Ajay Kumar Garg Engineering College, Ghaziabad
Department of Applied Sciences & Humanities
Model solution Sessional Test-II

Course: B.Tech
Session: 2017-18
Subject: Engg. Mathematics-I

Semester: I
Section: All
Sub. Code: RAS-103

Section-A

Q1→ Find approximate value of $\left[(0.98)^2+(2.01)^2\right]^{1/2}$

Sol.→ Let $f(x,y)=(x^2+y^2)^{1/2}$ —— ①

Taking $x=1,\ \delta x=-0.02,\ y=2,\ \delta y=0.01$

from ①, $\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$

$= (x^2+y^2)^{-1/2}(x\delta x + y\delta y) = \frac{1}{\sqrt{5}}(0) = 0$

$\therefore \left[(0.98)^2+(2.01)^2\right] = f(1,2)+\delta f = \sqrt{5}+0=\sqrt{5}$

Q2→ If the base radius and height of a cone are measured as 4 and 8 inches with a possible error of 0.04 and 0.08 inches respectively, calculate the Percentage (%) error in calculating volume of the cone.

Sol.→ Volume $V=\frac{1}{3}\pi r^2 h$, or $\log V = \log\frac{\pi}{3}+2\log r+\log h$

diff. $\frac{\delta V}{V} = 2\frac{\delta r}{r}+\frac{\delta h}{h} = 2\left(\frac{0.04}{4}\right)+\left(\frac{0.08}{8}\right) = 0.03$

$\therefore$ Percentage error in volume $= 0.03\times 100 = 3\%$

Q3→ Investigate for consistency of the following equations:

$4x-2y+6z=8,\ x+y-3z=-1,\ 15x-3y+9z=21$

Sol.→ Augmented matrix $[A:B] = \left[\begin{array}{ccc:c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$

Apply row transformations, to reduce it to echelon form, we get,

$$[A:B] = \begin{bmatrix} 1 & 1 & -3 & \vdots & -1 \\ 0 & -6 & 19 & \vdots & 12 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$\therefore \rho(A:B) = \rho(A) = 2 < 3$ (No. of variables)

Hence the given system of equations are consistent and ~~possess~~ have infinitely many solutions.

Q4→ If rank of the matrix $A = \begin{bmatrix} 4 & -2 & -2 \\ -5 & 3 & 1 \\ +1 & -1 & b \end{bmatrix}$ is 2, then find the value of b.

Sol. → $\rho(A) = 2 \Rightarrow |A| = 0 \Rightarrow \begin{vmatrix} 4 & -2 & -2 \\ -5 & 3 & 1 \\ 1 & -1 & b \end{vmatrix} = 0.$

$\Rightarrow 4(3b+1) + 2(-5b-1) - 2(5-3) = 0$

$\Rightarrow 2b - 2 = 0 \Rightarrow \boxed{b = 1}$

Q5→ If the vectors $(0,1,a)$, $(1,a,1)$ and $(a,1,0)$ are linearly dependent, then find the value of a.

Sol→ Coff. matrix $A = \begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix}$

The given vectors are Linearly dependent if
$\rho(A) <$ no. of unknowns (=3)

Now $\rho(A) < 3$ is Possible only if $|A| = 0$

Hence $|A| = \begin{vmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix} = 0$

$\Rightarrow 2a - a^3 = 0$

$\Rightarrow \boxed{a = 0, \pm\sqrt{2}}$

Section-B

6). In a plane triangle ABC, find the maximum value of $\cos A \cos B \cos C$.

Solⁿ In a plane triangle ABC, $A+B+C=\pi$

or $A+B=\pi-C$ or $C=\pi-(A+B)$

$\therefore \cos A\cos B\cos C = \cos A\cos B\cos[\pi-(A+B)] = -\cos A\cos B\cos(A+B)$

Let $f(A,B) = -\cos A\cos B\cos(A+B)$

$\therefore \frac{\partial f}{\partial A} = \cos B\sin(2A+B)$; $\frac{\partial f}{\partial B} = \cos A\sin(2B+A)$

$\frac{\partial^2 f}{\partial A^2} = r = 2\cos B\cos(2A+B)$

$s = \frac{\partial^2 f}{\partial A\partial B} = \cos(2A+2B)$

$t = 2\cos A\cos(A+2B)$.

for maxima & minima $\frac{\partial f}{\partial A}=0$ & $\frac{\partial f}{\partial B}=0$

$\cos B\sin(2A+B)=0$ — ① & $\cos A\sin(2B+A)=0$ — ②

in ① if $\cos B=0$ then $B=\pi/2$

$\therefore$ from ② $\cos A\sin(\pi+A)=0$

or $-\sin A\cos A=0$

i.e. $\cos A=0 \Rightarrow A=\pi/2$

or $\sin A=0 \Rightarrow A=0$

Which is not possible.

$\therefore$ $\cos B\neq 0$. Similarly $\cos A\neq 0$ in eqⁿ ②.

$\therefore$ $\sin(2A+B)=0 \Rightarrow 2A+B=\pi$ — ③

& $\sin(2B+A)=0 \Rightarrow 2B+A=\pi$ — ④

solving ③ & ④ we get $A=\pi/3=B$

$\therefore r=-1,\ s=-\frac{1}{2},\ t=-1$

$\therefore rt-s^2=\frac{3}{4}>0$ i.e. +ive. & $r<0 \Rightarrow f(A,B)$ is maximum at $A=B=\pi/3$.

Maximum value $f(\pi/3,\pi/3)=\frac{1}{8}$ Ans.

7.). Expand x^y in powers of (x-1) and (y-1) up to the third degree terms and hence evaluate $(1.1)^{1.02}$

Solⁿ Expansion of $f(x,y)$ in powers of $(x-a)$ & $(y-b)$ is given by-

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y} + \frac{1}{\lfloor 2}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] + \frac{1}{\lfloor 3}\left[(x-a)^3\frac{\partial^3 f(a,b)}{\partial x^3} + \cdots\right] + \cdots$$

here $f(x,y) = x^y$

$x=1,\ y=1$

$f(x,y)$	x^y	1
$f_x(x,y)$	yx^{y-1}	1
$f_y(x,y)$	$x^y \log x$	0
$f_{xx}(x,y)$	$y(y-1)x^{y-2}$	0
$f_{yx}(x,y)$	$x^{y-1} + yx^{y-1}\log x$	1
$f_{yy}(x,y)$	$x^y(\log x)^2$	0
$f_{xxx}(x,y)$	$y(y-1)(y-2)x^{y-3}$	0
$f_{xxy}(x,y)$	$(y-1)x^{y-2} + y(y-1)x^{y-2}\log x + yx^{y-2}$	1
$f_{xyy}(x,y)$	$yx^{y-1}(\log x)^2 + 2x^{y-1}\log x$	0
$f_{yyy}(x,y)$	$x^y(\log x)^3$	0

$$\therefore f(x,y) = x^y = 1 + (x-1) + 0 + \frac{1}{\lfloor 2}[0 + 2(x-1)(y-1) + 0] + \frac{1}{\lfloor 3}[3(x-1)^2(y-1)] + \cdots$$

$$= 1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + \cdots$$

Ans.

$\therefore f(1.1, 1.02) = 1.1021$

8). **Find the characteristic equation of the matrix** $A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$ **and hence, compute** A^{-1}**. Also find the matrix represented by** $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$.

Solⁿ The ch. eqⁿ. of A is $|A-\lambda I|=0$

or $\lambda^3-5\lambda^2+7\lambda-3=0$.

by Cayley-Hamilton theorem, $A^3-5A^2+7A-3I=0$ — ①

Pre-multiplying ① by A^{-1}, we get

$A^2-5A-7I-3A^{-1}=0$

or $A^{-1}=\frac{1}{3}(A^2-5A+7I)$

Now $A^2=A.A.=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{3}\begin{bmatrix}2&-1&-1\\0&3&0\\-1&-1&2\end{bmatrix}$ Ans.

Now $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$.

$=A^5(A^3-5A^2+7A-3I)+A(A^3-5A^2+7A-3I)+A^2+A+I$

$=A^2+A+I$ (using ①)

$=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}$ Ans.

9). If $u = x + 2y + z,\ v = x - 2y + 3z,\ w = 2xy - xz + 4yz - 2z^2,$ check whether they are dependent or not. Find the relation between them if possible.

Solⁿ.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix} = 4x - 8y + 12z - 4x + 8y - 12z = 0.$$

$\therefore$ u, v & w are not independent $\left(\because \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0\right)$

Now

$$u + v = 2x + 4z$$

$$u - v = 4y - 2z$$

$$\therefore (u+v)(u-v) = 4(2xy + 4yz - zx - 2z^2)$$

or $\quad u^2 - v^2 = 4w$ <u>Ans.</u>

Q10). Determine the values of λ & μ such that the system $2x-5y+2z=8$, $2x+4y+6z=5$ & $x+2y+\lambda z=\mu$ has (i) no solution (ii) a unique solution (iii) infinite number of solutions. Find all possible solⁿ.

Solⁿ Augmented matrix $[A:B] = \begin{bmatrix} 2 & -5 & 2 & : & 8 \\ 2 & 4 & 6 & : & 5 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & -5 & 2 & : & 8 \\ 0 & 9 & 4 & : & -3 \\ 0 & 9 & 2\lambda-2 & : & 2\mu-8 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to 2R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & -5 & 2 & : & 8 \\ 0 & 9 & 4 & : & -3 \\ 0 & 0 & 2\lambda-6 & : & 2\mu-5 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Case I If $\lambda = 3$, $\mu \neq 5/2$.

$\rho(A) = 2$, $\rho[A:B] = 3$.

$\therefore \rho[A] \neq \rho[A:B]$

$\therefore$ The system has no solution.

Case II If $\lambda \neq 3$ & μ have any value.

then $\rho(A) = \rho[A:B] = 3 =$ number of unknowns

$\therefore$ the system has unique solⁿ.

Case III If $\lambda = 3$, $\mu = 5/2$

$\rho[A] = \rho[A:B] = 2 <$ number of unknowns.

$\therefore$ the system has an infinite number of solⁿ.

possible solⁿ in Case III is —

equivalent System $\begin{bmatrix} 2 & -5 & 2 \\ 0 & 9 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 0 \end{bmatrix}$ let $z = k$

$2x - 5y + 2z = 8$

$9y + 4z = -3 \Rightarrow y = \frac{-3-4k}{9}$

$2x = 8 - 5\frac{(3+4k)}{9} - 2k$

$x = 4 - \frac{5}{18}(3+4k) - k.$

In Case II If $\lambda \neq 3$ & μ have ~~any~~ any value.

then · Equivalent System

$$\begin{bmatrix} 2 & -5 & 2 \\ 2 & 4 & 6 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ \mu \end{bmatrix}$$

Can be written as $\begin{bmatrix} 2 & -5 & 2 \\ 0 & 9 & 4 \\ 0 & 0 & 2\lambda-6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 2\mu-5 \end{bmatrix}$

$$z = \frac{2\mu-5}{2\lambda-6} = \frac{\mu-5/2}{\lambda-3} \quad (\lambda \neq 3)$$

$$9y+4z = -3$$

$$\therefore 9y = -3 - 4\left(\frac{\mu-5/2}{\lambda-3}\right)$$

$$\therefore y = -\frac{1}{3} - \frac{4}{9}\left(\frac{\mu-5/2}{\lambda-3}\right)$$

$$2x-5y+2z = 8$$

$$2x = 8+5y-2z$$

$$x = 4+\frac{5}{2}y-z$$

$$= 4 - \frac{5}{2}\left[\frac{1}{3}+\frac{4}{9}\left(\frac{\mu-5/2}{\lambda-3}\right)\right] - \frac{\mu-5/2}{\lambda-3} = \frac{19}{3}\left[\frac{1}{2}-\frac{1}{3}\left(\frac{\mu-5/2}{\lambda-3}\right)\right]$$

for given $\lambda (\neq 3)$ & μ we get unique value of x, y & z.

Section - C

Q 11) Find the dimensions of a rectangular box of maximum capacity whose surface area is given when (i) box is open at the top (ii) box is closed.

Sol. → Let x, y, z be the dimensions of the rectangular box, so volume $V = xyz$, — (1)

Total surface area $S = nxy + 2yz + 2zx$ = given const. — (2)

Here (i) $n=1$ if box is open at the top
(ii) $n=2$ if box is closed.

∴ Lagrangian's func. $f = xyz + \lambda(nxy + 2yz + 2zx)$ — (3)

for stationary points $df = 0$,

$$\Rightarrow \quad yz + \lambda(ny + 2z) = 0 \quad \text{— (4)}$$
$$xz + \lambda(nx + 2z) = 0 \quad \text{— (5)}$$
$$xy + \lambda(2y + 2x) = 0 \quad \text{— (6)}$$

multiply eqns. (4) ~~by~~, (5), (6) by x, y, z successively and add, we get

$$3xyz + \lambda\{2(nxy + 2yz + 2zx)\} = 0$$

$$\Rightarrow 3V + \lambda(2S) = 0 \Rightarrow \boxed{\lambda = -\frac{3V}{2S}}$$

Put in (4), $yz - \frac{3V}{2S}(ny + 2z) = 0$.

$$\Rightarrow yz - \frac{3xyz}{2S}(ny + 2z) = 0$$

$$\Rightarrow yz\left[1 - \frac{3zx(ny+2z)}{2S}\right] = 0 \Rightarrow nxy + 2xz = \frac{2S}{3} \quad \text{— (7)}$$

Similarly from (5) & (6), we obtain

$$nxy + 2yz = \frac{2S}{3} \quad \text{— (8)}$$

and $$2yz + 2xz = \frac{2S}{3} \quad \text{— (9)}$$

Subtracting ⑧ from ⑦, we get,

$$2z(x-y)=0 \Rightarrow \boxed{x=y} \quad \text{--- (10)}$$

Subtracting ⑨ from ⑧, we get

$$\boxed{ny = 2z} \quad \text{--- (11)}$$

Substituting ⑩ & ⑪ in eqn. ②,

$$\Rightarrow nx \cdot x + 2(x)\left(\frac{nx}{2}\right) + 2\left(\frac{nx}{2}\right)x = S.$$

$$\Rightarrow 3nx^2 = S \Rightarrow \boxed{x^2 = \frac{S}{3n}}$$

(i) when box is open at the top, $n=1$

$$\therefore x^2 = \frac{S}{3} \Rightarrow \boxed{x = \sqrt{\frac{S}{3}},\ y = \sqrt{\frac{S}{3}},\ z = \frac{1}{2}\sqrt{\frac{S}{3}}}$$

(ii) when box is closed, $n=2$.

$$\therefore x^2 = \frac{S}{6} \Rightarrow \boxed{x = \sqrt{\frac{S}{6}},\ y = \sqrt{\frac{S}{6}},\ z = \sqrt{\frac{S}{6}}}$$

Q12) Diagonalise the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Sol → $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

The charecteristic equation of A are

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$, solving we get $\boxed{\lambda = 1, \sqrt{5}, -\sqrt{5}}$

(i) for $\lambda_1 = 1$, let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be eigen vector s.t.

$$[A - \lambda_1 I] X_1 = 0.$$

$$\Rightarrow \begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0 \xrightarrow[R_3 + R_1]{R_1 + 2R_2} \begin{bmatrix} 0 & 4 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$$\Rightarrow 4y_1 = 0 \Rightarrow \boxed{y_1 = 0}$$

$$x_1 + y_1 + z_1 = 0 \Rightarrow x_1 + z_1 = 0.$$

let $z_1 = k_1$, $x_1 = -k_1$

$$\therefore X_1 = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(ii) for $\lambda_2 = \sqrt{5}$, let $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be eigen vector s.t

$$[A - \lambda_2 I] X_2 = 0.$$

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$$\xrightarrow[R_3 + R_2]{R_1 + 2R_2} \begin{bmatrix} 1-\sqrt{5} & 6-\sqrt{5} & 0 \\ 1 & 2-\sqrt{5} & 1 \\ 0 & 1-\sqrt{5} & 1-\sqrt{5} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$$\xrightarrow[\frac{1}{1-\sqrt{5}} R_3]{\frac{1}{1-\sqrt{5}} R_1} \begin{bmatrix} 1 & 1-\sqrt{5} & 0 \\ 1 & 2-\sqrt{5} & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0 \qquad \left(\because 6-\sqrt{5} = (1-\sqrt{5})^2\right)$$

$$\Rightarrow \left.\begin{aligned} x_2 + (1-\sqrt{5}) y_2 &= 0 \\ x_2 + (2-\sqrt{5}) y_2 + z_2 &= 0 \\ y_2 + z_2 &= 0 \end{aligned}\right\} \quad \text{let } z_2 = k_2,\ y_2 = -k_2 \Rightarrow x_2 = (1-\sqrt{5}) k_2.$$

$$\therefore X_2 = \begin{bmatrix} (1-\sqrt{5}) k_2 \\ -k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1-\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$$

Simillarly for $\lambda = -\sqrt{5}$

eigen vector $X_3 = k_3 \begin{bmatrix} 1+\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ modal matrix, $m = \begin{bmatrix} -1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$\therefore m^{-1} = \dfrac{\text{Adj}\, m}{|m|} = \dfrac{1}{2\sqrt{5}} \begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ -1 & -2-\sqrt{5} & -1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}$

Now

$$m^{-1}Am = \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ -1 & -2-\sqrt{5} & -1 \\ 1 & 2-\sqrt{5} & +1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ -\sqrt{5} & -5-2\sqrt{5} & -\sqrt{5} \\ -\sqrt{5} & 5-2\sqrt{5} & -\sqrt{5} \end{bmatrix} \begin{bmatrix} -1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix} = D.$$