V.3

INTERSECTION OF TWO LINES IN THREE-SPACE

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Let each line be defined by a point P_k and a unit direction vector $\mathbf{V}_k$, $k = 1,2$. Then we can express each line parametrically by writing

$$L_1(t) = P_1 + \mathbf{V}_1 t \quad and \quad L_2(s) = P_2 + \mathbf{V}_2 s.$$

The intersection occurs when $L_1(t) = L_2(s)$ or equivalently when

$$P_1 + \mathbf{V}_1 \mathrm{t} = P_2 + \mathbf{V}_2 s$$

Subtracting P_1 from both sides and crossing with $\mathbf{V}_2$ yields

$$(\mathbf{V}_1 \times \mathbf{V}_2)t = (P_2 - P_1) \times \mathbf{V}_2.$$

Now dotting with $(\mathbf{V}_1 \times \mathbf{V}_2)$ and dividing by $|\mathbf{V}_1 \times \mathbf{V}_2|^2$ give us

$$t = \mathrm{Det}\{(P_2 - P_1)\ ,\ \mathbf{V}_2,\ \mathbf{V}_1 \times \mathbf{V}_2\} / |\mathbf{V}_1 \times \mathbf{V}_2|^2.$$

Symmetrically, solving for s, we obtain

$$s = \mathrm{Det}\{(P_2 - P_1),\ \mathbf{V}_1,\ \mathbf{V}_1 \times \mathbf{V}_2\} / |\mathbf{V}_1 \times \mathbf{V}_2|^2.$$

Two important observations follow:

- If the lines are parallel, the denominator $|\mathbf{V}_1 \times \mathbf{V}_2|^2 = 0$.
- If the lines are skew, s and t represent the parameters of the points of closest approach.