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Computing the Intersection of a Line and a Cylinder

Ching-Kuang Shene
Department of Mathematics and Computer Science
Northern Michigan University
Marquette, MI 49855-5340
facs@nmumus.bitnet

Computing the intersection of a line and a surface is a common operation in graphics applications. Traditional methods usually assume that the surface is given by an implicit equation and reduce the intersection problem to solving a single-variable equation. However, in many graphics applications, a cylinder or a cone is represented by some geometric form like the one used in this Gem. Although a geometric form can be transformed to an implicit equation (Shene 1994) so that traditional methods could be applied, a direct geometric method would be more efficient and robust. In this Gem, we shall present a simple geometric technique to compute the intersection of a line and a circular cylinder. The following notations will be used throughout this Gem:

- Upper- (*resp.*, lower-) case vectors are position (*resp.*, direction) vectors. Position vectors are sometimes referred to as points. Therefore, $\vec{P}$ and P are equivalent. All direction vectors are of unit length. $|\vec{U}|$ is the length of vector $\vec{U}$.
- $\overleftrightarrow{PQ}$ and $\overline{PQ}$ are the line and the segment, respectively, determined by points $\vec{P}$ and $\vec{Q}$.
- $\vec{u} \times \vec{v}$ denotes the cross product of vectors $\vec{u}$ and $\vec{v}$.
- $\vec{u} \otimes \vec{v}$ is the normalized $\vec{u} \times \vec{v}$. That is, $\vec{u} \otimes \vec{v} = \vec{u} \times \vec{v}/|\vec{u} \times \vec{v}|$.
- $\ell(\vec{A}, \vec{u})$ is the line defined by base point $\vec{A}$ and direction $\vec{u}$.
- $\mathcal{C}(\vec{A}, \vec{u}, r)$ is the circular cylinder with axis $\ell(\vec{A}, \vec{u})$ and radius r.

Let $\ell(\vec{A}, \vec{u})$ and $\mathcal{C}(\vec{B}, \vec{v}, r)$ be a line and a circular cylinder. If $\vec{u}$ and $\vec{v}$ are parallel, then we have two cases to consider based on the distance from $\vec{B}$ to ℓ. If this distance is not equal to r, ℓ does not intersect $\mathcal{C}$; otherwise, ℓ lies on $\mathcal{C}$.

Suppose ℓ and the axis of $\mathcal{C}$ are not parallel. Let θ be the acute angle between $\vec{u}$ and $\vec{v}$. Thus, $\cos\theta = |\vec{u} \cdot \vec{v}|$. Let $\overleftrightarrow{OP}$ be the common perpendicular of ℓ and the axis of the cylinder, where $\vec{O}$ is on the cylinder's axis and $\vec{P}$ is on ℓ. Let $|d|$ be the length of the segment $\overline{OP}$. Then, the plane containing $\ell(\vec{A}, \vec{u})$ and $\overleftrightarrow{OP}$ cuts $\mathcal{C}$ in an ellipse with

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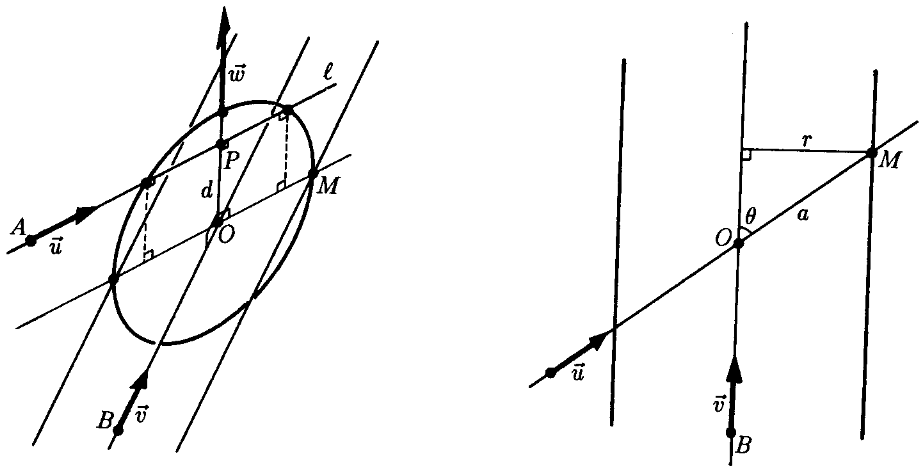

Figure 1. Computing the intersection point of a line and a circular cylinder.

semi-major axis length $a = |\vec{M} - \vec{O}| = r/\sin\theta = r/\sqrt{1 - (\vec{u}\cdot\vec{v})^2}$ and semi-minor axis length r, where M is the intersection point of $\mathcal{C}$ and the line through O and parallel to ℓ (see Figure 1). If $\overleftrightarrow{OM}$ and $\overleftrightarrow{OP}$ are chosen to be the x- and the y-axes, respectively, and O the origin, the intersection ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{r^2} = 1$. Since $\ell(\vec{A}, \vec{u})$ is parallel to the x-axis at a distance of $|d|$, its intersection points with the ellipse can be determined by computing the x-coordinates corresponding to $y = |d|$. Hence, we have

$$x = \pm\frac{a}{r}\sqrt{r^2 - d^2} = \pm\sqrt{\frac{r^2 - d^2}{1 - (\vec{u}\cdot\vec{v})^2}}$$

If $r < |d|$, ℓ intersects $\mathcal{C}$ at two points,

$$\vec{P} \pm \sqrt{\frac{r^2 - d^2}{1 - (\vec{u}\cdot\vec{v})^2}}\vec{u}$$

If $r = |d|$, ℓ is tangent to $\mathcal{C}$ at $\vec{P}$; otherwise, ℓ does not intersect $\mathcal{C}$.

Remark. $\vec{P}$ and $|d|$ are not difficult to compute. Since the common perpendicular of ℓ and the cylinder's axis has direction $\vec{w} = \vec{u} \otimes \vec{v}$, we have

$$\vec{A} + r\vec{u} + d\vec{w} = \vec{B} + s\vec{v} \tag{1}$$

for some appropriate r and s. Since both $\vec{u}$ and $\vec{v}$ are perpendicular to $\vec{w}$, computing the inner product of Equation (1) with $\vec{w}$ gives $d = (\vec{B} - \vec{A}) \cdot \vec{w}$. Computing the cross product of Equation (1) with $\vec{v}$ gives $(\vec{B} - \vec{A} - d\vec{w}) \times \vec{v} = r(\vec{u} \times \vec{v})$. Computing the inner product of this result with $\vec{u} \times \vec{v}$ delivers $r = [(\vec{B} - \vec{A} - d\vec{w}) \times \vec{v}] \cdot (\vec{u} \times \vec{v}) / |\vec{u} \times \vec{v}|^2$. Therefore, $\vec{P} = \vec{A} + r\vec{u}$ is determined. □

Using some results from classic theory of conic sections (Drew 1875, Macaulay 1895), we can apply the same technique to compute the intersection of a line and a cone; however, the resulting formulæ are more involved. The interested reader should refer to (Johnston and Shene 1992) for the details.

See also the other article on ray-cylinder intersection in this volume (Cychosz and Waggenspack 1994).

◇ Bibliography ◇

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