MATH 112A Homework 0 - October 3, 2025

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1 Problem 1

Which is the following operators are linear? Provide reasons.

(1) $\mathscr{L}u = u_x + e^x u_y$

(2) $\mathscr{L}u = u_x + |u_y|$

(3) $\mathscr{L}u = u_x + u_y + 1$

(4) $\mathscr{L}u = u_t + u_{xxx}$

Solution There are two requirements that need to be fufilled in order for $\mathscr{L}$ to be linear. The first requirement is that $\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v$, and that $\mathscr{L}(cu) = c\mathscr{L}u$.

According to this defintiion, this means that only (1) and (4) are linear.

To show that (1) is linear, we have to show that both $\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v$ and that $\mathscr{L}(cu) = c\mathscr{L}u$. We find that

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_x + e^x(u+v)_y \\ &= u_x + v_x + e^x(u_y + v_y) \\ &= u_x + e^x u_y + v_x + e^x v_y \\ &= \mathscr{L}u + \mathscr{L}v\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}(cu) &= (cu)_x + e^x(cu)_y \\ &= c(u_x) + ce^x(u_y) \\ &= c(u_x + e^x u_y) \\ &= c\mathscr{L}u\end{aligned}$$

To show that (4) is linear, we have to show that both $\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v$ and that $\mathscr{L}(cu) = c\mathscr{L}u$. We find that

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_t + (u+v)_{xxx} \\ &= u_t + v_t + u_{xxx} + v_{xxx} \\ &= (u_t + u_{xxx}) + (v_t + v_{xxx}) \\ &= \mathscr{L}u + \mathscr{L}v\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}(cu) &= (cu)_t + (cu)_{xxx} \\ &= c(u_t) + c(u_{xxx}) \\ &= c(u_t + u_{xxx}) \\ &= c\mathscr{L}u\end{aligned}$$

Now, (2) and (4) are not linear since they do not meet the two criteria mentioned earlier.

2 Problem 2

For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous.

(1) $yu_x + xu_y = 0$.

(2) $u_t - u_{xx} + 1 = 0$.

(3) $u_t + u_{xxt} + uu_x = 0$.

(4) $u_{tt} - u(u_x)^3 = 0$.

(5) $u_{tt} - u_{xx} + x^2 = 0$.

Solution

(1) This is a linear homogeneous equation with order 1.

(2) This is a linear inhomogeneous equation with order 2.

(3) This is a nonlinear equation with order 3.

(4) This is a nonlinear equation with order 2.

(5) This is a linear inhomogeneous equation with order 2.

3 Problem 3

Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same g is a solution of the homogeneous equation $\mathscr{L}u = 0$.

Solution Let $\mathscr{L}$ be a linear operator and let a and b be functions that satisfy $\mathscr{L}a = g$ and $\mathscr{L}b = g$. Then, $\mathscr{L}a - \mathscr{L}b = 0$. Since $\mathscr{L}$ is a linear operator this means that $\mathscr{L}(a - b) = 0$. Therefore, $a - b$ is a solution to the homogeneous equation $\mathscr{L}u = 0$.

4 Problem 4

Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE

$$uu_{xy} = u_x u_y$$

for all pairs of (differentiable) functions f and g of one variable.

Solution If $u(x, y) = f(x)g(y)$ for some functions f and g, then

$$\begin{aligned} u_x &= f'(x)g(y) \\ u_y &= f(x)g'(y) \\ u_{xy} &= f'(x)g'(y) \end{aligned}$$

Substituting these into the PDE, we find that

$$\begin{aligned} (f(x)g(y))(f'(x)g'(y)) &= (f'(x)g(y))(f(x)g'(y)) \\ f(x)f'(x)g(y)g'(y) &= f(x)f'(x)g(y)g'(y) \end{aligned}$$

Therefore $u(x, y) = f(x)g(y)$ is a valid solution of the PDE, regardless of what $f(x)$ or $g(y)$ is.

5 Problem 5

Verify by direct substitution that

$$u_n(x, y) = \sin nx + \sinh ny$$

is a solution of the PDE:

$$u_{xx} + u_{yy} = 0$$

for every $n > 0$.

Solution We find that

$$\begin{aligned} u_{xx} &= -\sin nx \sinh ny \\ u_{yy} &= \sin nx \sinh ny \end{aligned}$$

and so

$$\begin{aligned} u_{xx} + u_{yy} &= (-\sin nx \sinh ny) + (\sin nx \sinh ny) \\ &= 0 \end{aligned}$$

So, $u_n(x, y) = \sin nx \sinh ny$ is a solution of the PDE $u_{xx} + u_{yy} = 0$ for every $n > 0$.