浙江工业大学2018/2019（一）期末试卷
《复变函数与积分变换》

《Final Exam for 'Functions of a Complex Variable and Integral Transforms'》

Class:____________ Name (in Chinese):____________ Student ID:____________

Please write in English! Good luck!

Question	1	2	3	4	5	6	7	8	Total
Score									

1. Choose only one correct answer to each queation （3 points each, total 15 points）

Score	

(1) $f(z) = u(x,y) + iv(x,y)$ is continuous at $z_0 = x_0 + iy_0$ if only if (B)

(A) $u(x,y)$ is continuous at (x_0, y_0)
(B) $u(x,y)$ and $v(x,y)$ are both continuous at (x_0, y_0)
(C) $v(x,y)$ is continuous at (x_0, y_0)
(D) $u(x,y) + v(x,y)$ is continuous at (x_0, y_0)

(2) If $f(z) = u + iv$ is analytic in domain D, which one of the followings is not correct? (C)

(A) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$
(B) $f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$
(C) $f'(z) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$
(D) $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

(3) If $C : |z+1| = 1$ in counterclockwise oriented, then $\oint_C \frac{dz}{(z-1)^3(z+1)^3} =$ (B)

(A) $\frac{3\pi}{8}i$ (B) $-\frac{3\pi}{8}i$ $\frac{\frac{1}{(z-1)^3}}{(z+1)^3}$ $= \frac{2\pi i}{2!}\left(\frac{1}{(z-1)^3}\right)''\Big|_{z=-1}$ (C) $\frac{3\pi}{4}i$ (D) $-\frac{3\pi}{4}i$

(4) $z = 2$ is _____ of $e^{\frac{1}{z-2}}$ (D)

(A) analytic point
(B) removable singularity
(C) pole
(D) essential singularity

(5) Let $\mathscr{F}[f(t)] = F(\omega)$, then which of the followings is not correct? (D)

(A) $\mathscr{F}[f(t) * f(t)] = [F(\omega)]^2$
(B) $\delta(-t) = \delta(t)$
(C) $\int_{-\infty}^{+\infty} e^{-i\omega t} dt = 2\pi\delta(\omega)$
(D) $\mathscr{F}[f(at)] = \frac{1}{a}F(\frac{w}{a}),\ a \neq 0,\ a \in \mathbb{R}$ $|a|$

2. Fill in each blank (3 points each, total 30 points)

Score	

C-R equations

(1) Determine the real constant $k =$ __1__ such that $f(z) = e^x(cosky + isinky)$ is analytic.

(2) The trigonometric and exponential forms of $z = \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}$ are __$\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10}$__ and __$e^{i\frac{3\pi}{10}}$__.

$1+i=\sqrt{2}\cdot e^{i(2k\pi+\frac{\pi}{4})}$

$2^{\frac{1}{8}}e^{i\frac{2k\pi+\frac{\pi}{4}}{4}}$, $k=0,1,2,3$

(3) $\sqrt[4]{1+i}=$ $2^{\frac{1}{8}}e^{i\frac{2k\pi+\frac{\pi}{4}}{4}}$ (in exponential form).

(4) Let C be the upper half of the circle $|z|=1$ joining 1 to -1, then $\int_C \frac{1}{\sqrt[3]{z}}dz=$ $-\frac{9}{4}+\frac{3\sqrt{3}}{4}i$.

$z=e^{i\theta}$, $\pi\geq\theta\geq0$ $\int_C\frac{1}{\sqrt[3]{z}}dz=\int_0^\pi e^{-\frac{1}{3}i\theta}ie^{i\theta}d\theta=i\int_0^\pi e^{\frac{2}{3}i\theta}d\theta=i\cdot\frac{e^{\frac{2}{3}i\theta}\big|_0^\pi}{\frac{2}{3}i}=\frac{3}{2}\left(e^{\frac{2\pi}{3}i}-1\right)$

(5) The power series of $\frac{1}{(1-z)^2}$ is $\sum_{n=1}^{\infty}nz^{n-1}$. $\left(\frac{1}{1-z}\right)'=\frac{1}{(1-z)^2}$

6. $\frac{e^{\frac{1}{z}}}{1-z}=(1+z+z^2+\dots+z^n+\dots)\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\dots+\frac{1}{n!z^n}+\dots\right)$
$=\dots+\frac{1}{z}\left(1+\frac{1}{2!}+\dots+\frac{1}{n!}+\dots\right)$

(6) Let $f(z)=\frac{e^{\frac{1}{z}}}{1-z}$, then $\mathrm{Res}[f(z),0]=$ $1+\frac{1}{2!}+\dots+\frac{1}{n!}$. (or $e-1$)

(7) The Fourier transform of $f(t)=\delta(t-1)(t-2)^2\cos t$ is $e^{-i\omega}\cos 1$. $\int_{-\infty}^{\infty}\delta(t-1)(t-2)^2\cos t\,e^{-i\omega t}dt$

(8) What kind of isolated singular point of $f(z)=\frac{z^4}{1+z^4}$ at $z=\infty$? ____.

What kind of isolated singular point of $f(z)=\cot z-\frac{1}{z}$ at $z=0$? removable.

(Please fill in removable, pole and its order, essential singularity, or not isolated singularity).

$\lim_{z\to0}\cot z-\frac{1}{z}$
$=\lim_{z\to0}\frac{z-\sin z}{z\sin z}$
$=\lim_{z\to0}\frac{z-\sin z}{z^2}$
$=\lim_{z\to0}\frac{z-z-\frac{1}{3!}z^3}{z^2}=0$

3. (7 points)

Score	

Verify that

$$f(z)=\begin{cases}0, & z=0\\ \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & z\neq0\end{cases}$$

satisfies C-R equations at the point $z=0$, but it is not analytic.

$f(z)=\frac{x^3-y^3}{x^2+y^2}+i\frac{x^3+y^3}{x^2+y^2}$

$\frac{\partial u}{\partial x}=\lim_{\Delta x\to0}\frac{\frac{\Delta x^3}{\Delta x^2}}{\Delta x}=1$

$\frac{\partial u}{\partial y}=\lim_{\Delta y\to0}\frac{\frac{-\Delta y^3}{\Delta y^2}}{\Delta y}=-1$

Similarly, $\frac{\partial v}{\partial x}=1$ $\frac{\partial v}{\partial y}=1$

$\therefore\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ $\therefore$ u satisfies C-R equations

$\lim_{\Delta z\to0}\frac{\frac{\Delta x^3-\Delta y^3}{\Delta x^2+\Delta y^2}}{\Delta z}\ \underset{\Delta y=k\Delta x}{=}\ \begin{cases}1, & \Delta y=0,\ \Delta x\to0\\ i, & \Delta x=0,\ \Delta y\to0\end{cases}$

$\therefore u(x,y)$ is not differentiable

$\therefore u(x,y)$ is not analytic

4. (5 points each)

Score	

Compute Laurent series of function $f(z) = \dfrac{1}{(z-1)(z-3)}$ in the assigned annulus:

(1) $0 < |z-1| < 1$

$$\frac{1}{z-1}\cdot\frac{1}{z-1-2} = -\frac{1}{2}\cdot\frac{1}{z-1}\cdot\frac{1}{1-\frac{z-1}{2}} = -\frac{1}{2}\cdot\frac{1}{z-1}\cdot\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z-1)^{n-1}$$

or $$\frac{1}{(z-1)(z-3)} = \frac{1}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{z-1-2} - \frac{1}{z-1}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n - \frac{1}{z-1}\right)$$

$$= -\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)} = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z-1)^{n-1}$$

(2) $2 < |z-3| < +\infty$

$$f(z) = \frac{1}{z-3}\cdot\frac{1}{z-3+2}$$

$$= \frac{1}{(z-3)^2}\cdot\frac{1}{1+\frac{2}{z-3}}$$

$$= \frac{1}{(z-3)^2}\cdot\sum_{n=0}^{\infty}(-2)^n\cdot\left(\frac{1}{z-3}\right)^n$$

$$= \sum_{n=0}^{\infty}(-2)^n\frac{1}{(z-3)^{n+2}}$$

$$= \sum_{n=2}^{\infty}(-2)^{n-2}\cdot\frac{1}{(z-3)^n}$$

$\frac{6 \cdot e^{\frac{1}{z}}}{1-z} =$

$(1+\frac{1}{z}+$

$= \cdots + \frac{1}{z}($

$\lim_{z\to 0} \cot z -$

$= \lim_{z\to 0} \frac{z-\sin}{z\sin}$

$= \lim_{z\to 0} \frac{z-}{}$

$= \lim_{z\to 0} \frac{z-}{}$

5. (6 points each)

Score	

Find Laplace transform of the following $f(t)$:

(1) $f(t) = t \sin kt$

$$\mathcal{L}[\sin kt] = \frac{k}{s^2+k^2}$$

$$\mathcal{L}[t\sin kt] = -\left(\frac{k}{s^2+k^2}\right)' = \frac{2ks}{(s^2+k^2)^2}$$

(2) $f(t) = \int_0^t \tau e^{-2\tau} \sin 3\tau d\tau$

$$f'(t) = te^{-2t}\sin 3t$$

$$\cancel{\mathcal{L}[f'(t)] =}$$

$$\mathcal{L}[\sin 3t \cdot e^{-2t}] = \frac{3}{(s+2)^2+9}$$

$$\mathcal{L}[f'(t)] = -\left(\frac{3}{(s+2)^2+9}\right)' = \frac{6(s+2)}{((s+2)^2+9)^2}$$

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

$$\therefore F(s) = \frac{6(s+2)}{s[(s+2)^2+9]^2}$$

6. (6 points each)

Score	

Evaluate the following integrals:

(1) $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} \quad (a>1).$

$z=e^{i\theta}$

$= \oint_{|z|=1} \frac{1}{a+\frac{z+z^{-1}}{2}} \cdot \frac{1}{iz}\, dz$

$= \oint_{|z|=1} \frac{1}{i} \cdot \frac{2}{z^2+2az+1}\, dz$

$= \frac{1}{i} \oint_{|z|=1} \frac{1}{(z-\alpha)(z-\beta)}\, dz$

$\alpha = -a+\sqrt{a^2-1} \qquad \beta = -a-\sqrt{a^2-1} \qquad |\alpha|<1 \quad |\beta|>1$

$\alpha \cdot \beta = 1$

$\therefore \mathrm{Res}\left[\frac{1}{(z-\alpha)(z-\beta)}, \alpha\right] = \frac{1}{2\sqrt{a^2-1}}$

$\therefore I = \frac{2}{i} \cdot 2\pi i \cdot \frac{1}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$

(2) $\int_{-\infty}^{+\infty} \frac{(\cos 3x)^2}{x^2+4x+5}\, dx$

$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1+\cos 6x}{(x+2)^2+1}\, dx$

$= \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{1}{(x+2)^2+1}\, dx + \int_{-\infty}^{\infty} \frac{\cos 6x}{(x+2)^2+1}\right.$

$f(z) = \frac{1}{(z+2)^2+1} = \frac{1}{(z+2+i)(z+2-i)}$

$z=-2+i$

$\therefore \int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\, dx = 2\pi i\, \mathrm{Res}\left[f(z), -2+i\right] = 2\pi i \cdot \frac{1}{2i} = \pi$

$\int_{-\infty}^{\infty} \frac{\cos 6x}{x^4+4x+5}\, dx = \mathrm{Re}\left\{2\pi i \mathrm{Res}\left[f(z)e^{6zi}, -2+i\right]\right\}$

$= \mathrm{Re}\left\{2\pi i \cdot \frac{e^{-6-12i}}{2i}\right\} = \pi e^{-6}\cos 12$

$\therefore I = \frac{1}{2}\left(\pi + \pi e^{-6}\cos 12\right)$

7. (6 points)

Score	

Solve the following differential equation using Laplace transform:

$y'' + 3y' + y = 3\cos t$, $y(0) = 0$, $y'(0) = 1$.

$\mathcal{L}[y'] = sY(s) - \cancel{s}y(0) = sY(s)$

$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1$

$\mathcal{L}[\cos t] = \frac{s}{s^2+1}$

$\therefore s^2Y(s) - 1 + 3sY(s) + Y(s) = \frac{3s}{s^2+1}$

$(s^2+3s+1)Y(s) = \frac{3s}{s^2+1} + 1$

$Y(s) = \frac{1}{s^2+1}$

$\therefore y(t) = \sin t$

8. (8 points)

Score	

Given $\mathscr{F}[e^{-\beta t^2}] = \sqrt{\frac{\pi}{\beta}}e^{-\frac{\omega^2}{4\beta}}, \beta > 0$. Let $f(t) = te^{-t^2}$, then

(1) Find the Fourier transform of $f(t)$.

(2) Prove that $\int_0^{+\infty} \omega e^{-\frac{\omega^2}{4}} \sin \omega t d\omega = 2\sqrt{\pi}te^{-t^2}$.

$F(w) = \mathcal{F}[te^{-\beta t^2}] = i \cdot \frac{d}{dw}(\mathcal{F}[e^{-t^2}]) = \sqrt{\pi}i(e^{-\frac{w^2}{4}})' = \sqrt{\pi}i(-\frac{w}{2})e^{-\frac{w^2}{4}}$

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwt}dt$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{\sqrt{\pi}i}{2} we^{-\frac{w^2}{4}}e^{iwt}dw$

$= \frac{-i}{4\sqrt{\pi}}\int_{-\infty}^{\infty} we^{-\frac{w^2}{4}}(\cos wt + i\sin wt)dw$

$= \frac{-i \cdot i}{\cancel{4}2\sqrt{\pi}}\int_0^{\infty} we^{-\frac{w^2}{4}}\sin wt\,dw$

$= \frac{1}{2\sqrt{\pi}}\int_0^{\infty} we^{-\frac{w^2}{4}}\sin wt\,dw = te^{-t^2}$

$\therefore \int_0^{+\infty} we^{-\frac{w^2}{4}}\sin wt\,dw = 2\sqrt{\pi}te^{-t^2}$