VII.2

RECOVERING THE DATA FROM THE TRANSFORMATION MATRIX

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In *Graphics Gems* ("Matrices and Transformations"), we showed how to construct the 4×4 matrices for affine and projective transformations—rigid motion, scaling, and projections—which were defined relative to some arbitrary positions and orientations described by scalars, points, and vectors. In this volume, we added the 4×4 matrices for shear and pseudo-perspective. Now we shall show how to retrieve the defining data—scalars, points, and vectors—from the 4×4 matrix when we know the type of transformation represented by the matrix. This is useful, for example, when we concatenate matrices for rotations around different axes and then want to know the axis and angle of the resulting rotation.

Most of the affine and projective transformations we discuss have fixed points and fixed directions—that is, values that are left invariant by the transformation. These fixed values show up as eigenvectors of the transformation matrix. (In keeping with standard usage, we shall use the term eigenvector even when the fixed value actually is a point rather than a vector). The data we seek to extract often are simply eigenvectors of the transformation matrix.

Briefly, an eigenvector v of a transformation T is any non~ero vector such that

$$\mathrm{T}(v) = \beta \mathrm{I}(v) = \beta v,$$

where I is the identity transformation. The scalar β is called an eigenvalue of T, and v is said to be an eigenvector of T corresponding to the eigenvalue β. The eigenvalues of T are the roots of the equation:

$$\mathrm{Det}(\mathbf{T} - \beta\mathbf{I}) = 0$$

where **I** is the identity matrix of the same size as the matrix **T**. There are well-known algorithms for computing eigenvalues and eigenvectors; readers not familiar with these concepts should consult a standard linear algebra text.

We shall adopt the following notation:

$$\mathbf{M} = 4 \times 4 \text{ matrix},$$

$$\mathbf{M}_{33} = \text{upper left } 3 \times 3 \text{ submatrix of } \mathbf{M},$$

$$\mathbf{M}_{34} = \text{upper } 3 \times 4 \text{ submatrix of } \mathbf{M},$$

$$\mathbf{M}_{43} = \text{left } 4 \times 3 \text{ submatrix of } \mathbf{M},$$

$$\mathbf{M}^{\mathrm{T}} = \text{transpose of } \mathbf{M},$$

$$\mathrm{Trace}\,(\mathbf{M}_{33}) = \sum_k \mathbf{M}_{33}(k, k) = \mathbf{M}(1, 1) + \mathbf{M}(2, 2) + \mathbf{M}(3, 3).$$

Armed with these concepts and this notation, we are ready now to extract the data from the transformation matrices. Note that in many cases, the data is not unique. For example, if we define a plane by a point and a unit vector, the point is not unique. Usually, this point is an eigenvector of the transformation matrix relative to some fixed eigenvalue. If we do not specify further, then any such eigenvector will suffice. Often, too, we will require a unit eigenvector corresponding to some eigenvalue β. Such an eigenvector is found readily by computing any eigenvector v corresponding to the eigenvalue β and then normalizing its length to one, since, by linearity, if v is an eigenvector corresponding to the eigenvalue β, then cv also is an eigenvector corresponding to the eigenvalue β.

Translation

Let

$$w = \text{Translation vector}$$

Given

$$\mathbf{T}(w) = \text{Translation matrix}$$

Compute

$$w = \text{Fourth row of } \mathbf{T}(w) = (0,\ 0,\ 0,\ 1) * \mathbf{T}(w)$$

Rotation

Let

$$\begin{aligned} L &= \text{Axis line} \\ w &= \text{Unit vector parallel to } L \\ Q &= \text{Point on } L \\ \phi &= \text{Angle of rotation} \end{aligned}$$

Given

$$\mathbf{R} = \mathbf{R}(w,\ \phi,\ Q) = \text{Rotation matrix}$$

Compute

$$\cos\phi = \frac{\left(\text{Trace}\left(\mathbf{R}_{33}\right) - 1\right)}{2}$$

$$\begin{aligned} w &= \text{Unit eigenvector of } \mathbf{R}_{33} \text{ corresponding to the eigenvalue 1} \\ Q &= \text{Any eigenvector of } \mathbf{R} \text{ corresponding to the eigenvalue 1} \end{aligned}$$

$$\sin\phi = \frac{\left\{\mathbf{R}(1,2) + (\cos\phi - 1)w_1 w_2\right\}}{w_3}$$

Notice that the sign of sin ϕ depends on the choice of w, since both w and $-w$ are eigenvectors of $\mathbf{R}_{33}$ corresponding to the eigenvalue 1. Therefore, we cannot find ϕ without first deciding on the choice of w.

Mirror Image

Let

$$S = \text{Mirror plane}$$

$$n = \text{Unit vector perpendicular to } S$$

$$Q = \text{Point on } S$$

Given

$$\mathbf{M} = \mathbf{M}(n, Q) = \text{Mirror matrix}$$

Compute

$$n = \text{Unit eigenvector of } \mathbf{M}_{33} \text{ corresponding to the eigenvalue } -1$$

$$Q = \text{Any eigenvector of } \mathbf{M} \text{ corresponding to the eigenvalue } +1$$

Scaling

Let

$$Q = \text{Scaling origin}$$

$$c = \text{Scaling factor}$$

$$w = \text{Scaling direction}$$

a. Uniform scaling
Given

$$\mathbf{S} = \mathbf{S}(Q, c) = \text{Scaling matrix}$$

Compute

$$c = \frac{\text{Trace}(\mathrm{S}_{33})}{3}$$

Q = Any eigenvector of $\mathbf{S}$ corresponding to the eigenvalue 1

b. Nonuniform scaling
Given

$$\mathbf{S} = \mathbf{S}(Q, c, w) = \text{Scaling matrix}$$

Compute

$c = \text{Trace}(\mathbf{S}_{33}) - 2$
w = Unit eigenvector of $\mathbf{S}_{33}$ corresponding to the eigenvalue c
Q = Any eigenvector of $\mathbf{S}$ corresponding to the eigenvalue 1

Shear

Let

S = Shearing plane

v = Unit vector perpendicular to S

Q = Point on S

w = Unit shearing direction vector

ϕ = Shearing angle

Given

$$\mathbf{S} = \text{Shear}(Q,\ v,\ w,\ \phi) = \text{Shearing matrix}$$

Compute

w_1, w_2 = independent eigenvectors of $\mathbf{S}_{33}$ corresponding to the eigenvalue 1

$$v = \frac{w_1 \times w_2}{|w_1 \times w_2|}$$

$$\tan\phi = |v * (\mathbf{S} - \mathbf{I})_{33}|$$

$$w = \frac{v * (\mathbf{S} - \mathbf{I})_{33}}{\tan\phi}$$

Q = any eigenvector of $\mathbf{S}$ corresponding to the eigenvalue 1

Projection

Let

S = Image plane

n = Unit vector perpendicular to S

Q = Point on S

w = Unit vector parallel to projection direction

R = Perspective point

a. Orthogonal projection
Given

$$\mathbf{O} = \mathbf{O}\ \text{proj}(n,\ Q) = \text{Projection matrix}$$

Compute

n = Unit eigenvector of $\mathbf{O}_{33}$ corresponding to the eigenvalue 0

Q = Any eigenvector of $\mathbf{O}$ corresponding to the eigenvalue 1

b. Parallel projection
Given

$$\mathbf{P} = \mathbf{P}\ \mathrm{proj}(n, Q, w) = \text{Projection matrix}$$

Compute

Q = Any eigenvector of $\mathbf{P}$ corresponding to the eigenvalue 1

w = Unit eigenvector of $\mathbf{P}_{33}$ corresponding to the eigenvalue 0

n = Unit eigenvector of $\mathbf{P}_{33}^{\mathrm{T}}$ corresponding to the eigenvalue 0

$$= \frac{w * \left\{(\mathbf{I} - \mathbf{P})_{33}^{\mathrm{T}}\right\}}{\left|w * \left\{(\mathbf{I} - \mathbf{P})_{33}^{\mathrm{T}}\right\}\right|}$$

c. Pseudo-perspective
Given

$$\mathbf{P} = \mathrm{Pseudo}(n, Q, R) = \text{Pseudo-perspective matrix}$$

Compute

Q = Any eigenvector of $\mathbf{P}$ not corresponding to the eigenvalue 0

n^{T} = First three entries of the fourth column of $\mathbf{P} = \mathbf{P}_{34} * (0, 0, 0, 1)^{\mathrm{T}}$

$$R = -\frac{\{(0,0,0,1) * \mathbf{P}_{43} + (Q \cdot n)n\}}{(Q \cdot n)}$$

d. Perspective
Given

$$\mathbf{P} = \mathrm{Persp}(n, Q, R) = \text{Perspective matrix}$$

Compute

Q = Any eigenvector of **P** not corresponding to the eigenvalue 0

n^T= –First three entries of the fourth column of **P** = $-\mathbf{P}_{34} * (0, 0, 0, 1)^T$

R = eigenvector of **P** corresponding to the eigenvalue 0

$$= \frac{(0,0,0,1) * \mathbf{P}_{43}}{(Q \cdot n)}$$

See also 7.1 Decomposing a Matrix into Simple Transformations, Spencer W. Thomas