MATH 112A Homework 1 - October 10, 2025

Benjamin Tong

1 Monday

1.1 Problem 1

Solve the following PDE for $u(x, t)$:

$$4u_t + 7u_x = 0,\ u(x, 0) = e^x.$$

Use the method based on directional derivative. Show all the details.

Solution First, we rewrite the PDE as $7u_x + 4u_t = 0$. Then, we note that $7u_x + 4u_t = Du \cdot \langle 7, 4 \rangle$. So, the solution to $u(x, t)$ must be something of the form

$$u(x, t) = f(4x - 7t)$$

Since our constraint is that $u(x, 0) = e^x$, we find that

$$\begin{aligned} u(x, 0) &= f(4x - 0t) \\ e^x &= f(4x) \\ f(4x) &= e^x \\ f\left(4\left(\frac{x}{4}\right)\right) &= e^{\frac{x}{4}} \\ f(x) &= e^{\frac{x}{4}} \end{aligned}$$

Therefore, $u(x, t) = e^{\frac{(4x-7t)}{4}}$ is the solution.

1.2 Problem 2

Solve the following PDE for $u(x, t)$:

$$-2u_t + 11u_x = 0, u(x, 0) = \sin 2x$$

Use the method based on the change of coordinates. Show all the details.

Solution First, we rewrite the PDE as $11u_x - 2u_t = 0$. Then, we define the variables x' and t' as

$$\begin{aligned} x' &= -2x + 11t \\ t' &= 11x + 2t \end{aligned}$$

Now, we can rewrite both u_x and u_t in terms of x' and t' instead using the chain rule. We find that

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} \\ &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial x} \\ &= -2u_{x'} + 11u_{t'} \end{aligned}$$

and

$$\begin{aligned} u_t &= \frac{\partial u}{\partial t} \\ &= \frac{\partial u}{\partial x'}\frac{\partial t'}{\partial t} + \frac{\partial u}{\partial t'}\frac{\partial x'}{\partial t} \\ &= 11u_{x'} + 2u_{t'} \end{aligned}$$

Now, we substitute this into the original equation to get

$$\begin{aligned}
-2u_t + 11u_x &= 0 \\
-2\left(-2u_{x'} + 11u_{t'}\right) + 11\left(11u_{x'} + 2u_{t'}\right) &= 0 \\
4u_{x'} - 22u_{t'} + 121u_{x'} + 22u_{t'} &= 0 \\
125u_{x'} &= 0 \\
u_{x'} &= 0
\end{aligned}$$

Integrating this gives us with respect to x' gives us $u = f(t') = f(11x + 2t)$. Now, since our constraint is $u(x, 0) = \sin 2x$ this means that

$$\begin{aligned}
u(x, 0) &= f(11x + 2(0)) \\
\sin 2x &= f(11x) \\
f(11x) &= \sin(2x) \\
f\left(11\left(\frac{x}{11}\right)\right) &= \sin\left(2\left(\frac{x}{11}\right)\right) \\
f(x) = \sin\left(\frac{2}{11}x\right)
\end{aligned}$$

So, $u(x, t) = \sin\left(\frac{2}{11}(11x + 2t)\right)$ is the solution to the PDE.

2 Wednesday

2.1 Problem 1

Solve the solution $u(x, t)$ to the equation

$$u_t + u_x = 1, \; u(x, 0) = 2x^2 - 3x + 5$$

Solution We define the variables x' and t' as

$$\begin{aligned}
x' &= x + t \\
t' &= -x + t
\end{aligned}$$

We find that

$$\begin{aligned}
u_t &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial t} \\
&= u_{x'} + u_{t'}
\end{aligned}$$

and

$$\begin{aligned}
u_x &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial x} \\
&= u_{x'} - u_{t'}
\end{aligned}$$

Integrating this with respect to x' gives us $u = \frac{1}{2}x' + f(t')$ In terms of x and t, $u(x,t) = \frac{1}{2}(x+t) + f(-x+t)$. Now, since our constraint is $u(x,0) = 2x^2 - 3x + 5$, it means that

$$\begin{aligned} u(x,0) &= 2x^2 - 3x + 5 \\ \frac{1}{2}(x+0) + f(-x+0) &= 2x^2 - 3x + 5 \\ \frac{1}{2}x + f(-x) &= 2x^2 - 3x + 5 \\ f(-x) &= 2x^2 - \frac{7}{2}x + 5 \\ f(-(-x)) &= 2(-x)^2 - \frac{7}{2}(-x) + 5 \\ f(x) &= 2x^2 + \frac{7}{2}x + 5 \end{aligned}$$

Therefore, the solution to the PDE is

$$u(x,t) = \frac{1}{2}(x+t) + \left(2(-x+t)^2 + \frac{7}{2}(-x+t) + 5\right)$$

2.2 Problem 2

Solve the equation

$$(1+t^2)u_t + u_x = 0,\ u(x,0) = \frac{1}{1+x^2}$$

Solution First of all, we find the equation of the characteristic curves. The characteristic curves must follow the equation

$$\frac{dt}{dx} = \frac{1+t^2}{1}$$

This ODE has the solution

$$x = \arctan(t) + C$$

This means that $u(\arctan(t) + C, t)$ will always have the same value regardless of what t is, and this value is only dependent on C. Since $C = x - \arctan(t)$, we say that

$$u(x,t) = f(x - \arctan(t))$$

An additional constraint states that $u(x,0) = \frac{1}{1+x^2}$, so

$$\begin{aligned} f(x - \arctan(0)) &= \frac{1}{1+x^2} \\ f(x) &= \frac{1}{1+x^2} \end{aligned}$$

Therefore,

$$u(x,t) = \frac{1}{1 + (x - \arctan(t))^2}$$

is the solution to the PDE.

2.3 Problem 3

Solve the PDE $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$. First, we define the variables x' and y' as

$$\begin{aligned} x' &= x + y \\ y' &= y - x \end{aligned}$$

We find that we can rewrite u_x and u_y in terms of x' and y' like so:

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} \\ &= 1u_{x'} - 1u_{y'} \end{aligned}$$

and

$$\begin{aligned} u_y &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} \\ &= 1u_{x'} + 1u_{y'} \end{aligned}$$

Now, substituting u_x and u_y with these values we get

$$\begin{aligned} (u_{x'} - u_{y'}) + (u_{x'} + u_{y'}) + u &= e^{x+2y} \\ 2u_{x'} + u &= e^{x+2y} \end{aligned}$$

Note that $x + 2y = \frac{3}{2}x' - \frac{1}{2}y'$, so we get the equation

$$\begin{aligned} 2u_{x'} + u &= e^{\left(\frac{3}{2}x' - \frac{1}{2}y'\right)} \\ &= e^{-\frac{1}{2}y'}e^{\frac{3}{2}x'} \end{aligned}$$

Now, we can solve this like a normal ODE. We find that

$$\begin{aligned} 2u_{x'} + u &= e^{-\frac{1}{2}y'}e^{\frac{3}{2}x'} \\ u_{x'} + \frac{1}{2}u &= \frac{1}{2}e^{-\frac{1}{2}y'}e^{\frac{3}{2}x'} \\ e^{\frac{1}{2}x'}\left(u_{x'} + \frac{1}{2}u\right) &= \frac{1}{2}e^{-\frac{1}{2}y'}e^{\frac{3}{2}x'} \\ e^{\frac{1}{2}x'}u_{x'} + \frac{1}{2}e^{\frac{1}{2}x'}u &= \frac{1}{2}e^{-\frac{1}{2}y'}e^{\frac{3}{2}x'} \\ \left(e^{\frac{1}{2}x'}u\right)' &= \frac{1}{2}e^{-\frac{1}{2}y'}e^{\frac{3}{2}x'} \\ e^{\frac{1}{2}x'}u &= \int \frac{1}{2}e^{-\frac{1}{2}y'}e^{\frac{3}{2}x'}\ dx' \\ e^{\frac{1}{2}x'}u &= \frac{1}{3}e^{-\frac{1}{2}y'}e^{\frac{3}{2}x'} + g(y') \\ u &= \frac{1}{3}e^{x' - \frac{1}{2}y'} + \frac{g(y')}{e^{\frac{1}{2}x'}} \end{aligned}$$

Substituting in back the values of x' and y' in terms of x and y we get

$$\begin{aligned} u &= \frac{1}{3}e^{x' - \frac{1}{2}y'} + \frac{g(y')}{e^{\frac{1}{2}x'}} \\ &= \frac{1}{3}e^{(x+y) - \frac{1}{2}(y-x)} + \frac{g(y-x)}{e^{\frac{1}{2}x + \frac{1}{2}y}} \\ &= \frac{1}{3}e^{\frac{3}{2}x + \frac{1}{2}y} + \frac{g(y-x)}{e^{\frac{1}{2}x + \frac{1}{2}y}} \end{aligned}$$

3 Friday

3.1 Problem 1

For any non-integer real number α, derive the binomial formula:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

where the coefficient for x^n is

$$\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

3.2 Problem 2

A flexible chain of length ℓ is hanging from one end $x = 0$ but oscillates horizontally. Let the x axis point downward and the u axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.