

# ON THE LAST QUESTION OF STEFAN BANACH

PASHA ZUSMANOVICH

**ABSTRACT.** We discuss the last question of Banach, posed by him in 1944, shortly before his death, about extension of a ternary map to superposition of a binary map. We try to put things into the context of Polish mathematics of that time, and touch upon subsequent developments in such diverse areas as multivalued logics, binary and ternary semigroups, theory of clones, and Hilbert’s 13th problem. Making almost a full circle in time, we show how variants of Banach’s question may be settled using a 1949 idea of Jacobson about envelopes of Lie triple systems.

## INTRODUCTION

At the end of 1944, shortly before his death in August 1945, Stefan Banach regularly met with Andrzej Alexiewicz, then a fresh PhD from the (underground at that time) Lvov University. During these meetings, they discussed a lot of mathematics, and Alexiewicz kept a diary whose mathematical part, including questions posed by Banach, was published posthumously by Alexiewicz’s son in [Al]. The last entry in this diary, dated December 29, 1944, reads:

“There exists a nontrivial example of ternary multiplication, which is not generated by binary multiplication (Banach). Can any finite set with ternary commutative multiplication be extended so that ternary multiplication is generated by a binary multiplication?”

In what follows, the claim and the question from this passage will be referred as “Banach’s claim” and “Banach’s question”. This seems to be the last question of Banach which appeared in the “literature” (broadly interpreted)<sup>†</sup>. It is interesting that among all other questions posed by Banach – in his papers, in the Scottish Book [Sc], in other diary entries of [Al], in the problem sections of *Fundamenta Mathematicae* and *Colloquium Mathematicum* (the latter being entered posthumously) – this is the only one which does not belong to the field of analysis. (The only possible exception is Question 47 from [Sc] about permutations of infinite matrices, which, incidentally, also deals with (im)possibility of building a certain class of maps from “simpler” ones). On the other hand, that Banach was interested in such sort of questions – belonging, somewhat vaguely, to a crossroad of universal algebra, discrete mathematics, logic<sup>‡</sup>, and combinatorics – is, perhaps, not accidental at all, as such a crossroad, along the functional analysis, the main Banach’s occupation, was another “Polish speciality” at that time (and long thereafter).

It is the purpose of this note to discuss (and answer) possible interpretations of this question. For the historical context we refer the reader to a variety of excellent sources: to [Ba3] for the last years of Banach in Lvov during two Russian and one German occupations; to Duda [D], Turowicz [CP], and to introductory chapters of [Sc] for the general atmosphere of mathematical Lvov; and

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<sup>†</sup> Four Banach’s papers were published posthumously in 1948, but they were prepared from manuscripts apparently dated around 1940 (cf. [Ba2]).

<sup>‡</sup> According to Hugo Steinhaus (cf. [St]), Banach “did not relish any logic research although he understood it perfectly”. On the other hand, in [Mu, p. 40] several instances of Banach’s involvement into contemporary logical activity are given. Anyhow, “logic” is a vast field, and, as we try to argue below, there are certain connections between Banach’s question and some logical investigations cultivated in Poland (especially Warsaw) between the two world wars.

to Murawski [Mu] and Woleński [Wo1], [Wo2] for a unique interaction of mathematics and logic in inter-war Poland.

### 1. A 1955 PAPER BY ŁOŚ, HILBERT’S 13TH PROBLEM, AND FUNCTIONAL COMPLETENESS

There is at least another mention of a variant of Banach’s question in the literature, namely by Jerzy Łoś, a student at the Lvov University during 1937–1939, in [Ł2, §16]. There, also with reference to Alexiewicz, Łoś writes:

“During the last war Banach showed that not every ternary semigroup is reducible and he put forward the problem whether every ternary semigroup may be extended to a reducible one”.

Here by a ternary semigroup one means a set  $X$  with a ternary map

$$(1) \quad f : X \times X \times X \rightarrow X,$$

subject to a ternary variant of associativity:

$$f(f(x, y, z), u, v) = f(x, f(y, z, u), v) = f(x, y, f(z, u, v))$$

for any  $x, y, z, u, v \in X$ , and by a reducible ternary semigroup one means a ternary semigroup  $(X, f)$  with multiplication given by

$$(2) \quad f(x, y, z) = (x * y) * z,$$

where  $(X, *)$  is an ordinary (i.e., binary) semigroup structure on the same underlying set  $X$ . Łoś answers the question in affirmative as a consequence of his general results about extensibility of first-order logical models. Later an alternative and more constructive proof was given by Monk and Sioson [MS, Theorems 1 and 2]. The latter paper contains also examples of ternary (in fact,  $n$ -ary for any positive integer  $n$ ) semigroups not representable in the form (2), thus validating this variant of Banach’s claim.

However, this variant of Banach’s question is more narrow in scope than those presented in [Al]. Can the latter question be interpreted in a different way? In the absence of additional qualifications, the most general reading is the following: “multiplication” on a set  $X$  means an arbitrary ternary map (1), without associativity, or any other, for that matter, constraint. But what does “generation” mean?

Virtually all Banach’s works are devoted to (proper) analysis and/or measure theory (including the famous Banach–Tarski paradox), with, perhaps, only two exceptions: a joint unpublished work with Stanisław Mazur on computable analysis (a constructive approach to analysis)<sup>†</sup>, and a short paper [Ba1], which touches, seemingly, a similar question. There, Banach gives a shorter proof of an earlier result of Waław Sierpiński [Si2]:

**Theorem 1** (Sierpiński). *For any countable set  $\mathcal{S}$  of unary maps  $X \rightarrow X$  on an infinite set  $X$ , there are two unary maps  $f, g : X \rightarrow X$  (not necessary belonging to  $\mathcal{S}$ ), such that any map from  $\mathcal{S}$  is a composition of  $f$  and  $g$ .*

This statement is, essentially, about 2-generation of a countable transformation semigroup, and was apparently rediscovered in the literature several times afterwards. It admits various generalizations to other semigroups and groups, some of them, especially in topological setting (where “generation” is understood up to the closure), were pursued by Sierpiński and two Banach’s students, Stanisław Ulam (PhD from the Lvov Polytechnic, 1933), and Józef Schreier (PhD from the Lvov University, 1934; cf. Guran and Prytula [GP] for an interesting discussion, and the

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<sup>†</sup> That it remained unpublished is probably not surprising, owing to Mazur’s (non)publishing habits, even of joint papers, cf., e.g., [CP, p. 14].

bestseller [U2, p. 82] for Ulam's account how his joint work with Schreier secured him a place in the Harvard Society of Fellows in 1936).

Yet Theorem 1 about unary maps proved by Sierpiński and Banach admits generalizations in another direction – to the multiary maps. (Note the drastic difference between superposition of unary maps, which reduces to a mere composition and hence is associative, and the general case of superposition of multiary maps for which associativity is not even well defined). This may suggest that “generation” in Banach's question can be interpreted as an arbitrary superposition of maps (with possible repetition of variables). For example, among the ternary maps obtained by superposition of a binary map  $f$  with itself, are the maps

$$f(f(x, y), z), \quad f(f(x, y), f(y, z)), \quad f(x, f(y, f(z, z))), \quad f(f(f(x, y), f(x, z)), x),$$

etc.

In this general setting, however, Banach's claim becomes false: any map of an arbitrary (finite) arity on any set – in fact, any countable set of such maps – can be generated in this fashion by one binary map (a multiary analog of Theorem 1). More precisely:

**Theorem 2** (Webb, Łoś). *For any countable set  $\mathcal{S}$  of multiary maps  $X \times \cdots \times X \rightarrow X$  (of, possibly, different arities) on a set  $X$ , there is a binary map  $f : X \times X \rightarrow X$  (again, not necessary belonging to  $\mathcal{S}$ ) such that any map from  $\mathcal{S}$  is a superposition of  $f$ .*

This was proved several times in the literature: for the first time, in [We1] for the case of  $X$  finite (the required binary map is a multiary generalization of the Sheffer stroke). In the case of  $X$  infinite, a different, and much shorter<sup>†</sup> proof was presented, again, by Łoś [L1] (a similar, and yet simpler, proof was rediscovered more than half century later by Goldstern [G]). These proofs, in its turn, are based on another result of Sierpiński [Si3] to the effect that any single multiary map on an infinite set can be obtained as a superposition of (possibly, several) binary maps.

The latter paper of Sierpiński appeared around the same time as Banach's question under discussion: the same 1945 issue of *Fundamenta Mathematicae* in which the paper was published, the first one after the 6-year break occurred during WWII, contains an announcement about Banach's death. The same paper contains also another elementary, but interesting for us result:

**Theorem 3** (Pólya–Szegő). *For any binary bijection  $g : X \times X \rightarrow X$  on a (necessary infinite) set  $X$ , and any ternary map  $f : X \times X \times X \rightarrow X$ , there is another binary map  $h : X \times X \rightarrow X$  such that*

$$f(x, y, z) = g(h(x, y), z)$$

for any  $x, y, z \in X$ .

(The statement readily generalizes to  $n$ -ary maps  $f$ ). The result was, however, not new at that time: it appeared as the solution to the problem 119, imaginatively entitled “Are there actually functions of 3 variables?”, of Part 2 in the first 1925 edition of the famous book [PS]. Sierpiński's interest in this topic stems, evidently, from (a variant of) Hilbert's 13th problem: does there exist a real continuous function in 3 variables which cannot be represented as a superposition of real continuous functions in 2 variables? See, for example, his earlier paper [Si1] where Theorem 3 is proved for the case where  $X$  is the set of real numbers (albeit without using axiom of choice which is necessary in the general case), with reference to the (in)famously, at that time, erroneous paper by Bieberbach about Hilbert's 13th problem.

Ulam, along with Mark Kac (PhD from the Lvov University, 1937, with Banach as a member of examining committee), also an active participant of the Lvov mathematical scene until the end

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<sup>†</sup> A quote from Erdős [E]: “Now it frequently happens in problems of this sort that the infinite dimensional case is easier to settle than the finite dimensional analogues. This moved Ulam and me to paraphrase a well known maxim of the American armed forces in WWII: ‘The difficult we do immediately, the impossible takes a little longer’, viz: ‘The infinite we do immediately, the finite takes a little longer’ ”.

of 1930s<sup>†</sup> (cf. [Sc], [U2], and [F]), has at least a cursory interest in Hilbert’s 13th problem as well. In the collection of Ulam’s problems<sup>‡</sup> [U1, Chapter IV, Problem 2 and Chapter VI, Problem 5], which he positions as a sort of successor to the Scottish Book, as well in the joint Kac’s and Ulam’s popular book [KU, p. 163], they note a famous 1956–1957 negative solution of the Hilbert’s 13th problem by Kolmogorov and Arnold:

**Theorem 4** (Kolmogorov, Arnold, cf. [Ar, Theorem 1]). *For any real continuous function in 3 variables  $f : [0, 1] \times [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ , there are 9 real continuous functions in 2 variables  $\varphi_{ij}, \psi_{ij}$ ,  $i, j = 1, 2, 3$ , such that*

$$f(x, y, z) = \sum_{i=1}^3 \sum_{j=1}^3 \varphi_{ij}(\psi_{ij}(x, y), z)$$

for any  $0 \leq x, y, z \leq 1$ .

Kac and Ulam proceed then by formulating various extensions of the problem (for other classes of functions, for functions acting on more complicated spaces, etc.). In [U1], Ulam acknowledges Banach, among others, for “the pleasure of past collaboration”, but it is unclear whether this interest in the circle of questions related to Hilbert’s 13th problem goes back to the Lvov years, and if yes, whether it has something to do with Banach’s question. At any rate, as was noted on several occasions, “the iteration and composition of functions” was one of the main leitmotifs of Ulam’s work during his life (cf., e.g., [U3, p. x]). Of course, there is a very long way from Theorems 1–3 to Theorem 4, but they represent development along the same circle of ideas (Arnold himself used to cite Theorem 3 as one of the early precursors in his popular expositions of the history of Hilbert’s 13th problem).

The questions whether that or another set of maps generates all maps within a given class, framed in terms of functional completeness of various multivalued propositional calculi (“multi-valued logics”, or “logistics”, as it was called then) and the corresponding truth tables, were also popular among Polish logicians at that time. In particular, Jan Łukasiewicz was concerned about functional completeness of his famous 3-valued logic – that is, “implication”, the binary map  $L$  on the 3-element set  $\{0, \frac{1}{2}, 1\}$  given by “multiplication table”

	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

together with “negation”, the unary map  $N$  defined by

$$0 \mapsto 1, \frac{1}{2} \mapsto \frac{1}{2}, 1 \mapsto 0,$$

and similar systems. (While Łukasiewicz puts emphasis on the philosophical significance of many-valued logics – cf., e.g., [Mu, pp. 75–76] – most of the questions related to these logics, including question of functional completeness, naturally have a purely formal, mathematical, character). His student Jerzy Śłupecki proved in his PhD thesis (cf. [Sl]) functional incompleteness of the set  $\{L, N\}$ , and, in the positive direction, established functional completeness of the set  $\{L, N, T\}$ , where  $T$  is the unary map sending all 3 elements to  $\frac{1}{2}$ :

**Theorem 5** (Śłupecki). *Let  $X = \{0, \frac{1}{2}, 1\}$  be a set of 3 elements.*

- (i) *There are multiary maps  $X \times \cdots \times X \rightarrow X$  which cannot be represented as a superposition of the maps  $L$  and  $N$ .*
- (ii) *Any multiary map  $X \times \cdots \times X \rightarrow X$  is a superposition of the maps  $L$ ,  $N$ , and  $T$ .*

<sup>†</sup> Among the two, Ulam was more actively engaged in the Lvov mathematical life, and not only because he was a few years Kac’s senior, but by a more prosaic reason. Speaks Kac (cf. [F]): “I was less of a habitue of the Scottish Café... I was financially somewhat less affluent than Stan – I was ... independently poor. And it did cost a little to visit in the Café”.

<sup>‡</sup> Dedicated to Schreier’s memory, who committed suicide to avoid being captured by Germans during WWII.

(It is interesting to note that the same 1936–1937 issue of *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie* in which Śłupecki’s paper was published, contains the paper [We2] of Donald L. Webb, a fresh PhD from Caltech and the author of the already mentioned [We1], in which he extends the results of the latter paper. This, among other things, suggests that developments in this branch of logical calculi on both sides of Atlantic did not proceed in isolation at that time<sup>†</sup>). Earlier, another Łukasiewicz student, Mordchaj Wajsberg has reported at the Łukasiewicz–Tarski seminar<sup>‡</sup> some related results – see a brief description at the very end (“Anmerkung”) of [Wa] (cf. also Surma [Su, §ix]). In the same paper, more binary maps generating all maps of arbitrary arity on a finite set are presented without proof. Some of the Wajsberg’s results apparently go back as early as 1927, i.e. almost a decade before Webb.

More logical calculi, both complete and incomplete, were developed around this time, first of all, by Emil Leon Post (born in Poland, otherwise not related to that country<sup>§</sup>), as well as by Łukasiewicz, Śłupecki, Bolesław Sobociński (another PhD student of Łukasiewicz), Eustachy Żyliński (professor of the Lvov University in 1919–1941), Zygmunt Zawirski, and others. Post’s purely formal treatment of many-valued logics (“a combinatorial scheme” in the words of Mostowski [Mo, p. 3]) is, perhaps, closer in spirit to mathematical questions considered here than the philosophical attitude of Łukasiewicz.

Another feature of the work of Łukasiewicz’s school and Polish logicians in general, was the constant quest for minimal, as far as possible, systems of axioms for that or another logical calculus. Minimality was understood both in terms of the number of axioms and their length (cf., e.g., [Wo1, pp. 390–391] and [Su, §v])<sup>¶</sup>. Banach’s claim and question resonate well with this line of thought, as they can be phrased as follows: if a certain 3-term operation in an  $n$ -valued logic is not generated by a 2-term one (alas, as we have seen, this is wrong), the next best thing to ask is to extend the 3-term operation so that it will be generated that way.

Of course, many further results about Hilbert’s 13th problem, from one side, and functional completeness and incompleteness of various logical calculi, from the other, were obtained afterwards (cf., e.g., the survey by Vitushkin [V] for the former, and the book by Lau [L] for the latter), but their discussion will bring us far away from our main topic.

## 2. CLONES OF ANALYTIC AND ORDERED MAPS, COUNTING SUPERPOSITIONS, EXAMPLES TO BANACH’S CLAIM

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<sup>†</sup> The personal contacts started, probably, with the visit of Willard Van Orman Quine (then at Harvard) to Warsaw in 1932. Ernest Nagel (then at Columbia) made public (cf. [Nag]) his interesting impressions of visiting Poland in 1935, but, written from the philosopher’s rather than logician’s standpoint (“the logical researches both at Warsaw and Lwów are extraordinary specialized and technical”), these impressions, probably, contributed little to interchange of logical and mathematical ideas between the countries. Ulam was moving back and forth between US (Princeton and then Harvard) and Lvov in 1935–1939, but his interests, at least at that time, were outside logic. Webb’s thesis advisor, Eric Temple Bell, an enthusiastic, albeit not always precise, writer of popular mathematical books, praised Łukasiewicz’s 3-valued logic as one of the four major contributions “on the nature of truth” during the last 6000 years (cf. [Be, pp. 258–262]), so it is, perhaps, not accidental that Webb, a young Californian, published in a relatively obscure Warsaw journal.

<sup>‡</sup> “Professor Łukasiewicz’s seminar at Warsaw was crowded with competent young men, incomparably better equipped in logic than students of like age in America, who were expected to write as seminar exercises papers which elsewhere would be thought important enough for publication”, reports Nagel.

<sup>§</sup> A quote from [Wo1]: “When Tarski met Emil Post for the first time (in 1939 or 1940) he told him: ‘You are the only logician who achieved something important in propositional calculus without having anything to do with Poland’. Post answered: ‘Oh, no, I was born in Białystok’ ”.

<sup>¶</sup> This fascination with minimal systems of logical axioms was not shared by everyone in Poland. An anti-utopian novel “Nienasycenie” by Stanisław Ignacy Witkiewicz (a close friend of Leon Chwistek, professor of logic at the Lvov University, as well as of Alfred Tarski), written in 1927 and depicting conquest of Poland by enemy forces and establishment there of a totalitarian regime by the end of XX century, features a grotesque figure of logician Afanasol Benz (a Jew, stresses Witkiewicz) who invented a single axiom that nobody but him could understand, and from which all mathematics follows by a mere formal combination of symbols (cf. [Wi, p. 93]).



The “generation” in the results above is understood in the sense of the theory of clones, i.e. when forming superposition of maps, repeated variables (and, hence, superpositions of arbitrary length) are allowed. (Recall that clone is a set of a multiary maps on a fixed set  $X$  which contains projections on each variable, and is closed with respect to superposition; cf., e.g., [L, §1.3]). Thus, Banach’s claim can not be interpreted in terms of the clone of all maps on the underlying set  $X$ , as Theorem 2 tells that all the clone, including its ternary fragment, is generated by its binary fragment. It should be noted that if we assume that  $X$  possess some additional – topological, analytic, order, etc. – structure, and consider not the clone of all maps, but the clone of maps on  $X$  preserving this structure – then Theorem 2 about generation of all the maps by binary ones is no longer true. Some sporadic examples of various degree of sophistication:

- (i) Not every real analytic function in 3 variables can be represented as a superposition of analytic functions in two variables (cf. [V, §3]) – a statement made already by Hilbert in his original formulation of the 13th problem, whose proof utilizes some counting similar to elementary counting in the case of  $X$  finite, and also in the case of polynomials over a finite field in Example 2, see below.
- (ii) Superpositions of smooth functions in 2 variables satisfy various differential equations, not satisfied by all smooth functions in 3 variables (cf. solution of Problem 119a, Part 2 of [PS] for the relevant calculations).
- (iii) Algebraic functions in sufficiently high number of variables can not be represented as superposition of algebraic functions in sufficiently low (in particular, 2) number of variables – a suite of deep results due to V.I. Arnold and his followers, obtained by interpreting cohomology classes of a suitable braid group as obstructions to such representation (cf. Napolitano [Nap] for references).
- (iv) There is an 8-element poset whose clone of monotone maps cannot be generated not only by its binary fragment, but by any finite set of maps (due to G. Tardos, cf. [L, §11.5]).

However, as Banach is apparently interested in the case of  $X$  finite, and does not impose on  $X$  any additional structure, such interpretation seems to be unlikely.

If, however, we will understand the superposition in a more “operadic-like”, “multilinear” fashion, where each variable occurs only once, the only possible ways to generate the ternary map (1) by a binary one  $*$  :  $X \times X \rightarrow X$ , are:

$$(3L) \quad f(x, y, z) = (x * y) * z,$$

i.e. the same as in the version of the question from the Łoś paper [Ł2] discussed above, and

$$(3R) \quad f(x, y, z) = x * (y * z).$$

This is also in line with Theorem 3.

Of course, any map representable in the form (3L) gives rise, via permutation of arguments, to a map representable in the form (3R), and vice versa. Indeed, the equality (3R) can be rewritten as

$$(4) \quad f^{(13)}(x, y, z) = (x *^{(12)} y) *^{(12)} z,$$

where

$$f^\sigma(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

and  $\sigma$  is a permutation from  $S_n$ , the symmetric group in  $n$  variables. Banach is concerned with the case of “commutative” maps, which, by all accounts, are  $n$ -ary maps  $f$  which are stable under any  $\sigma \in S_n$ :  $f = f^\sigma$  (usually, such maps are called symmetric). As it follows from (4), a

commutative map  $f$  is representable in the form (3L) if and only if it is representable in the form (3R).

In this sense, Banach's claim becomes true. In the case of a finite set  $X$ , this is obvious from an elementary counting: the number of binary maps is  $|X|^{|X|^2}$ , so the number of ternary maps of the form (3L) and (3R) is less than  $2|X|^{|X|^2}$  (less, because these maps may coincide for different  $*$ 's), while the number of all ternary maps is  $|X|^{|X|^3}$ .

As an aside note, it seems to be an interesting question to estimate more exactly the number of different maps of the form (3L) and (3R) on an  $n$ -element set. A computer count produces the following table, where the second column contains the number of all binary maps on an  $n$ -element set,  $T_L(n)$  denotes the number of ternary maps of the form (3L),  $T_{LR}(n)$  denotes the number of ternary maps both of the form (3L) and (3R), and  $T_{comm}(n)$  denotes the number of commutative ternary maps of the form (3L) (which coincides with the number of such maps both of the form (3L) and (3R)) (so, for  $n > 1$  we have the obvious inequalities  $T_{comm}(n) < T_L(n) < T_{LR}(n) < 2T_L(n)$ ):

$n$	$n^{n^2}$	$T_L(n)$	$T_{LR}(n)$	$T_{comm}(n)$
1	1	1	1	1
2	16	14	21	5
3	19683	19292	38472	48

In the given range,  $T_{comm}(n)$  turns out to be equal to  $T_{comm}^{comm}(n)$ , the number of commutative ternary maps of the form (3L) with *commutative*  $*$ . We do not know the general formulas for  $T_L(n)$ ,  $T_{LR}(n)$ ,  $T_{comm}(n)$ , and  $T_{comm}^{comm}(n)$ , and refrain here from making any conjectures, but this seems to be an interesting topic to study further<sup>†</sup>.

It is easy to manufacture examples of a ternary map  $f$  which cannot be generated, in the sense of (3L), by any binary operation  $*$ , thus explicitly confirming Banach's claim (in examples below, the ternary maps are commutative, but they are not representable as superposition, in the sense of (3L), of any binary map, commutative or not).

**Example 1.** Let  $X$  be a set containing more than 2 elements, and  $a, b \in X$  be two distinct elements of  $X$ . Define a ternary map  $f$  on  $X$  by

$$f(x, y, z) = \begin{cases} a, & \text{if all } x, y, z \text{ are distinct from } a \\ b, & \text{if at least one of } x, y, z \text{ coincides with } a. \end{cases}$$

Suppose (3L) holds for some binary map  $*$  on  $X$ . If for any two elements  $x, y \in X$ , both distinct from  $a$ ,  $x * y \neq a$ , then for any 3 elements  $x, y, z \in X$ , each distinct from  $a$ , we have  $a = f(x, y, z) = (x * y) * z \neq a$ , a contradiction. Hence there are  $u, v \in X$ , both distinct from  $a$ , such that  $u * v = a$ . Then  $a * a = (u * v) * a = f(u, v, a) = b$ , and then for any  $x \in X$ ,  $b * x = (a * a) * x = f(a, a, x) = b$ , and, finally,  $a = f(b, u, v) = (b * u) * v = b * v = b$ , a contradiction.

**Example 2.** Let  $X$  be a set of  $q = p^n$  elements, where  $p$  is a prime, and  $f$  a ternary map on  $X$ . Endow  $X$  with the structure of the finite field  $\text{GF}(q)$ . The question is whether there exists or not a binary map  $g$  on  $\text{GF}(q)$  such that

$$(5) \quad f(x, y, z) = g(g(x, y), z)$$

for any  $x, y, z \in \text{GF}(q)$ . Since, due to Lagrange interpolation formula, each  $k$ -ary function on  $\text{GF}(q)$  can be represented as a polynomial in  $k$  variables with coefficients in  $\text{GF}(q)$ , and the degree

<sup>†</sup> A simple Perl program, `binary-ternary.pl`, which computes these numbers for small values of  $n$ , is available as an ancillary file in the arXiv version of this note. At the time of this writing, the 3-term sequences for  $T_L(n)$  and  $T_{LR}(n)$  are absent in the Online Encyclopedia of Integer Sequences.

$< q$  in each variable (cf., e.g., Lidl–Niederreiter [LN, pp. 368–369]), and, moreover, two such polynomial maps are equal if and only if their coefficients are equal, the condition (5) can be rewritten as a system of polynomial equations in (polynomial) coefficients of  $g$ . Suitably choosing  $f$ , one can obtain a system not having solutions in  $\text{GF}(q)$  (in fact, “most” of the  $f$ ’s will do, as the system consists of  $q^3$  equations in  $q^2$  unknowns). For example, defining a ternary map  $f : \text{GF}(2) \times \text{GF}(2) \times \text{GF}(2) \rightarrow \text{GF}(2)$  by  $f(x, y, z) = xy + xz + yz$ , and writing  $g(x, y) = a + bx + cy + dxy$  for some  $a, b, c, d \in \text{GF}(2)$ , we arrive at the system

$$a + ab = 0, \quad b^2 = 0, \quad bc = 0, \quad c + ad = 0, \quad bd = 1, \quad cd = 1, \quad d^2 = 0$$

which, evidently, does not have solutions.

Of course, nothing in these examples is specific to 3 variables, and they can be easily extended to  $n$ -ary maps for arbitrary  $n$ , and, moreover, to non-representability in the form  $(x * y) \circ z$  for two binary maps  $*$  and  $\circ$ , and similar  $n$ -ary expressions.

### 3. ANSWER TO BANACH’S QUESTION

The following answers the question of Banach interpreted as above – i.e. about extensions of arbitrary ternary maps to those having the form (3L) – in affirmative. (We deal with arbitrary, not necessary commutative, maps).

**Theorem A.** *For any (finite) set  $X$  and a ternary map  $f : X \times X \times X \rightarrow X$ , there is a (finite) set  $Y \supset X$  and a binary map  $* : Y \times Y \rightarrow Y$  such that*

$$f(x, y, z) = (x * y) * z$$

for any  $x, y, z \in X$ .

*Proof* (Referee). Define  $Y$  as a disjoint union  $X \cup (X \times X)$ , and the binary map  $*$  on it as follows:

$$\begin{aligned} x * y &= (x, y) \\ (x, y) * z &= f(x, y, z), \end{aligned}$$

where  $x, y, z \in X$ . The rest, i.e.  $x * (y, z)$  and  $(x, y) * (z, t)$ , is defined in an arbitrary way.  $\square$

This elementary construction has a drawback that it cannot be easily modified to obtain similar statements in classes of maps satisfying various conditions. For example, the so constructed binary map  $*$  is, generally, neither commutative, nor associative, even if the initial ternary map  $f$  is. To remedy this, one may adopt just a slightly more complicated approach, based on various, related, constructions of envelopes of Lie and other triple systems, an idea going back to the pioneering paper [J] of Nathan Jacobson (born in Warsaw, but otherwise not related to Poland) published only a few years later than the question was posed. In this way, we get an alternative proof of a positive answer to Banach’s question in the narrower – “associative” – version, given in the papers [L2] and [MS] as mentioned at the beginning of §. Our proof differs from both of them and, as we hope, is shorter and simpler.

**Theorem B** (Łoś, Monk–Sioson). *For any (finite) commutative ternary semigroup  $(X, f)$  there is a (finite) commutative binary semigroup  $(Y, *)$  such that  $Y \supset X$  and*

$$f(x, y, z) = (x * y) * z$$

for any  $x, y, z \in X$ .

*Proof.* Let  $M$  be the set consisting of maps  $X \rightarrow X$  of the form

$$m_{x,y} : z \mapsto f(x, y, z)$$

for all  $x, y \in X$ , and  $\mathcal{M}$  the subsemigroup of the semigroup of all maps  $X \rightarrow X$  with respect to composition, generated by  $M$ . Note that commutativity and associativity of  $f$  imply

$$m_{x,y} \circ m_{u,v} = m_{u,v} \circ m_{x,y}$$

for any  $x, y, u, v \in X$ , so  $\mathcal{M}$  is a commutative semigroup.



Define  $Y$  as a disjoint union  $X \cup \mathcal{M}$ , and the binary map  $*$  on it as follows:

$$\begin{aligned} x * y &= m_{x,y} \\ x * g &= g * x = g(x) \\ g * h &= g \circ h \end{aligned}$$

for  $x, y \in X$ ,  $g, h \in \mathcal{M}$  ( $\circ$  denotes the composition of maps). As  $f$  is commutative,  $m_{x,y} = m_{y,x}$ , and hence  $*$  is commutative.

For any  $x, y, z \in X$ , we have

$$(x * y) * z = m_{x,y}(z) = f(x, y, z).$$

It remains to check the associativity of  $*$ . For 3 terms which all belong to  $X$ , the associativity of  $*$  follows from the commutativity of  $f$ . Similarly, for 3 terms which all belong to  $\mathcal{M}$ , the associativity of  $*$  follows from the associativity of  $\circ$ . If two terms, say,  $g$  and  $h$ , belong to  $\mathcal{M}$ , and one, say  $x$ , belongs to  $X$ , the associativity of  $*$  follows from the commutativity of  $\circ$  in  $\mathcal{M}$ :

$$\begin{aligned} (g * h) * x &= (g \circ h) * x = g(h(x)) = g * (h(x)) = g * (h * x), \\ (g * x) * h &= g(x) * h = h(g(x)) = g(h(x)) = g * (h(x)) = g * (x * h). \end{aligned}$$

In the remaining cases, where one term,  $g = m_{u,v}$  ( $u, v \in X$ ), belongs to  $\mathcal{M}$ , and two terms,  $x$  and  $y$ , belong to  $X$ , assuming additionally  $z \in X$ , and utilizing commutativity and associativity of  $f$ , we have:

$$\begin{aligned} ((x * y) * g)(z) &= (m_{x,y} * g)(z) = (m_{x,y} \circ g)(z) = f(x, y, g(z)) = f(x, y, f(u, v, z)) \\ &= f(x, f(u, v, y), z) = f(x, g(y), z) = m_{x,g(y)}(z) = (x * g(y))(z) = (x * (y * g))(z) \end{aligned}$$

and

$$\begin{aligned} ((x * g) * y)(z) &= (g(x) * y)(z) = m_{g(x),y}(z) = f(g(x), y, z) = f(f(u, v, x), y, z) \\ &= f(x, f(u, v, y), z) \stackrel{\text{as above}}{=} (x * (g * y))(z), \end{aligned}$$

what completes the proof.  $\square$

One can deal similarly with not necessary commutative ternary semigroups (one needs then to consider, instead of  $m_{x,y}$ , both “left” and “right” multiplications), with ternary groups (thus recovering a part of Post’s results [Pos]), etc.

Some of the statements of this section may be suitably extended, via a straightforward iterative procedure, to the maps  $f$  of arbitrary arity, but we will not venture into this.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSTRAVA, OSTRAVA, CZECH REPUBLIC

*Email address:* [pasha.zusmanovich@osu.cz](mailto:pasha.zusmanovich@osu.cz)