Qf 2023/2024. April exam

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r, q = 0

1 The CEV model

We consider the model (CEV)

$$dS(t) = \sigma_X S^{\beta}(t) dW_t, \quad \beta > \frac{1}{2}, \tag{1}$$
$$S(t_0) = S_o$$

represents the elasticity of volatility

Let the integration trajectory

$$0 = t_0, t_1, \ldots, t_N = T. \quad t_n = ndt$$

and, in order to semplify notation we define $S_n := S(t_n)$ We can suggest two integration schemes:

1.

$$S_{n+1} = S_n + \sigma S_n^{\beta} \sqrt{dt} \eta_n \quad \eta_n \stackrel{d}{=} \mathcal{N}_{0,1} \tag{2}$$

2. Let $X(t) = \log(\, S(t)\,)$ then:

$$\begin{aligned} dX(t) &= \frac{dS(t)}{S(t)} - \frac{1}{2}\left(\frac{dS(t)}{S(t)}\right)^2 \\ &= -\frac{1}{2}\sigma^2 S^{2(\beta-1)} dt + \sigma S^{\beta-1}(t) dW_t \end{aligned}$$

and the integration scheme will be:

$$S_{n+1} = S_n \exp\left(-\frac{1}{2}\sigma^2 S_n^{2(\beta-1)} dt + \sigma S_n^{\beta-1}\sqrt{dt}\,\eta_n\right) \tag{3}$$

Things are actually not that simple given that as we can see clearly if for some t_n we get $S_n = 0$, it will remain zero from that point on. When $\beta < 1$ the zero will be reached for sure. Nothing wrong with that given that we could look at this as a model that implements default. Nonetheless we do not want that. If we rewrite the model as a modified log normal process we have

$$\frac{dS_t}{S_t} = \sigma_X S_t^{\beta-1} dW_t$$

and we can think of it as a model with a stochastic volatility

$$\sigma = \sigma_X S^{\beta-1},$$

and we procedd imposing a cut off. When $S_t < \epsilon$ we make the replacement

$$\sigma \to \sigma_X \epsilon^{\beta-1}$$

There are problems also when $\beta > 1.0$. In this case the algorithm could explode. In this sesion we will be interested only for the case $\beta < 1$ given that

$$\sigma = \frac{\sigma_X}{S_X^{1-\beta}}$$

captures the empirical evidence that we see increased volatility when the market goes down.

(epsilon) represents a cutoff point for the price of the underlying asset. This cutoff point is introduced to prevent unrealistic behavior in the model, particularly in situations where extreme movements in the price of the underlying asset could lead to numerical instability or impractical results.

In cases where the CEV model incorporates a beta parameter (β) equal to 0.5 or less, indicating decreasing volatility with increasing asset prices, it's likely that the asset price will approach zero over time. In such situations, it becomes crucial to introduce a cutoff point (ε) to prevent the asset price from reaching zero, as allowing the asset price to fall to zero would not accurately reflect market dynamics.

By setting a small cutoff value for ε, practitioners ensure that the CEV model maintains realistic behavior by preventing the asset price from becoming negative or approaching zero. The cutoff value serves as a lower boundary that constrains the asset price, preventing it from reaching implausible levels.

Choosing an appropriate cutoff value for ε involves balancing the need to prevent unrealistic asset price behavior with computational efficiency and model accuracy. The cutoff value should be small enough to prevent the asset price from reaching zero but large enough to avoid introducing excessive numerical instability into the model.

Practitioners may determine the cutoff value through empirical analysis, sensitivity testing, or calibration to market data. By carefully selecting the cutoff value for ε, practitioners can ensure that the CEV model accurately captures the dynamics of decreasing volatility while maintaining stability and reliability in its results.

Without specific market data or context, it's challenging to provide an exact numerical value for the cutoff (ε). However, typical values for ε in the context of CEV models with decreasing volatility (β < 1) are often on the order of fractions of the current asset price or standard deviations of price movements. For example, ε might be set to a small percentage of the current asset price or a few standard deviations of historical price volatility.

2 The displaced diffusion model

The displaced diffusion model is a very light modificaton of BS. Given a log normal process

$$dY = \sigma_Y Y dW_t \tag{4}$$

We define

$$S_t = X_t - \Delta,$$

with Δ a constant. In order to define completely the problem we should supplement the initial condition

$$X_o = S_o + \Delta$$

3 The Heston model

The Heston model is the last model we do consider in this exam: no need to describe this given that we saw it in class.

Exercise 3.1. *For each of the three models describe above consider the grid* (t_n, k_j) *where* t_n *takes the values*

$$1M,\ 2M,\ 3M,\ 6M,\ 12M,\ 18M$$

and for each t_n *conder the strikes*

$$0.8,\ 0.9,\ 0.95,\ 0.975,\ 1,\ 1.025,\ 1.05,\ 1.1,\ 1.2$$

For each pair (t_n, k_j) *compute the price of the associated call option, and the impled volatility. The set of volatility* $\sigma(t_n, k_j)$ *is the volatility surface implied bu the model. Provide the results for the implied volatility and, possibly, a graphica representation.*

4 Parameters

For every model we always have $S_0 = 1.$. For the CEV and displaced diffusion use a range of sigma (let's say 4 values)

$$.2 \leq \sigma \leq .6$$

For the CEV model explore ethe range

$$.55 \leq \beta <= .9$$

while for the displaced diffusion, explore the range

$$.2 \leq \Delta \leq .8$$

As far as Heston is concerned we will use the parameters:

$$\lambda = 7.7648, \quad \overline{\nu} = 0.0601, \quad \eta = 2.0170, \quad \rho = -0.6952, \quad \nu_0 = 0.0475 \tag{5}$$

5 Super Bonus

For all of the models there are available in literature analytical solutions (or quasi analytical). Particularly simple is the expression for the displaced diffusion. Compare your MC results with the analytical solution