EE 605-A / CS 505-A: Probability and Stochastic Processes I

Stevens Institute of Technology

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Homework 1 - 200 points

1) A group of 10 gray pigeons named *GP_0, GP_1, …., GP_i, …, GP_9* (where *i* takes on integer values from 0 to 9) in a town home *TH1* in Hoboken, NJ have 4 gray pigeon holes *GPH_1, GPH_2, GPH_3,* and *GPH_4* for feeding in different corners of the home. Each gray pigeon hole can allow only one pigeon to occupy the space available in the hole. (40 points)
   a. Find the number of ways in which these 10 gray pigeons can occupy these 4 pigeon holes.
   b. If the gray pigeon *GP_7* occupies the first pigeon hole *GPH_1*, find the number of ways in which the remaining pigeons can occupy the remaining 3 holes.
   c. If the gray pigeon *GP_7* occupies the second pigeon hole *GPH_2*, find the number of ways in which the remaining pigeons can occupy the remaining 3 holes.
   d. If gray pigeon *GP_7* occupies the third pigeon hole *GPH_3*, find the number of ways in which the remaining pigeons can occupy the remaining 3 holes.
   e. If gray pigeon *GP_7* occupies the fourth pigeon hole *GPH_4*, find the number of ways in which the remaining pigeons can occupy the remaining 3 holes.
   f. Find the number of ways in which the gray pigeon *GP_7* occupies at least one of the 4 pigeon holes in two ways:
      i. Use the results from parts b through e to find your answer to part f.
      ii. Find the number of ways in which the gray pigeon *GP_7* does not occupy any of the 4 pigeon holes. Hence find the answer to part f.
   g. Find the probability that the gray pigeon *GP_7* occupies at least one of the 4 pigeon holes.

2. A group of 9 yellow pigeons named *YP_0, YP_1, …., YP_j, …, YP_8* (where *j* takes on integer values from 0 to 9) in an adjacent town home *TH2* have 3 yellow pigeon holes *YPH_1, YPH_2,* and *YPH_3* for feeding in different corners of the home. Each yellow pigeon hole can allow only one pigeon to occupy the space available in the hole. (5 points)
   a. Find the number of ways in the which the yellow pigeons can occupy the 3 yellow pigeon holes in their home.

3. The town homes *TH1* and *TH2* share a wall. A door in the wall is opened so that each of the pigeons can fly and roam around in either of the two town homes and occupy any of the 7 pigeon holes now available to them. (25 points)
   a. Find the number of ways in which the pigeons in both homes can occupy the 7 pigeon holes available to them.
   b. In this context, find the probability that the gray pigeons continue to occupy the gray pigeon holes, and the yellow pigeons continue to occupy the yellow pigeon holes.
   c. Alternatively, in this context, find the probability that the gray pigeons occupy the yellow pigeon holes, and that the yellow pigeons occupy the gray pigeon holes.

4. A group of 8 lavender pigeons named *LP_0, LP_1, ….., LP_k, ….., LP_7* (where *k* takes on integer values from 0 to 7) in a neighboring home decide to host a neighborhood pigeon-mingling party and they invite the gray pigeons and the yellow pigeons to the party. All pigeons are asked to wear number tags to the party. The gray pigeons decide to wear a number tag at the party where the tag simply takes on values $GT(i) = (i+1)$. The yellow pigeons decide to make it a little bit more interesting and wear a number tag at the party where the tag takes on values $YT(j) = (2j+1)$. The lavender pigeons go crazy and decide to wear a number tag at the party where the tag takes on values $LT(k) = 3^k$. Here, *i, j, k* are the aforementioned subscripts to the names for the respective pigeons. The pigeons play a game at the party to see if any pigeons share the same number tag. (total 70 points: 10 points for a, 30 points for b-g, 30 points for h)
   a. How many triplet groups of pigeons consisting of a gray pigeon, a yellow pigeon and a lavender pigeon share the same number tag? What is the number tag value that each such triplet group shares?
   b. How many pairs of pigeons consisting of a gray pigeon and a yellow pigeon share the same number tag (excluding the triplet groups that share the same number)?
   c. How many pairs of pigeons consisting of a gray pigeon and a lavender pigeon share the same number tag (excluding the triplet groups that share the same number)?
   d. How many pairs of pigeons consisting of a yellow pigeon and a lavender pigeon share the same number tag (excluding the triplet groups that share the same number)?
   e. How many gray pigeons have a number tag that is not shared with either a yellow pigeon or a lavender pigeon?
   f. How many yellow pigeons have a number tag that is not shared with either a gray pigeon or a lavender pigeon?
   g. How many lavender pigeons have a number tag that is not shared with either a gray pigeon or a yellow pigeon?
   h. At the end of the party, the pigeons decide to form a neighborhood committee of 3 pigeons to plan for future get-togethers with one representative from the gray pigeons, one representative from the yellow pigeons, and one representative from the lavender pigeons. The probability of selection of a pigeon from each of the groups is chosen to be proportional within each group to the number tag that each pigeon is wearing in that group.
      i. Let each of the gray pigeons have a probability of selection within the group of gray pigeons given by $PG(i) = \alpha\, GT(i)$ where α is a constant of proportionality. Find α. *(Please note that the sum of probabilities should be equal to 1.)*
      ii. Let each of the yellow pigeons have a probability of selection within the group of yellow pigeons given by $PY(j) = \beta\, YT(j)$ where β is a constant of proportionality. Find β. *(Please note that the sum of probabilities should be equal to 1.)*
      iii. Let each of the lavender pigeons have a probability of selection within the group of lavender pigeons given by $PL(k) = \gamma\, LT(k)$ where γ is a constant of proportionality. Find γ. *(Please note that the sum of probabilities should be equal to 1.)*

5. Let the set A = {1, 2, 3, 4, 5, 6, 7, 8}. Let the set B = {1, 2, 3, 4, 5, 6, 7, 8} as well. (total 60 points)
    a. Create the cartesian product set C = A x B.
        i. How many elements consisting of tuples (a,b) are in the set C = A x B, where a ε A, and b ε B?
        ii. If we compute the sum of the values (a + b) for these tuples, what is the range of such values?
        iii. Show a matrix with an ordering of the elements in the set C such that all the tuples (a,b) share the same sum value (a + b) in the off-diagonals. (10 points)
    b. Create a different set D represented by elements a/b where a ε A, and b ε B.
        i. How many elements consisting of elements (a/b) are in the set D where a ε A, and b ε B?
        ii. If we compute the sum of the values (a + b) for these elements in the set D, what is the range of such values?
        iii. Show a matrix with an ordering of the elements in the set D such that the elements (a/b) share the same sum value (a + b) in the off-diagonals. (10 points)
    c. Now, let us extend both sets A and B, and let A consist of the set of natural numbers N, and let B consist of the set of natural numbers N as well.
        i. Construct the cartesian product set E = A x B. Explain in your own words, why the set E is countably infinite.
        ii. Construct the set F with elements (a/b) where a ε A, and b ε B. Explain in your own words, why the set F is countably infinite.
    d. Now, let us further extend set A, and let A consist of the set of integers Z. Let set B continue to consist of the set of natural numbers N.
        i. Find a mapping between elements of set A with the elements of set B, to show that A is countably infinite.
        ii. Construct the set Q of rational numbers with elements (a/b) where a ε A, and b ε B. Explain in your own words, why the set Q is countably infinite.