◊ VI.2

R.E versus N.H Specular Highlights

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◊ Abstract ◊

Phong's illumination model is one popular method for calculating the specular contribution from a light source, which involves taking the dot product of the light's reflection vector R and the eye vector E (Phong 1975). Another method involves finding the vector halfway between the eye and the light vector (call this H), and taking the dot product of H and the surface normal vector N (Blinn 1977). In either case, the resulting dot product is raised to an exponent to control the shininess of the specular contribution. This Gem will discuss some commonalities and misconceptions regarding the two models and show that similar highlights result if Blinn's exponent is four times Phong's exponent, that is: $(N \cdot H)^{4r} \approx (R \cdot E)^r$.

◊ Organization and Definitions ◊

This Gem is organized as follows:

- Definitions for each lighting model are presented along with some history about the origins of each model, and some common misconceptions.
- Motivation for using the $N.H$ model and why it is faster in some situations.
- Relationships between the two models are derived for the cases where L, N, and E are coplanar. Associated errors are examined.
- Images relating the two lighting models.
- Pixel errors when N is not coplanar with L and E.
- A general relationship between the two lighting models.
- Characterizing the maximum error and its location.
- Summarizing the pixel errors.
- Summary.

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For the following, refer to Figure 1. All vectors are normalized and anchored at the point to be lit. For convenience, some vector examples may be specified in their unnormalized form, such as $[-1, 1, 0]$ instead of $[-\sqrt{2}, \sqrt{2}, 0]$. This paper uses a right-handed coordinate system.

L = Vector from point to light
N = Surface normal at the point
E = Vector from point to eye
R = Reflected light vector = $2(L \cdot N)N - L$
M = Reflected eye vector = $2(E \cdot N)N - E$
H = Half angle vector = $Normalize(L + E) = (L + E)/\text{sqrt}(2 + 2(L \cdot E))$

Let α	be the angle between L and N, which is the same as between N and R
Let β	be the angle between L and H, which is the same as between H and E
Let $R.E$	be the name of the lighting model which uses the dot product of R and E to calculate the specular contribution
Let $N.H$	be the name of the lighting model which uses the dot product of H and N to calculate the specular contribution
Let ρ	be the angle from R to E (rho for reflection); $\cos\rho = R \cdot E$
Let ν	be the angle from N to H (nu for normal); $\cos\nu = N \cdot H$
Let r	be the shininess exponent for the $R.E$ model
Let n	be the shininess exponent for the $N.H$ model
Let f	be the factor required to make the following true: $\cos^r\rho = \cos^{fr}\nu$, $n = fr$
LNR	refers to the plane containing the vectors L, N, and R
LHE	refers to the plane containing the vectors L, H, and E
$[a, b]$	means the angle between vector a and b

In this Gem, all pixel errors are represented on a scale from -255 to 255. These represent the maximum error in typical images, because the peak pixel value (radiance) of a highlight is usually chosen to be 255 or less for frame buffers with 8 bits per channel.

◇ Some History ◇

Bui-Tuong Phong was the first to propose a model for specular reflections (Phong 1975). This model basically evaluates the highlight intensity at $(R \cdot E)^r$ (see Figure 1). Blinn later proposed a variation of this highlight evaluation: $(N \cdot H)^n$, where $H = (L + E)/|L + E|$ (Blinn 1977). Both models have often been referred to as Phong's illumination model (not to be confused with Phong shading).

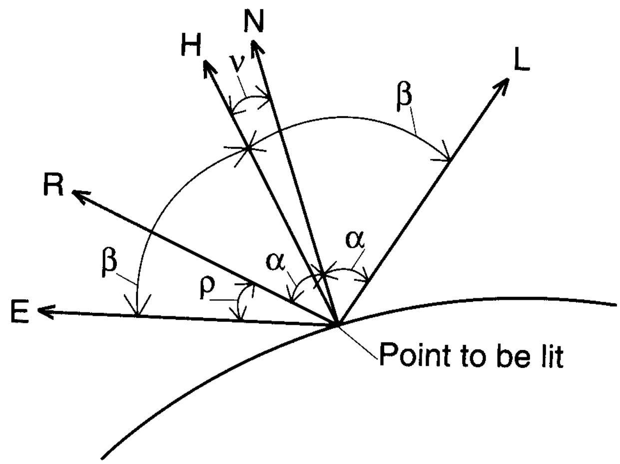

Figure 1. Vectors associated with the ***R.E*** and ***N.H*** lighting models in coplanar case.

There are numerous misconceptions about these two models such as the notion that the $R.E$ and $N.H$ models are actually the same (Hearn and Baker 1986, pp. 281–282) (this was corrected in a later edition of the book). In a later section, it will be shown that $[R, E]$ (ρ) is in certain cases equal to twice $[N, H]$ (ν). Watt and Watt stated this relationship as generally true in their textbook (Watt and Watt 1992, p. 45). However, this is not generally correct since it assumes that all the vectors described in Figure 1 are coplanar.

There are two extreme sets of conditions where ρ is constant and ν can vary over a wide range. Consider: $E = [0, 0, 1]$, $L = [-\epsilon, 0, -1]$, and $N = [x, y, 0]$. Various combinations of x and y will all produce a reflection vector such that ρ is small, yet ν ranges from 0° to 90°. If $N = [-1, 0, 0]$, then ν is also about 0°. If $N = [-1, 1, 0]$, then ν is about 45°. If $N = [0, 1, 0]$, then ν is about 90°. Conditions of this type produce a number of visual artifacts.

A second set of extreme conditions are situations like $E = [0, 0, 1]$, $L = [-1, 0, 0]$, and $N = [x, y, 0]$. In this case R is always in the xy plane and ρ is always 90°. When $N = [0, 1, 0]$, then $H = [-1, 0, 1]$ and ν is also 90°. As N is rotated about the z-axis from $[0, 1, 0]$ to $[-1, 0, 0]$, then ν changes from 90° to 45°. Conditions of this type do not produce visual artifacts if the exponent is large. For example, even when n is as small as 10, the maximum pixel error is $255 \cos^{10} 45° = 8$.

Clearly $\rho \neq 2\nu$ for many non-coplanar conditions. Errors associated with this fact will be examined later. However, note that for all cases, the maximum highlight for each model resides at exactly the same location. Proof of this is simple and left to the reader.

Other references on this topic can be found, though lightly discussed, in textbooks (Foley *et al.* 1990, pp. 738–739, 813), (Hall 1989, pp. 76–78, 87). Also, Max mentioned that $\rho \approx 2\nu$ and the use of $n = 4r$ for some limited conditions (Max 1988, p. 39).

◇ Motivations for Using the *N.H* Model ◇

It can be argued that the $N.H$ evaluation is computationally faster than $R.E$. If we have a directional light with a constant viewing vector (perhaps an orthographic view or the eye at infinity for lighting), then the H vector is constant for the entire rendered scene. Thus the $N.H$ evaluation only requires finding H when the light vectors change. If H is not constant, then $N \cdot H$ can be evaluated as $(N \cdot L + N \cdot E)/\text{sqrt}(2 + 2(L \cdot E))$. Generally, $N \cdot L$ is already calculated for the diffuse component, while $N \cdot E$ is constant for all lights being considered with N.

When using the $R.E$ model, it's not necessary to compute R for every light. By computing the eye's reflection vector (M), then the value of $R \cdot E$ is equivalent to $L \cdot M$. Therefore, only one reflection calculation is required for each normal. This implies that M is constant for each shading pass (computed only once per shading pass), and can be reused as the mirror reflection ray during ray tracing as well. As a result, it can be seen that for different assumptions, the $R.E$ model is faster than the $N.H$ model.

We also need to consider which of the two models is more physically based. Hall plotted and compared the distribution functions of $R \cdot E$ and $N \cdot H$ (Hall 1989). Hall showed that the $N \cdot H$ distribution matched the more physically based Torrance-Sparrow model much more closely than $R \cdot E$. In addition, renderings were done by the authors that supported the conclusions of these function plots, in which $N \cdot H$ highlights were much more similar in shape to the Torrance-Sparrow model than the $R \cdot E$ highlights.

Understanding the relationship between these two lighting models is helpful in matching software between an Application Programming Interface (API) that uses the $R.E$ model (such as PEX) and ones that use the $N.H$ lighting model (such as Silicon Graphics' GL).

◇ Coplanar *L, N,* and *E* ◇

When L, N, and E are coplanar, it can be shown that:

$$\rho = 2\nu, \quad 0^\circ \leq \rho < 180^\circ \tag{1}$$

$$\begin{aligned} \rho + 2\alpha &= 2\beta; \text{ and } \beta = \nu + \alpha \\ \rho + 2\alpha &= 2(\nu + \alpha) = 2\nu + 2\alpha \\ \rho &= 2\nu \end{aligned}$$

Although this relationship is valid as long as $N \cdot H > 0$, keep in mind that this is only meaningful for $\rho < 90^\circ$, otherwise the $R.E$ lighting model implies that the specular contribution should be zero while the $N.H$ specular contribution may exist until ρ is almost 180°. This difference can often be ignored because the specular contribution from the $N.H$ lighting model becomes negligible for typical exponents with $\rho > 90°$.

One way to simulate $R.E$ lighting with the $N.H$ model would be to calculate $N \cdot H$ and convert it to $R \cdot E$ before raising the value to the exponent. Using standard trigonometric identities produces:

$$\cos\rho = \cos 2\nu = 2\cos^2\nu - 1 \tag{2}$$

So, if $N \cdot H$ is available, it could be converted to $R \cdot E$ with:

$$R \cdot E = 2(N \cdot H)(N \cdot H) - 1 \tag{3}$$

The drawback of this approach is that it requires modifying the internals of whatever is calculating $N \cdot H$ and raising it to the power of r.

Alternatively, what is the value of f that would satisfy the following?

$$(R \cdot E)^r = (N \cdot H)^{fr} \tag{4}$$

In other words, given that the specular contribution will be calculated using $N \cdot H$ raised to some power, how should the exponent for the $R.E$ model be modified such that (4) is true.

To solve for f, start by taking the rth root of both sides of Equation (4):

$$\begin{aligned}
&(R \cdot E) = (N \cdot H)^f \\
&\text{Substitute } (R \cdot E) = 2(N \cdot H)^2 - 1 \\
&2(N \cdot H)^2 - 1 = (N \cdot H)^f \\
&\text{Take the log of both sides} \\
&\log(2(N \cdot H)(N \cdot H) - 1) = f\ \log(N \cdot H) \\
&\text{And solve for } f
\end{aligned}$$

$$f = \log(2(N \cdot H)(N \cdot H) - 1)\ /\ \log(N \cdot H) \tag{5}$$

Thus, we see that the f is independent of the exponent, depending solely on the $N \cdot H$ result. Table 1 shows some substitutions of $N \cdot H$ into (5).

From this we hypothesize that using a constant f of 4 might provide an acceptable approximation. Comparing the Taylor series expansion of $\cos^r 2\nu$ and $\cos^{4r}\nu$ shows that this is quite accurate for small ρ, and $\rho \approx 2\nu$.

By definition, $\cos(a) = 1 - \frac{a^2}{2} + O(a^4)$ and $(1+x)^p = 1 + px + O(x^2)$ so for small ν

$$\begin{aligned}
\cos^r 2\nu &\approx (1 - 2\nu^2)^r \approx 1 - 2r\nu^2 \\
\cos^{4r}\nu &\approx (1 - \tfrac{\nu^2}{2})^{4r} \approx 1 - 2r\nu^2
\end{aligned}$$

thus $\cos^r 2\nu \approx \cos^{4r}\nu$

Table 1. Required f for ρ given any exponent, coplanar *LNE*

ρ (degrees)	5	20.0	40.0	60.0	80.0	85.0	87.00	89.00	89.50	89.90
Required f	4.0038	4.06	4.28	4.82	6.57	8.01	9.19	11.98	13.85	18.37

Given $\rho = 2\nu$, this table shows the value of f required to make the following true: $\cos^r \rho = \cos^{fr} \nu$.

Table 2. Pixel value errors for r from 5 to 45, coplanar *LNE*

	Shininess exponent r								
ρ (deg.)	5.00	10.00	15.00	20.00	25.00	30.00	35.00	40.00	45.00
10.00	0.069	0.128	0.179	0.221	0.255	0.284	0.307	0.325	0.339
20.00	0.906	1.330	1.466	1.435	1.318	1.161	0.995	0.835	0.690
30.00	3.252	3.210	2.376	1.564	0.965	0.571	0.329	0.186	0.103
40.00	6.227	3.437	1.424	0.525	0.181	0.060	0.019	0.006	0.002
50.00	7.668	1.913	0.360	0.060	0.010	0.001	0.000	0.000	0.000
60.00	6.391	0.560	0.038	0.002	0.000	0.000	0.000	0.000	0.000
70.00	3.525	0.082	0.002	0.000	0.000	0.000	0.000	0.000	0.000
80.00	1.195	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000
85.00	0.573	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Given $\rho = 2\nu$, and pixel values that range from 0 to 255, this table shows the error associated with calculating the specular contribution using: $\cos^{4r} \nu$ instead of $\cos^r \rho$.
Specifically, table value $= 255(\cos^{4r} \nu - \cos^r \rho)$.

As ρ increases beyond 90°, the $R.E$ result would be less than zero, but $N \cdot H$ is still greater than zero because the $N \cdot H$ curve does not fall off as rapidly as $R \cdot E$. This means that for large angles from the eye to the reflection vector, the error increases without bounds. Fortunately, when the value of $(N \cdot H)^{fr}$ is calculated, and combined with the pixel range of $[0, 255]$, the maximum pixel value error is limited. We measure the absolute error using the following definition:

$$\text{pixel error} = 255(\cos^{4r} \nu - \cos^r \rho) \tag{6}$$

For exponents from 5 to 45, Table 2 shows the error in pixel values. Note that using $f = 4$ produces very little error. Although the required f from Table 1 increases with ρ, the final pixel values are smaller, and the overall effect is very little pixel value errors even when using a constant f of 4. For exponents greater than 45 (which are typical values), the error remains less than one pixel value. When the exponent is less than 5 (not typical), and ρ approaches 90°, the maximum pixel error may be from 10 to greater than 60.

◇ Images Relating the Two Lighting Models ◇

In order to test the notion of using $f = 4$ in the general case, we rendered some spheres under varying conditions. Figure 2 was rendered with the values of $E = [0, 0, 1]$, $L = [Lx, 0, Lz]$, and $N = [Nx, Ny, Nz]$.

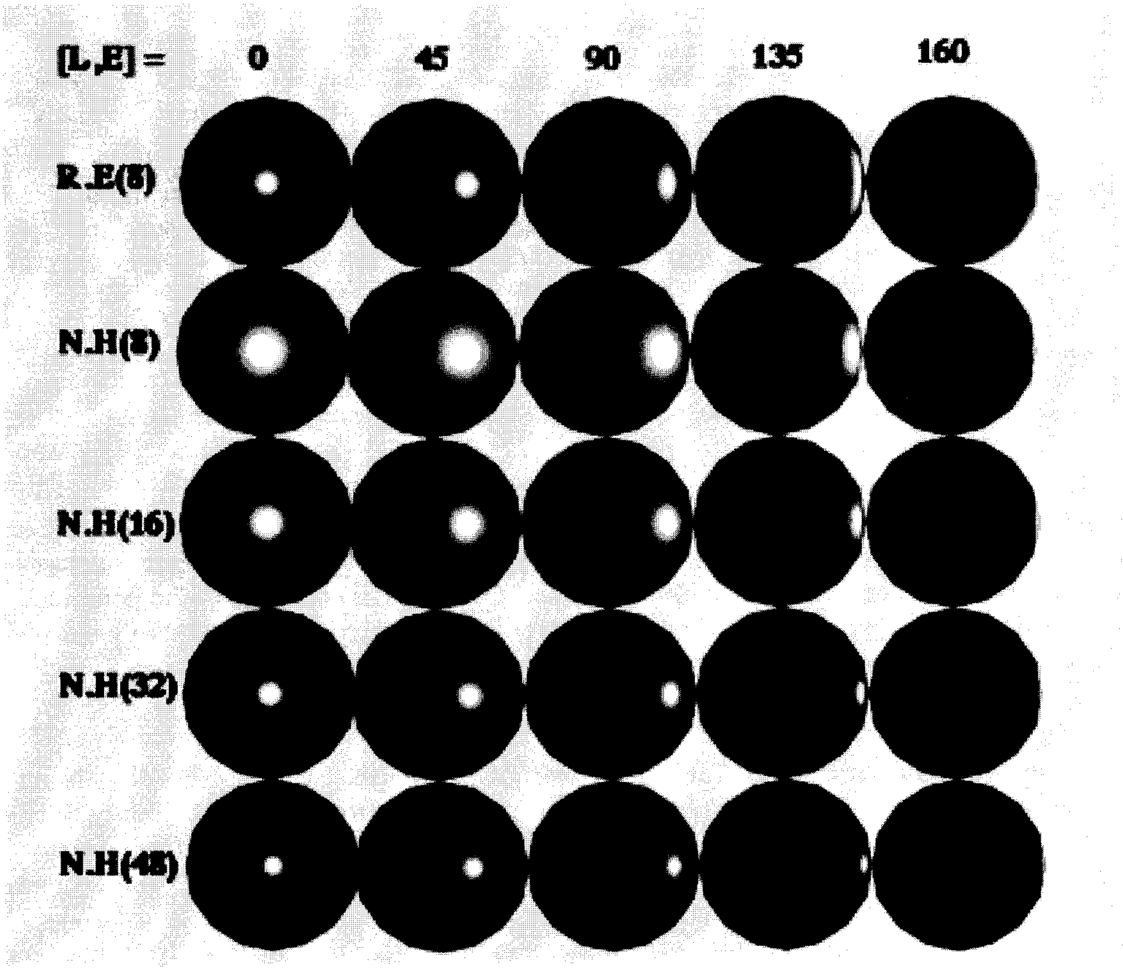

Figure 2. Normalized images comparing *R.E* and *N.H* specular highlights.

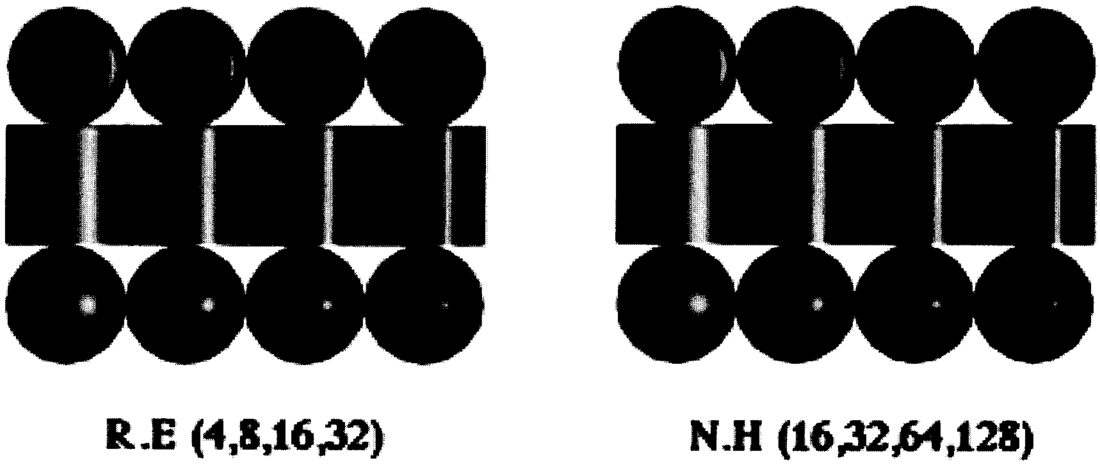

Figure 3. Some quadric surfaces comparing $R.E$ and $N.H$ specular highlights.

Columns of images from left to right correspond to $[L, E]$ of: 0°, 45°, 90°, 135°, and 160°. From top to bottom, each row of spheres was rendered with: $R.E$ model, $N.H$ model with $f = 1, 2, 4$, and 6. Note that for $[L, E] \leq 45°$, the $R.E$ model and $N.H$ model with $f = 4$ exhibit almost identical highlights. However, for $[L, E] > 45°$, their highlights look noticeably different. This is when ρ is small and $[L, E]$ becomes large. Even more interesting are the general shapes of the differing highlights: the $N.H$ model appears to generate more circular highlights regardless of the geometry, whereas the $R.E$ model produces highlights that appear to wrap around the geometry. The $N.H$ highlight effect more closely corresponds to the physically based lighting effects observed in nature.

Figure 3 shows three distinct geometry types (bottle cap, cylinder, sphere) lit by a single directional light source.

Geometries exhibit different patterns of highlight (from different surface normal orientations). The left image was produced with the $R.E$ model, using cosine exponents of 4, 8, 16 and 32. The right image was produced with the $N.H$ model, with the exponent values multiplied by $f = 4$. The highlights look to be very similar, and hardly distinguishable from each other, except for the left-most cylinders in each image.

A final note: it is possible to generate results where both models will exhibit a sharp discontinuity in highlights, as can be seen in Figure 2 when $[N, H]$'s exponent is 8, and $[L, E] = 135°$. This occurs when $N.L < 0$, but the highlight dot products ($R.E$ or $N.H$) are still positive. In most rendering software, the $N \cdot L < 0$ check determines if any diffuse or specular light is received. Thus, this is a common flaw in both models.

◇ Pixel Errors When *N* Is Not Coplanar with *L* and *E* ◇

Unfortunately, when E is not coplanar with L and N, the relationship of $\rho = 2\nu$ can only be an approximation. However, the associated errors are minimized by using f equal to four.

In order to visualize the errors, note that L, N, R are always coplanar, and L, H, E are also coplanar. Given any system of L, N, and E, it's possible to transform the original coordinates to a normalized right-handed reference coordinate system such that:

- The eye vector is pointing along the +z axis. $E = [0, 0, 1]$.
- The LHE plane is transformed such that L, H, and E lie in the $-x$ half space, on the xz plane.
- The L vector is in the xz plane where $x \leq 0$. $L = [Lx, 0, Lz]$, $Lx^2 + Lz^2 = 1$.
- The normal can be anywhere in the $+z$ halfspace. This implies that the reflection vector may be anywhere. To characterize the position of the normal, we define the angles ϕ and θ. See Figure 4. ϕ is measured from the $-x$ axis. θ is measured from the $+y$ axis. Both these angles range from 0° to 180°. Definitions for the normal angles were chosen such that plots and tables enumerating information have roughly the same orientation.
  $N = [-\sin\theta\cos\phi, \cos\theta, \sin\theta\sin\phi]$
- All possible normal positions are representable with: $0° \leq \phi \leq 180°$ and $0° \leq \theta \leq 180°$.

This scenario corresponds to looking at the specular highlight on a unit sphere, using directional lights with the eye at infinity (constant angle between L and E). These assumptions leave three degrees of freedom: two for the direction of the normal and one for the angle between the light and eye.

Figure 5 shows the pixel errors for different N vectors, given $[L, E] = 90°$, $f = 4$, and $r = 10$.

◇ A General Relationship between the Two Lighting Models ◇

It was possible to derive the following generalization which relates ρ and ν:

$$\cos\rho = \cos 2\nu \; + \; 2\sin^2\beta\cos^2\theta \tag{7}$$

The coplanar cases correspond to $\theta = 90°$. Note the error increases as β increases or the normal moves from the coplanar cases. Equation 7 was perceived by a stroke of intuition, preceded by hours of looking at function plots of various sub-terms associated with $N.H$ and $R.E$.

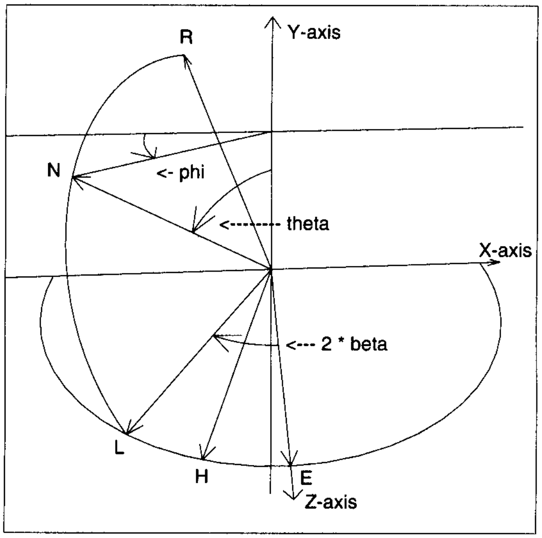

Figure 4. Reference coordinate system for visualizing lighting model parameters.

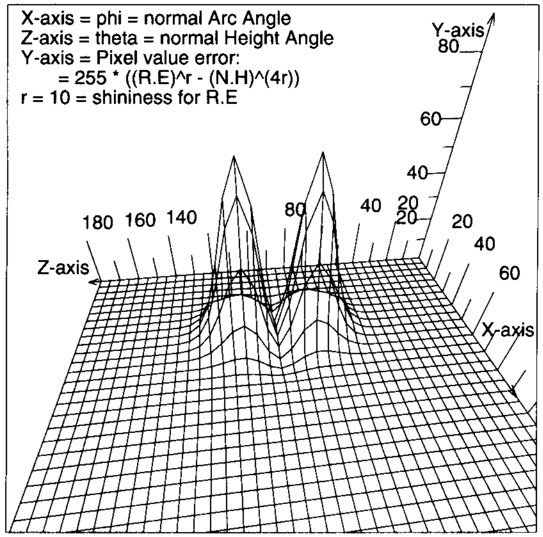

Figure 5. Pixel errors: $f = 4$, $[L, E] = 90°$, $Y = 255$ $((R \cdot E)^r - (N \cdot H)^{4r})$.

Equation 7 can be completely generalized by noting that:

$$
\begin{aligned}
\cos\theta &= \frac{L \times H \cdot N}{|L \times H|} = \frac{L \times E \cdot N}{|L \times E|} \\
|L \times H| &= \sin\beta \\
|L \times E| &= \sin 2\beta \\
1 + L \cdot E &= 1 + \cos 2\beta = 2\cos^2\beta
\end{aligned}
$$

Appropriate substitutions into 7 will lead to:

$$\cos\rho = \cos 2\nu + 2(L \times H \cdot N)^2 = \cos 2\nu + \frac{(L \times E \cdot N)^2}{1 + L \cdot E} \tag{8}$$

$$R \cdot E = 2(N \cdot H)(N \cdot H) - 1 + 2(L \times H \cdot N)^2 \tag{9}$$

Equation 9 is not strictly a function of $N.H$ because of the cross product term. Such a function may not exist because of the degrees of freedom associated with α and β. Further substitutions of expressions that only involve L, E, or N will result in:

$$(L \cdot E)^2 + (E \cdot N)^2 + (N \cdot L)^2 - 2(L \cdot E)(E \cdot N)(N \cdot L) - 1 + (L \times E \cdot N)^2 = 0 \tag{10}$$

Although Equations 7 through 9 are more useful when considering the lighting models, the proof of their validity has only been verified by showing that 10 is an identity. Equation 10 is also interesting because it is valid for any three normalized vectors.

Equation 10 may be proved by expanding terms and recombining. A more elegant proof was found by Alan Paeth. For square matrices, the following identity is true:

$$|M^T M| = |M|^2 \tag{11}$$

Let $M = [L|E|N]$, the matrix formed by augmenting column vectors L, E, and N. Substituting M into Equation 11 will lead to Equation 10, which may be done with unnormalized vectors to get a similar identity.

Points along the plane where $L \times E \cdot N = 0$ will generate pixel values with little difference between the two models, even though $R \cdot E$ falls off more rapidly than $N \cdot H$ when moving away from the highlight. This corresponds to the coplanar LNE cases.

As N changes from the case producing the highlight along the plane where $L \times E \times N$ points in the same direction as $L \times E \times H$, then $N \cdot H$ falls off more rapidly than $R \cdot E$. These are the cases which produce the error bumps as depicted in Figure 5. Somewhere in between this case and coplanar LNE there is a contour of points where the error is zero.

Visual differences occur when two conditions are met: ρ is small *and* $[L, E]$ is large, i.e., when looking at points near a highlight caused by a light vector that is nearly opposite the eye vector. See Figure 2.

A more complex interaction is introduced by considering the eye at varying distances from a unit sphere. Again the shape of the error function is similar to the previous examples, but the error increases as the eye is brought closer to the unit sphere. When the eye is farther than about 5 units from the unit sphere, then the error function roughly corresponds to the one with the eye at infinity.

Similar complications occur when the light is not directional, i.e., it is a spot or positional light.

◇ Characterizing the Maximum Error and Its Location ◇

This section is intended to provide a feel for how the bumps in the pixel error plot of Figure 5 change with $[L, E]$ and shininess. Although the error function is quite complex for small shininess values (less than 10), some generalizations can be made for larger exponents.

It's possible to derive closed form error functions for some combinations of f and shininess; however, many conditions produce intractable equations to solve. The following observations were obtained using numerical techniques.

When $f = 4$, the shape of the plots depicting the pixel errors is roughly the same for other values of $[L, E]$, except when $[L, E]$ is very small or very large. As $[L, E]$ becomes

Table 3. Maximum pixel error given shininess and [L, E], with f = 4

	2β = angle between L and E (degrees)						
r	0.00	30.00	60.00	90.00	120.00	150.00	180.00
10.00	-3.63	4.73	25.56	63.75	122.24	198.52	255.00
20.00	-1.77	5.53	26.23	63.75	121.35	197.23	255.00
40.00	-0.87	6.00	26.56	63.75	120.91	196.59	255.00
60.00	-0.58	6.16	26.67	63.75	120.77	196.37	255.00
80.00	-0.44	6.24	26.73	63.75	120.69	196.27	255.00
100.00	-0.35	6.29	26.76	63.75	120.65	196.20	255.00
200.00	-0.18	6.40	26.83	63.75	120.56	196.08	255.00
300.00	-0.12	6.43	26.86	63.76	120.54	196.03	255.00
400.00	-0.09	6.45	26.87	63.76	120.52	196.01	255.00

small, the peaks merge to form a ring around the maximum highlight. At the same time, the maximum pixel value error decreases to less than 4 for a shininess of 10.

As $[L, E]$ becomes large, the peaks grow to a maximum of 255 pixel values.

For a given $[L, E]$, the location of the maximum error moves toward the point of maximum highlight as the shininess increases.

Curiously, the maximum pixel error is relatively independent of the exponent, depending mostly on the angle $[L, E]$. While this may appear to be a problem with the approximation, closer examination reveals that the volume of the error bumps decreases significantly as the shininess increases. Table 3 shows the maximum pixel error for some values of $[L, E]$.

◇ Summarizing the Pixel Errors ◇

For each error surface like the one depicted in Figure 5, it's possible to characterize the pixel error by calculating the average and standard deviation. A summary of sampled pixel error averages is not presented since the average is always close to zero because of the many samples, only a few are non-zero near the highlight. A summary of the standard deviations would show error values less than 10 for exponents of 40 and $[L, E]$ less than 120°, or with exponents of 180 and $[L, E]$ less than 140°.

While the volume under the error region decreases as the shininess increases, the peak error values remain nearly the same. This is due to the nature of the equations which produce zero error along a line through the center of the highlight.

◇ Summary ◇

The $R.E$ and $N.H$ lighting models are different, although similar highlights result if Blinn's exponent is four times Phong's exponent, that is: $(N \cdot H)^{4r} \approx (R \cdot E)^r$.

Visual differences occur when two conditions are met: ρ is small and $[L, E]$ is large, i.e., when looking at points near a highlight caused by a light vector that is nearly opposite the eye vector.

An identity was derived which relates angles involved in the $R.E$ and $N.H$ lighting models:

$$\cos\rho = \cos 2\nu + \frac{(L \times E \cdot N)^2}{1 + L \cdot E} \tag{12}$$

Demonstrating the validity of Equation 12 resulted in proving an interesting identity that is valid for any three normalized vectors, L, E, and N:

$$(L \cdot E)^2 + (E \cdot N)^2 + (N \cdot L)^2 - 2(L \cdot E)(E \cdot N)(N \cdot L) - 1 + (L \times E \cdot N)^2 = 0 \tag{13}$$

◇ Acknowledgments ◇

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