III.3

DECOMPOSING LINEAR AND AFFINE TRANSFORMATIONS

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Goal

Every nonsingular linear transformation of three-dimensional space is the product of three scales, two shears, and one rotation. The goal of this Gem is to show how to decompose any arbitrary, singular or nonsingular, linear or affine transformation of three-dimensional space into simple, geometrically meaningful, factors. For an alternative approach to similar problems (see Thomas, 1991).

Nonsingular Linear Transformations

Linear transformations of three-dimensional space are generally represented by 3×3 matrices. To decompose an arbitrary nonsingular linear transformation, consider, then, an arbitrary nonsingular 3×3 matrix L. We shall show that L can be factored into the product of three scales, two shears, and one rotation matrix.

Let the rows of L be given by the vectors u, v, w. Since the matrix L is nonsingular, the vectors u, v, w are linearly independent. Therefore, using the Gram-Schmidt orthogonalization procedure, we can generate 3 orthonormal vectors u^*, v^*, w^* by setting

$$u^* = u / |u|,$$

$$v^* = (v - (v \cdot u^*)u^*) / |v - (v \cdot u^*)u^*|,$$

$$w^* = (w - (w \cdot u^*)u^* - (w \cdot v^*)v^*) / |w - (w \cdot u^*)u^* - (w \cdot v^*)v^*|.$$

This orthogonalization procedure can be used to decompose the matrix L into the desired factors.

Begin with the rotation. By construction, the matrix R whose rows are u^*, v^*, w^* is an orthogonal matrix. If $\mathrm{Det}(R) = -1$, replace w^* by $-w^*$. Then R is the rotation matrix we seek. Using the results in Goldman (1991a), we can, if we like, retrieve the rotation axis and angle from the matrix R.

The three scaling transformations are also easy to find. Let

$$s_1 = |u|,$$

$$s_2 = |v - (v \cdot u^*)u^*|,$$

$$s_3 = |w - (w \cdot u^*)u^* - (w \cdot v^*)v^*|.$$

That is, s_1, s_2, s_3 are the lengths of u^*, v^*, w^* before they are normalized. Now let S be the matrix with s_1, s_2, s_3 along the diagonal and with zeroes everywhere else. The matrix S represents the three independent scaling transformations that scale by s_1 along the x-axis, s_2 along the y-axis, and s_3 along the z-axis. (If $\mathrm{Det}(R)$ was originally -1, then replace s_3 by $-s_3$. In effect, this mirrors points in the xy-plane.)

Before we can introduce the two shears, we need to recall notation for the identity matrix and the tensor product of two vectors.

Identity:

$$I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Tensor Product:

$$v \otimes \omega = \begin{vmatrix} v_1\omega_1 & v_1\omega_2 & v_1\omega_3 \\ v_2\omega_1 & v_2\omega_2 & v_2\omega_3 \\ v_3\omega_1 & v_3\omega_2 & v_3\omega_3 \end{vmatrix} = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} * |\omega_1 \omega_2 \omega_3| = v^t * \omega.$$

Here $*$ denotes matrix multiplication and the superscript t denotes transpose. Observe that for all vectors μ

$$\mu \cdot I = \mu,$$

$$\mu * (v \otimes \omega) = (\mu \cdot v)\omega.$$

Now we are ready to define the two shears. Recall from Goldman (199lb) that a shear H is defined in terms of a unit vector v normal to a plane Q, a unit vector ω in the plane Q, and an angle ϕ by setting

$$H = I + \tan\phi(v \otimes \omega).$$

Let H_1 be the shear defined by unit normal v^*, the unit vector u^*, and the angle θ by setting

$$H_1 = I + \tan\theta(v^* \otimes \omega^*),$$

$$\tan\theta = v \cdot u^* / s_2.$$

Similarly, let H_2 be the shear defined by unit normal w^*, the unit vector r^*, and the angle ψ by setting

$$H_2 = I + \tan\psi(w^* \otimes r^*),$$

$$\tan\psi = SQRT\{(w \cdot u^*)^2 + (w \cdot v^*)^2\} / s_3,$$

$$r^* = \{(w \cdot u^*)u^* + (w \cdot v^*)v^*\} / s_3 \tan\psi$$

Then it is easy to verify that

$$u^* * H_1 = u^*, \qquad v^* * H_1 = v^* + (v \cdot u^* / s_2)u^*, \qquad w^* * H_1 = w^*,$$

$$u^* * H_2 = u^*, \qquad v^* * H_2 = v^*,$$

$$w^* * H_2 = w^* + \{(w \cdot u^*)u^* + (w \cdot v^*)v^*\} / s_3.$$

Finally, we shall show that

$$L = S * R * H_1 * H_2.$$

Since the transformation L is linear, we need only check that both sides give the same result on the canonical basis **i**, **j**, **k**. By construction we know that

$$\mathbf{i} * L = u, \qquad \mathbf{j} * L = v, \qquad \mathbf{k} * L = w,$$

so we need to verify that we get the same results for the right-hand side. Let us check.

First,

$$\begin{aligned} \mathbf{i} * S * R * H_1 * H_2 &= (s_1)\mathbf{i} * R * H_1 * H_2 \\ &= (s_1)u^* * H_1 * H_2 \\ &= u. \end{aligned}$$

since by construction the two shears H_1 and H_2 do not affect u^*.

Next,

$$\begin{aligned} \mathbf{j} * S * R * H_1 * H_2 &= (s_2)\mathbf{j} * R * H_1 * H_2 \\ &= (s_2)v^* * H_1 * H_2 \\ &= \{s_2 v^* + (v \cdot u^*)u^*\} * H_2 \\ &= s_2 v^* + (v \cdot u^*)u^* \\ &= v. \end{aligned}$$

Finally,

$$\begin{aligned} \mathbf{k} * S * R * H_1 * H_2 &= (s_3)\mathbf{k} * R * H_1 * H_2 \\ &= (s_3)w^* * H_1 * H_2 \\ &= (s_3)w^* * H_2 \\ &= s_3 w^* + (w \cdot u^*)u^* + (w \cdot v^*)v^* \\ &= w. \end{aligned}$$

Although we have succeeded in factoring an arbitrary nonsingular linear transformation, notice that this decomposition is not unique. Indeed, the Gram-Schmidt orthogonalization procedure depends upon the ordering of the vectors to which it is applied. We could, for example, have applied the Gram-Schmidt procedure to the vectors in the order w, u, v instead of u, v, w. We would then have retrieved a different decomposition of the same matrix. Nevertheless, this procedure is still of some value since it allows us to decompose an arbitrary nonsingular linear transformation into simple, geometrically meaningful factors.

Singular Linear Transformations

Now let L be an arbitrary singular 3×3 matrix. There are three cases to consider, depending on the rank of L. If $\mathrm{rank}(L) = 0$, there is nothing to do since L simply maps all vectors into the zero vector. The case $\mathrm{rank}(L) = 1$ is also essentially trivial, since all vectors are simply appropriately scaled and then projected onto a single fixed vector. Therefore, we shall concentrate on the case where $\mathrm{rank}(L) = 2$.

We will show that when $\mathrm{rank}(L) = 2$, we still need one rotation, but we require only two scales, one shear, and one parallel projection. Thus, the number of scales is reduced by one and a shear is replaced by a parallel projection. Moreover, we shall shovv that the parallel projection can be replaced by a shear followed by an orthogonal projection.

Again, let the rows of L be given by the vectors u, v, w. Since the matrix L is singular, the row vectors u, v, w are linearly dependent, but since $\mathrm{rank}(L) = 2$, two rows of L are linearly independent. For simplicity and without loss of generality, we will assume that u and v are linearly independent.

Modifying the Gram-Schmidt orthogonalization procedure, we can generate three orthonormal vectors u^*, v^*, w^* by setting

$$u^* = u / |u|,$$

$$v^* = (v - (v \cdot u^*)u^*) / \left| v - (v \cdot u^*)u^* \right|,$$

$$w^* = u^* \times v^*.$$

This orthogonalization procedure can again be used to decompose the matrix L into the desired factors.

By construction, the matrix R whose rows are u^*, v^*, w^* is an orthogonal matrix and $\text{Det}(R) = 1$. The matrix R is the rotation matrix that we seek. We can recover the axis and angle of rotation from the matrix R using the techniques described in Goldman (199la).

The two scaling transformations are also easy to find. Let

$$s_1 = |u|,$$

$$s_2 = \left| v - (v \cdot u^*)u^* \right|.$$

Note that s_1, s_2 are, respectively, the lengths of u^*, v^* before they are normalized. Now let S be the matrix with s_1, s_2, 1 along the diagonal and with zeroes everywhere else. The matrix S represents the two independent scaling transformations that scale by s_1 along the x-axis and s_2 along the y-axis.

The shear H is the same as the first of the two shears that we used to decompose a nonsingular linear transformation. Using the notation for the identity matrix and the tensor product of two vectors that we established above,

$$H = I + \tan\theta(\mathrm{v}^* \otimes \mathrm{u}^*),$$

$$\tan\theta = v \cdot u^*/s_2.$$

Thus, H is the shear defined by the unit normal vector v^*, the unit vector u^*, and the angle θ. Again, it is easy to verify that

$$u^* * H = u^*, \qquad v^* * H = v^* + (v \cdot u^*/s_2)u^*, \qquad w^* * H = w^*.$$

Last, we define the parallel projection P to be projection into the u^*v^*-plane parallel to the vector$(w^* - w)$. According to Goldman(1990), the matrix P is given by

$$P = I - w^* \otimes (w^* - w).$$

Notice that if $w = 0$, this parallel projection reduces to orthogonal projection into the u^*v^*-plane (Goldman, 1990). In any event, it is easy to verify that

$$u^* * P = u^*, \qquad v^* * P = v^*, \qquad w^* * P = w.$$

Finally, let us show that

$$L = S * R * H * P$$

by checking that both sides give the same result on the canonical basis **i**, **j**, **k**. By construction we know that

$$\mathbf{i} * L = u, \qquad \mathbf{j} * L = v, \qquad \mathbf{k} * L = w,$$

so we need to verify that we get the same results for the right-hand side. Let us check.

First,

$$\begin{aligned} \mathbf{i} * S * R * H * P &= (s_1)\mathbf{i} * R * H * P \\ &= (s_1)u^* * H * P \\ &= s_1 u^* \\ &= u, \end{aligned}$$

since by construction the two linear transformations H and P do not affect u^*.

Next,

$$\begin{aligned} \mathbf{j} * S * R * H * P &= (s_2)\mathbf{j} * R * H * P \\ &= (s_2)v^* * H * P \\ &= \{s_2 v^* + (v \cdot u^*)u^*\} * P \\ &= s_2 v^* + (v \cdot u^*)u^* \\ &= v. \end{aligned}$$

Finally,

$$\begin{aligned}\mathbf{k} * S * R * H * P &= \mathbf{k} * R * H * P \\ &= w^* * H * P \\ &= w^* * P \\ &= w.\end{aligned}$$

By the way, every parallel projection can be written as the product of a shear followed by an orthogonal projection. To see this, recall that a projection P parallel to the vector ω into the plane with normal vector n is given by Goldman (1990):

$$P = I - (n \otimes \omega)/n \cdot \omega.$$

Consider the orthogonal projection O into the plane perpendicular to the unit vector n (Goldman, 1990),

$$O = I - (n \otimes n),$$

and the shear K defined by the unit normal vector n, the unit vector $v = (n - \omega/\omega \cdot n)/\left|n - \omega/\omega \cdot n\right|$, and the angle θ given by $\tan\theta = \left|n - \omega/\omega \cdot n\right|$:

$$K = I + \tan\theta(n \otimes v).$$

Since $v \cdot n = 0$, it follows that $(n \otimes v) * (n \otimes n) = (v \cdot n)(n \otimes n) = 0$. Therefore,

$$I - (n \otimes \omega)/n \cdot \omega = \{I + \tan\theta(n \otimes v)\} * \{I - (n \otimes n)\},$$

or, equivalently,

$$P = K * O$$

In our case $\omega = w^* - w$, $n = w^*$, and $v = w$. Thus, we have shown that when $\mathrm{rank}(L) = 2$, we can factor L into the product of two scales one rotation, two shears, and one orthogonal projection. This decomposition is the same as in the nonsingular case, except that the number of scales is reduced by one and the standard nonsingular factors—scales, rotation, and shears—are followed by an orthogonal projection.

Notice again that this decomposition is not unique. Indeed, the modified Gram-Schmidt orthogonalization procedure also depends upon the ordering of the vectors to which it is applied. We could, for example, have applied the Gram-Schmidt procedures to the vectors in the order v, u instead of u, v. We would then have retrieved a different decomposition of the same matrix. Nevertheless, this procedure is still of some value since it allows us to decompose an arbitrary singular linear transformation into simple, geometrically meaningful factors.

Affine Transformations

Finally, recall that every affine transformation A is simply a linear transformation L followed by a translation T. If the affine transformation is represented by a 4×3 matrix

$$A = \begin{vmatrix} L \\ T \end{vmatrix},$$

then the upper 3×3 submatrix L represents the linear transformation and the fourth row T represents the translation vector. Thus, to decompose an arbitrary affine transformation into simpler, geometrically meaningful factors, simply factor the associated linear transformation L and append the translation T. Thus, every nonsingular affine transformation of three-dimensional space can be factored into the product of three scales, two shears, one rotation, and one translation. Similarly, every singular affine transformation of three-dimensional space can be factored into the product of two scales, two shears, one rotation, one orthogonal projection, and one translation. Again, these decompositions are not unique, since the decompositions of the associated linear transformations are not unique. Nevertheless, these procedures are still of some value since they allow us to decompose arbitrary affine transformations into simple, geometrically meaningful factors.

See also G2, 319; G2, 320; G3, C.2.