# MATH 112A Homework 0 - October 3, 2025

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# 1 Monday

## 1.1 Problem 1

Solve the following PDE for $u(x,t)$:

$$4u_t + 7u_x = 0,\ u(x,0) = e^x.$$

Use the method based on directional derivative. Show all the details.

**Solution** First, we rewrite the PDE as $7u_x + 4u_t = 0$. Then, we note that $7u_x + 4u_t = Du \cdot \langle 7, 4\rangle$. So, the solution to $u(x,t)$ must be something of the form

$$u(x,t) = f(4x - 7t)$$

Since our constraint is that $u(x,0) = e^x$, we find that

$$\begin{aligned} u(x,0) &= f(4x - 0t) \\ e^x &= f(4x) \\ f(4x) &= e^x \\ f\left(4\left(\frac{x}{4}\right)\right) &= e^{\frac{x}{4}} \\ f(x) &= e^{\frac{x}{4}} \end{aligned}$$

Therefore, $u(x,t) = e^{\frac{(4x-7t)}{4}}$ is the solution.

## 1.2 Problem 2

Solve the following PDE for $u(x,t)$:

$$-2u_t + 11u_x = 0, u(x,0) = \sin 2x$$

Use the method based on the change of coordinates. Show all the details.

**Solution** First, we rewrite the PDE as $11u_x - 2u_t = 0$. Then, we define the variables $x'$ and $t'$ as

$$\begin{aligned} x' &= -2x + 11t \\ t' &= 11x + 2t \end{aligned}$$

Now, we can rewrite both $u_x$ and $u_t$ in terms of $x'$ and $t'$ instead using the chain rule. We find that

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} \\ &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial x} \\ &= -2u_{x'} + 11u_{t'} \end{aligned}$$

and

$$\begin{aligned} u_t &= \frac{\partial u}{\partial t} \\ &= \frac{\partial u}{\partial x'}\frac{\partial t'}{\partial t} + \frac{\partial u}{\partial t'}\frac{\partial x'}{\partial t} \\ &= 11u_{x'} + 2u_{t'} \end{aligned}$$

Now, we substitute this into the original equation to get

$$\begin{aligned}
-2u_t + 11u_x &= 0 \\
-2\left(-2u_{x'} + 11u_{t'}\right) + 11\left(11u_{x'} + 2u_{t'}\right) &= 0 \\
4u_{x'} - 22u_{t'} + 121u_{x'} + 22u_{t'} &= 0 \\
125u_{x'} &= 0 \\
u_{x'} &= 0
\end{aligned}$$

Integrating this gives us with respect to $x'$ gives us $u = f(t') = f(11x + 2t)$. Now, since our constraint is $u(x,0) = \sin 2x$ this means that

$$\begin{aligned}
u(x,0) &= f(11x + 2(0)) \\
\sin 2x &= f(11x) \\
f(11x) &= \sin(2x) \\
f\left(11\left(\frac{x}{11}\right)\right) &= \sin\left(2\left(\frac{x}{11}\right)\right) \\
f(x) = \sin\left(\frac{2}{11}x\right)
\end{aligned}$$

So, $u(x,t) = \sin\left(\frac{2}{11}\left(11x + 2t\right)\right)$ is the solution to the PDE.

# 2 Wednesday

## 2.1 Problem 1

Solve the solution $u(x,t)$ to the equation

$$u_t + u_x = 1,\ u(x,0) = 2x^2 - 3x + 5$$

**Solution** We define the variables $x'$ and $t'$ as

$$\begin{aligned}
x' &= x + t \\
t' &= -x + t
\end{aligned}$$

We find that

$$\begin{aligned}
u_t &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial t} \\
&= u_{x'} + u_{t'}
\end{aligned}$$

and

$$\begin{aligned}
u_x &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial x} \\
&= u_{x'} - u_{t'}
\end{aligned}$$

Integrating this with respect to $x'$ gives us $u = \frac{1}{2}x' + f(t')$ In terms of $x$ and $t$, $u(x,t) = \frac{1}{2}(x+t) + f(-x+t)$. Now, since our constraint is $u(x,0) = 2x^2 - 3x + 5$, it means that

$$\begin{aligned}
u(x,0) &= 2x^2 - 3x + 5 \\
\frac{1}{2}(x+0) + f(-x+0) &= 2x^2 - 3x + 5 \\
\frac{1}{2}x + f(-x) &= 2x^2 - 3x + 5 \\
f(-x) &= 2x^2 - \frac{7}{2}x + 5 \\
f(-(-x)) &= 2(-x)^2 - \frac{7}{2}(-x) + 5 \\
f(x) &= 2x^2 + \frac{7}{2}x + 5
\end{aligned}$$

Therefore, the solution to the PDE is

$$u(x,t) = \frac{1}{2}(x+t) + \left(2(-x+t)^2 + \frac{7}{2}(-x+t) + 5\right)$$

### 2.2 Problem 2

Solve the equation

$$(1+t^2)u_t + u_x = 0, \;\; u(x,0) = \frac{1}{1+x^2}$$

### 2.3 Problem 3

Solve the PDE $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

## 3 Friday

### 3.1 Problem 1

For any non-integer real number $\alpha$, derive the binomial formula:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

where the coefficient for $x^n$ is

$$\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

### 3.2 Problem 2

A flexible chain of length $\ell$ is hanging from one end $x = 0$ but oscillates horizontally. Let the $x$ axis point downward and the $u$ axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.