IX.8

MATRICES AND TRANSFORMATIONS

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People often struggle to find the 4×4 matrices for affine or projective transformations that are not relative to the origin or the coordinate axes, but rather relative to some arbitrary points or lines. Often they proceed by transforming the problem to the origin and coordinate axes, finding the transformation matrix relative to this canonical position, and then transforming back to an arbitrary location. All this extra work is unnecessary. Here we describe the 4×4 matrices for the following transformations:

- Translation
- Rotation
- Mirror Image
- Scaling
  —Uniform
  —Nonuniform
- Projections
  —Orthogonal
  —Parallel
  —Perspective

Each of these transformations in any arbitrary position can be defined in terms of three basic matrices: the tensor product, the cross product, and the identity matrix. We include the definitions of these basic building blocks along with the affine and projective transformation matrices.

Notation

a. Identity

$$I = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

b. Tensor Product

$$v \otimes w = \begin{vmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{vmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} * \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}$$

c. Cross Product

$$w\mathrm{x}\_ = \begin{vmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{vmatrix}$$

Observations

a. $u * I = u$

b. $u * (v \otimes w) = (u \cdot v) w$

c. $u * (w\mathrm{x}\_) = w \mathrm{x} u$

Translation

w = translation vector

$$T(w) = \begin{vmatrix} I & 0 \\ w & 1 \end{vmatrix}$$

Rotation

L = Axis line
w = Unit vector parallel to L
Q = Point on L
Θ = Angle of rotation

$$R(w, \Theta) = (\cos \Theta) I + (1 - \cos \Theta) w \otimes w + (\sin \Theta) w \times\_$$

$$R(w,\Theta,Q) = \begin{vmatrix} R(w,\Theta) & 0 \\ Q - Q * R(w,\Theta) & 1 \end{vmatrix}$$

Mirror Image

S = Mirror plane
n = Unit vector perpendicular to S
Q = Point on S

$$M(n,Q) = \begin{vmatrix} I - 2(n \otimes n) & 0 \\ 2(Q \cdot n)n & 1 \end{vmatrix}$$

Scaling

Q = Scaling origin
c = Scaling factor
w = Unit vector parallel to scaling direction

a. Uniform scaling

$$S(Q,c) = \begin{vmatrix} cI & 0 \\ (1-c)Q & 1 \end{vmatrix}$$

b. Nonuniform scaling

$$S(Q,c,w) = \begin{vmatrix} \mathrm{I}-(1-c)(w \otimes w) & 0 \\ (1-c)(Q \cdot w)w & 1 \end{vmatrix}$$

Projection

S = Image plane

n = Unit vector perpendicular to S

Q = Point on S

w = Unit vector parallel to projection direction

R = Perspective point

a. Orthogonal projection

$$Oproj(n,Q) = \begin{vmatrix} \mathrm{I} - n \otimes n & 0 \\ (Q \cdot n)n & 1 \end{vmatrix}$$

b. Parallel projection

$$Pproj(n,Q,w) = \begin{vmatrix} \mathrm{I} - (n \otimes w) / (w \cdot n) & 0 \\ [(Q \cdot n) / (w \cdot n)]w & 1 \end{vmatrix}$$

c. Perspective projection

$$Persp(n,Q,R) = \begin{vmatrix} [(R - Q) \cdot n]\mathrm{I} - n \otimes R & -{}^{t}n \\ (Q \cdot n)R & R \cdot n \end{vmatrix}$$