Homework Assignment 1

I don't tell u who i am

2018.10.6

1 Chapter 2

1.1 Question 1

Using the Laplace of the determinant of an $S \times S$ matrix, prove that

$$X = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S & S-1 & \cdots & 1 \end{pmatrix} \tag{1}$$

Has determinant equal to 1.

Answer : Expand the matrix by the first row. We have

$$X = 1 \times \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S-1 & S-2 & \cdots & 1 \end{pmatrix} \tag{2}$$

Keep expanding the matrix and we have

$$X = 1 \times 1 \times 1 \times \cdots \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

Expand the last matrix and we can conclude that the X has determinant to 1.

1.2 Question 2

Now repeat exercise 1 by adding to the securit structure put options on the stock, following the same steps as explained in the text (for call options). Are markets complete?

Answer: Let us consider a specific example in which stock value at time $t = 1$ are equal to the index of the state of the world: $s = (1, 2, \cdots, S)$. We can introduce $S-1$ put options with payoff $(k-s)^+$ for $k = 1, \cdots, S-1$: we obtain the securities

$$c_2 = (1, 0, 0, \cdots, 0, 0)' \tag{4}$$

$$c_3 = (2, 1, 0, \cdots, 0, 0)' \tag{5}$$

$$\vdots \tag{6}$$

$$c_S = (S-1, S-2, S-3, \cdots, 1, 0)' \tag{7}$$

Which together with the stock give rise to the security structure

$$X = \begin{pmatrix} 1 & 2 & \cdots & S \\ 0 & 1 & \cdots & S-1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{8}$$

This is an upper triangular $S \times S$ matrix whose determinant is one (the product of the terms on the diagnal). It can be proved by following simple Laplace steps just like what we do in Question 1. Therefore X is full rank and markets are complete.

1.3 Question 3

Suppose there exists only a risk-free asset $x^1 = (1, 1, \cdots, 1)'$ and a risky asset $x^2 \neq x^1$ and S states of the world. Let p_1 amd p_2 be the prices of these two assets. A forward contract on the stock is an agreement to pay an amount F at a future date $t = T$ in exchange for the payment x_s^j when the state $s \in \{1, 2, \cdots, S\}$ realizes, with no cash flow exchange at time $t = 0$. Assuming arbitrage opportunities are ruled out, find the fair value of F.

Answer Let x^j be an equivalent of $a \times x^1 + b \times x^2$. The payoff of the forward contract is $x_s^j - F$. To replicate this, we buy $\frac{F}{T}x^1$ and x^j at time $t = 0$ and sell $\frac{F}{T}x^1$, which equals to F, at time $t = T$. Therefore we have $x_s^j - F$, and hence F will be equal to the value of the portfolio of long a x^j and short $\frac{F}{T}x^1$ today. So,we have

$$F = (a + \frac{F}{T}) \times p_1 + b \times p_2 \tag{9}$$

This function equals to

$$F = \frac{aTp_1 + bTp_2}{T - p_1} \tag{10}$$

where a and b fit $x^j = a \times x^1 + b \times x^2$

Note: In fact in this question x^j is not a particular security (accoring to the teacher), so the answer should be changed. But the concept is the same.

2 Chapter 3

2.1 Question 1

Determine whether the following statements are true or false. Provide a proof or a counter-example.

2.1.1 1.

Law of one price and complete markets imply no strong arbitrage.

Answer True. We prove the contrapositive "if NSA doesn't hold, LOOP or complete markets don't hold".

When NSA doesn't hold, if $Xh \geq 0$, we have $p \cdot h < 0$. Assume that LOOP holds. Let $h = j - k$, where j and k are two portfolios. So we have

$$X(j-k) \geq 0 \tag{11}$$

which indicates $Xj - Xk \geq 0$. So $Xj \geq Xk$. Also we have

$$p(j-k) < 0 \tag{12}$$

which indicates $pj - pk < 0$. So $pj < pk$.

It is clearly that when $Xj = Xk$, we have $pj < pk$. Thus LOOP doesn't hold. Thus the statement is true.

2.1.2 2.

Law of one price and complete markets imply no arbitrage.

Answer False. Let us consider a security structure that looks like:

$$X = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{13}$$

It's a complete market. We have two portfolios $h = k = (-1, 1)'$. It definitely satisfies LOOP since $h = k$. And $Xh = Xk = (1, 2) > 0$. If we have $p = (11, 1)$, then $p \cdot h = -10$. So this statement is false.

2.1.3 3.

No strong arbitrage and complete markets imply no arbitrage.

Answer False. Consider a complete market with a security portfolio X to be

$$X = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \tag{14}$$

Consider $h = (-2, 1)'$ and $p = (1, 2)$. Thus $Xh > 0$, however $p \times h = 0$. It fits NSA, but doesn't fit NA. So this statement is false.

2.2 Question 2

Suppose there exists 3 states of the world $s = 1, 2$ and 2 assets x^1, x^2.

2.2.1 1.

Suppose $x^1 = (2, 1, 0)'$ and $x^2 = (0, 1, 0)'$. Describe the asset plan. Are markets complete?

Answer The security structure looks like:

$$X = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{15}$$

There are three states. So whatever the portfolio is, if the state turns out to be 3, the payoff is always 0. If we remove state 3, it is clear that x^1 and x^2 are linearly independent. Hence the asset span

$$\langle X \rangle = \{z = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in R\} \tag{16}$$

So markets are incomplete.

2.2.2 2.

Suppose $p_1 = 4$ and $p_2 = 3$. What type of no-arbitrage requirements does this markets satisfy?

Answer It satisy NA. If we have a portfolio like $h = (a, b)'$, and $p \times h$ equals to $3a + 4b$. Thus

$$Xh = \begin{pmatrix} 2a \\ a+b \\ 0 \end{pmatrix} \tag{17}$$

If $Xh > 0$, we have $a > 0$ and $a + b > 0$, which results in $a > 0$ and $a > -b$. Hence $4a + 3b > 0$. We can say this market satisfy NA.

2.2.3 3.

What are the restrictions on p_1 and p_2 such that this market satisfies LOOP, NSA and NA? (Write each restriction separately)

Answer Let's consider LOOP first. if we define $h = (a, b)'$ and $k = (c, d)'$ then

$$Xh = \begin{pmatrix} 2a \\ a+b \\ 0 \end{pmatrix}, Xk = \begin{pmatrix} 2c \\ c+d \\ 0 \end{pmatrix} \tag{18}$$

if $Xh = Xk$ then $a = c$ and $a + b = c + d$, which means $a = c$ and $b = d$. So $p \times h = p \times k$ no matter what p_1 and p_2 is. **LOOP is satisfied in any p_1 and p_2 cases.**

Let's consider NSA then. Still, we define $h = (a, b)'$. When $Xh \geq 0$, we have $a \geq 0$ and $a + b \geq 0$. If NSA is satisfied, $p_1 \times a + p_2 \times b \geq 0$. Let it be divided by p_1, we have $a + \frac{p_2}{p_1} b \geq 0$. Since $a \geq 0$ and $a + b \geq 0$, we can conclude that $0 \leq \frac{p_2}{p_1} \leq 1$. Since $p_1 \geq 0 and p_2 \geq 0$, **hence $p_1 \geq p_2$ is the restriction of NSA**.

Let's consider NA finally. Still, we define $h = (a, b)'$. When $Xh > 0$, we have $a > 0$ and $a + b > 0$. If NA is satisfied, $p_1 \times a + p_2 \times b > 0$. Let it be divided by p_1, we have $a + \frac{p_2}{p_1} b > 0$. Since $a > 0$ and $a + b > 0$, we can conclude that $0 < \frac{p_2}{p_1} < 1$. Since $p_1 > 0$ and $p_2 > 0$, **hence $p_1 > p_2$ is the restriction of NA**.

2.2.4 4.

Repeat 1), 2) and 3) for $x^1 = (1, 1, 0)'$ and $x^2 = (0, 2, 0)'$.

Answer The security structure looks like:

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 0 \end{pmatrix} \tag{19}$$

There are three states. So whatever the portfolio is, if the state turns out to be 3, the payoff is always 0. If we remove state 3, it is clear that x^1 and x^2 are linearly independent. Hence the asset span

$$\langle X \rangle = \{z = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in R\} \tag{20}$$

So markets are incomplete.

Now $p_1 = 4$ and $p_2 = 3$. Define $h = (a, b)'$,we have

$$Xh = \begin{pmatrix} a \\ a+2b \\ 0 \end{pmatrix} \tag{21}$$

When $Xh > 0$, we have $a > 0$ and $a + 2b > 0$. So we have $1.5a + 3b > 0$.Thus $ph = 4a + 3b > 0$. **The market satisfies NA.**

Let's consider LOOP. if we define $h = (a, b)'$ and $k = (c, d)'$ then

$$Xh = \begin{pmatrix} a \\ a + 2b \\ 0 \end{pmatrix}, Xk = \begin{pmatrix} c \\ c + 2d \\ 0 \end{pmatrix} \tag{22}$$

when $Xh = Xk$ is satisfied, we have $h = k$, so **LOOP is satisfied no matter what p_1 and p_2 are.**

Let's consider NSA. When $XH \geq 0$, we have $a \geq 0$ and $a + 2b \geq 0$. $p_1 Let \times a + p_2 \times b$ be divided by p_1, we have $a + \frac{p_2}{p_1} b$. If we want it to be greater than 0, we have $\frac{p_2}{p_1} \leq 2$. **So the condition for NSA is $p_2 \leq 2p_1$.**

Let's consider NA. When $XH > 0$, we have $a > 0$ and $a + 2b > 0$. $p_1 Let \times a + p_2 \times b$ be divided by p_1, we have $a + \frac{p_2}{p_1} b$. If we want it to be greater than 0, we have $\frac{p_2}{p_1} < 2$. **So the condition for NA is $p_2 < 2p_1$.**

2.2.5 5.

Repeat 1), 2) and 3) for $x^1 = (1, 1, 0)'$ and $x^2 = (0, 2, 0)'$ and $x^3 = (0, 1, 1)'$.

Answer The security structure looks like:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{23}$$

It is clear that $rank(X) = 3$, which means the market is full and the asset span $\langle X \rangle$ is R^3.

There is not enough price information so we skip question 2 and jump to question 3.

Let's consider LOOP. If we define $h = (a, bc)'$ and $k = (d, e, f)'$ then

$$Xh = \begin{pmatrix} a \\ a + 2b + c \\ c \end{pmatrix}, Xk = \begin{pmatrix} d \\ d + 2e + f \\ f \end{pmatrix} \tag{24}$$

When $Xh = Xk$,we have $h = k$, thus **LOOP is satisfied no matter what p_1 and p_2 are.**

Let's consider NSA. When $Xh \geq 0$, we have $a \geq 0, c \geq 0$ and $a + 2b + c \geq 0$. We have ph to be $p_1 a + p_2 b + p_3 c$. To make every a, b, c fit NSA, **The condition is $p_2 \leq 2p_1$ and $p_2 \leq 2p_3$.**

Let's consider NA finally. When $Xh > 0$, we have $a > 0, c > 0$ and $a+2b+c > 0$. We have ph to be $p_1 a + p_2 b + p_3 c$. To make every a, b, c fit NSA, **The condition is $p_2 < 2p_1$ and $p_2 < 2p_3$.**

3 Additional Question

3.1 1

Prove that if v is a linear functional, LOOP holds.

$$v(z) \equiv \{p \cdot h : z = Xh\} \tag{25}$$

Answer We prove the contrapositive "if LOOP doesn't hold, $v(z)$ is not a linear functional".

Define h and k to be two portfolios that satisy $Xh = Xk$. LOOP doesn't hold implies that $p \cdot h \neq p \cdot k$. Assume that $v(z)$ is a linear functional. We have

$$v(0) = v(Xh - Xk) = v(Xh) - v(Xk) = p \cdot h - p \cdot k \tag{26}$$

Also,

$$v(0) = v(Xk - Xh) = v(Xk) - v(Xh) = p \cdot k - p \cdot h \tag{27}$$

Since $p \cdot h \neq p \cdot k$, we have $v(0) \neq v(0)$, which is of course wrong. Thus, $v(z)$ can't be a linear functional. So the contrapositive proves true. The statement proves true, too.

$$0 = p \cdot h - p \cdot k \tag{28}$$

So we have $p \cdot h = p \cdot k$. Then LOOP holds.