◇ IV.1

Identities for the Univariate and Bivariate Bernstein Basis Functions

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◇ Introduction ◇

Bézier curves and surfaces are essential to a wide variety of applications in computer graphics and geometric modeling, and the Bernstein basis functions play a central role in the construction and analysis of these curve and surface schemes. Here we shall adopt the standard notation $B_k^n(t), 0 \le k \le n$, and $B_{i,j}^n(s,t), 0 \le i+j \le n$, to represent the univariate and bivariate Bernstein basis functions of degree n.

Let $\{P_k\}$ and $\{P_{i,j}\}$ be arrays of control points. Then Bézier curves and surfaces are defined in the following fashion.

Bézier Curve

$$C(t) = \sum_k B_k^n(t) P_k, \quad t \in [0,1]$$

Tensor Product Bézier Surface

$$P(s,t) = \sum_i \sum_j B_i^m(s) B_j^n(t) P_{i,j}, \quad s,t \in [0,1]$$

Triangular Bézier Surface

$$T(s,t) = \sum_{0 \le i+j \le n} B_{i,j}^m(s,t) P_{i,j}, \quad (s,t) \in \Delta^2 = \{(s,t) \mid s,t \ge 0 \text{ and } s+t \le 1\}$$

The purpose of this gem is to assemble in one place those identities involving the univariate and bivariate Bernstein basis functions that help to facilitate the symbolic and numeric manipulation of Bézier curves and surfaces. This gem presents these identities in a consistent framework and may serve both as a compact reference and as a subject for further study.

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The formulas are organized into twenty-five categories. In general, each category begins with the formulas for the univariate bases and then lists the corresponding formulas for the bivariate bases, though in some cases no direct univariate (xvii) or bivariate (xi) analogues exist.

Currently these identities are widely scattered throughout the literature. For all of the more complicated identities, citations have been provided where proofs of these formulas or analogous formulas may be found. However, some of the simpler identities are so well known or so easy to derive from other identities that no citation is supplied.

◇ Identities for the Bernstein Basis Functions ◇

(i) Definitions

(a)

$$B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad 0 \le k \le n$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

(b)

$$B_{i,j}^n(s,t) = \binom{n}{i \quad j} s^i t^j (1-s-t)^{n-i-j}, \quad 0 \le i+j \le n$$

$$\binom{n}{i \quad j} = \frac{n!}{i!\,j!\,(n-i-j)!}$$

(ii) Non-negativity

(a)

$$B_k^n(t) \ge 0, \quad 0 \le t \le 1$$

(b)

$$B_{i,j}^n(s,t) \ge 0, \quad (s,t) \in \Delta^2$$

(iii) Symmetries

(a)

$$B_k^n(t) = B_{n-k}^n(1-t)$$

(b)

$$B_{i,j}^n(s,t) = B_{i,n-i-j}^n(s, 1-s-t)$$

(c)

$$B_{i,j}^n(s,t) = B_{n-i-j,j}^n(1-s-t, t)$$

(d)

$$B_{i,j}^n(s,t) = B_{j,i}^n(t,s)$$

(iv) Corner Values

(a)

$$\begin{aligned} B_k^n(0) &= 0, k \neq 0 \\ &= 1, k = 0 \end{aligned}$$

(b)

$$\begin{aligned} B_k^n(1) &= 0, k \neq n \\ &= 1, k = n \end{aligned}$$

(c)

$$\begin{aligned} B_{i,j}^n(0,0) &= 0, (i,j) \neq (0,0) \\ &= 1, (i,j) = (0,0) \end{aligned}$$

(d)

$$\begin{aligned} B_{i,j}^n(1,0) &= 0, (i,j) \neq (n,0) \\ &= 1, (i,j) = (n,0) \end{aligned}$$

(e)

$$\begin{aligned} B_{i,j}^n(0,1) &= 0, (i,j) \neq (0,n) \\ &= 1, (i,j) = (0,n) \end{aligned}$$

(v) Boundary Values

(a)

$$\begin{aligned} B_{i,j}^n(s,0) &= 0, & j \neq 0 \\ &= B_i^n(s), & j = 0 \end{aligned}$$

(b)

$$\begin{aligned} B_{i,j}^n(0,t) &= 0, & i &\neq 0 \\ &= B_j^n(t), & i &= 0 \end{aligned}$$

(c)

$$\begin{aligned} B_{i,j}^n(s,1-s) &= 0, & i+j &\neq n \\ &= B_i^n(s), & i+j &= n \end{aligned}$$

(vi) Partitions of Unity (Farin 1988)

(a)

$$\sum_k B_k^n(t) = 1$$

(b)

$$\sum_i \sum_j B_{i,j}^n(s,t) = 1$$

(vii) Alternating Sums

(a)

$$\sum_k (-1)^k B_k^n(t) = (1-2t)^n$$

(b)

$$\sum_i \sum_j (-1)^{i+j} B_{i,j}^n(s,t) = (1-2s-2t)^n$$

(viii) Conversion to Monomial Form (Polya and Schoenberg 1958)

(a)

$$B_k^n(t)/(1-t)^n = \binom{n}{k} u^k, \quad u = t/(1-t)$$

(b)

$$B_k^n(t)/t^n = \binom{n}{k} u^{n-k}, \quad u = (1-t)/t$$

(c)

$$B^n_{i,j}(s,t)/(1-s-t)^n = \binom{n}{i \quad j} u^i v^j, \quad u = s/(1-s-t), v = t/(1-s-t)$$

(d)

$$B^n_{i,j}(s,t)/s^n = \binom{n}{i \quad j} u^{n-i-j} v^j, \quad u = (1-s-t)/s, v = t/s$$

(e)

$$B^n_{i,j}(s,t)/t^n = \binom{n}{i \quad j} u^i v^{n-i-j}, \quad u = s/t, v = (1-s-t)/t$$

(ix) Representation in Terms of Monomials (Farouki and Rajan 1988)

(a)

$$B^n_k(t) = \sum_{k \le j \le n} (-1)^{j-k} \binom{n}{k} \binom{n-k}{j-k} t^j$$

(b)

$$B^n_{i,j}(s,t) = \sum_{k \ge i} \sum_{l \ge j} (-1)^{i+j+k+l} \binom{n}{i \quad j} \binom{n-i-j}{k-i \quad l-j} s^k t^l, \quad 0 \le k+l \le n$$

(x) Representation of Monomials (Farouki and Rajan 1988)

(a)

$$t^j = \sum_{j \le k \le n} \frac{\binom{k}{j}}{\binom{n}{j}} B^n_k(t), \quad 0 \le j \le n$$

(b)

$$s^i t^j = \sum_{k \ge i} \sum_{l \ge j} \frac{\binom{k}{i}\binom{l}{j}}{\binom{n}{i \quad j}} B^n_{k,l}(s,t), \quad 0 \le i+j \le n$$

(xi) Linear Independence

(a)

$$\sum_k c_k B_k^n(t) = 0 \Longleftrightarrow c_k = 0 \text{ for all } k$$

(b)

$$\sum_i \sum_j c_{i,j} B_{i,j}^n(s,t) = 0 \Longleftrightarrow c_{i,j} = 0, \text{ for all } i, j$$

(xii) Descartes' Law of Signs (Polya and Schoenberg 1958)

(a)

$$\text{Zeros in } (0,1) \text{ of } \left\{ \sum_k c_k B_k^n(t) \right\} \leq \text{Sign alternations of } (c_0, c_1, \ldots, c_n)$$

(b) There is no known analogous formula for the bivariate Bernstein basis functions.

(xiii) Recursion (Farin 1988)

(a)

$$B_k^n(t) = (1-t)B_k^{n-1}(t) + tB_{k-1}^{n-1}(t)$$

(b)

$$B_{i,j}^n(s,t) = (1-s-t)B_{i,j}^{n-1}(s,t) + sB_{i-1,j}^{n-1}(s,t) + tB_{i,j-1}^{n-1}(s,t)$$

(xiv) Discrete Convolution

(a)

$$(B_0^n(t), \ldots, B_n^n(t)) = \underbrace{\{(1-t), t\} * \cdots * \{(1-t), t\}}_{n \text{ factors}}$$

(b)

$$(B_{0,0}^n(t), \ldots, B_{0,n}^n(t)) = \underbrace{\{(1-s-t), s, t\} * \cdots * \{(1-s-t), s, t\}}_{n \text{ factors}}$$

(xv) Subdivision (Goldman 1982, 1983)

(a)

$$B_i^n(rt) = \sum_{i \le k \le n} B_i^k(r) B_k^n(t)$$

(b)

$$B_i^n((1-t)r+t) = \sum_{0 \le k \le i} B_{i-k}^{n-k}(r) B_k^n(t)$$

(c)

$$B_i^n((1-t)r+ts) = \sum_k \left\{ \sum_{p+q=i} B_p^{n-k}(r) B_q^k(s) \right\} B_k^n(t)$$

(d)

$$B_{i,j}^n(su, sv+t) = \sum_k \sum_l B_{i,j-l}^k(u,v) B_{k,l}^n(s,t)$$

(e)

$$B_{i,j}^n(tu+s, tv) = \sum_k \sum_l B_{i-k,j}^l(u,v) B_{k,l}^n(s,t)$$

(f)

$$B_{i,j}^n((1-s-t)u+s, (1-s-t)v+t) = \sum_k \sum_l B_{i-k,j-l}^{n-k-l}(u,v) B_{k,l}^n(s,t)$$

(g)

$$\begin{aligned} &B_{i,j}^n((1-s-t)u_1+sv_1+tw_1, (1-s-t)u_2+sv_2+tw_2) \\ &\quad = \sum_k \sum_l \left\{ \sum_{a+c+e=i, b+d+f=j} B_{a,b}^{n-k-l}(u_1,u_2) B_{c,d}^k(v_1,v_2) B_{e,f}^l(w_1,w_2) \right\} B_{k,l}^n(s,t) \end{aligned}$$

(xvi) Partial Derivatives (Farin 1988)

(a)

$$dB_k^n(t)/dt = n\{B_{k-1}^{n-1}(t) - B_k^{n-1}(t)\}$$

(b)

$$d^p B_k^n(t)/dt^p = \frac{n!}{(n-p)!} \sum_{0 \le j \le p} (-1)^{p-j} \binom{p}{j} B_{k-j}^{n-p}(t)$$

(c)

$$\partial B_{i,j}^n(s,t)/\partial s = n\{B_{i-1,j}^{n-1}(s,t) - B_{i,j}^{n-1}(s,t)\}$$

(d)

$$\partial B_{i,j}^n(s,t)/\partial t = n\{B_{i,j-1}^{n-1}(s,t) - B_{i,j}^{n-1}(s,t)\}$$

(e)

$$\partial^{p+q} B_{i,j}^n(s,t)/\partial s^p \partial t^q = \frac{n!}{(n-p-q)!} \sum_\alpha \sum_\beta (-1)^{p+q+\alpha+\beta} \binom{p}{\alpha} \binom{q}{\beta} B_{i-\alpha,j-\beta}^{n-p-q}(s,t)$$

(xvii) Directional Derivatives (Farin 1986)

(a)

$$D_{\boldsymbol{u}}\{B_{i,j}^n(s,t)\} = n\{u_1 B_{i-1,j}^{n-1}(s,t) + u_2 B_{i,j-1}^{n-1}(s,t) - (u_1+u_2) B_{i,j}^{n-1}(s,t)\}$$

(b)

$$D_{\boldsymbol{u}}^m\{B_{i,j}^n(s,t)\} = \frac{n!}{(n-m)!} \sum_\alpha \sum_\beta \binom{m}{\alpha \quad \beta} u_1^\alpha u_2^\beta \{-u_1 - u_2\}^{m-\alpha-\beta} B_{i-\alpha,j-\beta}^{n-m}(s,t)$$

($D_{\boldsymbol{u}}^m$ denotes the mth directional derivative in the direction $\boldsymbol{u} = (u_1, u_2)$.)

(xviii) Integrals (Farin 1988)

(a)

$$\int_0^t B_k^n(\tau)d\tau = \sum_{k+1 \le j \le n+1} \frac{B_j^{n+1}(t)}{n+1}$$

(b)

$$\int_t^1 B_k^n(\tau)d\tau = \sum_{0 \le j \le k} \frac{B_j^{n+1}(t)}{n+1}$$

(c)

$$\int_0^1 B_k^n(\tau)d\tau = \frac{1}{n+1}$$

(d)

$$\int_0^s B_{i,j}^n(\sigma,t)d\sigma = \sum_{h\geq i+1} \frac{B_{h,j}^{n+1}(s,t)}{n+1}$$

(e)

$$\int_s^{1-t} B_{i,j}^n(\sigma,t)d\sigma = \sum_{h\leq i} \frac{B_{h,j}^{n+1}(s,t)}{n+1}$$

(f)

$$\int_0^{1-t} B_{i,j}^n(\sigma,t)d\sigma = \frac{B_j^{n+1}(t)}{n+1}$$

(g)

$$\int_0^t B_{i,j}^n(s,\tau)d\tau = \sum_{k\geq j+1} \frac{B_{i,k}^{n+1}(s,t)}{n+1}$$

(h)

$$\int_t^{1-s} B_{i,j}^n(s,\tau)d\tau = \sum_{k\leq j} \frac{B_{i,k}^{n+1}(s,t)}{n+1}$$

(i)

$$\int_0^{1-s} B_{i,j}^n(s,\tau)d\tau = \frac{B_i^{n+1}(s)}{n+1}$$

(j)

$$\iint_{\Delta^2} B_{i,j}^n(\sigma,\tau)d\sigma d\tau = \frac{1}{(n+1)(n+2)}$$

(xix) Degree Elevation (Farin 1988)

(a)

$$(1-t)B_k^n(t) = \frac{n+1-k}{n+1}B_k^{n+1}(t)$$

(b)

$$tB_k^n(t) = \frac{k+1}{n+1}B_{k+1}^{n+1}(t)$$

(c)

$$B_k^n(t) = \frac{n+1-k}{n+1}B_k^{n+1}(t) + \frac{k+1}{n+1}B_{k+1}^{n+1}(t)$$

(d)

$$(1-s-t)B_{i,j}^n(s,t) = \frac{n+1-i-j}{n+1}B_{i,j}^{n+1}(s,t)$$

(e)

$$sB_{i,j}^n(s,t) = \frac{i+1}{n+1}B_{i+1,j}^{n+1}(s,t)$$

(f)

$$tB_{i,j}^n(s,t) = \frac{j+1}{n+1}B_{i,j+1}^{n+1}(s,t)$$

(g)

$$\begin{aligned}B_{i,j}^n(s,t) &= \frac{n+1-i-j}{n+1}B_{i,j}^{n+1}(s,t)\\ &\quad + \frac{i+1}{n+1}B_{i+1,j}^{n+1}(s,t) + \frac{j+1}{n+1}B_{i,j+1}^{n+1}(s,t)\end{aligned}$$

(xx) Products and Higher-Order Degree Elevation (Farouki and Rajan 1988)

(a)

$$B_j^m(t)B_k^n(t) = \frac{\binom{m}{j}\binom{n}{k}}{\binom{m+n}{j+k}}B_{j+k}^{m+n}(t)$$

(b)

$$B_k^n(t) = \sum_{0 \le j \le m} \frac{\binom{m}{j}\binom{n}{k}}{\binom{m+n}{j+k}} B_{j+k}^{m+n}(t)$$

(c)

$$B_{i,j}^m(s,t) B_{k,l}^n(s,t) = \frac{\binom{m}{i \quad j}\binom{n}{k \quad l}}{\binom{m+n}{i+k \quad j+l}} B_{i+k,j+l}^{m+n}(s,t)$$

(d)

$$B_{k,l}^n(s,t) = \sum_i \sum_j \frac{\binom{m}{i \quad j}\binom{n}{k \quad l}}{\binom{m+n}{i+k \quad j+l}} B_{i+k,j+l}^{m+n}(s,t)$$

(xxi) Generating Functions

(a)

$$\sum_k B_k^n(t) x^k = \{(1-t) + tx\}^n$$

(b)

$$\sum_k B_k^n(t) e^{ky} = \{(1-t) + te^y\}^n$$

(c)

$$\sum_i \sum_j B_{i,j}^n(s,t) x^i y^j = \{(1-s-t) + sx + ty\}^n$$

(d)

$$\sum_i \sum_j B_{i,j}^n(s,t) e^{ix} e^{jy} = \{(1-s-t) + se^x + te^y\}^n$$

(xxii) Marsden Identities (Cavaretta and Micchelli 1992, Marsden 1970)

(a)

$$(x-t)^n = \sum_k \frac{(-1)^k}{\binom{n}{k}} B_{n-k}^n(x) B_k^n(t)$$

(b)

$$(sx+ty+1)^n = \sum_i \sum_j (x+1)^i (y+1)^j B_{i,j}^n(s,t)$$

(xxiii) de Boor–Fix Formulas
(de Boor and Fix 1973, Lodha and Goldman 1994, Zhao and Sun 1988)

(a)

$$\left\{\frac{1}{n!}\binom{n}{i}\right\} \sum_p (-1)^{j+p} \{B_i^n(t)\}^{(p)} \{B_{n-j}^n(t)\}^{(n-p)} = \delta_{i,j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

(b)

$$\left(\frac{1}{n!}\right) \sum_p \sum_q \binom{i}{p}\binom{j}{q} (n-p-q)!\, \partial^{p+q} B_{k,l}^n(0,0)/\partial s^p \partial t^q = \delta_{i,k}\delta_{j,l}$$

(xxiv) Relationships between Univariate and Bivariate Basis Functions
(Goldman and Filip 1987, Goldman 1983)

(a)

$$B_i^n(s) = \sum_{0 \le j \le n-i} B_{i,j}^n(s,t)$$

(b)

$$B_j^n(t) = \sum_{0 \le i \le n-j} B_{i,j}^n(s,t)$$

(c)

$$B_k^n(s+t) = \sum_{i+j=k} B_{i,j}^n(s,t)$$

(d)

$$B^n_{i,j}(s,t) = \sum_p \sum_q (-1)^{n-p-q} \binom{n}{p\ \ q} B^p_i(s) B^q_j(t)$$

(e)

$$B^n_{i,j}(su,tv) = \sum_k \sum_l B^k_i(u) B^l_j(v) B^n_{k,l}(s,t)$$

(xxv) Conversion between Bivariate and Tensor Product Bases
(Goldman and Filip 1987, Brueckner 1980)

(a)

$$B^m_i(s)B^n_j(t) = \sum_k \sum_l \frac{\binom{k}{i}\binom{l}{j}\binom{m+n-k-l}{m-i+j-l}}{\binom{m+n}{n}} B^{m+n}_{k,l}(s,t)$$

(b)

$$B^n_{i,j}(s,t) = \sum_{k,l} \sum_{p,q} (-1)^{n-p-q} \frac{\binom{n}{p\ \ q}\binom{p}{i}\binom{q}{j}\binom{n-p}{k-i}\binom{n-q}{l-j}}{\binom{n}{k}\binom{n}{l}} B^n_k(s) B^n_l(t)$$

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