# Ajay Kumar Garg Engineering College, Ghaziabad

## Department of ECE

### Model Solution (Sessional Test-2)

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|---|---|
| Course: B. Tech | Semester: V |
| Session: 2017-18 | Section: EN-1, EN-2 |
| Subject: Fundamentals of E.M Theory | Sub. Code: NEC-508 |
| Max Marks: 50 | Time: 2 hour |

*Note*: Answer **all** the sections.

**Section-A**

A. Attempt **all** the parts. **(5x2 =10)**

1 Tabulate the analogy between Electric and Magnetic fields.

Ans-

| Term | Electric fields | Magnetic fields |
|---|---|---|
| Basic Law | $F = \frac{Q_1 Q_2}{4\pi \epsilon r^2} \hat{a}_r$ | $dB = \frac{\mu_0 I dl \times \hat{a}_R}{4\pi R^2}$ |
| Force Law | $F = QE$ | $F = Qu \times B$ |
| Source element | $dQ$ | $Qu = I dl$ |
| Flux density | $D = \frac{\Psi}{S}$ (C/m2) | $B = \frac{\Psi}{S}$ (Wb/m2) |
| Poisson's equation | $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$ | $\nabla^2 A = -\mu J$ |

2. For the current density $J = 10z \sin^2\phi \, \hat{a}_\rho$ A/m2. Find the current through the cylindrical surface $\rho = 2, 1 \le z \ge 5$ m.

Ans: We know $I = \int J . ds$ where $ds = \rho \, d\phi \, dz \, \hat{a}_\rho$

$$= \int_{z=1}^{5} \int_{\phi=0}^{2\pi} (10 z \sin^2\phi \, \hat{a}_\rho) . \rho \, d\phi \, dz \, \hat{a}_\rho$$

$$= 10\rho \int_{z=1}^{5} z \, dz . \int_{\phi=0}^{2\pi} \sin^2\phi \, d\phi$$

$$= 10\rho \left[\frac{z^2}{2}\right]_1^5 \int_0^{2\pi} \frac{1-\cos 2\phi}{2}\, d\phi$$

$$= 10\rho \cdot 12 \cdot \frac{1}{2} \cdot 2\pi$$

$$= 10 \times 2 \times 12 \times \pi$$

$$\boxed{I = 754 \text{ A}}$$ Ans.

3. Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$, find the electric flux density $D$ at $(2, \frac{\pi}{2}, 0)$.

Ans. $D = \varepsilon_0 E$

But $E = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right]$

$$= \frac{20}{r^3}\sin\theta\cos\phi\,\hat{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\hat{a}_\theta + \frac{10}{r^3}\sin\phi\,\hat{a}_\phi$$

At $(2, \pi/2, 0)$

$$D = \varepsilon_0 E \;(r = 2,\ \theta = \frac{\pi}{2},\ \phi = 0)$$

$$D = 2.5\,\varepsilon_0\,\hat{a}_r \text{ C/m}^2 = 22.1\,\hat{a}_r \text{ pC/m}^2$$ Ans.

4. State and explain Poisson's and Laplace's equation.

Ans. Poisson's & Laplace's equation can derived from Gauss's law

$$\nabla \cdot D = \nabla \cdot \varepsilon E = \rho_v$$

and $E = -\nabla V$

$$\nabla \cdot (-\varepsilon \nabla V) = \rho_v$$

for inhomogeneous medium

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$ This is Poisson's Equation.

spectial case, when $\rho_v = 0$

$$\boxed{\nabla^2 V = 0}$$ This is Laplace's equation.

5. Define continuity equation ?

Ans. Continuity Equation. From the principle of charge conservation

$$I_{out} = \oint J \cdot ds = -\frac{dQ_{in}}{dt}$$

Invoking the divergence theorem

$$\oint_S J \cdot ds = \int_v \nabla \cdot J \, dv$$

But $$-\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_v \rho_v \, dv = -\int \frac{\partial \rho_v}{\partial t} dv$$

$$\therefore \int_v \nabla \cdot J \, dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

$$\boxed{\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}}$$ This is continuity equation

## Section - B

B. Attempt all the parts (5×5=25)

6. State the Coulombs law and derive the electric field intensity for the infinite line charge distribution.

Ans - Coulomb's law - Force F b/w two point charges $Q_1$ and $Q_2$ is:

i > Along the line joining them

ii > Directly proportional to the product $Q_1 Q_2$ of the charges

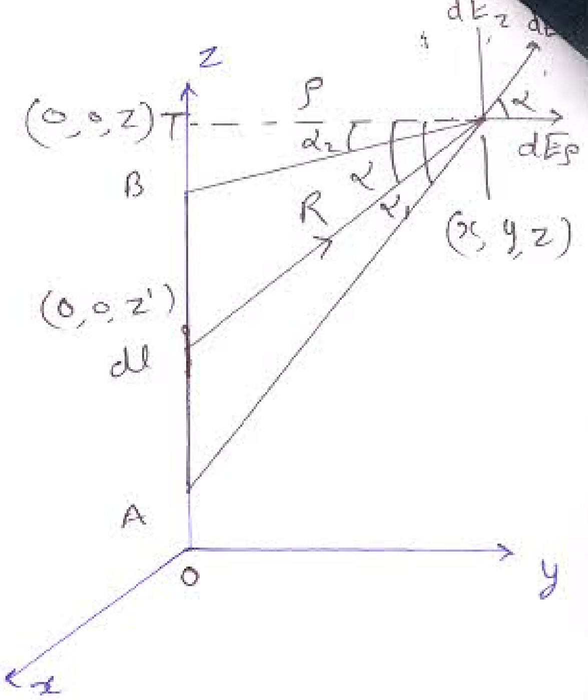
iii > Inversely proportional to the square of the distance R b/w them

$$\boxed{F = \frac{k Q_1 Q_2}{R^2}}$$

where $\varepsilon_0 = 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}$ F/m

or $k = \frac{1}{4\pi\varepsilon_0} \simeq 9 \times 10^{9}$ m/F

A Line Charge—




we know $dQ = \rho_L \, dL = \rho_L \, dz$

According to fig

$dl = dz'$

$$\bar{R} = (x, y, z) - (0, 0, z')$$

$$= x\,\hat{a}_x + y\,\hat{a}_y + (z - z')\,\hat{a}_z$$

$$\bar{R} = \rho\,\hat{a}_\rho + (z - z')\,\hat{a}_z$$

$$\bar{E} = \frac{\rho_L}{4\pi\epsilon_0}\int_A \frac{\rho\,\hat{a}_\rho + (z - z')\,\hat{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}\,dz'$$

To evaluate this, $R = [\rho^2 + (z - z')]^{1/2} = \rho \sec\alpha$

$z' = OT - \rho\tan\alpha, \quad dz' = -\rho\sec^2\alpha\,d\alpha$

Hence $$E = -\frac{\rho_L}{4\pi\epsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{\rho\sec^2\alpha\,[\cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z]\,d\alpha}{\rho^2\sec^2\alpha}$$

$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\,\hat{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\,\hat{a}_z\right]$$

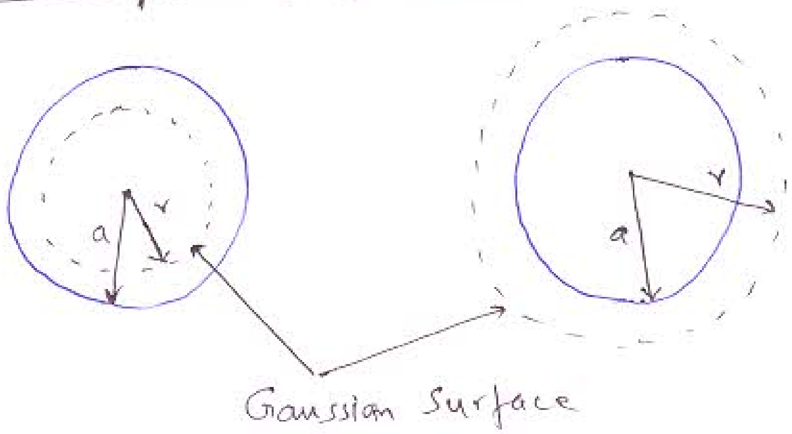
For Infinite line, $\alpha_1 = \frac{\pi}{2}, \ \alpha_2 = -\frac{\pi}{2}$

$$\boxed{E = \frac{\rho_L}{2\pi\epsilon_0\rho}\,\hat{a}_\rho}$$

7. Calculate the electric flux density everywhere for a Uniformly Charged sphere (application of Gauss's law) of radius 'a' with a uniform charge $\rho_0$.

Given that $D = z\rho\cos^2\phi\,\hat{a}_z$ C/m². Calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \leq z \leq 2$ m.

Ans. Uniformly Charged Sphere




For $r \leq a$, $Q_{enc} = \int_v \rho_v \, dv = \rho_0 \int_v dv$

$$= \rho_0 \frac{4}{3} \pi r^3$$

and $\Psi = \oint_s D \cdot ds = D_r \cdot \oint_s ds = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$

$$= D_r \cdot 4\pi r^2$$

Hence, $\Psi = Q_{enc}$ gives

$$D_r \cdot 4\pi r^2 = \frac{4\pi r^3}{3} \rho_0$$

or $$D = \frac{r}{3} \rho_0 \hat{a}_r \quad 0 < r \leq a$$

For $r \geq a$. $Q_{enc} = \int_v \rho_v \, dv = \rho_0 \int_v dv$

$$= \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

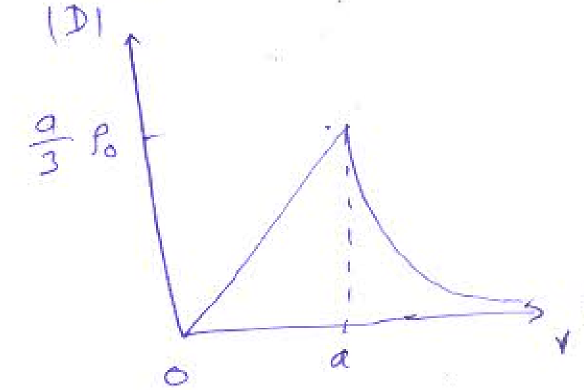
$$= \rho_0 \frac{4}{3} \pi a^3$$

while $$\Psi = \oint_s D \cdot ds = D_r \cdot 4\pi r^2$$

$$D_r \, 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_o$$

$$D_r = \frac{a^3}{3r^2}\rho_o \qquad r \geq a$$

$$D = \begin{cases} \frac{r}{3}\rho_o \, \hat{a}_r & 0 \leq r \leq a \\ \frac{a^3}{3r^2} \cdot \rho_o \, \hat{a}_r & r \geq a \end{cases}$$




$$\rho_v = \nabla \cdot D = \frac{\partial D_z}{\partial z} = \rho \cos^2\phi$$

At $(1, \pi/4, 3)$ $\rho_v = 1.\cos^2(\pi/4) = 0.5 \ C/m^3$.

Total charge enclosed, $Q = \int_V \rho_v \, dv = \int_V \rho \cos^2\phi \, \rho \, d\phi \, d\rho \, dz$

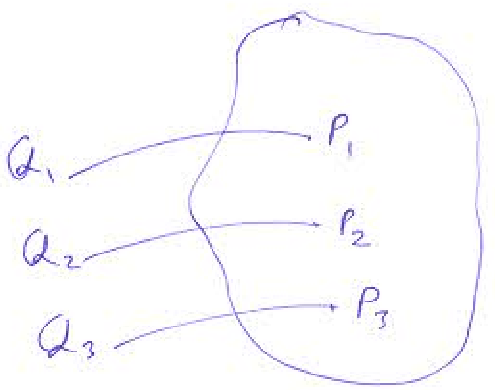
$$Q = \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho$$

$$\boxed{Q = \frac{4\pi}{3} C}$$

8. Derive an expression for energy density in electrostatic fields.

Ans. $W_E \rightarrow$ The total work done by the charges

1, 2 and 3.




$$W_E = W_1 + W_2 + W_3$$

$$= Q_1(0) + Q_2(V_{21}) + Q_3(V_{32} + V_{31})$$

$$\boxed{W_E = Q_2 V_{21} + Q_3 (V_{32} + V_{31})} \quad — (1)$$

In reverse order

$$W_E = W_3 + W_2 + W_1$$

$$= Q_3 . 0 + Q_2 (V_{23}) + Q_1 (V_{13} + V_{12})$$

$$W_E = Q_2 V_{23} + Q_1 (V_{13} + V_{12}) \quad —— (2)$$

Adding (1) + (2)

$$2W_E = Q_1 (V_{13} + V_{12}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$$

$$W_E = \frac{1}{2} [Q_1 V_1 + Q_2 V_2 + Q_3 V_3]$$

for n charges, we have

$$W_E = \frac{1}{2} \sum_{k=1}^{n} Q_K V_K$$

using equn. (3)

$$W_E = \frac{1}{2} \int \rho_v V \, dv$$

$$= \frac{1}{2} \int (\nabla . D) V \, dv$$

$$= \frac{1}{2} \int (\nabla . VD) \, dv - \frac{1}{2} \int_v (D . \nabla V) \, dv$$

$$W_E = \frac{1}{2} \oint_s (V D) . ds - \frac{1}{2} \int (D . \nabla V) \, dv$$

For large surface, Ist integral tend to zero

So, $$W_E = -\frac{1}{2}\int (D \cdot \nabla V)\, dv$$

$$= \frac{1}{2}\int (D \cdot E)\, dv$$

and since $E = -\nabla V$ and $D = \varepsilon_0 E$

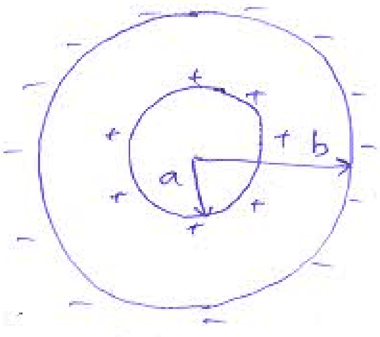
$$\boxed{W_E = \frac{1}{2}\int_v D \cdot E\, dv = \frac{1}{2}\int_v \varepsilon_0 E^2\, dv}$$

9. A spherical capacitance with $a = 1.5$ cm, $b = 4$ cm has an inhomogeneous dielectric of $\varepsilon = \frac{10\varepsilon_0}{r}$. Calculate the capacitance of the capacitance.

Ans: $a = 1.5$ cm

$b = 4$ cm has an inhomogeneous diel

$C = ?$




$$C = \frac{Q}{V}$$

$$Q = \int D\, ds$$

$$D = \varepsilon E$$

$$\therefore Q = \varepsilon \int E \cdot ds$$

$$Q = \varepsilon E_r (4\pi r^2) \quad E_r = \frac{Q}{4\pi \varepsilon r^2}$$

as $$E_r = \frac{Q}{4\pi \varepsilon r^2} \quad \text{as } \varepsilon = \frac{10\varepsilon_0}{r}$$

$$= \frac{Q}{4\pi \times \frac{10\varepsilon_0}{r} r^2} = \frac{Q}{4\pi\varepsilon_0}\hat{a}_r$$

As, $$V = -\int E_r\, dl \quad dl = dr\,\hat{a}_r$$

$$V = -\frac{Q}{40\pi\varepsilon_0}\int_b^a \frac{1}{r}\hat{a}_r\, dr\,\hat{a}_r$$

$$V = \frac{-Q}{40\pi \varepsilon_0} \left[\log r\right]_b^a$$

$$= \frac{-Q}{40\pi \varepsilon_0} \left[\log a - \log b\right]$$

$$V = \frac{Q}{40\pi \varepsilon_0} \log(b/a)$$

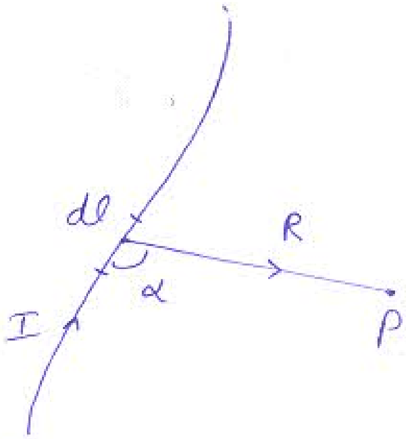
$$C = \frac{Q}{V} = \frac{40\pi \varepsilon_0}{\log(b/a)}$$

$$= \frac{40 \times 3.14 \times 8.85 \times 10^{-12}}{\log(4/1.5)}$$

$$\boxed{C = 1.33\ nF}$$ Ans

10. State Biot Savart's law and derive an expression for magnetic field intensity due to infinite straight line current carrying conductor.

Ans. Biot. Savart's law can be defined as differential magnetic field for the current carrying element is directly proportional to $(I\,dl)$ and the angle of sine between the point P and current carrying element and is inversely proportional to the square of the distance




$$dH \propto \frac{I\,dl \sin\alpha}{R^2}$$

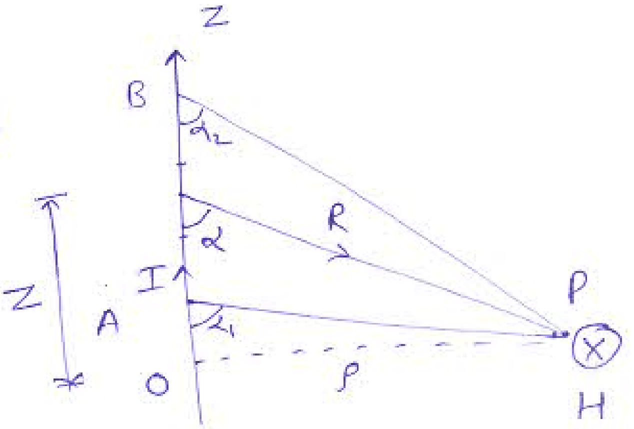
$$dH = \frac{K I dl \sin\alpha}{R^2}$$

where $K = \frac{1}{4\pi}$

$$dH = \frac{1}{4\pi} \cdot \frac{I dl \times \vec{R}}{R^3}$$

$$\boxed{I\, dl = K\, ds = J.\, dv}$$

Magnetic field Intensity due to a straight line current carrying conductor:




dH at P due to an element dl at (0, 0, z).

$$dH = \frac{I\, dl \times \hat{a}_R}{4\pi R^2}$$

$$= \frac{I\, dl \times \bar{R}}{4\pi R^3}$$

But $dl = dz\, \hat{a}_z$ and $\bar{R} = \rho\, \hat{a}_\rho - z\, \hat{a}_z$, so

$$dl \times R = \rho\, dz\, \hat{a}_\phi$$

Hence

$$H = \int \frac{I \rho\, dz}{4\pi [\rho^2 + z^2]^{3/2}}\, a_\phi$$

Letting $z = \rho \cot\alpha$, $dz = -\rho \operatorname{cosec}^2\alpha\, d\alpha$

$$H = -\frac{I}{4\pi\rho} \cdot \hat{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha$$

$$\boxed{H = \frac{I}{4\pi\rho} \cdot (\cos\alpha_2 - \cos\alpha_1)\, \hat{a}_\phi}$$

For infinite straight line current carrying conductor. $\alpha_1 = \pi$, $\alpha_2 = 0$

$$\bar{H} = \frac{I}{4\pi\rho}\cdot\left[\cos 0 - \cos(\pi)\right]\hat{a}_\phi$$

$$= \frac{I}{4\pi\rho}\cdot\left[1-(-1)\right]$$

$$\boxed{\bar{H} = \frac{I}{2\pi\rho}\hat{a}_\phi}$$

## Section - C

C. Attempt all the parts.

(2 × 7.5 = 15)

11. Derive all the Maxwell Equation with their physical significance for time invariant electric and magnetic field.

Ans: First Maxwell's Equ$^n$: (Gauss's law).

As Gauss law state that the total flux in the closed surface is equal to the charge enclosed into the closed surface.

$$\psi = Q_{enclosed}$$

$$\psi = \oint_s D.ds = Q_{enclosed}$$

$$= \text{total charge enclosed } Q = \int_v \rho_v \, dv$$

$$Q = \oint_s D.ds = \int \rho_v \, dv$$

By applying divergence theorem,

$$\oint_s D.ds = \int_v \nabla.D \, dv$$

$$\boxed{\rho_v = \nabla D}$$

2) $2^{nd}$ Maxwell's Equation : $(\nabla \times E = 0)$

As the electric potential is given by : $V_{AB}$.

Hence

$$V_{BA} = -V_{AB}$$

that is,
$$V_{BA} = -V_{AB}.$$

or
$$V_{BA} + V_{AB} = \oint E.dl = 0$$

or

$$\boxed{\oint E.dl = 0}$$

Applying Stoke's theorem,

$$\oint E.dl = \int (\nabla \times E).ds = 0$$

$$\boxed{\nabla \times E = 0}$$

3) $3^{rd}$ Maxwell's equation $(\nabla \times H = J)$ :-

Ampere's Circutal law, the circulation of H equals $I_{enc}$.

ie
$$\oint_L H.dl = I_{enc} \qquad \text{--- (1)}$$

Applying Stokes's theorem to the left-hand side

$$I_{enc} = \oint_L H.dl = \int_S (\nabla \times H).ds \qquad \text{--- (2)}$$

But

$$I_{enc} = \int J.ds$$

Compare (1) & (2)

$$\boxed{\nabla \times H = J}$$

$\nabla \times H = J \neq 0$, ie is a magnetostatic field is not conservative.

4) ## 4th Maxwell Equation ($\nabla . B = 0$)

Magnetic flux density : $B = \mu_0 H$

$$\mu_0 = 4\pi \times 10^{-7} \ H/m$$

The magnetic flux through a surface S is given by

$$\psi = \int B . ds$$

Total flux through a closed surface in a magnetic field must be zero; that is

$$\oint_s B . ds = 0$$

this equation is referred to as the law of conservative nature of magnetic flux.

Applying the divergence theorem. we obtain

$$\oint_s B . ds = \int_v \nabla . B \, dv = 0$$

$$\boxed{\nabla . B = 0}$$

This is maxwell's 4th equation.

12. Discuss various boundary conditions as applied to electric field b/w dielectric - dielectric, conductor - dielectric and conductor - free space.

Ans : 1) Dielectric - Dielectric -

To determine the boundary conditions, we need to use Maxwell's equations.



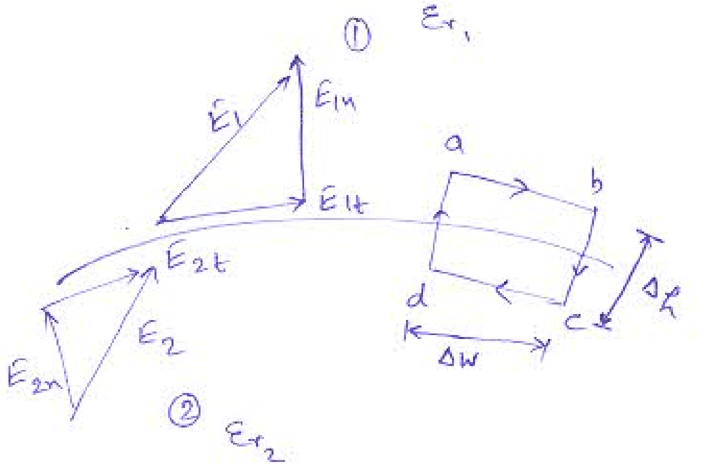
7.

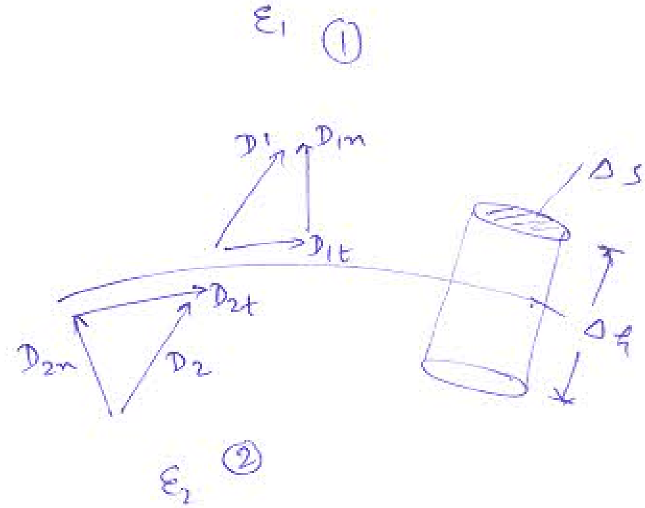
$$\oint E \cdot dl = 0$$

and

$$\oint D \, ds = Q_{enc}$$

$$E = E_t + E_n.$$




$$E_1 = E_{1t} + E_{1n}$$

$$E_2 = E_{2t} + E_{2n}$$

Apply on closed path

$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

$$0 = (E_{1t} - E_{2t})\,\Delta w$$

$$\boxed{E_{1t} = E_{2t}}$$ ← Continuan in nature at interface

$$D = \varepsilon E$$

$$\therefore \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Similarly on fig. ②

$$\Delta Q = \rho_s \Delta s = D_{1n}\Delta s - D_{2n}\Delta s$$

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

If $\rho_s = 0$ $\quad \boxed{D_{1n} = D_{2n}}$

B. Conductor - Dielectric Boundary Conditions :-

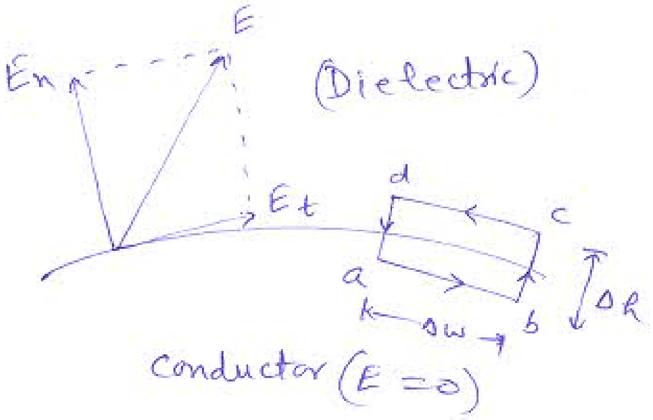



fig. ③

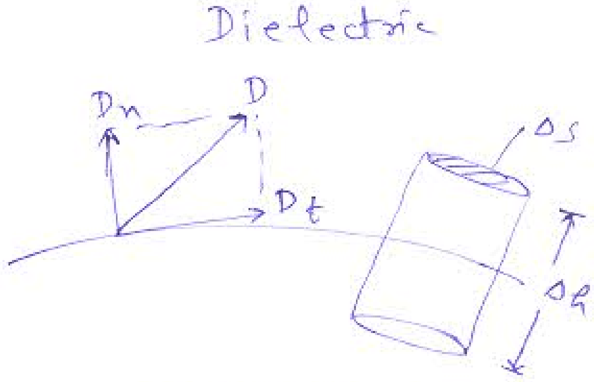



fig. ④

from fig. ③.

$$0 = 0.\Delta w + 0.\frac{\Delta h}{2} + E_n \frac{\Delta h}{2} - E_t.\Delta w - E_n.\frac{\Delta h}{2} - 0.\frac{\Delta h}{2}$$

As $\Delta h \to 0$

$$\boxed{E_t = 0}$$

Similarly, by applying to fig ④.

$$\Delta Q = D_n.\Delta S - 0.\Delta S$$

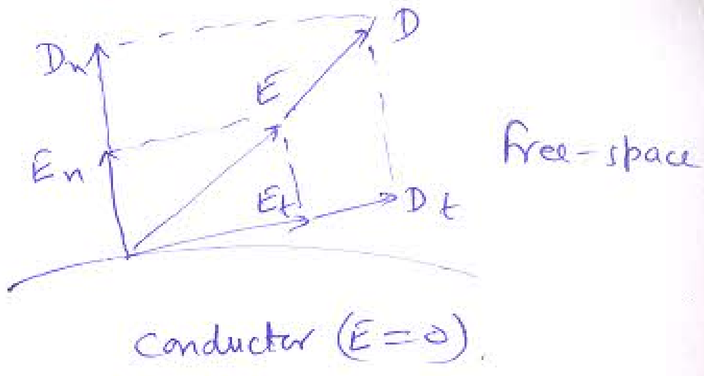
$$D_n = \frac{\Delta Q}{\Delta S} = \rho_s$$

$$\boxed{D_n = \rho_s}$$

c). Conductor - Free space Boundary Conditions -

$$D_t = \varepsilon_0 E_t = 0$$

$$D_n = \varepsilon_0 E_n = \rho_s$$




E field must approach a conducting surface normally.