Mean Curvature Flow with Surgery for Low Entropy Mean Convex Hypersurfaces

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Abstract. In this article, we extend the mean curvature flow with surgery to mean convex hypersurfaces with low entropy. In particular, 2-convexity is not assumed.

1. Introduction

In [**13**] Colding and Minicozzi introduced a quantity $\lambda(M)$ of a submanifold $M \subset \mathbb{R}^N$ they called the entropy; its an especially interesting quantity for a number of reasons, one of which being that it in a strong sense captures information about a submanifold (see for example the landmark work by Bernstein and Wang [**5**]) but on the other hand the set of submanifolds is a robust set under perturbations compared to an apriori curvature condition, which constrain the geometry of a submanifold pointwise. We denote by $\Lambda_k = \lambda(\mathbb{S}^k) = \lambda(\mathbb{S}^k \times \mathbb{R}^{n-k})$. According to Stone's computation [**29**]:

$$\Lambda_1 > \frac{3}{2} > \Lambda_2 > ... > \Lambda_n \to \sqrt{2}$$

.

In $\mathbb{R}^4$, from work in [**6**] by Bernstein and the second named author that when the entropy is below $\Lambda_2 = \lambda(\mathbb{S}^2 \times \mathbb{R})$ any closed hypersurface M with $\lambda(M) < \Lambda_2$ is topologically a 3-sphere and the level-set flow such surfaces with stay connected until extinction. It is a natural question to ask what can be said of flows of surfaces with the next highest level of entropy, namely those with entropy below Λ_1 (or in higher dimensions, Λ_{n-2}).

As a step towards answering this question, note that mean convex self shrinkers of entropy bounded by Λ_1 are either S^n or $S^{n-1} \times \mathbb{R}$ and hence are 2-convex. In addition, mean curvature flow with surgery has been established for globally 2-convex hypersurfaces. This suggests a mean curvature flow with surgery for mean convex low entropy (but not necessarily 2-convex) hypersurfaces is possible and is the topic of this article. Namely, in this article we show how to extend the mean curvature flow with surgery as defined by Haslhofer and Kleiner to mean convex surfaces of low entropy:

THEOREM 1.1. *Let $\mathcal{M} = \mathcal{M}(\alpha, n, \Pi)$ be the set of α noncollapsed closed hypersurfaces in $\mathbb{R}^{n+1}$ with entropy less than $\Pi < \Lambda_{n-2}$, then for any $M \in \mathcal{M}$ there is a mean curvature flow with surgery for a uniform choice of parameters $H_{th}, H_{neck}, H_{trig}$.*

Note that if $\Pi < \Lambda_{n-1}$ then any $M \in \mathcal{M}$ must shrink to a point; we will assume throughout that $\Pi \gneq \lambda_{n-1}$. In the process of proving our theorem we construct a mean convex hypersurface of low entropy (which in this article refers to surfaces M with $\lambda(M) < \Lambda_{n-2}$ unless otherwise stated) that develops a neckpinch, showing that surgeries are to be expected for $M \in \mathcal{M}$ above; see section 3.2 below. There are several corollaries of the surgery; the first consequence was first noted in the original mean curvature flow with surgery paper by Huisken and Sinestrari [**23**]:

COROLLARY 1.2. *If $M \in \mathcal{M}$, then $M \cong S^n$ or a finite connect sum of $S^{n-1} \times S^1$*

The second corollary, an extension of the first corollary, was observed in the 2-convex case by the first named author in [**27**]; only set monotonicity of the MCF was required so the proof immediately adapts to the low entropy case:

COROLLARY 1.3. *Let $\Sigma(d, C, \mathcal{M})$ be the set of hypersurfaces $M \in \mathcal{M}$ with diam$(M) < d$ (or equivalently up to translation, $\mathcal{M} \subset B_d(0)$) and $H < C$. Then $\Sigma(d, C, \mathcal{M})$ up to isotopy consists of finitely many hypersurfaces, in fact at most $2^{2n\frac{(12dC\sqrt{n})^n}{\alpha^n}}$.*

The last corollary relates the mean curvature flow with surgery to the level set flow, another weak solution to the mean curvature flow; it was shown by Laurer [**25**] and independently Head [**21**]:

COROLLARY 1.4. *Let $M \in \mathcal{M}$ and denote by $(M_H)_t$ the mean curvature flow with surgery of M with $H_{neck} = H$. Then as $H \to \infty$ the spacetime track of $(M_H)_t$ Hausdorff converges to the spacetime track of the level set flow of M.*

The structure of the article is as follows: first we give preliminary information on the mean curvature flow with surgery and Colding and Minicozzi's entropy, then we describe the proof of theorem 1.1 above. We end with some concluding remarks about future avenues of investigation and a supplementary appendix justifying the need for the roundabout construction of the "low entropy" surgery cap.

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2. Preliminaries.

The first subsection introducing the mean curvature flow we borrow quite liberally from the first named author's previous paper [**27**]. The second subsection concerns the mean curvature flow with surgery as constructed by Haslhofer and Kleiner in [**18**]. The third subsection introduces some basic facts and definitions concerning Colding and Minicozzi's entropy introduced in [**13**].

In this subsection we start with the differential geometric, or "classical," definition of mean curvature flow for smooth embedded hypersurfaces of $\mathbb{R}^{n+1}$; for a nice introduction, see [**26**]. Let M be an n dimensional manifold and let $F : M \to \mathbb{R}^{n+1}$ be an embedding of M realizing it as a smooth closed hypersurface of Euclidean space - which by abuse of notation we also refer to M. Then the mean curvature flow of M is given by $\hat{F} : M \times [0,T) \to \mathbb{R}^{n+1}$ satisfying (where ν is outward pointing normal and H is the mean curvature):

$$\frac{d\hat{F}}{dt} = -H\nu,\ \hat{F}(M,0) = F(M) \tag{2.1}$$

(It follows from the Jordan separation theorem that closed embedded hypersurfaces are oriented). Denote $\hat{F}(\cdot,t) = \hat{F}_t$, and further denote by M_t the image of $\hat{F}_t$ (so $M_0 = M$). It turns out that (2.1) is a degenerate parabolic system of equations so take some work to show short term existence (to see its degenerate, any tangential perturbation of F is a mean curvature flow). More specifically, where g is the induced metric on M:

$$\Delta_g F = g^{ij}\left(\frac{\partial^2 F}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F}{\partial x^k}\right) = g^{ij}h_{ij}\nu = H\nu \tag{2.2}$$

Now one could apply for example deTurck's trick to reduce the problem to a nondegenerate parabolic PDE (see for example chapter 3 of [**4**]) or similarly reduce the problem to an easier PDE by writing M as a graph over a reference manifold by Huisken and Polden (see [**26**]). At any rate, we have short term existence for compact manifolds.

Now that we have established existence of the flow in cases important to us, let's record associated evolution equations for some of the usual geometric quantities:

- $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}$
- $\frac{\partial}{\partial t} d\mu = -H^2 d\mu$
- $\frac{\partial}{\partial t} h^i_j = \Delta h^i_j + |A|^2 h^i_j$
- $\frac{\partial}{\partial t} H = \Delta H + |A|^2 H$
- $\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4$

So, for example, from the heat equation for H one sees by the maximum principle that if $H > 0$ initially it remains so under the flow. There is also a more complicated tensor maximum principle by Hamilton originally developed for the Ricci flow (see

[**17**]) that says essentially that if M is a compact manifold one has the following evolution equation for a tensor S:

$$\frac{\partial S}{\partial t} = \Delta S + \Phi(S) \tag{2.3}$$

and if S belongs to a convex cone of tensors, then if solutions to the system of ODE

$$\frac{\partial S}{\partial t} = \Phi(S) \tag{2.4}$$

stay in that cone then solutions to the PDE (2.2) stay in the cone too (essentially this is because Δ "averages"). So, for example, one can see then that convex surfaces stay convex under the flow very easily this way using the evolution equation above for the Weingarten operator. Similarly one can see that **2-convex hypersurface** (i.e. for the two smallest principal curvatures λ_1, λ_2, $\lambda_1 + \lambda_2 > 0$ everywhere) remain 2-convex under the flow.

Another important curvature condition in this paper is α **non-collapsing**: a mean convex hypersurface M is said to be 2-sided α non-collapsed for some $\alpha > 0$ if at every point $p \in M$, there is an interior and exterior ball of radius $\alpha/H(p)$ touching M precisely at p. This condition is used in the formulation of the finiteness theorem. It was shown by Ben Andrews in [**1**] to be preserved under the flow for compact surfaces. (a sharp version of this statement, first shown by Brendle in [**8**] and later Haslhofer and Kleiner in [**20**], is important in [**9**] where MCF+surgery to $n = 2$ was first accomplished).

Finally, perhaps the most geometric manifestation of the maximum principle is that if two compact hypersurfaces are disjoint initially they remain so under the flow; this fact is used in section 4.2 below. So, by putting a large hypersphere around M and noting under the mean curvature flow that such a sphere collapses to a point in finite time, the flow of M must not be defined past a certain time either in that as $t \to T$, M_t converge to a set that isn't a manifold. Note this implies as $t \to T$ that $|A|^2 \to \infty$ at a sequence of points on M_t; if not then we could use curvature bounds to attain a smooth limit M_T which we can then flow further, contradicting our choice of T. Thus weak solutions to the flow are necessitated; one type of weak solution is the mean curvature flow with surgery:

First we give the definition of α controlled:

Definition 2.1. *(Definition 1.15 in* [**18**]*) Let* $\mathbb{a} = (\alpha, \beta, \gamma) \in (0, N-2) \times (0, \frac{1}{N-2}) \times (0, \infty)$. *A smooth compact closed subamnifold* $M^n \subset \mathbb{R}^{n+1}$ *is said to be* $\mathbb{a}$*-controlled if it satisfies*

(1) is α*-noncollapsed*
(2) $\lambda_1 + \lambda_2 \geq \beta H$ *(*β *2-convex)*
(3) $H \leq \gamma$

Speaking very roughly, for the mean curvature flow with surgery approach of Haslhofer and Kleiner, like with the Huisken and Sinestrari approach there are three main constants, $H_{th} \leq H_{neck} \leq H_{trig}$. If H_{trig} is reached somewhere during the mean curvature flow M_t of a manifold M it turns out the nearby regions will be "neck-like" and one can cut and glue in appropriate caps (maintaining 2-convexity, etc) so that after the surgery the result has mean curvature bounded by H_{th}. The high curvature regions have well understood geometry and are discarded and the mean curvature flow with surgery proceeds starting from the low curvature leftovers. Before stating a more precise statement we are forced to introduce a couple more definitions. First an abbreviated definition of the most general type of piecewise smooth flow we will consider.

Definition 2.2. *(see Definition 1.3 in* [**18**]*) An* $(\alpha, \delta) -$ *flow* M_t *is a collection of finitely smooth* α*-noncollapsed flows* $\{M_t^i \cap U\}_{t \in [t_{i-1}, t_i]}$, $(i = 1, \ldots k; t_0 < \ldots t_k)$ *in an open set* $U \subset \mathbb{R}^{n+1}$*, such that:*

(1) for each $i = 1, \ldots, k-1$*, the final time slices of some collection of disjoint strong* δ*-necks (see below) are replaced by standard caps, giving* $M_{t_i}^{\#} \subset M_{t_i}^i =: M_{t_i}^-$ *(in terms of the regions they bound).*
(2) the initial time slice of the next flow, $M_{t_i}^{i+1} =: M_{t_i}^+$*, is obtained from* $M_{t_i}^{\#}$ *by discarding some connected components.*

Of course, now we should define what we mean by standard caps, cutting and pasting, and strong δ-necks. Since we will need them in the sequel, we will give the full definitions; these are essentially definitions 2.2 through 2.4 in [**18**]:

Definition 2.3. *A standard cap is a smooth convex domain that coincides with a smooth round half-cylinder of radius 1 outside a ball of radius 10.*

The model we give for a standard cap will morally agree with the definition given above although the radius outside which it will agree with the round cylinder will potentially need to be taken larger than 10. In the next definition that in practice (considering a neck point p on M) will be $\frac{1}{H(p)}$; in particular after rescaling it will be equal to 1:

Definition 2.4. *We say than an (α,δ)-flow M_t has a strong δ-neck with center p and radius s at time $t_0 \in I$, if $\{s^{-1}\cdot(M_{t_0+s^2t}-p)\}_{t\in(-1,0]}$ is δ-close in $C^{[1/\delta]}$ in $B^U_{1/\delta}\times(-1,0]$ to the evolution of a round cylinder $S^n\times\mathbb{R}$ with radius 1 at $t=0$, where $B^U_{1/\delta}=s^{-1}\cdot((B(p,s/\delta)\cap U)-p)\subset B(0,1/\delta)\subset\mathbb{R}^{n+1}$.*

Now is the definition of cutting and pasting:

Definition 2.5. *We say that a final time slice of a strong δ-neck ($\delta\le\frac{1}{10\Gamma}$) with center p and radius s is replaced by a pair of standard caps if the pre-surgery domain $M^{\#}$ is replaced by a post surgery domain M^{+} such that*

(1) the modification takes place inside a ball $B=B(p,5\Gamma s)$
(2) there are bounds for the second fundamental form and its derivatives:

$$\sup_{M^+\cap B}|\nabla^\ell A|\le C_\ell s^{-1-\ell}\ (\ell=0,1,2,\ldots)$$

(3) if $B\subset U$, then for every point $p_+\in M^+\cap B$ with $\lambda_1(p_+)<0$, there is a point $p_\#\in M^\#\cap B$ with $\frac{\lambda_1}{H}(p_+)<\frac{\lambda_1}{H}(p_\#)$
(4) if $B(p,10\Gamma s)\subset U$, then $s^{-1}(M^+-p)$ is $\delta'(\delta)$-close in $nB(0,10\Gamma)$ to a pair of disjoint standard saps that are at distance Γ from the origin.

With these definitions in mind before moving on we state an important set of properities that standard caps satisfy. As long as the cap we construct satisfies the defintion of standard cap above and that after the gluing the postgluing domain adheres to definition 2.5 above the proposition will be true. We include this though for completeness sake since it is used, as one may check, many times in the proof of the canonical neighborhood theorem.

Proposition 2.1. *Let C be a standard cap with $\alpha,\beta>0$. There is a unique mean curvature flow $\{C_t\}_{t\in[0,1/2(N-2))}$ starting at C. It has the following properties.*

(1) It is α-noncollapsed, convex, and β-uniformly 2-convex.
(2) There are continuous increasing functions $\underline{H},\overline{H}:[0,\frac{1}{2(N-2)}\to\mathbb{R}$, with $H(t)\to\infty$ as $t\to\frac{1}{2(N-2)}$ such that $\underline{H}(t)\le H(p,t)\le\overline{H}(t)$ for all $p\in C_t$ and $t\in[0,1/2(N-2))$.
(3) For every $\epsilon>0$ and $\tau<\frac{1}{2(N-2)}$ there exists an $R=R(\epsilon,\tau)<\infty$ such that outside $B(0,R)$ the flow C_t, $t\in[0,\tau]$, is ϵ close the the flow of the round cylinder.
(4) For every $\epsilon>0$, there exists a $\tau=\tau(\epsilon)<\frac{1}{2(N-2)}$ such that every point $(p,t)\in\partial K_t$ with $t\ge\tau$ is ϵ-close to a β-uniformly 2-convex ancient α-noncollapsed flow.

We sketch the proof of canonical neighborhood theorem below (of course, full details are in [**18**]). Before that we finally state the main existence result of Haslhofer and Kleiner; see theorem 1.21 in [**18**]

THEOREM 2.2. *(Existence of mean curvature flow with surgery). There are constants $\overline{\delta} = \overline{\delta}(\alpha) > 0$ and $\Theta(\delta) = \Theta(\alpha, \delta) < \infty$ ($\delta \leq \overline{\delta}$) with the following significance. If $\delta \leq \overline{\delta}$ and $\mathbb{H} = (H_{trig}, H_{neck}, H_{th})$ are positive numbers with $H_{trig}/H_{neck}, H_{neck}/H_{th}, H_{neck} \geq \Theta(\delta)$, then there exists an $(\alpha, \delta, \mathbb{H})$-flow $\{M_t\}_{t \in [0,\infty)}$ for every α-controlled surface M.*

The reason we choose to employ the scheme set out by Haslhofer and Kleiner because the surgery problem is then reduced to showing ancient α-nonocollapsed flows of suitably low entropy are β 2-convex for some $\beta > 0$. Without going into more details than necessary, we recall on last theorem we will need in the sequel, see theorem 1.22 in [**18**]:

THEOREM 2.3. *(Canonical neighborhood theorem) For all $\epsilon > 0$, there exists $\overline{\delta} = \overline{\delta}(\alpha) > 0$, $H_{can}(\epsilon) = H_{can}(\alpha, \epsilon) < \infty$ and $\Theta_\epsilon(\delta) = \Theta_\epsilon(\alpha, \delta) < \infty$ ($\delta \leq \overline{\delta}$) with the following signifigance. If $\delta < \overline{\delta}$ and M is an $(\alpha, \delta, \mathbb{H})$-flow with $H_{trig}/H_{neck}, H_{neck}/H_{th} \geq \Theta_\epsilon(\delta)$, then any $(p,t) \in \delta M$ with $H(p,t) \geq H_{can}(\epsilon)$ is ϵ-close to either (a) a β-uniformly 2-convex ancient α-noncollapsed flow, or (b) the evolution of a standard cap preceeded by the evolution of a round cylinder.*

The above theorem is roughly proven by letting the surgery ratios above degenerate to infinity for a sequence of flows and analyizing the possibilities for the limits, which are guaranteed by a convergence theorem of Haslhofer and Kleiner. In the case of no surgeries the limit that is ancient and β two convex and α non collapsed, so that the theorem follows since the convergence is in a suitably strong topology. If there are surgeries, then it follows that the limit contains a line (more specifically, see claim 4.3 and the discussion afterwards in [**18**]), from which (b) follows. this part uses the properties of the cap that are satisfied in proposition 2.1 above.

The last case does not employ two convexity, so to see that the canonical neighborhood theorem is true in our setting it suffices to show that ancient, α-noncollapsed, low entropy flows are in fact β 2-convex for some $\beta > 0$ and that our cap is suitably constructed to satisfy proposition 2.1; these are both attended to in the next section.

Now, to prove the existence of the surgery, Haslhofer and Kleiner proceed by finding regions which seperate high curvature regions, where some points have $H = H_{trig}$, and low curvature regions where $H \leq H_{th}$; see claim 4.6 in [**18**]. These will be strong neck points in the sense above on which they can do surgery; see claim 4.7 in [**18**].

If the ancient flow found in the canonical neighborhood theorem is compact, it will be diffeomorphic to a sphere, see the discussion after claim 4.8 in [**18**]. Furthermore as long as H_{th} is taken large enough (roughly large enough to employ the canonical neighborhood theorem for appropriately small ϵ as we do in section 3.3 below) all points in the intermediate region between H_{th} and H_{neck} can be forced to be neck points; we will also refer to this region as the neck region below. This is essentially also contained in the argument in the proof of corollary 1.25 in [**18**] following claim 4.8 therein.

For readers perhaps more familiar with the approach to surgery of Huisken and Sinestrari in [**23**], this is essentially the content of their neck continuation theorem (more precisely, theorem 8.1 in [**23**]); one starts by finding a neck point, and the statement is essentially that one may continue the neck as long as H is large (in our context, $H > H_{th}$), λ_1/H, and there are no previous surgeries in the way. If the second or third conditions are violated the case then is that the neck is ended by a convex cap.

In [**13**] Colding and Minicozzi discovered a useful new quantity called the entropy to study the mean curvature flow. To elaborate, consider a hypersurface $\Sigma^k \subset \mathbb{R}^\ell$; then given $x_0 \in \mathbb{R}^\ell$ and $r > 0$ define the functional $F_{x_0,r}$ by

$$F_{x_0,r}(\Sigma) = \frac{1}{(4\pi t_0)^{k/2}} \int_\Sigma e^{\frac{-|x-x_0|^2}{4r}} d\mu \tag{2.5}$$

Colding and Minicozzi then define the entropy $\lambda(\Sigma)$ of a submanifold to be the supremum over all $F_{x_0,r}$ functionals:

$$\lambda(\Sigma) = \sup_{x_0,r} F_{x_0,r}(\Sigma) \tag{2.6}$$

Important for below is to note that equivalently $\lambda(\Sigma)$ is the supremum of $F_{0,1}$ when we vary over rescalings (changing r) and translations (choice of x_0). For hypersurfaces with polynomial growth this supremum is attained and, for self shrinkers Σ, $\lambda(\Sigma) = F_{0,1}(\Sigma)$. In fact, self shrinkers are critical points for the entropy so it is natural next to ask what the stable ones are. If Σ_2 is a normal variation of Σ and x_s, t_s are variations with $x_0 = 0, r = 1$,

$$\partial_s \mid_{s=0} \Sigma_s = f\nu, \partial_s \mid_{s=0} x_s = y, \text{ and } \partial_s \mid_{s=0} t_s = h \tag{2.7}$$

The second variation formula one find is:

$$F_{0,1} = (4\pi)^{-n/2} \int_\Sigma (-fLf + 2fhH - h^2H^2f\langle y, \nu\rangle - \frac{\langle y, \nu\rangle^2}{2})e^{\frac{-|x|^2}{4}} d\mu \tag{2.8}$$

where L is given by the following:

$$L = \Delta + |A|^2 - \frac{1}{2}\langle x, \nabla(\cdot)\rangle + \frac{1}{2} \tag{2.9}$$

One can easily check that, where v is a constant vector field on $\mathbb{R}^n$, both $\langle v, \nu\rangle$ and H are eigenfunctions with eigenvalues $-1, -\frac{1}{2}$ respectively for L; $LH = H$ and $L\langle v, \nu\rangle = \frac{1}{2}\langle v, \nu\rangle$. L is self adjoint in the weighted space $L^2(e^{\frac{-|x|^2}{4}})$, so has a discrete set of eigenvalues with corresponding orthogonal sets of eigenfunctions. If a self shrinker isn't mean convex H switches signs on Σ, so by the minmax characterization for eigenvalues on a surface Σ must not be the lowest eigenvalue, and that there is a positive function f that is $L^2(e^{\frac{-|x|^2}{4}})$ orthogonal to both H and $\langle v, \nu\rangle$. It is clear from the second variation formula that f gives rise to an entropy decreasing variation of Σ, so that namely Σ is not stable. Thus all stable self shrinkers are mean convex and must be spheres and cylinders; using this; more precisely:

THEOREM 2.4. *(Theorem 0.12 in [**13**]) Suppose that Σ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth.*

(1) If Σ is not equal to $S^k \times \mathbb{R}^{n-k}$, then there is a graph $\widetilde{\Sigma}$ over Σ of a function with arbitrarily small C^m norm (for any fixed m) so that $\lambda(\widetilde{\Sigma}) < \lambda(\Sigma)$

(2) If Σ is not S^n and does not split off a line, then the function in (1) can be taken to have compact support.

Furthermore the entropy is monotone decreasing under the flow by Huisken monotonicity [**22**] so, if the entropy of a surface is lower than that of a certain self shrinker, that self shrinker won't be the singularity model for any singularities of the surface under the flow later on - this of course is essential and a surgery flow for low entropy surfaces wouldn't be sensible otherwise.

We end this discussion with a lemma which restricts which F-functionals we will need to consider when estimating the entropy. This is contained in the argument of lemma 7.7 of [**13**] which says that the entropy is achieved by an F functional for a smooth closed embedded hypersurface.

LEMMA 2.5. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be smooth and embedded. For a given $r > 0$, the supremum over x_0 of $F_{x_0,r}(\Sigma)$ is achieved within the convex hull of Σ*

To see this, Colding and Minicozzi note that from the first variation $F_{x_0,r}$ must be a critical point for fixed r when the integral $x - x_0$ vanishes, which couldn't occur if x_0 wasn't in the convex hull of Σ.

3. Proof of Theorem 1.1.

The proof of theorem 1.1 amounts to showing the following two things, most of the work in the article being to establish (2):

(1) Ancient mean convex solutions of low entropy are in fact uniformly β 2-convex for some $\beta > 0$, and
(2) The low entropy condition is preserved across surgeries.

Using the first item one can proceed exactly as in [**18**] to establish the canonical neighborhood theorem and so on as discussed in section 2.2. To elaborate, since all the low entropy mean convex hypersurface are uniformly 2-convex, all the statements in section 3 of [**18**] are true in our setting.

In the following (this concerns the second step) without loss of generality we will assume there is only one surgery performed at a time for a time slice T; if there are multiple to be performed at once the argument below works if they are considered successively (within a fixed time slice).

In this section we establish item (1) above. Before proceeding we recall that α-noncollapsing, entropy, and β 2-convexity are all scale invariant conditions/quantities. We apply the next proposition with $\epsilon_0 = \Lambda_{n-2} - \Pi$:

Proposition 3.1. *Pick $\epsilon_0 > 0$. There exists $\beta > 0$, depending only on α, n, and ϵ_0 such that if M_t^n be an α-noncollapsed ancient flow in $\mathbb{R}^{n+1}$ and $\lambda(M_t) < \Lambda_{n-2} - \epsilon_0$ then there exists some β so that M_t is β 2-convex.*

Proof. Suppose not, there exists a sequence of ancient α-noncollapsed flows $\{M_{i,t}\}$ with $(p_i, t_i) \in \{M_{i,t}\}$ such that $\frac{\lambda_1(p_i)+\lambda_2(p_i)}{H} < \frac{1}{i} \to 0$. We can translate and rescale to get a sequence of new flows $\{\widetilde{M_{i,t}}\}$ so that $\lambda_1(0)+\lambda_2(0) < \frac{1}{i}$ and $H(0,0) = 1$ for all i.

By the global convergence theorem (Theorem 1.12 of [**19**]), after passing to a subsequence, the sequence of rescaled flows $\{\widetilde{M_{i,t}}\}$ converge locally smoothly to an α-Andrews flow $\{M_{\infty,t}\}$ with convex time slices. And the limit flow satisfies $\lambda_1(0,0) = \lambda_2(0,0) = 0$. By the strong maximum principle for tensors (see for example the appendix of [**30**]), the limit flow splits of a plane. By Fatou's lemma applied to each of the $F_{x_0,r}$ functionals individually, we see that the limit flow is also low entropy $(\lambda(M_{\infty,t}) < \Lambda_{n-2})$.

Now take the blow-down of this limit flow at $t = -\infty$; by Huisken's monotonicity formula, we get a nontrivial (because $H(0,0) = 1$)) self-shrinker which splits off a plane and which is mean convex. By the classification of mean-convex self-shrinkers [**13**] the entropy is then at least Λ_{n-2}; by the above argument though the blowdown should as well be low entropy, so we get a contradiction. □

In Haslhofer and Kleiner, when they perform surgery at necks they only have to ensure the resulting surface stays uniformly 2-convex and do not worry about the affect on entropy at all. But one can see by a straightforward computation (see the appendix) that in a toy model of the surgery similar to their construction, where a round cylinder $S^{n-1} \times \mathbb{R}$ is replaced with a half cylinder $S^{n-1} \times (-\infty, 0]$ and a cap, the entropy of the postsurgery model must be strictly greater than that of the round cylinder.

Estimating exactly how much the entropy increases directly seems to be nontrivial, even in the toy case. To eschew this problem, in this section we construct a cap model C of low entropy, in a precise sense, by making use of the monotonicity of the entropy under the mean curvature flow and constructing a low entropy hypersurface, denoted below by Σ, that develops a neck pinch and is approximately cylindrical just away from the neckpinch in a precise way; as pointed out above, this example also shows that singularities are indeed a real possiblity for hypersurface M^n with $\lambda(M) < \Lambda_{n-2}$. Namely, the main result in this section is the following:

PROPOSITION 3.2. *For any $\epsilon > 0$, there exists R_1 such that if $R > R_1$, there exists a rotational symmetric n-dimensional cap model C, such that:*

(1) $C \subset B(0, 4R) \subset \mathbb{R}^{n+1}$,
(2) $\lambda(C) \leq \Lambda_{n-1} + \epsilon$,
(3) $C \cap \mathbb{R}^{n+1} \setminus B(0, 2R)$ agrees with a round half cylinder of radius 1 centered at the origin, and
(4) C is mean-convex and α non-collapsed for some $\overline{\alpha} > 0$.

Of course if M is α-noncollapsed for $\alpha > \overline{\alpha}$ it is also $\overline{\alpha}$-noncollapsed so the exact value of $\overline{\alpha}$ above is immaterial (although it will be close of that of a cylinder). Before proving the proposition we will need some lemmas. The first lemma says for some cases at least only $F_{x_0,r}$ of certain scales are relevent in the estimation of entropy.

LEMMA 3.3. *For any surface $\Sigma \subset \mathbb{R}^{n+1}$ contained in $B^1(0,1) \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$ with $\lambda(\Sigma) \leq \Lambda_{n-2}$, there exists $r_1 > 0$ (depending on the growth rate and constant) such that*

$$
\begin{aligned}
\lambda(\Sigma) = \sup_{x_0 \in \mathbb{R}^{n+1}, r>0} F_{x_0,r}(\Sigma) &= \sup_{x_0 \in B^1(0,1)\times\mathbb{R}^n, r<r_1} F_{x_0,r}(\Sigma) \\
&= \sup_{x_0 \in B^1(0,1)\times\mathbb{R}^n, r<r_1} \int_\Sigma \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} d\mu_x
\end{aligned} \tag{3.1}
$$

namely the entropy will only be approximated by F-functionals on a bounded range of scales.

Proof. That the sup only needs to be taken with $x_0 \in B^1(0,1) \times \mathbb{R}^n$ follows from lemma 2.5 above and that the surface is supported in this solid round cylinder.

By the entropy bound, we can get a uniform Euclidean volume bound on the surface Σ. $\mathrm{Vol}(\Sigma \cap B^{n+1}(p,r)) \leq Cr^n$ for any p, r and C is a universal constant. The lemma follows if we can show

$$
\lim_{r\to\infty} \sup_{\mathbb{R}^{n+1}} \int_\Sigma \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} d\mu_x = 0
$$

By breaking the integral up into integration on concentric annuli, it can be estimated by:

$$
\begin{aligned}
&\int_\Sigma \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} d\mu_x \\
=&\int_{\frac{\Sigma}{r}} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4}} d\mu_x \\
=&\sum_{k=1}^{\infty} \int_{\frac{\Sigma}{r}\cap[B^{n+1}(0,k)\setminus B^{n+1}(0,k-1)]} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4}} d\mu_x \\
\leq&\sum_{k=0}^{\infty} \left(\frac{C}{r}\right)^n \cdot e^{-(k-1)^2/4} \\
=&\frac{\widetilde{C}}{r^n} \\
\to& \ 0
\end{aligned} \tag{3.2}
$$

as $r \to \infty$. The volume bound $\mathrm{Vol}(\frac{\Sigma}{r} \cap [B^{n+1}(0,k) \setminus B^{n+1}(0,k-1)]) \leq \frac{C}{r}$ is because after rescaling $\frac{\Sigma}{r}$ is contained in a round solid cylinder of radius $\frac{1}{r}$. ☐

In the next lemma we observe that the integral in the defintion of F-functionals is concentrated within a bounded set for a given bounded range of scales; essentially if the scales aren't let to be large the ${}_{x_0,r}F$ functionals must be concentrated near x_0:

Lemma 3.4. *For any $\epsilon > 0, r_1 > 0$, there exists $R_0 >> 1$ such that if $R > R_0$, then for any $M^n \subset \mathbb{R}^{n+1}$ with entropy $\lambda(M) \leq \Lambda_{n-2}$*

$$\sup_{x_0 \in \mathbb{R}^{n+1}, r<r_1} F_{x_0,r}(M \cap B(x_0,R)^c) = \sup_{x_0 \in \mathbb{R}^{n+1}, r<r_1} \int_{M \setminus B^{n+1}(x_0,R)} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} d\mu \leq \epsilon \tag{3.3}$$

Proof. As above the entropy bound implies Euclidean volume bound and

$$\begin{aligned}
&\sup_{r<r_1} \int_{\{M-x_0\} \setminus B^{n+1}(0,R)} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4r}} d\mu \\
=&\sum_{k=1}^{\infty} \int_{\{M-x_0\} \cap (B^{n+1}(0,(k+1)R) \setminus B^{n+1}(0,k \cdot R)} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4r_1}} d\mu \\
\leq&\sum_{k=1}^{\infty} C[((k+1)R)^n - (kR)^n] e^{-|kR|^2/(4r_1)} \\
\leq&\sum_{k=1}^{\infty} \widetilde{C} k^{n-1} R^n e^{-|k(R-1)|^2/4r_1} \cdot e^{-k^2(2R-1)/(4r_1)} \\
=&e^{-k^2(2R-1)/(4r_1)} \sum_{k=1}^{\infty} \widetilde{C} k^{n-1} R^n e^{-|k(R-1)|^2/4r_1} \\
\leq&\bar{C} e^{-k^2(2R-1)/(4r_1)} \\
\to& \, 0
\end{aligned} \tag{3.4}$$

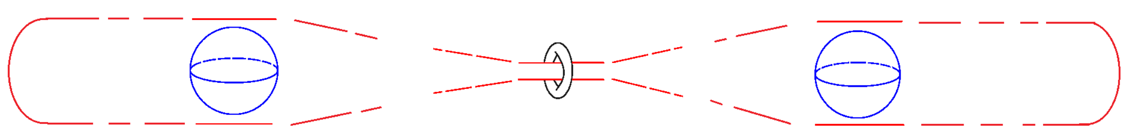
as $R \to \infty$.
So the lemma follows by choosing R_0 large enough. □

We also need to consider the following fact, which is a consequence of the continuity of each of the F functionals having bounded gradient within a C^3 bounded family of submanifolds; see [**28**] section 5.

Lemma 3.5. *For any $\epsilon > 0$, and $R > R_0$ chosen above, there exists $\delta(\epsilon, R) > 0$ such that if $\overline{C}$ is the graph of u over a round cylinder C_{r_0} of radius r_0 centered at origin and $||u||_{C_3(B(0,R))} \leq \delta$, then*

$$|\lambda(\overline{C} \cap B(0,R)) - \lambda(C_{r_0} \cap B(0,R))| < \epsilon \tag{3.5}$$

With this in mind we describe how to construct Σ. First, consider a part of a round cylinder "threaded" through a self shrinking torus. On either side, gradually start to let the cylinder flare out. Provided it doesn't change radius too quickly, by the lemma above its entropy will be very close to that of a cylinder. On the other hand, we can ensure very far away from the neck pinch that it will not be singular, because we will be able to fit very large spheres within it. Once we have flared the cylinders out enough to fit large spheres that are still smooth before the self shrinking torus, let the cylinder radius level off (suitably gradually). In summary consider the following schematic diagram of the construction:



More precisely, let $\rho(x) : \mathbb{R} \to [0,1]$ be a heavyside function, namely $\rho \in C_0^\infty$, $\rho(x) = 0$ when $x \leq 0$ and $\rho(x) = 1$ when $x \geq 1$. Let m be chosen so that $||\frac{1}{m}\rho||_{C_3} \leq \delta$ in the condition of Lemma 3.5. Let W be the width of Angenent's shrinking torus at the time slice with inner radius 1. Define $\eta(x) = \frac{1}{m}\rho(x - 2W)$. Denote the time $t_1 > 0$ to be the time when the self shrinking torus of width W shrinks to a point.

Define $\eta_k(x) = \frac{1+\sum_{j=1}^{km}(\eta(\frac{x}{2R_0}-j)+\eta(-\frac{x}{2R_0}-j))}{k+1}$ and choose k large enough so that $k^2m^2 > 2nt_1$. Then the surface of revolution $\Sigma_r \subset \mathbb{R}^{n+1}$ defined by rotating the graph of η_k around the x_1 axis must develop a neck-pinch singularity by the comparison principle for the mean curvature flow (as described in the background material, this is a consequece of the maximum principle).

This surface Σ_r is contained in a solid round cylinder of radius 1 because $\eta_k(x) \leq 1$ and it agrees with the round cylinder of radius 1 outside the ball of radius $2R_0(2W + mk+1)$. So by Lemma 3.3, the entropy of Σ_r are only approximated by F functionals with bounded scales. Moreover, by our choice of R_0, m, using lemma 3.4 and lemma 3.5, we have $\lambda(\Sigma_r) \leq \Lambda_{n-2} + \epsilon$.

Now for any $\widetilde{R} >> R_1, R_0$, choose $R' >> \max(\widetilde{R}, R_1, R_0)$ and cap of Σ_r by spherical caps outside the ball of radius $\widetilde{R}$ to get $\widetilde{\Sigma}_r$, which is of the shape of a long pill. By choosing our m large enough then $\widetilde{\Sigma}_r$ will be as small a perturbation of round cylinders in any neighborhood. So it will be mean convex and α non-collapsed.

By the lemma below, which one can interpret as a pseudolocality result of sorts, if R' is large enough, the evolution of $\widetilde{\Sigma}_r$ will be as close as we want to the evolution of Σ_r in $B^{n+1}(0, 4\widetilde{R})$ and thus will develop a neck-pinch singularity as well.

LEMMA 3.6. *Suppose M_1, M_2 are two submanifolds of $\mathbb{R}^N$ whose mean curvature flow exists on the interval $[0, T]$ and $|A|^2$ is uniformly bounded initially by say C. Picking ϵ and R, there exists $R'(\epsilon, C, R) > R$ so that if $M_1 \cap B(0, R') = M_2 \cap B(0, R')$ then $(M_1)_t \cap B(0, R)$ is ϵ close in C^2 local graphical norm to $(M_2)_t \cap B(0, R)$ for all $t \in [0, T]$.*

PROOF. Without loss of generality $R = 1$. Suppose the statement isn't true; then there is a sequence of hypersurfaces $\{M_{1i}, M_{2i}\}$, $R_i \to \infty$ and times $T_i \in [0, T]$ so that $M_{1i} = M_{2i}$ on $B(0, R_i)$ but $||M_{T_i} - M_{iT_i}||_{C^2} > \epsilon$ in $B_0(1)$. By passing to subsequences by Arzela-Ascoli via the curvature bounds we get limits $M_{1\infty}$, $M_{2\infty}$ so that $M_{1\infty} = M_{2\infty}$ (the flows of these manifolds will exist on $[0, T]$) but the flows don't agree at some time $T_1 \in [0, T]$; this is a contradiction. $\square$

Now we can give the construction of low-entropy cap C:

PROOF. (of Proposition 3.2) A result then by Angenant, Aschuler, and Giga [**3**] will ensure that the singular times of the level set flow starting from $\widetilde{\Sigma}_r$ are discrete, so immediately after the neckpinch time $t_{neckpinch} < t_1$ (and because the entire surface doesn't go singular before t_1) our surface will be consist of two smooth components. In addition, the flow is nonfattening, and one can see that the smooth points will move by their mean curvature vector at all times. Our choice of cap model then is one connected component of a time slice immediately after the first neck-pinch singularity that lies inside the ball $B^{n+1}(0, 4\widetilde{R})$. By a result due also to Haslhofer and Kleiner (see theorem 1.5 in [**19**]) the post-singular surface will be α noncollapsed as well.

Also by lemma 3.6, choosing $\widetilde{R}$ large, the evolution of $\widetilde{\Sigma}_r$ is as close as we want in C^2 norm to the evolution of a round cylinder in $B^{n+1}(0, 4\widetilde{R}) \setminus B^{n+1}(0, \frac{3}{2}\widetilde{R})$. By deforming it in $B^{n+1}(0, 4\widetilde{R}) \setminus B^{n+1}(0, \frac{3}{2}\widetilde{R})$, we can make it agree with a round cylinder in $B^{n+1}(0, 4\widetilde{R}) \setminus B^{n+1}(0, 2\widetilde{R})$ and keep the entropy bound $\Lambda_{n-1} + \epsilon$ by lemma 3.5 above. Then we extend this hypersurface by a half cylinder to get out cap C. $\square$

To describe how we glue it in, note an upshot of the canonical neighborhood theoem above is that if the mean curvature at the locations we intend to do surgery is large enough, after rescaling to make the mean curvature one the surface will be as close as we want (in C^3 norm, say) in as large a neighborhood as we want to a round cylinder of radius one. Meanwhile, without loss of generality (by applying a suitable rescaling) our surgery cap candidate constructed in the previous subsection agrees with a round cylinder far enough away from the origin.

The locations that we intend to do surgery at will have $H = H_{neck}$, as in [**19**], so choose $H_{neck} > H_{can}(\epsilon)$ with ϵ so that $1/\epsilon >> 2R_1$ and $\epsilon < \delta/2$. Denote the rescaled flow about the surgery spot by $\widetilde{M} = \frac{M-p}{H_{neck}}$, by our choice of ϵ let us perform the cap gluing by smoothly transitioning from $\widetilde{M} \cap (B(0,3R_0)/B(0,2R_0))$ to $C \cap (B(0,3R_0)/B(0,2R_0))$. In particular the surgery only will change the hypersurface in the region $\widetilde{M} \cap (B(0,3R_0)$ for the rescaled flow. Following the notation of Haslhofer and Kleiner we denote the surfaces pre and post-gluing by $\widetilde{M}^+$ and $\widetilde{M}^\#$ respecitvely for the rescaled surface and M^+, $M^\#$ for the original (spatial) scaled surfaces.

With regards to proposition 3.8, β 2-convexity isn't strictly necessary (its only included in [**18**] to preserve apriori curvature conditions) so we ignore that condition. Since the transition is taken where both surfaces are nearly cylindrical, α-noncollapsing is preserved. Items (1), (2) and (4) are clear as well; for the third point we note we may slightly bend the cylinder inwards without affecting the entropy in light of lemma 3.5 to make the postgluing domain satisfy (3).

Now that we have the cap and how to glue it in, we analyze the change in entropy due to surgery and ensure that, if surgery parameters are picked correctly, the post surgery surface will still have low entropy. Morally speaking, since the cap was constructed to have entropy very close to that of the cylinder, the post gluing domain should have low entropy as well. The rub is that the contribution to the F functionals near the surgery cap from the rest of the manifold could concievably be large, so that somehow even after placing caps the entropy is pushed over the low entorpy threshold. We show with a careful choice of surgery parameters that this won't occur.

To analyze the affect desingularization has on the entropy of the whole domain, for organizational conveinence we consider two domains, one about the surgery region centered at q and the other "far" from the surgery, which we denote U_e and U_f resceptively. More precisely, let $U_e = B(q, \frac{5R_0}{H_{neck}})$, R_0 as specified above, and let U_f be its complement. We see the surgery takes place entirely within U_e.

There are two types of F-functionals to consider for us, those which are concentrated near x_0, or roughly when r is small, and the diffuse ones where r is roughly large. We start by showing we can find $c_0 > 0$ so that all $F_{x_0,r}$ functionals with $r < c_0$ have $F_{x_0,r}(M_T) < \Pi$. We then show by taking H_{neck} large enough that we can arrange $F_{x_0,r}(M_T) < \Pi$ for $r > c_0$ as well.

Note as a byproduct of lemma 3.4, which one can see by rescaling, for every $s, \epsilon_0 > 0$, there exists $c = c(\epsilon_0)$ so that if $r < c$, $F_{x_0,r}(M) - F_{x_0,r}(M_T \cap B(x_0, s)) < \epsilon_0$. Also by α-noncollapsedness we know if H has an upper bound B, $|A|^2$ does as well. Hence in sufficiently small balls it can made as close one wants to a plane, giving us as a consequence that for every $\epsilon_0, B > 0$ there is s so that $F_{x_0,r}(M_T \cap B(x_0, s)) < 1+\epsilon_0$.

As a corollary of this observation we see that for every $H_{can}(\epsilon_0)$, there is a $1 >> s_1 > 0$ so that if $F_{x_0,r}(M_T \cap B(x_0, s_1)) > 1 + \epsilon_0$, then $H(y) > H_{can}(\epsilon_0)$ for some $y \in B(x_0, s_1) \cap M_T$. Of course, the rough plan is to estimate the value of $F_{x_0,r}$ in terms of the canonical neighborhood models of these points in some manner, at least the ones that will be represented by parts of the surface that persist after surgery.

Let $\epsilon_1 > \epsilon_0$ to be picked later (this will also restrict ϵ_0 of course) and suppose $H_{can}(\epsilon_1) < H_{th} < \frac{1}{2}H_{can}(\epsilon_0)$. We see that for a given ϵ_1, there is an $s_2 \leq s_1$ such that $H(x) > H_{can}(\epsilon_0) > 2H_{th}$ for all $x \in B(x_0, s_2)$, s_2 small enoughm by the gradient estimates (see theorem 1.10 in [**18**]) applied at points where $H(y) = H_{can}(\epsilon_0)$ (the other points will be "deeper" in the high curvature region and so the assertion also holds).

In particular, $x \in B(x_0, s_2) \cap M_T$ has $H(x) > H_{can}(\epsilon_1)$ for s_2 sufficiently small; intuitively speaking since we found one very high curvature point the point must be "deep" in the neck so only surrounded by high curvature points. From here on unless otherwise stated we abbreviate $H_{can} = H_{can}(\epsilon_1)$.

With all this being said, as a first pass consider $F_{x_0,r}$ functionals such that the following hold:

(1) $0 < r < c(s_2)$
(2) $F_{x_0,r}(B(x_0, s_2) \cap M_T) > 1 + \epsilon_0$
(3) $B(x_0, r) \cap M_T$ only contains "neck points" in the sense discussed in section 2.2 above.
(4) Furthermore, no surgeries are done in $B(x_0, r) \cap M_T$.

Of course, if the second point is satisfied then taking ϵ_1 small enough $1 + \epsilon_0 < \Pi$, so these $F_{x_0,r}$ functionals will not potentially ruin the low entropy condition. We will discuss the complement of the other three cases below in the order of (4), (3), and then (1).

We see by unpacking definitions the canonical neighborhood theorem and hypothesis (3) say above for any point $x \in B(x_0, s_2) \cap M_T$ we have $H(x) \cdot (M - x)$ is ϵ_1 close in $C^{\frac{1}{\epsilon_1}}$ topology to the round cylinder of radius 1 in the ball $B(\frac{1}{\epsilon_1})$.

So, taking $s = \frac{1}{\epsilon_1 \cdot H_{can}(\epsilon_1)} << s_2$, depending on H_{can}, and relabeling $s = s_2$ we have $M \cap B(x, s_2)$ is ϵ_1 close to a round cylinder after rescaling (with our previous choice of s_2, instead this was true in every neighborhood of every point) when $H = H_{can}$ somewhere within the ball. Of course, we continue for the time being to assume our stipulations (1) - (3) listed above.

By properties of Gaussian distribution, there exists a universal constant $N >> 1$ independent of scale or basepoint so that if $\sqrt{r} < \frac{4s_2}{N}$, then $\int_{M\cap B^c(x_0,r)} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{4r}} < \epsilon_3$. By making ϵ_3 small, we can force $\frac{s_2 \cdot H_{can}}{N} = \frac{1}{N\cdot\epsilon_3} >> 100$.

Now we estimate $F_{x_0,r}(M)$ for some point x on the flow with $H(x) > H_{can}$, and we want to show that if $\sqrt{r} \leq c_0 = \frac{s_2}{N}$, then the F functionals with centers at x_0 must be small in the sense they can be bounded above by $\Lambda_{n-1} + \epsilon_1 + \epsilon_2$. We note the following consequnece of the continunity of the flow and that on the initial time slice, $H < H_{can}(\epsilon_1)$ (for ϵ_1 small enough):

Lemma 3.7. *Those points with mean curvature $H(x,t) > H_{can}$ must be covered by union of balls* $\bigcup_{\tilde{x}, H(\tilde{x},\tilde{t}) = H_{can} \text{for some } \tilde{t} \leq t} [B(\tilde{x}, s_2)]$.

We have for any point x_0 with $H(x_0, t) \geq H_{can}$, and $r \leq c_0^2$

$$\begin{aligned} &F_{x_0,r}(M) \\ &= \int_M \frac{1}{(2\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} \\ &= \int_{H_{can}(M-x_0)} \frac{1}{(2\pi r H_{can}^2)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4rH_{can}^2}} \end{aligned} \tag{3.6}$$

By lemma 3.7 above, there are some points in the ball $B(x_0, s_2)$ of some previous time slice of the unscaled flow that with mean curvature exactly equal to H_{can}. After rescaling, there must some point $\widetilde{x}$ with mean curvature exactly 1 in the ball $B(0, s_2 H_{can}^2)$ of some previous time $t_1 < \widetilde{T}$ of the rescaled flow.

By the standard comparison argument with the self shrinking torus (threading the cylinder through the shrinking torus), the rescaled flow will develop a singularity by at least $t_1 + 100$, or in other words so that $\delta = \widetilde{T} - t_1 < 100$. Thus we have the following:

$$\begin{aligned}
&\int_{M_T} \frac{1}{(2\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} \\
&= \int_{\{H_{can}(M-x_0)\}_{\widetilde{T}}} \frac{1}{(2\pi r H_{can}^2)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4rH_{can}^2}} \\
&\leq \int_{\{H_{can}(M-x_0)\}_{\widetilde{T}-\delta}} \frac{1}{(2\pi(rH_{can}^2+\delta))^{\frac{n}{2}}} e^{\frac{-|x|^2}{4(rH_{can}^2+\delta)}}
\end{aligned} \tag{3.7}$$

Where the last line is by Huisken's monotonicity formula. By lemma 3.7 above there exists some point $\tilde{x}$ with mean curvature exactly equal to 1 after rescaling.

Moreover by our choice of parameter above

$$rH_{can}^2 \leq \frac{s_2^2 H_{can}^2}{N^2}$$

and

$$\delta < 100 < \frac{s_2^2 H_{can}^2}{N^2}$$

And thus

$$\begin{aligned}
rH_{can}^2 + \delta &\leq 2\frac{s_2^2 H_{can}^2}{N^2} \\
\sqrt{rH_{can}^2 + \delta} &\leq 2\frac{s_2 H_{can}}{N}
\end{aligned} \tag{3.8}$$

So

$$\begin{aligned}
&\int_{\{H_{can}(M-x_0)\}_{\widetilde{T}-\delta}} \frac{1}{(2\pi(rH_{can}^2+\delta))^{\frac{n}{2}}} e^{\frac{-|x|^2}{4(rH_{can}^2+\delta)}} \\
&\leq \int_{B(0,sH_{can})\cap\{H_{can}(M-x_0)\}_{\widetilde{T}-\delta}} \frac{1}{(2\pi(rH_{can}^2+\delta))^{\frac{n}{2}}} e^{\frac{-|x|^2}{4(rH_{can}^2+\delta)}} + \epsilon_2
\end{aligned} \tag{3.9}$$

By lemma 3.7 there're some point $\tilde{x} \in B(0, s_2 H_{can})$ in this time slice with mean curvature exactly 1 (or, unscaled, where $H = H_{can}$). By our choice of s_2 then:

$$\begin{aligned}
&\int_{B(0,s_2H_{can})\cap\{H_{can}(M-x_0)\}_{\widetilde{T}-\delta}} \frac{1}{(2\pi(rH_{can}^2+\delta))^{\frac{n}{2}}} e^{\frac{-|x|^2}{4(rH_{can}^2+\delta)}} + \epsilon_2 \\
&\leq \int_{B(\tilde{x},s_2H_{can})\cap\{H_{can}(M-x_0)\}_{\widetilde{T}-\delta}} \frac{1}{(2\pi(rH_{can}^2+\delta))^{\frac{n}{2}}} e^{\frac{-|x|^2}{4(rH_{can}^2+\delta)}} + \epsilon_2
\end{aligned} \tag{3.10}$$

$4s_3 H_{can} = \frac{4}{\epsilon_2}$ by the choice of parameters above, so in this ball is ϵ_1 close to a round cylinder and hence has entropy is at most $\Lambda_{n-1} + \epsilon_1$, thus:

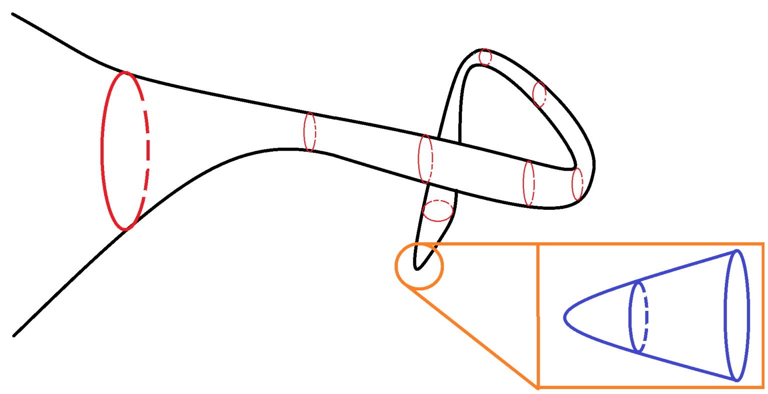
$$\int_{B(\tilde{x}, 4s_3 H_{can}) \cap \{H_{can}(M - x_0)\}_{\widetilde{T}-\delta}} \frac{1}{(2\pi(rH_{can}^2 + \delta))^{\frac{n}{2}}} e^{\frac{-|x|^2}{4(rH_{can}^2 + \delta)}} + \epsilon_2 \\ \leq \Lambda_{n-1} + \epsilon_1 + \epsilon_2 \tag{3.11}$$

Of course if ϵ_1 and ϵ_2 are sufficiently small, $\Lambda_{n-1} + \epsilon_1 + \epsilon_2 < \Pi$. Now suppose a surgery is done in $B(x_0, s_2)$ and let $F_{x_0,r}$ be an F functional still satisfying properties (2), (3) and $r < c_0$ above. Note we still have $F_{x_0,r}(M_T) < \Lambda_{n-1} + \epsilon_1 + \epsilon_2$ by the work above before a surgery is done. [1]

With that in mind, the surgery occurs in a small ball $B(q, r_0)$ about the center of the surgery region, where $r_0 = \frac{5R_0}{H_{neck}}$, and by the design of the surgery caps $|F_{x_0,r}(M_T^+ \cap B(q, r_0)) - F_{q,r}(M_T^\# \cap B(q, r_0))| < \epsilon$. Thus after surgery is done, $F_{x_0,r}(M_T^\# \cap B(q, r_0)) < \Lambda_{n-1} + \epsilon + \epsilon_1 + \epsilon_2$ and if ϵ is taken small enough this will be less than Π.

We claim that in fact without loss of generality only the base points with $B(x_0, s_2)$ containing neck regions possibly with surgery caps are the only ones one needs to consider. Suppose that the ancient flow one finds does lay in the region connecting H_{th} to a point where $H = H_{trig}$. Well it must not lay in the low curvature region, since $H_{can}(\epsilon_0) > H_{th}$, so we see it must be discarded after all the surgeries at the surgery time (we are considering them one at a time) are complete.

[1] The $F_{x_0,r}$ functional we see will increase most under the surgery if it is situated right at the center q of the surgery region, at least prior to the deletion of the high curvature region (by the symmetry of the Gaussian distribution); of course the subsequent deletion of the high curvature regions will only decrease each of the F functionals hence the entropy.



For example, in the diagram above (although in practice the circle might be quite a bit bigger relative to the scale of surgery), the circled tip of a high curature region is essentially modeled on a self translator and is not a neck point, but these are in the region of $M^{\#}$ that will be thrown away under surgery by the time is allowed to continue again. Note that the choice of c only relies α, Π, and γ (initial curvature bound on the surface when start MCF with surgery), since we use the canonical neighborhood theorem, and that $H_{can}(\epsilon_1) < H_{th} < \frac{1}{2}H_{can}(\epsilon_0)$. Namely, we may take H_{neck} freely in the next part of the argument.

The next case is when $c_0 \leq r$. In this case we will see the F functionals are in fact nonincreasing if H_{neck} is taken large enough. To do this we will show if the surgery region U_e is sufficiently small the Gaussian weights of F functionals with a lower bound on r are nearly constant within it in a sense made precise below. Then to conclude we use the following observation corresponding to the cap having less volume than the cylinder:

Lemma 3.8. *For $R > R_0$ chosen above, there is an $0 < \eta(R) < 1$ depending only on the surgery cap and R such that*

$$Vol(\widetilde{M}_T^{+} \cap B(0, 3R)) < \eta Vol(\widetilde{M}_T^{\#} \cap B(0, 3R))$$

where $\widetilde{M}_T = \frac{M-q}{H_{neck}}$

Where above $\widetilde{U}_e = H_{neck}(U_e - q)$, $\widetilde{U}_f = H_{neck}(U_f - q)$, and $\widetilde{M}_T = H_{neck}(M_T - q)$ denote the rescaled versions of U_e, U_f, and M_T where the surgery center q is translated to the origin.

To begin, note that the graduent of the Gaussian weight $e^{\frac{-|x-x_0|^2}{4r}}$ of a $F_{x_0,r}$ functional is given by:

$$\nabla e^{\frac{-|x-x_0|^2}{4r}} = \frac{-2(x-x_0)}{4r} e^{\frac{-|x-x_0|^2}{4r}} \tag{3.12}$$

Since by lemma 2.5 the entropy for a compact hypersurface will be attained by an F functional centered in its convex hull, without loss of generality x_0 is in the convex hull of $M^{\#}$. Since for such x_0 we have $|x - x_0| \leq D < \infty$[2], we see for a lower bound c on r we have $\nabla e^{\frac{-|x-x_0|^2}{4r}}| \leq \frac{D}{2c} < \infty$ for any choice of $x \in M_T$. Denote this upper bound by ρ.

We also note similarly for $r > c$ that the Gaussian weight of a $F_{x_0,r}$ functional (with x_0 in the convex hull of M) is bounded below by $e^{\frac{-D^2}{4c}} > 0$; denote this lower bound by σ. Also denote by $m_{x_0,r}$ and $M_{x_0,r}$ the minimum and maximum respectively of the Gaussian weight of $F_{x_0,r}$ in U_e. Then the following is true:

$$1 \geq \frac{m_{x_0,r}}{M_{x_0,r}} \geq \frac{m_{x_0,r}}{m_{x_0,r} + r_3\rho} \geq \frac{\sigma}{\sigma + r_e\rho} = 1 - \frac{r_e\rho}{\sigma + r_e\rho} \tag{3.13}$$

Since $\sigma > 0$ and $\rho < \infty$ we can make this quotient as close to one as we like by making r_e sufficiently small; in other words we can make the ratio of the minimum to the maximum of the weight in these F functionals as close to 1 as we want in U_e by increasing H_{neck}. Switching to the translated and rescaled picture (the ratio persists under rescaling), we have for $x_0 \in \widetilde{U}_f$ and for $r > c_1$ the following:

$$\begin{aligned}
&F_{x_0,r}(\widetilde{M}^+) \\
&= \int_{\widetilde{M}^+} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} \\
&\leq \int_{\widetilde{M}^+ \setminus \widetilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} + \int_{\widetilde{M}^+ \cap \widetilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} \\
&\text{(because surgery only happens in } \widetilde{U}_e) \\
&= \int_{\widetilde{M}^{\#} \setminus \widetilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} + \int_{\widetilde{M}^+ \cap \widetilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} \\
&\leq \int_{\widetilde{M}^{\#} \setminus \widetilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} + \int_{\widetilde{M}^+ \cap \widetilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} M_{x_0,r}
\end{aligned} \tag{3.14}$$

[2]of course, the diameter is decreasing under the flow so is uniformly bounded by the diameter of the initial time slice

$$
\begin{aligned}
&\text{(by Lemma 3.8)}\\
&\leq \int_{\widetilde{M}^{\#}\setminus\tilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} + \int_{\widetilde{M}^{\#}\cap\tilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} \eta M_{x_0,r}\\
&\text{(by choice of } r_e \text{ and that } r > 1/c_1)\\
&\leq \int_{\widetilde{M}^{\#}\setminus\tilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}} + \int_{\widetilde{M}^{\#}\cap\tilde{U}_e} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{\frac{-|x-x_0|^2}{4r}}\\
&= F_{x_0,r}(\widetilde{M^{\#}})
\end{aligned}
\tag{3.15}
$$

So that these F functionals don't increase under the surgery as claimed. In all

cases then we see the F functionals $F_{x_0,r}$ either didn't increase after the surgery or they are bounded after the surgery by $\lambda(S^{n-1}\times\mathbb{R})+\epsilon+\epsilon_1+\epsilon_2=\Lambda_{n-1}+\epsilon+\epsilon_1+\epsilon_2$ from above, where $\epsilon,\epsilon_1,\epsilon_2>0$, with a prudent choice of surgery parameters. Possibly taking $\epsilon,\epsilon_1,\epsilon_2$ even smaller gives $\Lambda_{n-1}+\epsilon+\epsilon_1+\epsilon_2<\Pi<\Lambda_{n-2}$ (we stipulated $\Lambda_{n-1}<\Pi$), so that the postsurgery surface is low entropy and we are done.

4. Concluding Remarks.

In [**10**] and [**11**], Buzano, Haslhofer, and Hershkovits prove connectedness results for the moduli space of 2-convex spheres in $\mathbb{R}^N$ respectively; by using the mean cuvature flow with surgery for surfaces of low entropy one sees that these results can be extended to show that the space of mean convex low entropy and spheres and tori are connected via mean convex paths. It would be interesting, though, to understand if honest connectedness results are possible; in other words if the low entropy assumption can be preserved long an isotopy to the round sphere.

Relatedly, we remark it is easy to see that low entropy, mean convex hypersurfaces and 2-convex hypersurfaces are different subsets of the space of all embedded hypersurfaces. For example, by replacing a small piece varying a small piece of a low entropy, mean convex hypersurface one can easily find a nearby hypersurface that is mean convex but not two convex, and the nearby hypersurface can be constructed to have low entropy as well by the continuity of the entropy under C^3 bounded perturbations as used above. So there are low entropy, mean convex hypersurfaces that aren't 2-convex; on the other hand, there are also 2-convex hypersurfaces that aren't low entropy. One way to construct such a 2-convex hypersurface that is not of low entropy is to construct a very fine lattice and construct a 2-convex hypersurface M around it so that it has very high surface area in a fixed ball (say $B(0,1)$). If the lattice is fine enough one can arrange $F_{0,1}(M)$ to be as large as one wants then.

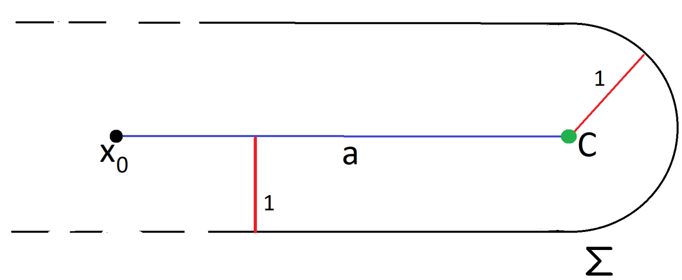
When one relaxes the mean convexity assumption, the singularity models in addition to the shrinking sphere and 2-convex cylinder are asymptotically conical self-shrinkers - see [**6**]. These correspond more or less to removable singularities but still they obstruct progress on defining a mean curvature flow with surgery with no curvature convexity assumption. If one could rule these cones out however, so the only singularities are mean convex, there is hope to provide a mean curvature flow with surgery for low entropy hypersurfaces with no curvature condition. One way to do so concievably is to show that mean convex singularities have parabolic mean convex neighborhoods, a folklore conjecture in the field, but to the author's knowledge no results one way or the other are known in this direction.

Appendix A. the need for a carefully designed cap

In this appendix we illustrate the need for a carefully designed cap by showing that the entropy of a capped off half cylinder will be strictly greater than that of a cylinder; in particular the potential increase of the F functionals near the surgery is a real issue and that care is warranted in understanding how much the entropy will increase.

In the following, we consider the toy surgery case of capping a half cylinder off with a hemisphere. We only consider F funcntionals for which the integral involved is particularly symmetrical; one quickly sees that calculating the F functionals directly for a cap (even in the toy case, where the capped off cylinder can be explicitly parameterized) is onerous; thats why in the above we opt instead to let the mean curvature flow create (in a sense) the cap for us.

That being said, consider the diagram of the cap below, where C is the center point of the hemisphere of radius 1, a is the distance of the point x_0 from C which lies in the center of the core of the cylinder, and the whole cap is denoted Σ:



Now, on one hand we can see by taking the derivative of $F_{x0(a),r}$ in a that the F functionals are a decreasing function of a; writing $F_{x_0,r}$ in terms of a:

$$
\begin{aligned}
F_{x_0,r}(\Sigma) &= \int_{-a}^{-\infty} \frac{1}{4\pi r} e^{\frac{-|x^2+1|}{4r}} 2\pi dx + \int_0^1 \frac{1}{4\pi r} e^{\frac{-|1-x^2+(a+x)^2|}{4r}} 2\pi\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}} dx \\
&= \int_{-a}^{-\infty} \frac{1}{2r} e^{\frac{-|x^2+1|}{4r}} dx + \int_0^1 \frac{1}{2r} e^{\frac{-(a^2+2ax+1)}{4r}} dx
\end{aligned}
\tag{A.1}
$$

The derivative in a then is given by:

$$
\begin{aligned}
\frac{d}{da} F_{x_0(a),r}(\Sigma) &= -\frac{1}{2r} e^{\frac{-(a^2+1)}{4r}} + \frac{d}{da}\int_0^1 e^{\frac{-(a^2+2ax+1)}{4r}} dx \\
&= \frac{-1}{2r} e^{\frac{-(a^2+1)}{4r}} + \frac{d}{da}\left(\frac{e^{\frac{-a^2-1}{4r}} - e^{\frac{-(a+1)^2}{4r}} a}{a}\right) \\
&= \frac{-1}{2r} e^{\frac{-(a^2+1)}{4r}} + \frac{e^{\frac{-(a+1)^2}{4r}} \frac{a^2+2r+a}{2r} - e^{\frac{-a^2-1}{4r}} \frac{a^2+2r}{2r}}{a^2} \\
&= \frac{-e^{\frac{-a^2-1}{4r}}}{a^2} + e^{\frac{-(a+1)^2}{4r}} \frac{a^2+2r+a}{2a^2 r}
\end{aligned}
\tag{A.2}
$$

We see for fixed choice of r that for a big enough, this is nonpositive. On the other hand, we see that as a tends to infinity for a fixed scale r that the F functionals must tend to that of the cylinder; implying the F functionals for a finite must be strictly greater than that of the round cylinder and hence the entropy of the postsurgery domain must be too.

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