Computational Finance, 2023/2024

June exam

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Volatility Surface

Abstract

This document reports the results for the Computational Finance final exam on June 21st.

For this assignment, we assume the markets behave according to the XX model: this can be represented by either the Heston model or the Variance Gamma model. Reasonably, as the reader might intuit, we are unaware that markets follow one of these two models and therefore we stick to the Black-Scholes model for option pricing. Thus, we start by computing the put option price according to the XX model (Heston and VG); this market price of the option can then be used to extract the implied volatility (the value of the volatility parameter to be used in BS model so that the BS option price matches the option market price). At this point, using the implied volatility just obtained, we can proceed with our portfolio analysis by first simulating N trajectories for the underlying asset acccording to the BS model. For each different scenario of the underlying price S^j, we compute the BS put price six months from now $\mathcal{P}^j_{BS}(t = 6\,m)$ as well as the profit/loss today $\mathcal{PL}^j$ as the difference between the market price and the BS price $\mathcal{P}^j_{BS}(t = 0)$. Finally, we show histograms for both random variables and the Value at Risk for $\mathcal{PL}^j$.

One will find this report divided into two main parts: the first one provides insights into the three models involved (Black-Scholes, Heston and Variance Gamma) as well as Monte Carlo methods and Variance Reduction techniques. The second part reports the portfolio analysis with both Heston and VG as the market models.

Contents

Part I

Market assumptions and Monte Carlo setup

1 Model insights

1.1 Portfolio analysis: the Black-Scholes model

In the risk neutral measure, the BS model is described by the SDE:

$$dS(t) = S(t)(rdt + \sigma dW_t),\, dW_t \sim N_{0,\sqrt{t}}$$

In order to simplify notation we define $S(t) = S_t$, $X(t) = X_t$. With a standard Euler scheme:

$$\begin{aligned}
&S_{t_{n+1}} = S_{t_n} e^{X_{t_{n+1}} - X_{t_n}} \\
&X_{t_{n+1}} - X_{t_n} = -\frac{\sigma^2[t_{n+1} - t_n]}{2} + \sigma[W_{t_{n+1}} - W_{t_n}],\, W_{t_{n+1}} - W_{t_N} \sim N_{-\sigma^2[t_{n+1}-t_n]/2, \sigma\sqrt{t_{n+1}-t_n}},\, X(0) = 0
\end{aligned}$$

Therefore, we implement Monte Carlo simulations generating N trajectories integrated over $0 = t_0 < t_1 < ... < t_N = T$ for the underlying asset. The N option payoffs for each of the underlying path are then used to calculate the MC option price and its statistical error.

We recall the closed-form analytical solution for the price of a put option as well:

$$\begin{aligned}
&P = ke^{-rT}N(-d_2) - S_0 N(-d_1) \\
&d_1 = \frac{ln(\frac{S_0}{k}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\
&d_2 = d1 - \sigma\sqrt{T}
\end{aligned}$$

1.2 Market models

1.2.1 The Heston model

We recall the the system of SDEs that define the Heston model in the risk neutral measure:

$$\begin{aligned}
&dS(t) = S(t)rdt + \sqrt{v_t}dW_t \\
&dv(t) = \lambda(\bar{v} - v(t))dt + \eta\sqrt{v(t)}dY_t \\
&dW_t = \rho dY_t + \sqrt{1-\rho^2}dZ_t
\end{aligned}$$

In order to simplify notation we define $S(t) = S_t$, $v(t) = v_t$.

Monte Carlo simulation

It is important to notice that, given the parameters

$$\lambda = 7.7648, \bar{v} = 0.0601, \eta = 2.017$$

the Feller condition

$$\eta^2 \leq 2\lambda\bar{v}$$

is violated.

This exposes the model to the risk that the process $v(t)$ might reach zero or even negative values. Therefore, we used the algorithm by Leif Andersen which uses the QE[1], known for its stability and efficiency, especially when dealing with the violation of the Feller condition.

Choosen an arbitrary level $\Psi_c \in [1,2]$ the equations below represent a summary of the *QE algorithm,* used for the simulation step from v_{t_n} to $v_{t_{n+1}}$.

$$
\begin{aligned}
&1.\, h = 1 - exp(-\lambda) \\
&2.\, m = \theta(v_{t_n} - \theta)(1-h) \\
&3.\, s^2 = (\frac{\eta^2 h}{\lambda})(v_{t_n}(1-h) + \frac{\theta h}{2}) \\
&4.\, Draw\, a\, uniform\, random\, number\, U_v \\
&5.\, if\, \Psi \leq \Psi_c \\
&\quad a = \frac{m}{1+b^2} \\
&\quad b^2 = 2\Psi^{-1} - 1 + \sqrt{2\Psi^{-1}}\sqrt{2\Psi^{-1} - 1},\, \Psi = s^2/m^2 \\
&\quad Z_v = \Phi^{-1}(U_v) \\
&\quad v_{t_{n+1}} = a(b + Z_v)^2 \\
&6.\, if\, \Psi > \Psi_c \\
&\quad \Psi = s^2/m^2 \\
&\quad \beta = \frac{2}{m(\Psi+1)} \\
&\quad p = \frac{\Psi - 1}{\Psi + 1} \\
&\quad v_{t_{n+1}} = \Psi^{-1}(U_v; p, \beta),\ \Psi^{-1} = \begin{cases} 0 & 0 \leq U_v \leq p \\ \beta^{-1} ln(\frac{1-p}{1-u}) & p < U_v \leq 1 \end{cases}
\end{aligned}
$$

Therefore, after setting the initial condition $v(0) = \nu_0$ and $X(0) = 0$, we proceed by generating NV trajectories integrated over $0 = t_0 < t_1 < ... < t_N = T$ for the volatility v_{t_n}.

For each of the volatility trajectory, we generate NS trajectories over $0 = t_0 < t_1 < ... < t_N = T$ for the underlying asset.

$$
\begin{aligned}
S_{t_{n+1}} &= S_{t_n} e^{X_{t_{n+1}} - X_{t_n}} \\
X_{t_{n+1}} - X_{t_n} &= -\frac{v_{t_{n+1}} - v_{t_n}}{2} dt + \sqrt{v_{t_{n+1}} - v_{t_n}}(\rho\xi_Y + \sqrt{1-\rho^2}\xi_Z),\ \xi_Y, \xi_Z \sim N_{0,1}
\end{aligned}
$$

Here it is important to remind that each block of $NS|nv_i$ cannot be considered fully independent. Thus we have to proceed by first calculating the average option payoff for each of the $NS|nv_i$ simulations with $i = 1, ..., NV$ and then using these NV variables to compute the Monte Carlo option price as well as the statistical error.

Quasi-analytical solution

Heston model's complexity arises from its stochastic volatility component, making direct solutions challenging. Instead, we use the characteristic function of the log-price of the asset, which provides a convenient way to handle the SDEs

[1] From: "Efficient Simulation of the Heston Stochastic Volatility Model" by Leif Andersen.

analytically. One of the key advantages of the Heston model is that the characteristic function of the log-price can be obtained in closed form. This property allows for efficient numerical pricing of European options using the Fourier transform approach.

The characteristic function $\phi(u)$ of the log-price $ln(S_T)$ is defined as:

$$
\begin{aligned}
&\phi(u) = E[e^{iuln(S_T)}] = exp\{iuln(S_0) + C(T,u) + D(T,u)v_0\} \\
&C(T,u) = \frac{\lambda\bar{v}}{\eta^2}[(\lambda - \rho\eta iu + d)T - 2ln(\frac{1 - ge^{dT}}{1-g})] \\
&D(T,u) = \frac{\lambda - \rho\sigma iu + d}{\eta^2}(\frac{1 - e^{dT}}{1 - ge^{dT}}) \\
&d = \sqrt{(\rho\eta iu - \lambda)^2 + \eta^2(iu + u^2)} \\
&g = \frac{\lambda - \rho\eta iu + d}{\lambda - \rho\eta iu - d}
\end{aligned}
$$

The characteristic function enables us to use the inverse Fourier transform to compute the price of a European call option.

$$
\begin{aligned}
&\mathcal{P}(S_0, K, T) = Ke^{-rT}P_2 - S_0P_1 \\
&P_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re(\frac{e^{-iuln(K)}\phi(u-i)}{iu\phi(-i)})du \\
&P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re(\frac{e^{-iuln(K)}\phi(u)}{iu})du
\end{aligned} \tag{1}
$$

To evaluate the integrals for P_1 and P_2 we use numerical techniques such as the Sinc method.

1.2.2 The Variance Gamma model

The Variance-Gamma process can be defined as

$$
\begin{aligned}
&X(t_n) = \theta\gamma(\Delta t_n) + \eta W_{\gamma(\Delta t_n)} \\
&\gamma(t + \Delta t_n) - \gamma(t) \sim \Gamma(\Delta t_n/v, v) \\
&W_{\gamma(\Delta t_n)} = \sqrt{\gamma(\Delta t_n)}Z_n,\ Z_n \sim N(0,1)
\end{aligned}
$$

$\gamma(t)$ is a Gamma process, which is a type of stochastic process with independent, gamma-distributed increments; $W_{\gamma(t)}$ represents a Brownian motion evaluated at the random time $\gamma(t)$.

Monte Carlo simulation

The Monte Carlo integration scheme can be defined as

$$
\begin{aligned}
&S_{t_{n+1}} = S_{t_n}e^{X_{t_n}} \\
&X(t_n) = \theta\gamma(t_{n+1} - t_n) + \eta\sqrt{\gamma(t_{n+1} - t_n)}Z_n + (t_{n+1} - t_n)\frac{\phi}{v} \\
&\phi = log(1 - \nu\theta - \frac{1}{2}\nu\eta^2)
\end{aligned}
$$

Therefore, we simulate N trajectories integrated over $0 = t_0 < t_1 < ... < t_N = T$ for the underlying asset. The N option payoffs for each of the underlying path are then used to calculate the MC option price and its statistical error.

Quasi-analytical solution

The characteristic function for the VG process

$$\phi_{X(t_n)}(u) = E[e^{iuX_n}] = (\frac{1}{1 - i\theta vu + (\eta^2 v/2)u^2})^{\frac{t_n}{v}}$$

can be used in the inverse Fourier transform as described in eq (1).

2 Variance Reduction

Reducing the MonteCarlo error can entail significant time and computational resources. Therefore, we present here two simple technics used to reduce the MC error without the need to increase the number of simulations.

2.1 Call Put Parity

Generally speaking, we know there are two ways of computing option prices. We can compute put option prices either via the put option formula or via the call put parity (i.e. computing call option prices and use them in the CP Parity to compute put option prices). In the context of Monte Carlo that means we can either discount the put average payoff or use the CP Parity with the call discounted average payoff.

Implementing MC for out of the money options, rather than in of the money options, will result in a lower number of positive payoffs which will contribute to the statistical error. As a result, the MC error for OTM options will be smaller than the MC error for ITM options.

Therefore, market values of put options (those obtained with Heston and VG models) used to extract the implied volatility are computed as follows:

$$\mathcal{P} = \begin{cases} e^{-rT}E[(k - S(T))^+], & if\, k \leq e^{(r-q)T}S(0) \\ e^{-rT}E[(S(T) - k)^+ - e^{-qT}S(0) + e^{-rT}k] & if\, k > e^{(r-q)T}S(0) \end{cases}$$

since $e^{-qT}S(0) - e^{-rT}k$ is deterministic and does not contribute to the statistical error.

From a practical point of view, the put option we intend to price is ITM and therefore its price will be computed via the the call put parity formula. Let's see how this approach impacts the MC error: the table below show the MC price and MC error for an ITM put option, comparing results obtained via the put option formula versus the call put parity formula. In order to make a point we will analyse a put option with strike = 1.10, so that it won't be very close to the ATM region as it is the one we will price in Part 2.

Heston		
$S_0 = 1.0$, $k = 1.10$, $T = 1.13$, $r = 0.01$, $q = 0$		
	MC price	MC error
put formula	0.1351	1.05e-03
c.p.p. formula	0.1353	2.03e-04

Variance-Gamma		
$S_0 = 1.0$, $k = 1.10$, $T = 1.13$, $r = 0.01$, $q = 0$		
	MC price	MC error
put formula	0.1435	7.60e-05
c.p.p. formula	0.1435	5.40e-05

One can see from the tables above that the MC error is higher when computing the ITM put price via the direct formula.

The MC price calculated using the call put parity formula has a MC error equal to the MC error of the MC price of a call option of the same type, as we are using the price of the call option to calculate the price of the put.

2.2 Control variates

We can further improve the accuracy of the estimate of the average of our target random variable by using a correlated auxiliary random variable that shares with our target the property of having the same expected value.

In practice, we can use the deviations of the simulated underlying price at maturity from their mean to eliminate part of the fluctuations of the simulated discounted payoffs.

Defined

$$X_n = e^{-rT} max(S_n(T) - k, 0)$$
$$Z_n = e^{rT} S_n(T)$$

The new option price will be given by the estimator

$$R := \frac{1}{N} \sum_{n=1}^{N} [X_n + c_{opt}(Z_n - E[Z])]$$

end the new MC error

$$err = \sqrt{\frac{\sum_{n=1}^{N} \{[X_n + c_{opt}(Z_n - E[Z])] - R\}}{N}}$$

where

$$c_{opt} = \frac{Cov(X, Z)}{Var(Z)}$$

The following example shows the impact of the control variates technic. Let's consider the option we will use to compute the implied volatility in the second part of this report

$$S_0 = 1.0, k = 1.03, T = 1.13, r = 0.01, q = 0$$

and compute its price with both Heston and VG models. Parameters values are those we will use in second part to compute the value of the put option and the related implied volatility

$$H: \ \lambda = 7.7648, \bar{v} = 0.0601, \eta = 2.017, v_0 = 0.0475, \rho = -0.6952 \ \ VG: \ \eta = 0.1494, v = 0.0626, \theta = -0.6635$$

Let's compare the prices and MC errors of both a call and a put options with and without applying the control variates technic. The following tables show both analytical and MC prices along with MC errors for both Heston and VG models, before and after applying the control variates technique. The number of simulations is set to $N = 2^{22}$ for the VG model and $NV = 2^{14}, NS = 2^{10}$ for the Heston model. Prices via the call put parity are shown as well (call_cp, put_cp).

Heston prices without CV				
	AS[2]	MC price	MC error	Op[3].
call	0.0478	0.0476	2.03e-04	True
call_cp	0.0478	0.0474	1.05e-03	True
put	0.1354	0.1351	1.05e-03	True
put_cp	0.1354	0.1353	2.03e-04	True

Heston prices with CV					
	AS	MC price	MC error	Op	ρ_{XZ}^2[4]
call	0.0478	0.0476	9.30e-05	True	0.88836
call_cp	0.0478	0.0474	9.30e-05	False	-
put	0.1354	0.1351	9.30e-05	False	-0.99605
put_cp	0.1354	0.1353	9.30e-05	True	-

VG prices without CV				
	AS	MC price	MC error	Op
call	0.0559	0.0559	5.40e-05	True
call_cp	0.0559	0.0559	7.60e-05	True
put	0.1435	0.1435	7.60e-05	True
put_cp	0.1435	0.1435	5.40e-05	True

VG prices with CV					
	AS	MC price	MC error	Op	ρ_{XZ}^2
call	0.0559	0.0559	3.24e-05	True	0.79909
call_cp	0.0559	0.0559	3.24e-05	True	-
put	0.1435	0.1435	3.24e-05	True	-0.90432
put_cp	0.1435	0.1435	3.24e-05	True	-

The control variates technique reduces the variance of the estimator R:

$$V\hat{ar}(R) = \sigma_X^2(1 - \rho_{XZ}^2)$$

Which means the MC error is reduced by a factor of $\sqrt{1 - \rho_{XZ}^2}$.

The Op condition isn't fulfilled for the Heston put (and call_cp) price when activating the control variates technique. With respect to this "undesired" result, the put call parity formula is a valuable tool as it computes a more precise put price which falls within the 99% confidence interval.

[2] Analytical solution.

[3] Op: boolean indicator that checks whether the Monte Carlo (MC) estimate of an option price is within the range $AS \pm 3 * MC\,error$

[4] Correlation between the variable of interest (e.g. payoff of an option) and the control variate.

Part II

Profit & Loss

In this second part of the document we report the profit & loss analysis. We start by computing the market price for the put option according to the XX model, as we assumed that market rules are set according to the XX model.

We will use The Black-Scholes to model the underlying asset dynamics. Since we believe in the BS model, the market price can then be used to reverse engineer the implied volatility - the volatility that, used in the BS model, returns the market price - which we can use as the volatility parameter in simulating N trajectories for the underlying asset and computing the put option payoffs and the related price with the BS model.

We then proceed to analyse the distribution of prices six months from now as well as the profit or loss today conditional on the trajectory. The analysis is carried out for both MC and (quasi)-analytical prices for the XX model as the option price used to compute the implied volatility, which will then be use in the Black-Scholes model .

3 Heston based market

3.1 Market put value and IV

In this third Section we perfom the analysis as if the market evolves according to the Heston model.

The first step requires the computation of today's value of the put option

$$\mathcal{P}_H(0) = P(0,T)E[(\kappa - S(T))^+]$$

with

$$k = 1.03,\, T = 1.13$$

on a stock with

$$S_0 = 1,\, q = 0$$

Also, interests rate are constant, set to $r = 0.01$.

For the Heston model we have this set of parameters:

parameter	value
λ	7.7648
$\bar{v}$	0.0601
η	2.017
v_0	0.0475
ρ	-0.6952

The value of the put option according to the Heston model is showed in the table below.

MC price	MC error	AS	Op
0.09733	1.50e-04	0.09755	True

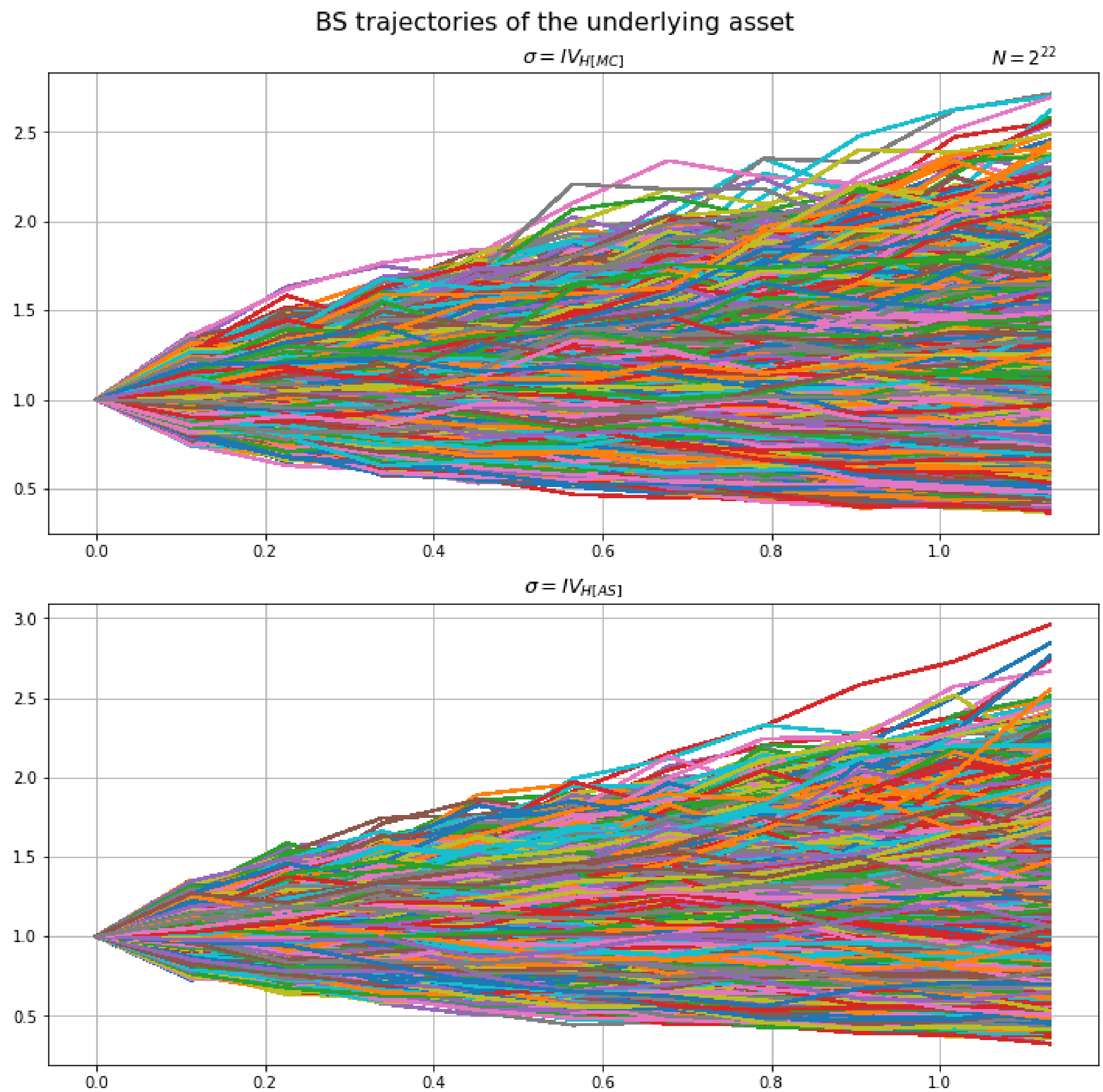
Secondly, we present the result for the volatility as implied by the market (Heston) value of the option. We present the results for the MC value as well as its analytical solution. We also show the BS price of the put option with $\sigma = IV_H$ to demonstrate the BS price matches the market price.

	$\mathcal{P}_{VG[MC]}(0)$	$\mathcal{P}_{VG[AS]}(0)$
IV	0.20557	0.20609
$mc\mathcal{P}_{BS}(0)$	0.09732	0.09765
$as\mathcal{P}_{BS}(0)$	0.09733	0.09733

We can use the IV as the σ parameter in the BS model.

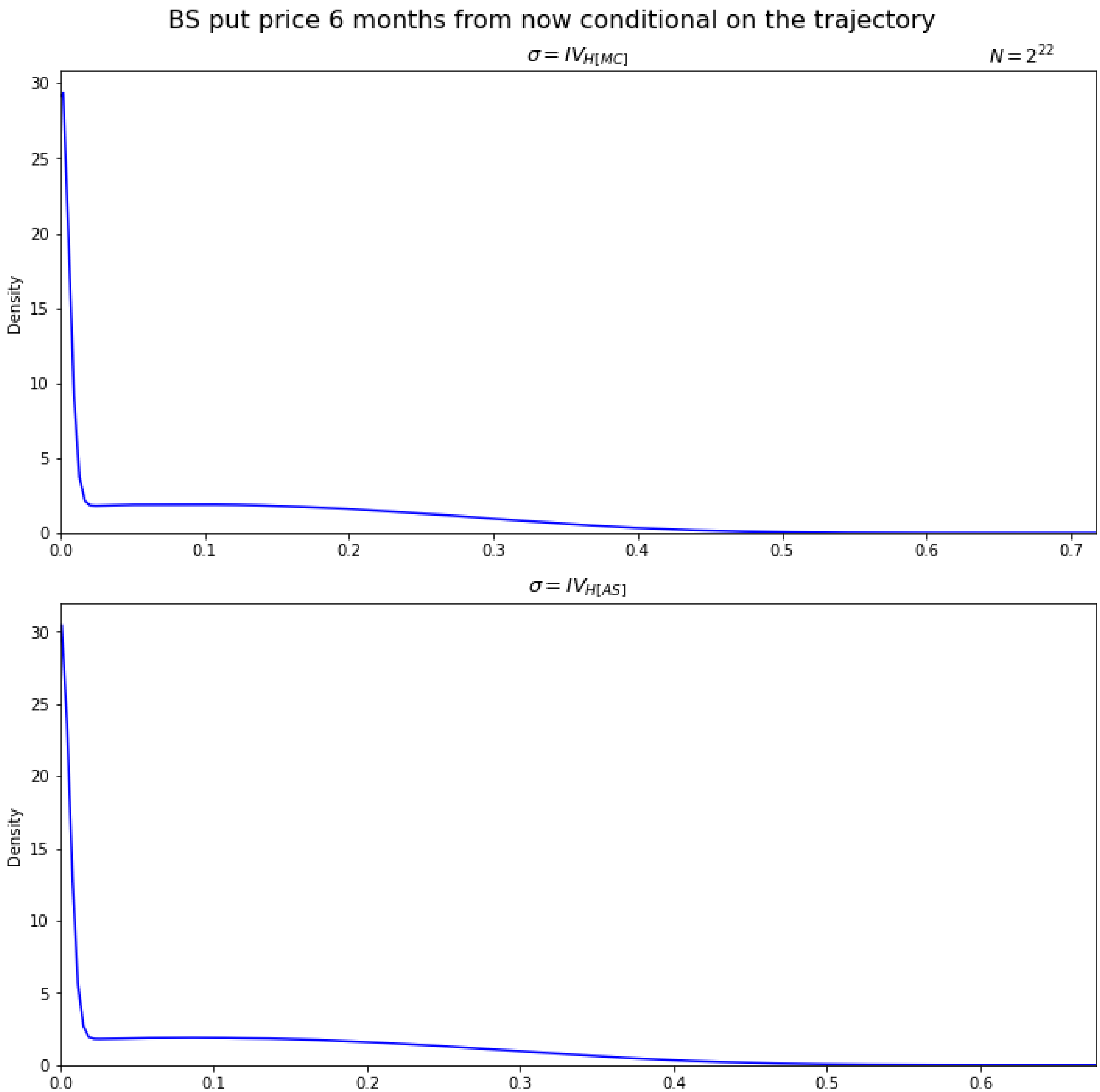
3.2 Portfolio analysis with BS

We now proceed to perform the analysis. Below we show the simulated BS trajectories.



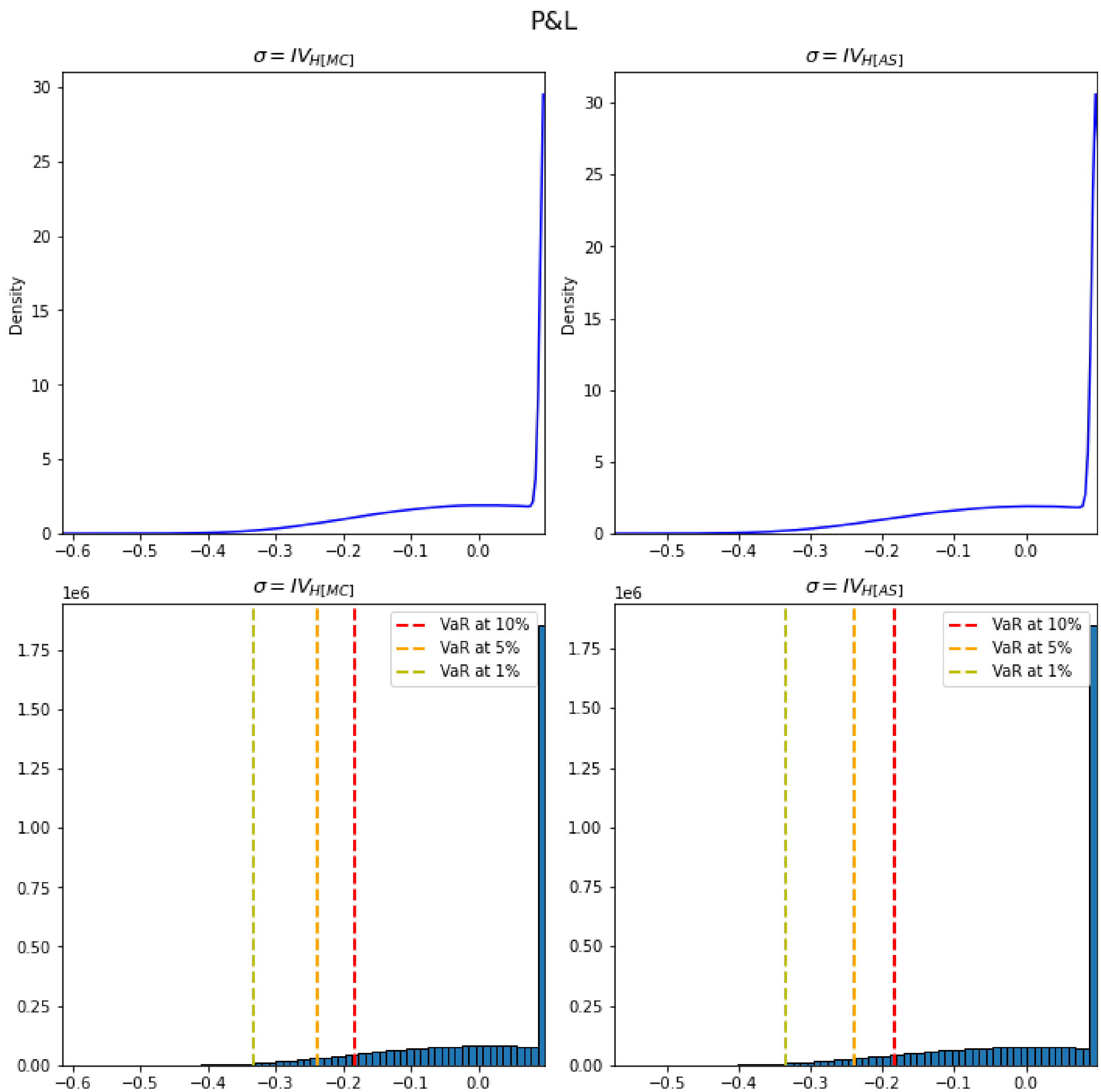
For each of trajectory S^j we can compute the price of the put option conditional on the trajectory 6 months from now, that is:

$$\mathcal{P}^j_{BS}(t_m) = P(t_m, T)E[(k - S^j(T))^+ | F_{t_m}],\ t_m = 6\, months$$



Finally we compute the profit/loss today conditional on the scenario S^j as:

$$\mathcal{PL}^j = \Pi(0) - P(0, t_m)\Pi^j(t_m)$$



	$\mathcal{PL}_{H[MC]}$	$\mathcal{PL}_{H[AS]}$
$Mean$	0.00001	-0.00010
$Profit(\%)$	0.60500	0.60498
$10\%VaR$	-0.18305	-0.18389
$5\%VaR$	-0.23922	-0.24011
$1\%VaR$	-0.33346	-0.33511

4 Variance Gamma based market

4.1 Market put value and IV

In this fourth Section we perfom the analysis as if the market evolves according to the Variance Gamma model.

The first step requires the computation of today's value of the put option

$$\mathcal{P}_{VG}(0) = P(0,T)E[(\kappa - S(T))^+]$$

with

$$k = 1.03,\ T = 1.13$$

on a stock with

$$S_0 = 1,\ q = 0$$

Also, interests rate are constant, set to $r = 0.01$.

For the Variance Gamma model we have this set of parameters:

parameter	value
η	0.1494
v	0.0626
θ	-0.6635

The value of the put option according to the VG model is showed in the table below.

MC price	MC error	AS	Op
0.10152	3.34e-05	0.10157	True

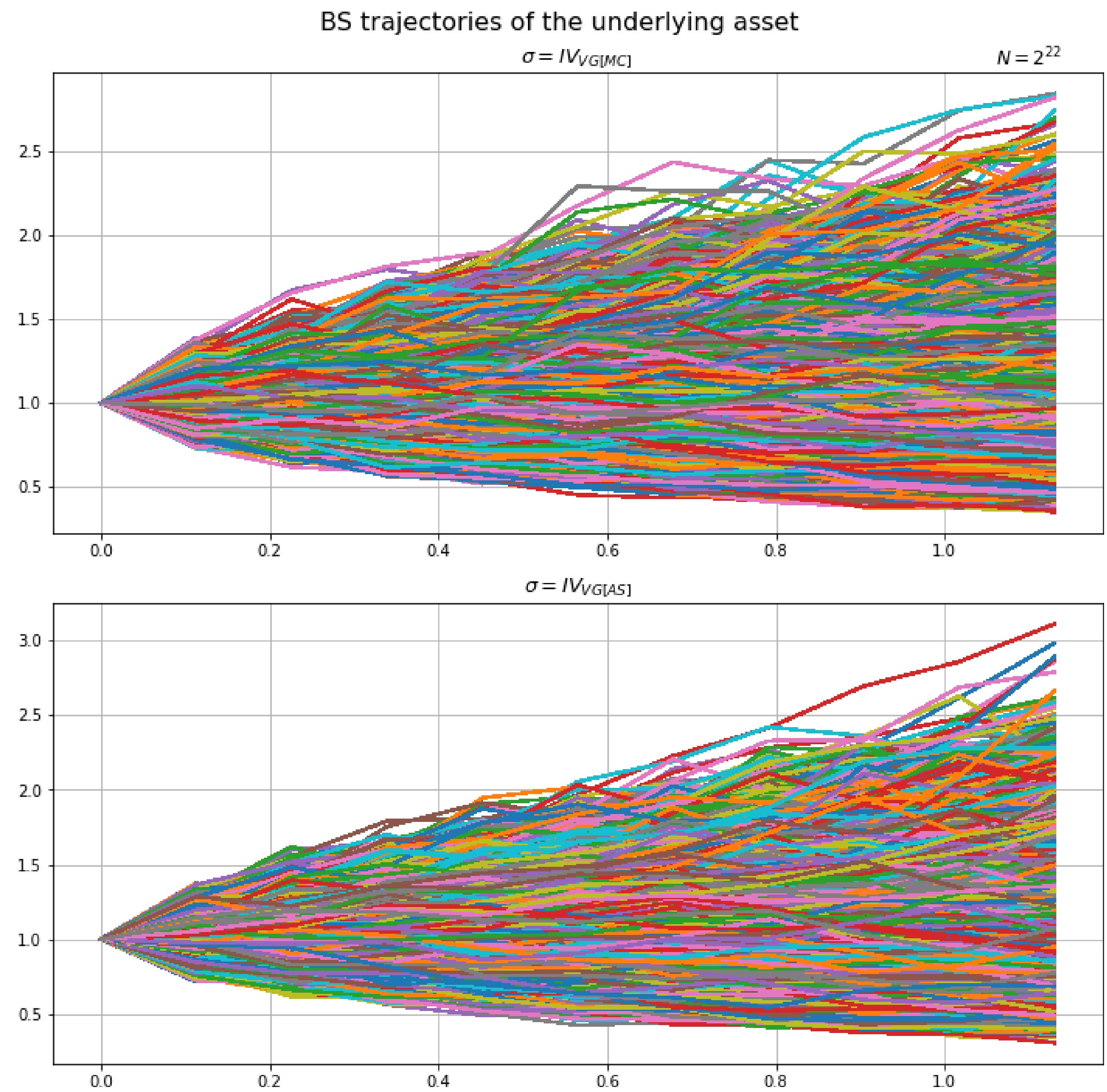
Secondly, we present the result for the volatility as implied by the market (Variance Gamma) value of the option. We present the results for the MC value as well as its analytical solution. We also show the BS price of the put option with $\sigma = IV_{VG}$ to demonstrate the BS price matches the market price.

	$\mathcal{P}_{VG[MC]}(0)$	$\mathcal{P}_{VG[AS]}(0)$
IV	0.21547	0.21559
$mc\mathcal{P}_{BS}(0)$	0.10150	0.10151
$as\mathcal{P}_{BS}(0)$	0.10152	0.10152

We can use the IV as the σ parameter in the BS model.

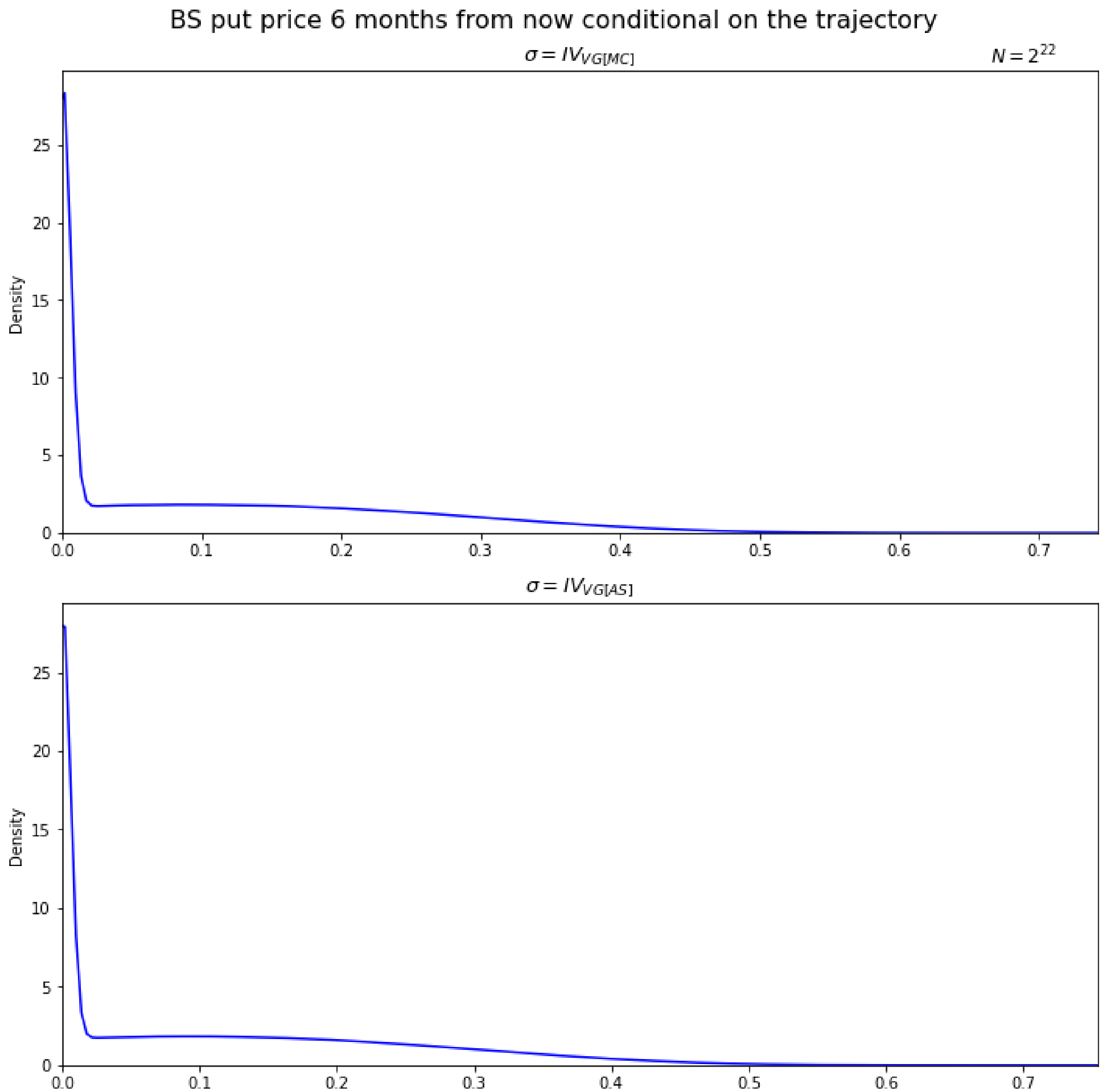
4.2 Portfolio analysis with BS

We now proceed to perform the analysis. Below we show the simulated BS trajectories.



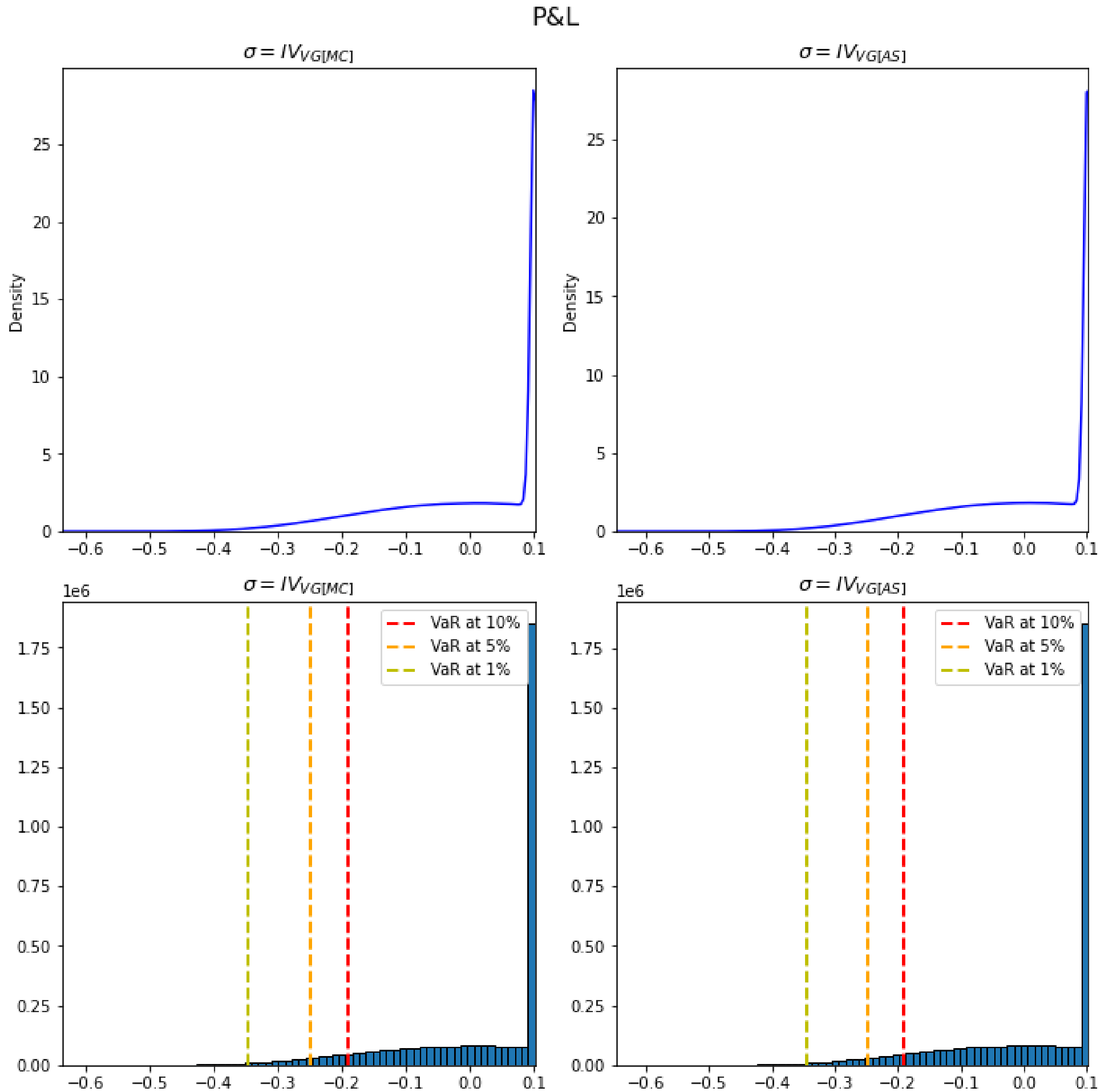
For each of trajectory S^j we can compute the price of the put option conditional on the trajectory 6 months from now, that is:

$$\mathcal{P}_{BS}^{j}(t_m) = P(t_m, T)E[(k - S^j(T))^+ | F_{t_m}],\ t_m = 6\,months$$



Finally we compute the profit/loss today conditional on the scenario S^j as:

$$\mathcal{PL}^j = \Pi(0) - P(0, t_m)\Pi^j(t_m)$$



	$\mathcal{PL}_{VG[MC]}$	$\mathcal{PL}_{VG[AS]}$
$Mean$	0.00002	0.00006
$Profit(\%)$	0.60444	0.60440
$10\%VaR$	-0.19060	-0.19059
$5\%VaR$	-0.24852	-0.24860
$1\%VaR$	-0.34531	-0.34526