V.5 NEWELL'S METHOD FOR COMPUTING THE PLANE EQUATION OF A POLYGON

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Here is a numerically robust way of computing the plane equation of an arbitrary 3-D polygon. This technique, first suggested by Newell (Sutherland *et al.*, 1974), works for concave polygons and polygons containing collinear vertices, as well as for nonplanar polygons, e.g., polygons resulting from perturbed vertex locations. In the last case, Newell's method computes a "best-fit" plane.

Newell's Method

It can be shown that the areas of the projections of a polygon onto the Cartesian planes, *xy*, *yz*, and *zx*, are proportional to the coefficients of the normal vector to the polygon. Newell's method computes each one of these projected areas as the sum of the "signed" areas of the trapezoidal regions enclosed between each polygon edge and its projection onto one of the Cartesian axes.

Let the n vertices of a polygon p be denoted by $V_1, V_2, \ldots, V_n$, where $V_i = (x_i, Y_i, z_i)$, $i = 1, 2, \ldots, n$. The plane equation $ax + by + cz + d = 0$ can be expressed as

$$(X - P) \cdot N = 0, \tag{1}$$

where $X = (x, y, z)$, $N = (a, b, c)$ is the normal to the plane, and P is an arbitrary reference point on the plane. The coefficients a, b, and c are given by

$$a = \sum_{i=1}^{n}(y_i - y_{i\oplus 1})(z_i + z_{i\oplus 1}),$$
$$b = \sum_{i=1}^{n}(z_i \quad z_{i\oplus 1})(x_i + x_{i\oplus 1}),$$
$$c = \sum_{i=1}^{n}(x_i - x_{i\oplus 1})(y_i + y_{i\oplus 1}),$$

where $\oplus$ represents addition modulo n.

The coefficient d is computed from Eq. (1) as

$$d = -P \cdot N, \tag{2}$$

where P is the arithmetic average of all the vertices of the polygon:

$$P = \frac{1}{n}\sum_{i=1}^{n} V_i. \tag{3}$$

It is often useful to "normalize" the plane equation so that $N = (a, b, c)$ is a unit vector. This is done simply by dividing each coefficient of the plane equation by $(a^2 + b^2 + c^2)^{1/2}$.

Newell's method may seem inefficient for planar polygons, since it uses all the vertices of a polygon when, in fact, only three points are needed to define a plane. It should be noted, though, that for arbitrary planar polygons, these three points must be chosen very carefully:

1. Three points uniquely define a plane if and only if they are not collinear; and
2. if the three points are chosen around a "concave" corner, the normal of the resulting plane will point in the direction opposite to the expected one.

Checking for these properties would reduce the efficiency of the three-point method as well as making its coding rather inelegant. A good strategy may be that of using the three-point method for polygons that are already known to be planar and strictly convex (no collinear vertices,) and using Newell's method for the rest.

See also G3, E, 4.