◊ II.5

The Pleasures of "Perp Dot" Products

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◊ Introduction ◊

While developing code to perform certain geometric tasks in computer graphics, we do a lot of pencil-and-paper calculations to work out the relationships among the various quantities involved. This often requires intricate manipulations involving individual components of points and vectors, which can be both confusing and error-prone. It's a boon, therefore, when a concise and expressive notational device is developed to "expose" key geometric quantities lurking beneath the surface of many problems. This Gem presents two such geometric objects and develops some algebraic tools for working with them. It then applies them to obtain compact explicit formulas for finding incircles, excircles, corner rounding, and other well-known, messy problems.

We work in 2D and make explicit a notation for a vector that lies perpendicular to a given vector. Figure 1 shows vector $\mathbf{a} = (a_x, a_y)$. There are two vectors that have the same length as $\mathbf{a}$ and are perpendicular to it; we give the name $\mathbf{a}^\perp$ (read as *a perp*) to the one that is rotated 90° counterclockwise (ccw). It is easy to see that its coordinates are

$$\mathbf{a}^\perp = \mathbf{a}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (-a_x, a_y) \tag{1}$$

formed by interchanging the components of $\mathbf{a}$ and negating the first. The "perp" symbol $\perp$ may be viewed as the "rotate 90° ccw" operator applied to any vector $\mathbf{a}$, whereupon

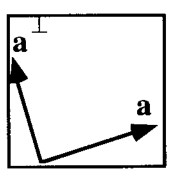

Figure 1. Vector a and its "perp."

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it enjoys the following easily proved properties:

Some Properties of $\mathbf{a}^\perp$

- Linearity: $(\mathbf{a}+\mathbf{b})^\perp = \mathbf{a}^\perp + \mathbf{b}^\perp$ and $(A\mathbf{a})^\perp = A\mathbf{a}^\perp$ for any scalar A
- Length of $\mathbf{a}^\perp$: $|\mathbf{a}| = |\mathbf{a}^\perp|$
- Applying $\perp$ twice: $\mathbf{a}^{\perp\perp} = (\mathbf{a}^\perp)^\perp = -\mathbf{a}$

(If we view a vector as a point in the complex plane, the perp operator is equivalent to multiplying by $i = \sqrt{-1}$, which makes these algebraic properties immediately apparent.)

As a simple example, we find the **perpendicular bisector** of the segment between points A and B, which arises in such studies as fractal curves and finding the circle through three points. The perpendicular bisector is the line that passes through the midpoint between A and B (given by $(A+B)/2$) and is perpendicular to the vector $B-A$. So using parameter t gives the compact parametric representation:

$$p(t) = \frac{1}{2}(A+B) + (B-A)^\perp t \tag{2}$$

By itself the perp notation is merely congenial. Its power emerges when we use it in conjunction with a second vector. Given 2D vectors $\mathbf{a}$ and $\mathbf{b}$, what is the nature of the dot product between $\mathbf{a}^\perp$ and $\mathbf{b}$? Just work out the usual component form to obtain

$$\mathbf{a}^\perp \cdot \mathbf{b} = a_x b_y - a_y b_x = \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \tag{3}$$

which shows that it is the determinant of the matrix with first row $\mathbf{a}$ and second row $\mathbf{b}$. We call $\mathbf{a}^\perp \cdot \mathbf{b}$ the *perp-dot* product of vectors $\mathbf{a}$ and $\mathbf{b}$. Some of its evident properties are

$$\mathbf{a}^\perp \cdot \mathbf{b} = -\mathbf{b}^\perp \cdot \mathbf{a},\ \ \mathbf{a}^\perp \cdot \mathbf{b}^\perp = \mathbf{a} \cdot \mathbf{b},\ \ \mathbf{a}^\perp \cdot \mathbf{a} = 0,\ \text{ and } \mathbf{a}^\perp \cdot \mathbf{a}^\perp = |\mathbf{a}^\perp|^2 = |\mathbf{a}|^2$$

(Aside: Pursuing the analogy with complex numbers, $\mathbf{a} \cdot \mathbf{b}$ corresponds to the real part of a^*b where a^* is the complex conjugate of a, and $\mathbf{a}^\perp \cdot \mathbf{b}$ corresponds to the imaginary part of a^*b. These correspondences also make the above properties readily apparent.)

From well-known properties of the dot product we see in Figure 2a that $\mathbf{a}^\perp \cdot \mathbf{b}$ has the value $|\mathbf{a}^\perp||\mathbf{b}|\cos(\phi)$, where ϕ is the angle between $\mathbf{a}^\perp$ and $\mathbf{b}$. Calling θ the angle from $\mathbf{a}$ to $\mathbf{b}$ (measured positive ccw), then $\cos(\phi)$ equals $\sin(\theta)$, so:

$$\mathbf{a}^\perp \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\theta) \tag{4}$$

This dot product is positive if $\mathbf{a}^\perp$ is less than 90° away from $\mathbf{b}$ (that is, if there is a left turn from $\mathbf{a}$ to $\mathbf{b}$), and is negative otherwise, as seen in Figure 2b. Thus it gives the *sense* of the turn from the direction of $\mathbf{a}$ to that of $\mathbf{b}$, which is a key ingredient in many geometric algorithms. If $\mathbf{a}^\perp \cdot \mathbf{b} = 0$, there is no turn, and $\mathbf{a}$ is parallel to $\mathbf{b}$.

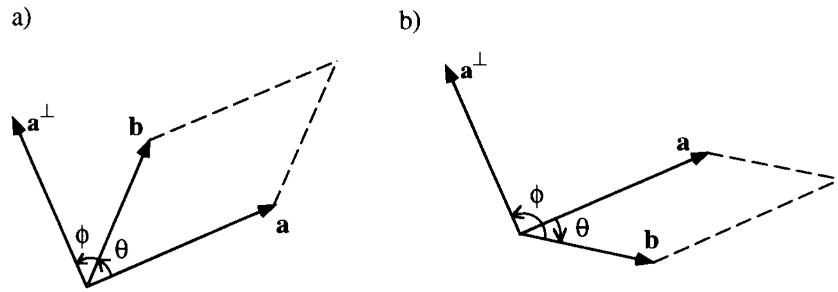

Figure 2. On the geometric nature of $\mathbf{a}^\perp \cdot \mathbf{b}$.

To summarize:

- $\mathbf{a}^\perp \cdot \mathbf{b} > 0$ if $\mathbf{b}$ is ccw from $\mathbf{a}$
- $\mathbf{a}^\perp \cdot \mathbf{b} < 0$ if $\mathbf{b}$ is cw from $\mathbf{a}$
- $\mathbf{a}^\perp \cdot \mathbf{b} = 0$ if $\mathbf{a}$ and $\mathbf{b}$ are parallel or antiparallel (linearly dependent)

The parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ has base $|\mathbf{a}|$ and altitude $|\mathbf{b}||\sin(\theta)|$, so its area is $|\mathbf{a}||\mathbf{b}||\sin(\theta)|$. Thus, using Equation 4,

$$|\mathbf{a}^\perp \cdot \mathbf{b}| = \text{area of parallelogram determined by } \mathbf{a} \text{ and } \mathbf{b} \tag{5}$$

which is twice the area of the triangle with vertices $(0,0)$, $\mathbf{a}$, and $\mathbf{b}$. Thus $\mathbf{a}^\perp \cdot \mathbf{b}$ is the familiar **signed area** determined by $\mathbf{a}$ and $\mathbf{b}$. It is the 2D analog of the *vector cross product* $\mathbf{a} \times \mathbf{b}$ that could be applied if $\mathbf{a}$ and $\mathbf{b}$ were 3D vectors. (More precisely, its value is seen from Equation 3 to be $\mathbf{a} \times \mathbf{b} \cdot \mathbf{k}$, where $\mathbf{k} = (0,0,1)$.) The power of the notation $\mathbf{a}^\perp \cdot \mathbf{b}$ in the 2D case is that it shows precisely how a signed area is "decomposed" into a *dot product* of readily recognizable geometric objects in the problem at hand. We see this power in action in the examples below.

Summary of Properties of $\mathbf{a}^\perp \cdot \mathbf{b}$

We list the principal properties of the perp-dot product $\mathbf{a}^\perp \cdot \mathbf{b}$. When the perp-dot product differs from the "regular" dot product $\mathbf{a} \cdot \mathbf{b}$ in an interesting way, we contrast them.

- $\mathbf{a}^\perp \cdot \mathbf{b}$ is linear in $\mathbf{a}$ and in $\mathbf{b}$ individually
- $\mathbf{a}^\perp \cdot \mathbf{b} = -\mathbf{b}^\perp \cdot \mathbf{a}$ versus $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a}^\perp \cdot \mathbf{b}^\perp = \mathbf{a} \cdot \mathbf{b}$, so $\mathbf{a}^\perp \cdot \mathbf{a}^\perp = |\mathbf{a}|^2$
- $\mathbf{a}^\perp \cdot \mathbf{a} = 0$ versus $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

- $\mathbf{a}^\perp \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\theta)$ versus $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$
- $(\mathbf{a}^\perp \cdot \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$
- $|\mathbf{a}^\perp \cdot \mathbf{b}|$ is the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$
- $\mathbf{a}^\perp \cdot \mathbf{b}$ is positive if and only if there is a ccw turn from $\mathbf{a}$ to $\mathbf{b}$

◇ Example Applications of the Perp-Dot Product ◇

The first example reiterates some classic uses of signed area and just casts them in terms of perp-dot products. For the subsequent examples, however, the perp-dot product appears in more surprising ways and leads to compact, explicit expressions for a number of geometric results. We present brief derivations of some of the formulas in order to show how the perp-dot product can be manipulated and its properties exploited.

Example 1: Convexity and area of polygons

Consider the simple polygon P with vertices P_i and edge vectors $\mathbf{v}_i = P_{i+1} - P_i$, for $i = 1, \ldots, N$ (where P_{N+1} is understood to equal P_1). This polygon is convex if and only if all turns from one edge vector to the next have the same sense. This rule is easily stated in terms of the perp-dot product as:

$$P \text{ is convex iff all } \mathbf{v_i}^\perp \cdot \mathbf{v_{i+1}} \geq 0 \text{ , or all } \mathbf{v_i}^\perp \cdot \mathbf{v_{i+1}} \leq 0$$

(Note: A more efficient convexity test is known that doesn't require a priori knowledge that P is simple; see e.g. (Moret and Shapiro 1991). It is also based on signed areas.)

Further, it is well known (e.g. (Goldman 1991, Hill 1990)) that to find the area of P just add up the N signed areas of its "component triangles" (with vertices (0,0), P_{i+1}, and P_i) and take the magnitude. In terms of perp-dot products:

$$\text{area} = \frac{1}{2}\left|\sum_{i=1}^{N} P_{i+1}^\perp \cdot P_i\right| \qquad \text{(area of polygon P)} \tag{6}$$

(Note that points appear in this expression where really only vectors should reside. Consider P_i as shorthand for the vector from $(0,0)$ to P_i, and similarly for $P_{i+1}^\perp$.)

Example 2: Find the intersection of two lines

The task of finding the intersection of two lines arises often in clipping and hidden line removal algorithms. Let the first line have parametric representation $A + \mathbf{a}t$, so that it passes through point A with direction $\mathbf{a}$. Similarly, the second line passes through

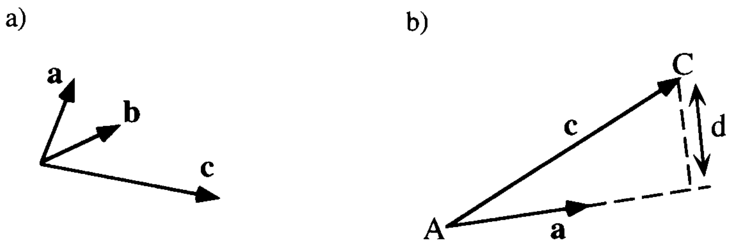

Figure 3. Finding the perpendicular projection of **c** onto **a**.

point B in direction $\mathbf{b}$, and so has parametric representation $B + \mathbf{b}u$. We seek values of the parameters t and u that make these points coincide: $A + \mathbf{a}t = B + \mathbf{b}u$. Calling $B - A = \mathbf{c}$, we must solve

$$\mathbf{a}t = \mathbf{c} + \mathbf{b}u \tag{7}$$

which gives a set of two equations in the two unknowns t and u. At this point one usually writes out the two equations in terms of components such as b_x and a_y, and invokes Cramer's rule. But the perp-dot product provides a much more direct route. Just form the dot product of both sides of the equation with $\mathbf{b}^\perp$. Since $\mathbf{b}^\perp \cdot \mathbf{b} = 0$, this eliminates the $\mathbf{b}u$ term and yields $(\mathbf{b}^\perp \cdot \mathbf{a})t = \mathbf{b}^\perp \cdot \mathbf{c}$, whereupon

$$t = \frac{\mathbf{b}^\perp \cdot \mathbf{c}}{\mathbf{b}^\perp \cdot \mathbf{a}} \tag{8}$$

as long as $\mathbf{b}^\perp \cdot \mathbf{a} \neq 0$; i.e., as long as $\mathbf{b}$ and $\mathbf{a}$ are not parallel. Hence the point of intersection is given explicitly by

$$A + \frac{\mathbf{b}^\perp \cdot \mathbf{c}}{\mathbf{b}^\perp \cdot \mathbf{a}}\mathbf{a} \qquad \text{(point of intersection)} \tag{9}$$

(Note: For lines in 3D space the explicit expression for t is more complex (Goldman 1990a): $t = (\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) / |\mathbf{a} \times \mathbf{b}|^2$.)

Example 3: Resolving a vector and orthogonal projections

The previous example is actually a special case of a more general problem: that of resolving a vector into the proper linear combination of two other vectors. (For instance, find "weights" R and S such that $(12, 9) = R(4, -6) + S(2, 7)$.) In general, given three 2D vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ as shown in Figure 3a, where $\mathbf{a}$ and $\mathbf{b}$ are not parallel ($\mathbf{a}^\perp \cdot \mathbf{b} \neq 0$), we seek R and S that solve $\mathbf{c} = R\mathbf{a} + S\mathbf{b}$.

First take dot products of both sides with $\mathbf{b}^\perp$ to obtain R, and then with $\mathbf{a}^\perp$ to obtain S.

$$\mathbf{c} = \frac{\mathbf{b}^\perp \cdot \mathbf{c}}{\mathbf{b}^\perp \cdot \mathbf{a}}\mathbf{a} + \frac{\mathbf{a}^\perp \cdot \mathbf{c}}{\mathbf{a}^\perp \cdot \mathbf{b}}\mathbf{b} \qquad (\mathbf{c} \text{ resolved into } \mathbf{a} \text{ and } \mathbf{b}) \tag{10}$$

This resolves $\mathbf{c}$ explicitly into the required portion "along" $\mathbf{a}$ and the required portion "along" $\mathbf{b}$. It's just Cramer's rule, of course, but written in a form that avoids dealing with individual components of the vectors involved. Note that it can be written in the symmetrical form:

$$(\mathbf{a}^\perp \cdot \mathbf{b})\mathbf{c} + (\mathbf{b}^\perp \cdot \mathbf{c})\mathbf{a} + (\mathbf{c}^\perp \cdot \mathbf{a})\mathbf{b} = \mathbf{0} \qquad \text{(relation between any three 2D vectors)} \tag{11}$$

which is true for *any* three 2D vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, even when some are parallel or even $\mathbf{0}$. This form is easily memorized, since each term involves $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in cyclic order, $\cdots \mathbf{b} \to \mathbf{c} \to \mathbf{a} \to \mathbf{b} \cdots$, and the three combinations appear once each. The truth of Equation 11 is obvious by dotting it with your choice of $\mathbf{a}^\perp$, $\mathbf{b}^\perp$, or $\mathbf{c}^\perp$.

As a special case of resolving one vector into two others, we often want the **orthogonal projection** of a vector $\mathbf{c}$ onto a given vector $\mathbf{a}$, as pictured in Figure 3b. But this is equivalent to resolving $\mathbf{c}$ into a portion along $\mathbf{a}$ and a portion perpendicular to $\mathbf{a}$, so just set $\mathbf{b} = \mathbf{a}^\perp$ in Equation 10, and use some of the properties summarized above to obtain

$$\mathbf{c} = \frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}|^2}\mathbf{a} + \frac{\mathbf{a}^\perp \cdot \mathbf{c}}{|\mathbf{a}|^2}\mathbf{a}^\perp \qquad (\mathbf{c} \text{ resolved into } \mathbf{a} \text{ and } \mathbf{a}^\perp) \tag{12}$$

The first term is the desired projection, and the second term gives the *error term* explicitly and compactly. This also immediately yields a formula for the distance from a point to a line. In Figure 3b the relevant point is C, and the line is $A + \mathbf{a}t$. The distance d is just the length of the second term in Equation 12:

$$d = \left| \frac{\mathbf{a}^\perp \cdot (C - A)}{|\mathbf{a}|^2}\mathbf{a}^\perp \right| = \left| \left(\frac{\mathbf{a}^\perp}{|\mathbf{a}|} \right) \cdot (C - A) \right| \qquad (\text{distance from } C \text{ to the line } A + \mathbf{a}t) \tag{13}$$

which has an appealing simplicity in terms of the unit vector in the direction of $\mathbf{a}^\perp$.

Example 4: Finding the excircle (and nine-point circle)

It is often of interest to locate the center S of the unique circle (called the *excircle* or *circumcircle*) that passes through three given points. Previous Gems (Goldman 1990b, Lopez-Lopez 1992) have offered closed forms for S. We derive a very simple one below, making ample use of the perp-dot product.

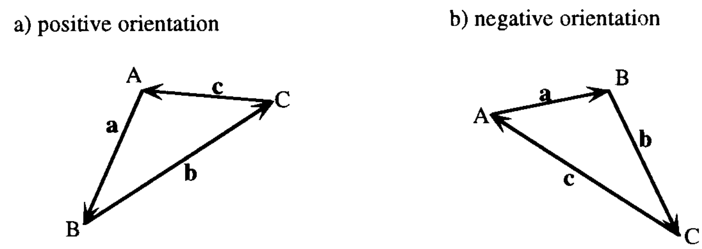

Figure 4. Orientation of triangles.

This and subsequent problems are based on a triangle, so we establish some notation for a triangle's ingredients. Figure 4 shows two triangles having vertices A, B, and C. The edges are labeled as vectors: side $\mathbf{a}$ emanates from A toward B, $\mathbf{b}$ from B toward C, etc., cyclically. In Figure 4a there is a ccw turn from each vector to the next (e.g., $\mathbf{a}^\perp \cdot \mathbf{b} > 0$, etc.) so the triangle has positive orientation and its signed area $\mathbf{a}^\perp \cdot \mathbf{b}/2$ is positive. The triangle in Figure 4b has negative orientation, since there is a cw turn from $\mathbf{a}$ to $\mathbf{b}$ ($\mathbf{a}^\perp \cdot \mathbf{b} < 0$), and its signed area $\mathbf{a}^\perp \cdot \mathbf{b}/2$ is negative. It is important that the formulas we develop below be correct for either orientation of the triangle involved, since a designer interacting with a CAD tool might specify its edges in different orders and directions at different moments.

Using either triangle of Figure 4, S must lie on the perpendicular bisector of each edge of triangle ABC. As stated above, the perpendicular bisector of AB is given parametrically by $(A + B)/2 + \mathbf{a}^\perp t$, and that of AC by $(A + C)/2 + \mathbf{c}^\perp u$. Point S lies where these meet, at the solution of: $\mathbf{a}^\perp t = \mathbf{b}/2 + \mathbf{c}^\perp u$. Again take the dot product of both sides, this time with $\mathbf{c}$, to obtain $t = 1/2(\mathbf{b} \cdot \mathbf{c})/(\mathbf{a}^\perp \cdot \mathbf{c})$ so the center S is given by the simple explicit form:

$$S = A + \frac{1}{2}(\mathbf{a} + \frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a}^\perp \cdot \mathbf{c}}\mathbf{a}^\perp) \tag{14}$$

The radius of the excircle follows as the length of $S - A$:

$$\text{radius} = \frac{|\mathbf{a}|}{2}\sqrt{\left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a}^\perp \cdot \mathbf{c}}\right)^2 + 1} \tag{15}$$

A similar version of S may be obtained based on edges AB and BC, and a third version can be based on AC and BC. It's a good exercise to obtain a form for S that is symmetrical in the points A, B, and C by averaging the three versions.

The exquisite **nine-point (Feuerbach) circle** contained in any triangle T passes through nine key points: the midpoints of the sides, the feet of the three altitudes, and the midpoints of the lines joining each vertex to the intersection of the altitudes

(Coxeter 1969). So it's just the excircle of the midpoint triangle M defined by the three midpoints of T. To find its center N apply Equation 14 with appropriate substitutions. The sides of M are parallel to those of T itself, so the result is very similar to Equation 14. It's a useful exercise to show that the center lies at

$$N = \frac{1}{2}(B + C) - \frac{1}{4}(\mathbf{a} + \frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a}^{\perp} \cdot \mathbf{c}}\mathbf{a}^{\perp}) \tag{16}$$

and that its radius is one half of the value cited in Equation 15.

Example 5: Finding the incircle

For completeness we mention a similar task: locating the center of the *incircle* (or *inscribed* circle) of a triangle. The incircle just touches each of the three sides, so it also locates the circle tangent to three given lines. Its center I lies where the three angle bisectors of the triangle meet. The angle bisector at vertex A of either triangle in Figure 4 has direction $\mathbf{m} = \hat{\mathbf{a}} - \hat{\mathbf{c}}$, where the ˆ symbol denotes that the vector has been normalized to unit length; i.e. $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$. Similarly, the angle bisector at B has direction $\mathbf{n} = \hat{\mathbf{b}} - \hat{\mathbf{a}}$. Finding the intersection as before, we obtain

$$I = A + \frac{\mathbf{n}^{\perp} \cdot \mathbf{a}}{\mathbf{n}^{\perp} \cdot \mathbf{m}}\mathbf{m} \tag{17}$$

Its radius is found as the distance from I to any of the three sides. This is easy to find explicitly, using Equation 13.

$$\text{radius} = \left| \frac{(\mathbf{n}^{\perp} \cdot \mathbf{a})(\mathbf{a}^{\perp} \cdot \mathbf{m})}{(\mathbf{n}^{\perp} \cdot \mathbf{m})\mathbf{a}} \right| \tag{18}$$

Example 6: Drawing rounded corners

A more challenging geometric problem is the following: given three points A, B, and C, and a distance r, draw the rounded curve shown in Figure 5a. It consists of a straight line from A toward B that blends smoothly into a circular arc of radius r, finally blending into a straight line to C. (It is implemented as the *arcto* operator in PostScript.)

The hard part is finding the center E of the circle (see Figure 5b) and the precise points D and F where the line blends with the arc. (The figure shows the case where there is a right turn from $\mathbf{a}$ to $\mathbf{b}$.) This becomes considerably easier when the perp-dot product is used. As before, the angle bisector at B is given parametrically by $B + \mathbf{n}u$ where $\mathbf{n} = \hat{\mathbf{b}} - \hat{\mathbf{a}}$ for unit vectors $\hat{\mathbf{b}}$ and $\hat{\mathbf{a}}$. E lies on the angle bisector at a distance

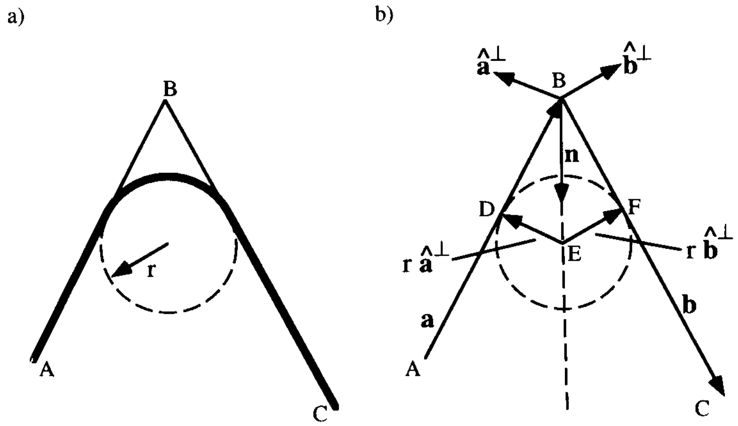

Figure 5. Drawing rounded corners.

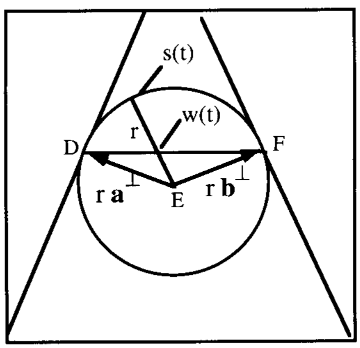

Figure 6. Finding points along the arc.

r from the line AB. Using appropriate ingredients in Equation 13 and simplifying the resulting expression, u must equal $u = r/|\mathbf{n} \cdot \hat{\mathbf{a}}^{\perp}| = r/|\hat{\mathbf{b}} \cdot \hat{\mathbf{a}}^{\perp}|$ so E is given explicitly by:

$$E = B + \frac{(\hat{\mathbf{b}} - \hat{\mathbf{a}})r}{\hat{\mathbf{b}} \cdot \hat{\mathbf{a}}^{\perp}} \tag{19}$$

When there is a right turn from $\mathbf{a}$ to $\mathbf{b}$ (so $\mathbf{b}^{\perp} \cdot \mathbf{a} > 0$ as in the figure), D and F are given by $D = E + r\hat{\mathbf{a}}^{\perp}$, and $F = E + r\hat{\mathbf{b}}^{\perp}$. Otherwise $\hat{\mathbf{a}}^{\perp}$ and $\hat{\mathbf{b}}^{\perp}$ point in the opposite directions. We capture both cases with the expression $D = E + \mathrm{sgn}(\mathbf{b}^{\perp} \cdot \mathbf{a})\hat{\mathbf{a}}^{\perp} r$ and $F = E + \mathrm{sgn}(\mathbf{b}^{\perp} \cdot \mathbf{a})\hat{\mathbf{b}}^{\perp} r$, where sgn(.) means the sign of its argument, $+1$ or -1.

To find points along the arc without recourse to awkward trigonometric functions, consider the point $w(t)$ that lies a fraction t of the way from D to F (see Figure 6), given by $w(t) = D(1 - t) + Ft$. The corresponding point $s(t)$ on the arc is at distance r along the line from E to $w(t)$, and so is given by $s(t) = E + r(w(t) - E)/|w(t) - E|$.

Suitable manipulations simplify this to:

$$s(t) = E + \operatorname{sgn}(\mathbf{b}^{\perp} \cdot \mathbf{a}) \frac{\hat{\mathbf{a}}^{\perp} + (\hat{\mathbf{b}}^{\perp} - \hat{\mathbf{a}}^{\perp})t}{|\hat{\mathbf{a}}^{\perp} + (\hat{\mathbf{b}}^{\perp} - \hat{\mathbf{a}}^{\perp})t|} r \tag{20}$$

The arc is drawn by making small increments dt in t, and drawing a line between successive values of $s(t)$, as suggested by the pseudocode:

```
moveto(A);
for(t = 0; t <= 1; t += dt) lineto(s(t));
lineto(C);
```

Although equal increments dt in t do not produce arc fragments of equal length, a smooth arc is drawn if dt is small.

◇ Conclusion ◇

We use vectors all the time in pencil-and-paper calculations while developing algorithms for computer graphics. They allow compact manipulations of geometric quantities, often without resort to coordinates. For some problems one can further delay the use of coordinates by giving a name $\mathbf{a}^{\perp}$ to one of the vectors perpendicular to a given vector $\mathbf{a}$. When such a vector is used in a dot product, the result has useful algebraic and geometric properties equivalent to the signed area of a triangle. Seeing signed area exposed as a dot product between readily interpreted vectors makes otherwise messy formulas more intelligible. In addition, Cramer's rule for solving vector equations arises simply by taking the dot product of both sides of the equation with the proper vector.

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