3.1. Discussion

A more naive approach would be to simply compute $\mathbf{v} = \mathbf{e_1} \times \mathbf{u}$ and $\mathbf{u} = \mathbf{v} \times \mathbf{w}$. This becomes ill-behaved when $\mathbf{u}$ and $\mathbf{e}_1$ are nearly parallel, at which point the naive approach substitutes $\mathbf{e_2}$ for $\mathbf{e_1}$. One could also choose a random vector instead of $\mathbf{e_1}$, and this works with high probability. Our algorithm simply systematically avoids the problem with these two approaches.

Another naive approach is to apply the Gram-Schmidt process (see, for example, [Hoffman, Kunze 71], Section 8.2) to the set $\mathbf{u}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$, discarding any vector whose projection onto the subspace orthogonal to the prior ones is shorter than, say, 1/10th. This works too—in fact, it can be used for any number of dimensions—but uses multiple square roots, and hence is computationally expensive.

References

[Milnor 65] John Milnor. *Topology from the Differentiable Viewpoint*, Charlottesville: University Press of Virginia, 1965.

[Hoffman, Kunze 71] Kenneth Hoffman and Ray Kunze. *Linear Algebra*, Englewood Cliffs, NJ: Prentice Hall, 1971.

Web Information:

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