B 2019.12.30

浙江工业大学2019/2020（一）期末试卷

《复变函数与积分变换》

《Final Exam for 'Functions of a Complex Variable and Integral Transforms'》

Class:__________ Name (in Chinese):____________ Student ID:____________

Please write in English! Good luck!

Question	1	2	3	4	5	6	7	8	Total
Score									

1. Choose only one correct answer to each question （3 points each, total 15 points）

(1) If $z = \frac{1}{\sqrt{2}}(1-i)$, then $z^{100} =$ (D)

$(e^{-i\frac{\pi}{4}})^{100} = e^{-i25\pi} = \cos 25\pi - i\sin 25\pi$
$= \cos\pi$

(A) -i (B) i (C) 1 (D) -1

(2) Let $f(t) = te^{iw_0t}$, then $\mathscr{F}[f(t)] =$ (C)

(A) $2\pi\delta'(w-w_0)$ (B) $2\pi\delta'(w+w_0)$ (C) $2\pi i\delta'(w-w_0)$ (D) $2\pi i\delta'(w+w_0)$

(3) If $C: |z-1| = \frac{1}{2}$ in counterclockwise oriented, then $\oint_C \frac{\cos z}{z^2}dz =$ (B)

(A) $2\pi i$ (B) 0 (C) πi (D) 1

(4) Let $F(s) = \frac{5}{(s+1)^2+25}$, then $f(t) = \mathscr{L}^{-1}[F(s)] =$ (B)

(A) $e^{-t}\cos 5t$ (B) $e^{-t}\sin 5t$ (C) $e^t\cos 5t$ (D) $e^t\sin 5t$

(5) Which of the followings is not correct? (B)

(A) $\delta(-t) = \delta(t)$ (B) $\mathscr{F}[u(t)] = \frac{1}{w} + \pi\delta(w)$

(C) $\mathscr{F}[e^{-at}] = \dfrac{1}{a+iw}$ (D) $\mathscr{F}[sgnt] = \frac{2}{iw}$

2. Fill in each blank (3 points each, total 30 points)

(1) $\sqrt[3]{-2+2i} =$ $\sqrt{2}e^{\frac{\frac{3}{4}\pi+2k\pi}{3}i}$. $k=0,1,2$ $\sqrt{4+4} = 2^{\frac{3}{2}}\cdot e^{\frac{3}{4}\pi i}$

(2) $\lim\limits_{z\to1+i}\frac{\bar{z}}{z} =$ $-i$. $\frac{1-i}{1+i} = \frac{(1-i)^2}{1+1} = \frac{1+i^2-2i}{2}$

(3) Let $C: |z| = 1$, then when $|a| < 1$, $\oint_C \frac{\sin z}{(z-a)^2}dz =$ $2\pi i\cos a$; $2\pi i\cdot(\sin z)'|_{z=a}$

when $|a| > 1$, $\oint_C \frac{\sin z}{(z-a)^2}dz =$ 0.

(4) The radius of the convergence of $\sum\limits_{n=0}^{\infty}\frac{n}{2^n}z^n$ is 2.

$\sqrt[n]{\frac{n}{2^n}} \to \frac{1}{2}$

(5) The Taylor series of $\frac{1}{1+z}$ around $z = 0$ is $\sum_{n=0}^{\infty}(-1)^n z^n$.

$\lim_{z \to -i} \frac{z^2}{z-i} = \frac{i^2}{-2i} = -\frac{i}{2}$

(6) Let $f(z) = \frac{z^2}{1+z^2}$, then $\text{Res}[f(z), -i] =$ $-\frac{i}{2}$, $\text{Res}[f(z), i] =$ $\frac{i}{2}$.

(7) What kind of isolated singular point of $\frac{\sin z - z}{z^3}$ at $z = 0$? removable.

What kind of isolated singular point of $f(z) = \frac{\sin z}{z^3}$ at $z = 0$? pole of order 2.

$\frac{z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - z}{z^3}$

$\frac{1 - \frac{z^2}{3!} + \cdots}{z^2}$

(Please fill in removable singularity, pole and its order, essential singularity, or not isolated singularity).

3. (7 points)

Given a function $v(x, y) = \arctan \frac{y}{x}$ $(x > 0)$, find an analytic function $f(z) = u(x, y) + iv(x, y)$.

$$\frac{\partial v}{\partial x} = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2+y^2}$$

$$u = \int \frac{\partial u}{\partial x} dx = \int \frac{x}{x^2+y^2} dx = \frac{1}{2}\ln(x^2+y^2) + c(y)$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} + c'(y) = \frac{y}{x^2+y^2}$$

$$c'(y) = 0 \qquad c(y) = c$$

$$\therefore u = \frac{1}{2}\ln(x^2+y^2) + c$$

$$\therefore f(z) = \frac{1}{2}\ln(x^2+y^2) + c + i\arctan\frac{y}{x}$$

$$= \frac{1}{2}\ln|z|^2 + i\arg z + c$$

$$= \ln|z| + i\arg z + c$$

$$= \ln z + c$$

4. (4 points each)

Compute Laurent series of function $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the assigned annulus:

(1) $0 < |z-1| < 1$

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z-1-1} = -\frac{1}{z-1} \cdot \frac{1}{1-(z-1)} = -\frac{1}{z-1} \cdot \sum_{n=0}^{\infty} (z-1)^n = -\sum_{n=0}^{\infty} (z-1)^{n-1}$$

(2) $1 < |z-1| < +\infty$

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z-1-1} = \frac{1}{(z-1)^2} \cdot \frac{1}{1-\frac{1}{z-1}} = \frac{1}{(z-1)^2} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{z-1}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(z-1)^{n+2}}$$

(3) $0 < |z-2| < 1$

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z-2} = \frac{1}{z-2} \cdot \frac{1}{z-2+1} = \frac{1}{z-2} \cdot \sum_{n=0}^{\infty} (-1)^n (z-2)^n = \sum_{n=0}^{\infty} (-1)^n (z-2)^{n-1}$$

5. (5 points each)

Find Laplace transform of the following $f(t)$:

(1) $f(t) = e^{-at} \sin kt$

$$\mathcal{L}[\sin kt] = \frac{k}{s^2+k^2}$$

$$\mathcal{L}[e^{-at}\sin kt] = \frac{k}{(s+a)^2+k^2}$$

(2) $f(t) = t\cos t$.

$$\mathcal{L}[\cos t] = \frac{s}{s^2+1}$$

$$\mathcal{L}[t\cos t] = -\left(\frac{s}{s^2+1}\right)' = -\frac{s^2+1-s\cdot 2s}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

6. (6 points each)

Evaluate the following integrals:

(1) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 2x + 2}$

$$f(z) = \frac{1}{z^2+2z+2} = \frac{1}{(z+1)^2+1} = \frac{1}{(z+1+i)(z+1-i)}$$

$$= 2\pi i \, \mathrm{Res}\left[f(z),\ i-1\right]$$

$$= 2\pi i \lim_{z \to i-1} \frac{1}{z+1+i} = 2\pi i \, \frac{1}{i-1+1+i} = \frac{2\pi i}{2i} = \pi$$

(2) $\displaystyle\int_{0}^{2\pi} \frac{d\theta}{2 + \cos\theta}.$

$$= \oint_{|z|=1} \frac{1}{2+\frac{z+\frac{1}{z}}{2}} \cdot \frac{1}{iz}\, dz$$

$$= \oint_{|z|=1} \frac{2}{i} \cdot \frac{1}{4z+z^2+1}\, dz$$

$$= \frac{2}{i} \oint_{|z|=1} \frac{1}{z^2+4z+4-3}\, dz$$

$$= \frac{2}{i} \oint_{|z|=1} \frac{1}{(z+2+\sqrt{3})(z+2-\sqrt{3})}\, dz$$

$$= \frac{2}{i} \cdot 2\pi i \, \mathrm{Res}\left\{f(z), \sqrt{3}-2\right\}$$

$$= 4\pi \cdot \lim_{z \to \sqrt{3}-2} \frac{1}{z+2+\sqrt{3}} = 4\pi \cdot \frac{1}{\sqrt{3}-2+2+\sqrt{3}} = \frac{4\pi}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

7. (7 points)

Solve the following differential equation using Laplace transform:
$y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$.

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 3Y(s) = \frac{1}{s+1}$$

$$(s^2+4s+3)Y(s) - s - 1 - 4 = \frac{1}{s+1}$$

$$(s+1)(s+3)Y(s) = s+5+\frac{1}{s+1}$$

$$Y(s) = \frac{s^2+6s+5+1}{(s+1)^2(s+3)} = \frac{1}{2}\cdot\frac{1}{(s+1)^2} + \frac{7}{4}\cdot\frac{1}{s+1} - \frac{3}{4}\frac{1}{s+3}$$

$$y(t) = \frac{1}{2}te^{-t} + \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t}$$

8. (7 points) Suppose that $f(z)$ and $g(z)$ are analytic at $z = a$ and $f(a) \neq 0$. If $g(z)$ has a zero of order 2 at $z = a$, then prove that $\text{Res}[\frac{f(z)}{g(z)}, a] = \dfrac{6f'(a)g''(a) - 2f(a)g'''(a)}{3[g''(a)]^2}$.

$$g(z) = (z-a)^2\phi(z), \quad \phi(a) = 0$$

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)^2\phi(z)}$$

$$\text{Res}\left[\frac{f(z)}{g(z)}, a\right] = \lim_{z\to a}\left(\frac{f(z)}{\phi(z)}\right)' = \frac{f'(z)\phi(z) - f(z)\phi'(z)}{\phi^2(z)} = \frac{f'(a)\phi(a) - f(a)\phi'(a)}{\phi^2(a)}$$

$$g(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots = (z-a)^2(c_2 + c_3(z-a) + \cdots)$$

$c_0 = 0$, $c_1 = 0$, $c_2 \neq 0$ $\qquad \phi(a) = c_2 = \frac{g''(a)}{2!}$ $\qquad \phi'(a) = c_3 = \frac{g'''(a)}{3!}$

$$\therefore \text{Res}\left[\frac{f(z)}{g(z)}, a\right] = \frac{f'(a)\frac{g''(a)}{2} - f(a)\frac{g'''(a)}{6}}{\left(\frac{g''(a)}{2}\right)^2} = \frac{6f'(a)g''(a) - 2f(a)g'''(a)}{3[g''(a)]^2}$$