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Department of ECE

Model Solution Sessional Test-2

Course:	B.Tech	Semester:	VII
Session:	2017-18	Section:	EC-1, 2, 3
Subject:	Information Theory &Coding	Sub. Code:	NEC-031
Max Marks: 50		Time:	2 hour

Note : Answer all sections

Section-A

Q.A Attempt all parts

Q.1. What is a DMC Channel?

Ans:- A Discrete Channel is a system consist of an Input alphabet X and output alphabet Y and a matrix $p(Y/X)$ that express probability of observing o/p w.r.t. ~~to~~ input. This channel is said to be DMC only when probability distribution of the output depends only on the input at that time & is independent of previous Inputs or outputs.

Q.2. Out of the following code which one is non-singular?

Source	S_1	S_2	S_3	S_4
Code A	00	001	101	110
Code B	00	100	111	00

Ans:- Code B is non-singular as S_1 & S_4 are having same codes to different symbols.

Q.3. State Source Coding Theorem.

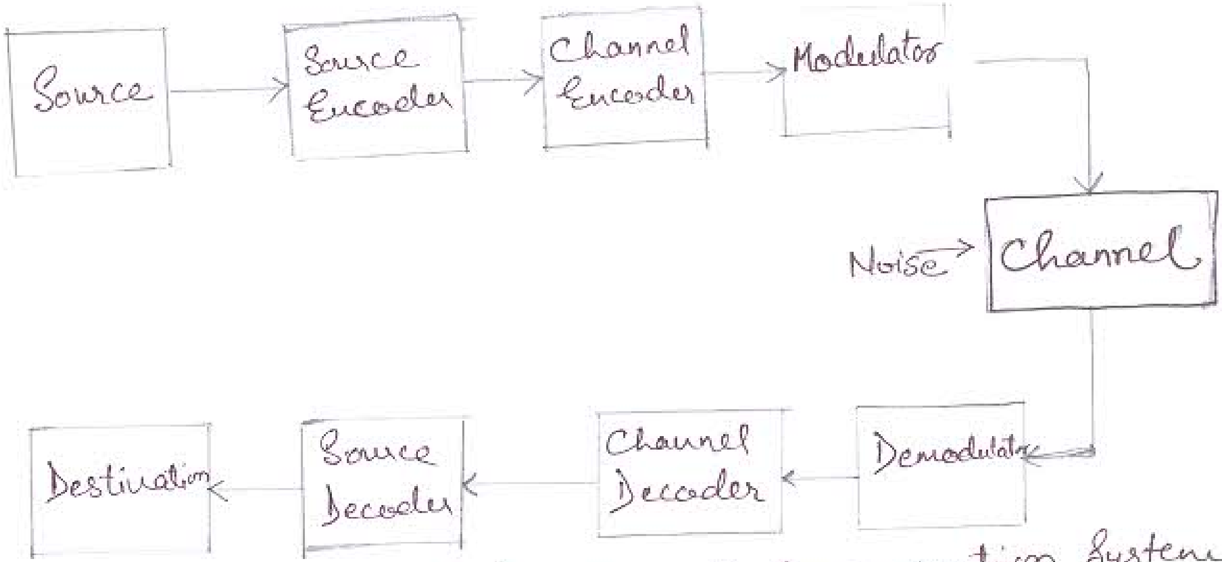
Ans:- A Source Coding Theorem or Shannon's First Theorem states that the bound on optimal code length should lie in between

$$H(X) \leq L < H(X)+1$$

where, L is codelength (expected) & H(X) is the entropy of source X.

Q.4. Explain briefly the block diagram of Communication System.

Ans :-



Block Diagram of Communication System

1. Source Encoder is used to reduce redundant bits so that more information travel over a channel.
2. Channel Encoder is used to add some redundant (parity) bits in controlled manner so that information can be reliably reconstructed at the receiver end.

Q.5. What is Joint Probability Matrix?

Ans :- JPM $\longrightarrow$

$$P(X,Y) = \begin{array}{c|c} & \begin{matrix} j=1 & j=2 & \cdots & j=m \end{matrix} \\ \begin{matrix} i=1 \\ i=2 \\ \vdots \\ i=n \end{matrix} & \begin{bmatrix} P(x_1,y_1) & P(x_1,y_2) & \cdots & P(x_1,y_m) \\ P(x_2,y_1) & P(x_2,y_2) & \cdots & P(x_2,y_m) \\ \vdots & \vdots & & \vdots \\ P(x_n,y_1) & P(x_n,y_2) & \cdots & P(x_n,y_m) \end{bmatrix} \end{array} \quad \text{—— (1)}$$

JPM has following properties –

(i) The sum of all the elements of the j^{th} column of JPM gives the probability of j^{th} output.

(ii) The sum of all the elements of the i^{th} rows of JPM gives the probability of i^{th} input.

(iii) The sum of all the elements of JPM is unity.

Section B.

B. Attempt all parts.

Q.6. State Channel Coding Theorem.
The Channel Matrix of a Communication Channel is

$$P(Y|X) = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

The Apriori probabilities are $P(x_1) = P(x_2) = 1/2$.

How much is the loss of information in this channel.

Ans:- Channel Coding Theorem states that all rates below capacity C are achievable, i.e. for every $\epsilon > 0$ and rate $R < C$, there exists a sequence of $(2^{nR}, n)$ codes with maximum probability of error

$$\lambda^{(n)} \leq \epsilon$$

for n sufficiently large. Conversely, if $\lambda^{(n)} \to 0$, then $R \leq C$.

We know that,
loss of information is given as $H(Y/X)$.

$$H(Y/X) = \sum_{X}\sum_{Y} p(x,y) \log_2 \frac{1}{P(Y/X)} \quad \text{--- (1)}$$

Now,

$$P(X,Y) = P(X) \cdot P(Y/X) = \begin{bmatrix} 1/4 & 1/4 \\ 1/8 & 3/8 \end{bmatrix} \quad \text{--- (2)}$$

From eq. (1) & (2), we get

$$H(Y/X) = \frac{1}{4}\log_2(2) + \frac{1}{4}\log_2(2) + \frac{1}{8}\log_2(4) + \frac{3}{8}\log_2(4/3)$$

$$= 0.906 \text{ bits/symbol}$$ Ans

Q.7. A source produces sequence of symbols having the following probabilities.

Symbol	A	B	C	D	E
probability	0.25	0.25	0.2	0.15	0.15.

Construct binary code using Shannon - Fano Elias Coding Procedure and find its Average length and efficiency.

Ans :-

Symbol	Probability	CDF $F(x)$	M. CDF $\bar{F}(x)$	Binary $\bar{F}(x)$	Codeword length $l(x) = \lceil \log \frac{1}{P(x)} \rceil + 1$	Codeword.
A	0.25	0.25	0.125	0.001000	3	001
B	0.25	0.50	0.375	0.011000	3	011
C	0.2	0.70	0.6	0.100100	4	1001
D	0.15	0.85	0.775	0.110001	4	1100
E	0.15	1.0	0.925	0.111011	4	1110

Average Length is given as

$$\bar{L} = \sum_x p(x)\, l(x)$$

$$\Rightarrow \bar{L} = 0.25 \times 3 + 0.25 \times 3 + 0.2 \times 4 + 0.15 \times 4 + 0.15 \times 4$$

$$\Rightarrow \bar{L} = 3.5 \text{ bits/symbol}$$

Entropy of source $H(X)$ is

$$H(X) = \sum_x p(x) \log_2 \frac{1}{P(x)}$$

$$\Rightarrow H(X) = \left(0.25 \log_2 \frac{1}{0.25}\right) \times 2 + 0.2 \log_2 \frac{1}{0.2} + 2 \times 0.15 \log_2\left(\frac{1}{0.15}\right)$$

$$H(X) = 1.875 \text{ bits/symbol.}$$

We know that,

$$\text{Efficiency} = \frac{H(x)}{L}$$

$$\Rightarrow \eta = \frac{1.875}{3.5} = 53.571\ \% \quad \text{Ans}$$

Q.8. State and Prove Asymptotic Equipartition Theorem

Ans:- In Information Theory, the analog of the law of large numbers is the Asymptotic Equipartition property (AEP). It is direct consequence of the weak law of large numbers. The law of large numbers states that for independent, identically distributed (i.i.d) random variables, $\frac{1}{n}\sum_{i=1}^{n} X_i$ is close to its expected value EX for large values of n.

The AEP states that $\frac{1}{n}\log\frac{1}{P(X_1, X_2 \cdots X_n)}$ is close to the entropy H, where $X_1, X_2 \cdots X_n$ are i.i.d random variables and $P(X_1, X_2 \cdots X_n)$ is the probability of observing the sequence $X_1 X_2 \cdots X_n$. Thus the probability $P(X_1 X_2 \cdots X_n)$ assigned to be an observed sequence will be close to 2^{-nH}.

Proof :—

Functions of independent random variables are also independent random variable. Thus, since the X_i are i.i.d, so are $\log P(X_i)$.

Hence by the weak law of large numbers,

$$-\frac{1}{n}\log P(X_1, X_2 \cdots X_n) = -\frac{1}{n}\sum_i \log P(X_i)$$

$$\rightarrow -E\log P(X) \text{ in probability}$$

$$= H(X)$$

which proves the theorem.

Q.9. What is Kraft Inequality ? What is the need of this Inequality ?

Ans:—

For any instantaneous code (prefix code) over an alphabet of size D, the codword lengths, $l_1, l_2 \ldots l_m$ must satisfy the inequality

$$\sum_i D^{-l_i} \leq 1. \quad \text{—— (1)}$$

Conversely, given a set of codeword lengths that satisfy this inequality, there exists an instantaneous code with these word lengths.

The above eq. (1) is known as <u>Kraft Inequality</u>.

We wish to construct instantaneous code of minimum expected length to describe a given source. It is clear that we cannot assign short codewords to all source symbols and still be prefix free.

In other words, all prefix codes satisfy Kraft McMillan Inequality, but all codes satisfying Kraft-McMillan inequality need not be prefix.

So any code word set that satisfies the prefix condition has to satisfy the Kraft inequality and that the Kraft inequality is a sufficient condition for the existence of a codword set with the specified set of codeword lengths.

Q.10. Explain the need for Source Coding in Communication System and Discuss about Compact Codes.

Ans:- Bandwidth is a very scarce resource in any of the communication systems. Thus it is always required to represent data with minimum number of bits. Hence, the source encoding also has to assure a better compression. Source encoding is, therefore, a process of assigning a minimal-length digital sequence to each of the information symbols emitted by the source. The codewords can be of either fixed or variable length.

In fixed length coding, all symbols are assigned the codewords of the same lengths.

However, in variable length coding, the length of the codewords is different for different symbols. It is advantageous in comparison to fixed length coding as we can assign a shorter codeword for the symbols that are more likely to occur compared to the rarely occuring symbols. Thus, a better compression can be achieved in such codes.

Compression algorithms are broadly classified as either lossy or lossless depending upon the applications.

In a digital Communication system, source encoding is the stage that follows the information source, the output of which is given to the source encoder. There is a needs to represent each information symbol in terms of a bit sequence called a codeword, to achieve communication. This codeword is termed as Compact Code.

For a binary encoder, the codeword takes the symbols (0,1); for ternary code it is (0,1,2); for quaternary it is (0,1,2,3) and so on.

Section C

C. Attempt all the parts.

Q. 11 Compare the Huffman and Shannon-Fano Elias Coding Algorithm for Data Compression

(i) Construct an optimal instantaneous code for the symbol probability given below:-

Symbols	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
Probability	0.25	0.15	0.15	0.10	0.10	0.09	0.06	0.06	0.04

(ii) Compute the efficiency of the code you have constructed.

Ans :-

Huffman Code	Shannon-Fano Elias
1. High Efficiency	Low Efficiency.
2. Not applicable in real time	Applicable in real time application.
3. Probability has to be arranged in descending order.	No need to arrange in descending order.
4. Time lagging	No time lag.
5. More storage is required	less storage required.

Entropy of Source X is given as

$$H(x) = \sum_{x} p(x) \log_2 \frac{1}{p(x)}$$

$$H(X) = -\left[0.25 \log_2(0.25) + 0.15 \times 2 \times \log_2(0.15) + 2 \times 0.10 \log_2(0.10) + 2 \times 0.06 \log_2(0.06) + 0.04 \log_2(0.04) + 0.09 \log_2(0.09)\right]$$

$$= \underline{2.971 \text{ bits / symbol}}$$

Optimal instantaneous Code is Huffman Coding.

Codeword	Symbols	Probability	Stage I	Stage II	III	IV	V	VI	VII
(01)	x_1	0.25	0.25	0.25	0.25	0.25 01	0.30 00	0.45 1	0.55 (0)
(000)	x_2	0.15	0.15	0.15 000	0.20 11	0.25 10	0.25 01	0.30 00	0.45 (1)
(001)	x_3	0.15	0.15	0.15 001	0.15 000	0.20 11	0.25 10	0.25 01	
(101)	x_4	0.10	0.10 101	0.15 100	0.15 001	0.15 000	0.20 11		
(110)	x_5	0.10	0.10 110	0.10 101	0.15 100	0.15 001			
(1000)	x_6	0.09 1000	0.10 111	0.10 110	0.10 101				
(1001)	x_7	0.06 1001	0.09 1000	0.10 111					
(1110)	x_8	0.06 1110	0.06 1001						
(1111)	x_9	0.04 1111							

Symbol	Codeword	length $l(x)$
x_1	01	2
x_2	000	3
x_3	001	3
x_4	101	3
x_5	110	3
x_6	1000	4
x_7	1001	4
x_8	1110	4
x_9	1111	4.

Average Codeword length $\bar{L} \rightarrow$

$$\bar{L} = \sum_x p(x)\, l(x)$$

$$\bar{L} = 0.25 \times 2 + (2 \times 0.15 \times 3) + (2 \times 0.10 \times 3) + 0.09 \times 4 + (2 \times 0.06 \times 4) + 0.04 \times 4 = 0.50 + 0.9 + 0.60 + 0.36 + 0.48 + 0.16$$

$$\Rightarrow \bar{L} = 3 \text{ bits / symbol}$$

$$\eta = \frac{H(x)}{\bar{L}} = \frac{2.971}{3} = 99.33\,\% \quad \text{Ans}$$

Q.12. Two Noisy channels are cascaded whose channel matrices are given by,

$$P(y_j/x_i) = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} \qquad P(z_j/y_i) = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

with $p(x_1) = p(x_2) = 1/2$. Show that $I(X;Y) > I(X;Z)$.

Ans:- Given.

$$P(Y/X) = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} \qquad P(Z/Y) = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}.$$

$P(x_1) = P(x_2) = 1/2$.

To prove $I(X;Y) \geq I(X;Z)$.

find $I(X;Y)$ i.e. $I(X;Y) = H(X) + H(Y) - H(X,Y)$ —(1)

$$H(X) = \sum_X p(x) \log_2 \frac{1}{P(X)} = \left(\frac{1}{2}\log_2 2\right) \times 2 = \underline{1 \text{ bits/symbol}}.$$

$$P(X,Y) = P(X) \cdot P(Y/X) = \begin{bmatrix} 1/8 & 1/4 & 1/8 \\ 1/4 & 1/8 & 1/8 \end{bmatrix}.$$

$$P(Y_1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}, \quad P(Y_2) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}, \quad P(Y_3) = \frac{1}{8} + \frac{1}{8} = 1/4.$$

$$H(Y) = \sum_Y P(y) \log_2\left[\frac{1}{P(Y)}\right] = \left[\frac{3}{8}\log_2 \frac{8}{3}\right] \times 2 + \frac{1}{4}\log_2 4.$$

$$H(Y) = \frac{3}{4}[3 - 1.585] = \underline{1.5613} \text{ bits/symbol}.$$

$$H(X,Y) = \sum P(x,y) \log_2 \frac{1}{P(X,Y)} = \left[\frac{1}{8}\log_2 8\right] \times 4 + \left[\frac{1}{4}\log_2 4\right] \times 2.$$

$$= \frac{3}{2} + 1 = \underline{2.5} \text{ bits/symbol}.$$

Substituting $H(X)$, $H(Y)$ & $H(X,Y)$ in eq. (1), we get

$$I(X;Y) = 1 + 1.5613 - 2.5 = \underline{0.061} \text{ bits/symbol}.$$

(2)

Now to find $I(X;Z)$, first we have to find $P(Z/X)$

i.e. $P(Z/X) = P(Y/X) \cdot P(Z/Y)$.

$$\Rightarrow P(Z/X) = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

$$P(Z/X) = \begin{bmatrix} (1/12 + 2/6) & (2/12 + 1/12) & (1/6 + 2/12) \\ (1/6 + 2/12) & (2/6 + 1/12) & (1/12 + 2/12) \end{bmatrix}$$

$$P(Z/X) = \begin{bmatrix} 5/12 & 3/12 & 4/12 \\ 4/12 & 5/12 & 3/12 \end{bmatrix}$$

$$P(X,Z) = P(X)\, P(Z/X) = \begin{bmatrix} 5/24 & 3/24 & 4/24 \\ 4/24 & 5/24 & 3/24 \end{bmatrix}.$$

$$P(Z_1) = \frac{5}{24} + \frac{4}{24} = \frac{9}{24} \qquad P(Z_2) = \frac{3}{24} + \frac{5}{24} = \frac{8}{24} \qquad P(Z_3) = \frac{4}{24} + \frac{3}{24} = \frac{7}{24}.$$

$$H(Z) = \frac{9}{24}\log_2\left(\frac{24}{9}\right) + \frac{8}{24}\log_2\left(\frac{24}{8}\right) + \frac{7}{24}\log_2\left(\frac{24}{7}\right)$$

$$H(Z) = 0.531 + 0.528 + 0.519 = \underline{\underline{1.578 \text{ bits/symbol}}}$$

$$H(X,Z) = \left(\frac{5}{24}\log_2\frac{24}{5}\right)\times 2 + \left(\frac{3}{24}\log_2\frac{24}{3}\right)\times 2 + \left(\frac{4}{24}\log_2\frac{24}{4}\right)\times 2.$$

$$H(X,Z) = 0.943 + 0.75 + 0.862$$

$$H(X,Z) = \underline{\underline{2.555 \text{ bits/symbol}}}.$$

$$I(X;Z) = H(X) + H(Z) - H(X,Z)$$

$$I(X;Z) = 1 + 1.578 - 2.555 = \underline{\underline{0.023 \text{ bits/symbol}}} \quad (3)$$

From equations (2) & (3)

$$I(X;Y) > I(X;Z)$$

<u>Hence proved</u>

— END —