# A Fluid Dynamic Model for the Movement of Pedestrians

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**Abstract**

A kind of fluid dynamic description for the collective movement of pedestrians is developed on the basis of a BOLTZMANN-like gaskinetic model. The differences between these pedestrian specific equations and those for ordinary fluids are worked out, for example concerning the mechanism of relaxation to equilibrium, the role of "pressure", the special influence of internal friction and the origin of "temperature". Some interesting results are derived that can be compared to real situations, for example the development of walking lanes and of pedestrian jams, the propagation of waves, and the behavior on a dance floor. Possible applications of the model to town- and traffic-planning are outlined.

## 1 Introduction

Former publications on the behavior of pedestrians have been mostly empirical (often in the sense of regression analyses) and had the intention to allow planning of efficient traffic [16, 24, 34]. Meanwhile there also exist theoretical approaches to pedestrian movement [2, 3, 7, 9, 12, 29, 32, 33]. However, most theoretical work has been done in the related topic of automobile traffic (see, for example, [1, 6, 8, 26, 27, 28]). Especially, there have been developed some BOLTZMANN-like (gaskinetic) approaches [1, 26, 27].

As far as pedestrian movement is concerned, the author has observed that footprints of pedestrian crowds in the snow or quick-motion pictures of pedestrians look similar to streamlines of fluids. It is the object of this paper to give a suitable explanation of the fluid dynamic properties of pedestrian crowds. HENDERSON was the first, who compared gaskinetic and fluid dynamic models to empirical data of pedestrian crowds [12, 13, 14, 15]. However, his work started from the conventional theory for ordinary fluids, and assumed a *conservation of momentum and energy* to be fulfilled. In contrast to HENDERSON's approach, this article develops a *special theory for pedestrians*—without making use of the unrealistic conservation assumptions.

We shall proceed in the following way: First the pedestrians will be distinguished into groups of different types $\mu$ of motion, normally representing different intended directions of walking. At a time $t$ the pedestrians of each type $\mu$ of motion can be characterized by several quantities like their place $\vec{x}$, their velocity $\vec{v}_\mu$ and their intended velocity $\vec{v}^0_\mu$ (that means the velocity they *want* to walk with). So, in a given area $A$ can be found a density $\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t)$ of pedestrians having a special type of motion $\mu$ and showing approximately the quantities $\vec{x}$, $\vec{v}_\mu$, $\vec{v}^0_\mu$ at time $t$. For the densities $\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t)$ equations of motion can be set up (section 2). From these equations we shall derive coupled differential equations for the spatial density $\langle \rho_\mu \rangle$ of pedestrians, their mean velocity $\langle v_\mu \rangle$ and velocity variance $\langle (\delta v_{\mu,i})^2 \rangle$ (section 3). The resulting equations show many similarities to the equations for ordinary fluids, but they contain some additional terms which take into account pedestrian intentions and interactions (sections 3.1, 4.1, 5.1 and 6). In section 4 we shall treat equilibrium situations and the propagation of density waves. However, in *non*equilibrium situations the final adaptation time to local equilibrium gives rise to internal friction (viscosity) and other additional terms (section 5). Effects of interactions (that means of avoiding maneuvers) between pedestrians will be discussed in section 6. These effects will lead to some conclusions applicable to town- and traffic-planning (section 7).

Readers who are not interested in the mathematical aspects may skip the formulas in the following. However, the mathematical results are important for analytical, computational or empirical evaluations.

## 2 Gaskinetic equations

Pedestrians can be distinguished into different *types $\mu$ of motion*, for example, by their different *intended directions* $\vec{e}_\mu := \vec{v}^0_\mu / \|\vec{v}^0_\mu\|$ of motion (normally 2 opposite directions, at crossings 4 directions). More exactly speaking, a pedestrian shall belong to a type $\mu$ of motion if it wants to walk with an *intended velocity*

$$\vec{v}^0 \in \mathcal{N}_\mu \, ,$$

where

$$\mathcal{N}_\mu := \{\vec{v}^0_\mu\}$$

is one of several disjoint and complementary sets. A type $\mu$ of motion still contains pedestrians with a variety of intended velocities $\vec{v}^0_\mu$, but the advantage resulting from a suitable choice of the sets $\mathcal{N}_\mu$ is to get approximately *unimodal* densities $\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t)$ and therefore to obtain appropriate mean value equations (see section 3). $\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t)$ describes the number $N_\mu$ of pedestrians of type $\mu$ within an area $A = A(\vec{x})$ around place $\vec{x}$ having approximately the *intended* velocity $\vec{v}^0_\mu$, but approximately the *actual* velocity $\vec{v}_\mu$. More exactly, $\hat{\rho}_\mu$ is defined by

$$\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t) \equiv \hat{\rho}_\mu(\vec{x}, \vec{u}_\mu, t) := \frac{N_\mu(\mathcal{U}(\vec{x}) \times \mathcal{V}(\vec{u}_\mu), t)}{A \cdot V} \, , \tag{1}$$

where $N_\mu$ is the number of pedestrians of type $\mu$ which are at time $t$ in a *state*

$$(\vec{x}', \vec{u}'_\mu) \in \mathcal{U}(\vec{x}) \times \mathcal{V}(\vec{u}_\mu)$$

out of the neighbourhood $\mathcal{U}(\vec{x}) \times \mathcal{V}(\vec{u}_\mu)$ of $\vec{x}$ and

$$\vec{u}_\mu := (\vec{v}_\mu, \vec{v}^0_\mu) \, .$$

State $(\vec{x}, \vec{u}_\mu)$ is an abbreviation for the property

$$(\vec{x}, \vec{u}_\mu) := (\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0) \,,$$

that an individual is at place $\vec{x}$ and wants to walk with the intended velocity $\vec{v}_\mu^0$, but in fact walks with velocity $\vec{v}_\mu$.

$$\mathcal{U}(\vec{x}) := \{\vec{x}^* \in \mathcal{M} : \|\vec{x}^* - \vec{x}\|_l \leq r\} \tag{2}$$

is a neighbourhood around the place $\vec{x}$ and belongs to the domain $\mathcal{M}$, which represents all *accessible* (or public) places $\vec{x}$. $A = A(\vec{x})$ denotes the area of $\mathcal{U}(\vec{x})$. Similarly,

$$\mathcal{V}(\vec{u}_\mu) := \{\vec{u}_\mu^* = (\vec{v}_\mu^*, \vec{v}_\mu^{0*}) : \|\vec{u}_\mu^* - \vec{u}_\mu\|_l \leq s, \vec{v}_\mu^0 \in \mathcal{N}_\mu\}$$

is a neighbourhood of $\vec{u}_\mu := (\vec{v}_\mu, \vec{v}_\mu^0)$ with a volume $V = V(\vec{u}_\mu)$.

We shall now establish a set of *continuity equations*, which are similar to the *ansatz* of Alberti and Belli [1]:

$$\begin{aligned}
\frac{d\hat{\rho}_\mu}{dt} &\equiv \frac{\partial \hat{\rho}_\mu}{\partial t} + \nabla_{\vec{x}} \left(\hat{\rho}_\mu \vec{v}_\mu\right) + \nabla_{\vec{v}_\mu} \left(\hat{\rho}_\mu \frac{\vec{f}_\mu}{m_\mu}\right) + \nabla_{\vec{v}_\mu^0} \left(\hat{\rho}_\mu \dot{\vec{v}}_\mu^0\right) \\
&:= \frac{\hat{\rho}_\mu^0 - \hat{\rho}_\mu}{\tau_\mu} + \sum_\nu \hat{S}_{\mu\nu} + \sum_\nu \hat{C}_{\mu\nu} + \hat{q}_\mu \,.
\end{aligned} \tag{3}$$

These equations can be interpreted as *gaskinetic equations* (see chapters 2.4, 2.7 of Ref. [18], and §3 of Ref. [19]). $m_\mu$ denotes the average mass of pedestrians belonging to type $\mu$ of motion. Apart from special situations it will not depend on $\mu$, that means $m_\mu \approx m_\nu$. The forces $\vec{f}_\mu := m_\mu \dot{\vec{v}}_\mu$ can often be neglected. However, they may be locally varying functions, depending on the attraction of the places $\vec{x}$. The temporal change $\dot{\vec{v}}_\mu^0$ of the intended velocity $\vec{v}_\mu^0$ can in principal be a function of place $\vec{x}$ and time $t$, but it is normally a small quantity ($\dot{\vec{v}}_\mu^0 \approx \vec{0}$). Otherwise, the type $\mu$ of motion is changed.

According to (3) the change of the density $\hat{\rho}_\mu$ with time is given by four effects:

- First by the tendency of the pedestrians to reach their intended velocity $\vec{v}_\mu^0$ [1, 9]. This causes $\hat{\rho}_\mu$ to approach

  $$\hat{\rho}_\mu^0(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t) := \delta(\vec{v}_\mu - \vec{v}_\mu^0)\rho_\mu^0(\vec{x}, \vec{v}_\mu^0, t) \tag{4}$$

  (the equilibrium density in the absence of disturbances) with a relaxation time

  $$\tau_\mu \equiv \frac{m_\mu}{\gamma_\mu}$$

  (see [9]). $\rho_\mu^0$ is the density of pedestrians with intended velocity $\vec{v}_\mu^0$ but arbitrary actual velocity $\vec{v}_\mu$. $\delta(.)$ denotes the Dirac delta function, which is different from zero only where its argument (.) vanishes.

- Second by the interaction of pedestrians, which can be modelled by a BOLTZMANN-like *stosszahlansatz* [10, 18, 19]. If we take into account that the interactions are of short range (in comparison with $r$, see (2)), we have:

$$\begin{aligned} \hat{S}_{\mu\nu} &= \iiint \hat{\sigma}_{\mu\nu}(\vec{u}^*_\mu, \vec{u}^*_\nu; \vec{u}_\mu, \vec{u}_\nu; \vec{x}, t)\hat{\rho}_\mu(\vec{x}, \vec{u}^*_\mu, t)\hat{\rho}_\nu(\vec{x}, \vec{u}^*_\nu, t) d^4\vec{u}_\nu \, d^4\vec{u}^*_\mu \, d^4\vec{u}^*_\nu \\ &- \iiint \hat{\sigma}_{\mu\nu}(\vec{u}_\mu, \vec{u}_\nu; \vec{u}^*_\mu, \vec{u}^*_\nu; \vec{x}, t)\hat{\rho}_\mu(\vec{x}, \vec{u}_\mu, t)\hat{\rho}_\nu(\vec{x}, \vec{u}_\nu, t) d^4\vec{u}_\nu \, d^4\vec{u}^*_\mu \, d^4\vec{u}^*_\nu \, . \end{aligned} \quad (5)$$

This term describes *pair interactions* between pedestrians of types $\mu$ and $\nu$ occuring with a total rate proportional to the densities $\hat{\rho}_\mu$ and $\hat{\rho}_\nu$ of both interacting types of motion. The *relative rate* for pedestrians of types $\mu$ and $\nu$ to change their states from $(\vec{x}, \vec{u}_\mu)$, $(\vec{x}, \vec{u}_\nu)$ to $(\vec{x}, \vec{u}^*_\mu)$, $(\vec{x}, \vec{u}^*_\nu)$ due to interactions is given by $\hat{\sigma}_{\mu\nu}(\vec{u}_\mu, \vec{u}_\nu; \vec{u}^*_\mu, \vec{u}^*_\nu; \vec{x}, t)$. Assuming that only the actual velocities $\vec{v}_\mu$, $\vec{v}_\nu$ and not the intended velocities $\vec{v}^0_\mu$, $\vec{v}^0_\nu$ are affected by interactions, we obtain

$$\hat{\sigma}_{\mu\nu}(\vec{u}^1_\mu, \vec{u}^1_\nu; \vec{u}^2_\mu, \vec{u}^2_\nu; \vec{x}, t) = \sigma_{\mu\nu}(\vec{v}^1_\mu, \vec{v}^1_\nu; \vec{v}^2_\mu, \vec{v}^2_\nu)\delta(\vec{v}^{0,2}_\mu - \vec{v}^{0,1}_\mu)\delta(\vec{v}^{0,2}_\nu - \vec{v}^{0,1}_\nu).$$

This results in

$$\begin{aligned} S_{\mu\nu}(\vec{x}, \vec{v}_\mu, t) &:= \int \hat{S}_{\mu\nu}(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t) d^2\vec{v}^0_\mu \\ &= \iint \sigma_{\mu\nu}(\vec{v}^*_\mu, \vec{v}^*_\nu; \vec{v}_\mu, \vec{v}_\nu)\rho_\mu(\vec{x}, \vec{v}^*_\mu, t)\rho_\nu(\vec{x}, \vec{v}^*_\nu, t) d^2\vec{v}_\nu \, d^2\vec{v}^*_\mu \, d^2\vec{v}^*_\nu \\ &- \iint \sigma_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu, \vec{v}^*_\nu)\rho_\mu(\vec{x}, \vec{v}_\mu, t)\rho_\nu(\vec{x}, \vec{v}_\nu, t) d^2\vec{v}_\nu \, d^2\vec{v}^*_\mu \, d^2\vec{v}^*_\nu \quad (6) \\ &= \iint \sigma^*_{\mu\nu}(\vec{v}^*_\mu, \vec{v}^*_\nu; \vec{v}_\mu)\rho_\mu(\vec{x}, \vec{v}^*_\mu, t)\rho_\nu(\vec{x}, \vec{v}^*_\nu, t) d^2\vec{v}^*_\mu \, d^2\vec{v}^*_\nu \\ &- \iint \sigma^*_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu)\rho_\mu(\vec{x}, \vec{v}_\mu, t)\rho_\nu(\vec{x}, \vec{v}_\nu, t) d^2\vec{v}_\nu \, d^2\vec{v}^*_\mu \quad (7) \end{aligned}$$

with

$$\sigma^*_{\mu\nu}(.,.;.) := \int \sigma_{\mu\nu}(.,.;.,\vec{v}) d^2\vec{v} \, , \quad (8)$$

which is used later on. (6) is similar to (5), and can be interpreted analogously. The explicit form of $\sigma^*_{\mu\nu}$ will be based on a *microscopic model* for the interactions and is discussed in section 6.

- Third by pedestrians changing their type $\mu$ of motion to another type $\nu$ of motion, for example when turning to the right or left at a crossing or when turning back (change of the intended direction). This can be modelled by

$$\begin{aligned} \hat{C}_{\mu\nu}(\vec{x}, \vec{u}_\mu, t) &= \int \hat{\sigma}^{\nu\mu}_\mu(\vec{u}_\nu; \vec{u}_\mu; \vec{x}, t)\hat{\rho}_\nu(\vec{x}, \vec{u}_\nu, t) d^4\vec{u}_\nu \\ &- \int \hat{\sigma}^{\mu\nu}_\mu(\vec{u}_\mu; \vec{u}_\nu; \vec{x}, t)\hat{\rho}_\mu(\vec{x}, \vec{u}_\mu, t) d^4\vec{u}_\nu \end{aligned}$$

with a transition rate proportional to the density of the changing type of motion.

If we assume that for the moment of changing the *intended* velocity $\vec{v}^0_\mu$ the *actual* velocity $\vec{v}_\mu$ still remains the same (but of course not thereafter), we have

$$\hat{\sigma}^{1,2}_\mu(\vec{u}_1; \vec{u}_2; \vec{x}, t) = \hat{\sigma}^{1,2}_\mu(\vec{v}^0_1; \vec{v}^0_2; \vec{x}, t)\delta(\vec{v}_2 - \vec{v}_1) \, .$$

This results in

$$C_{\mu\nu} := \int \hat{C}_{\mu\nu}(\vec{x}, \vec{u}_\mu, t) d^2\vec{v}^{\,0}_\mu = \sigma^{\nu\mu}_\mu(\vec{x}, \vec{v}_\mu, t)\rho_\nu(\vec{x}, \vec{v}_\mu, t) - \sigma^{\mu\nu}_\mu(\vec{x}, \vec{v}_\mu, t)\rho_\mu(\vec{x}, \vec{v}_\mu, t)\,, \quad (9)$$

where

$$\sigma^{1,2}_\mu(\vec{x}, \vec{v}_\mu, t) := \iint \hat{\sigma}^{1,2}_\mu(\vec{v}^{\,0}_1; \vec{v}^{\,0}_2; \vec{x}, t) \frac{\hat{\rho}_1(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_1, t)}{\rho_1(\vec{x}, \vec{v}_\mu, t)} d^2\vec{v}^{\,0}_1\, d^2\vec{v}^{\,0}_2$$

and

$$\rho_\mu(\vec{x}, \vec{v}_\mu, t) := \int \hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t) d^2\vec{v}^{\,0}_\mu\,. \quad (10)$$

- Fourth by the density gain $\hat{q}^+_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t)$ or density loss $\hat{q}^-_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t)$ per time unit, caused by pedestrians which enter or leave the system $\mathcal{M}$ at a marginal place $\vec{x} \in \partial\mathcal{M}$ (for example a house) with the intended velocity $\vec{v}^{\,0}_\mu := \vec{v}^{\,0} \in \mathcal{N}_\mu$ and the actual velocity $\vec{v}_\mu$:

$$\hat{q}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t) := \hat{q}^+_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t) - \hat{q}^-_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t)\,. \quad (11)$$

# 3 Macroscopic equations

For further discussion we need the notations

$$\begin{aligned}
\langle\rho_\mu\rangle &:= \int \rho_\mu(\vec{x}, \vec{v}_\mu, t) d^2\vec{v}_\mu = \int \hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t) d^2\vec{v}_\mu\, d^2\vec{v}^{\,0}_\mu\,, \\
\langle\psi_\mu(\vec{v}_\mu, \vec{v}^{\,0}_\mu)\rangle &:= \int \psi(\vec{v}_\mu, \vec{v}^{\,0}_\mu) \frac{\hat{\rho}(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t)}{\langle\rho_\mu\rangle} d^2\vec{v}_\mu\, d^2\vec{v}^{\,0}_\mu\,, \\
\delta\vec{v}_\mu &:= \vec{v}_\mu - \langle\vec{v}_\mu\rangle\,, \\
\delta\vec{v}^{\,0}_\mu &:= \vec{v}^{\,0}_\mu - \langle\vec{v}^{\,0}_\mu\rangle\,, \\
\varrho_\mu(\vec{x}, \vec{v}_\mu, t) &:= m_\mu \rho_\mu(\vec{x}, \vec{v}_\mu, t)\,, && (12) \\
p_{\mu,\alpha\beta} &:= \langle\varrho_\mu\rangle \langle\delta v_{\mu,\alpha}\delta v_{\mu,\beta}\rangle = \int \delta v_{\mu,\alpha}\delta v_{\mu,\beta}\varrho_\mu(\vec{x}, \vec{v}_\mu, t) d^2\vec{v}_\mu\,, && (13) \\
\vec{j}_{\mu,i} &:= \langle\varrho_\mu\rangle \left\langle \delta\vec{v}_\mu \frac{(\delta v_{\mu,i})^2}{2} \right\rangle = \int \delta\vec{v}_\mu \frac{(\delta v_{\mu,i})^2}{2} \varrho_\mu(\vec{x}, \vec{v}_\mu, t) d^2\vec{v}_\mu\,, && (14) \\
\chi_{\mu\nu}\left(\psi_\mu(\vec{v})\right) &:= \iiint \psi_\mu(\vec{v})\sigma^*_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^{\,*}_\mu) \frac{\rho_\mu(\vec{x}, \vec{v}_\mu, t)}{\langle\rho_\mu\rangle} \frac{\rho_\nu(\vec{x}, \vec{v}_\nu, t)}{\langle\rho_\nu\rangle} d^2\vec{v}_\mu\, d^2\vec{v}_\nu\, d^2\vec{v}^{\,*}_\mu\,, && (15) \\
\chi^{1,2}_\mu\left(\psi_\mu(\vec{v}_1)\right) &:= \int \psi_\mu(\vec{v}_\mu)\sigma^{1,2}_\mu(\vec{x}, \vec{v}_\mu, t) \frac{\rho_1(\vec{x}, \vec{v}_\mu, t)}{\langle\rho_1\rangle} d^2\vec{v}_\mu \\
&= \int \psi_\mu(\vec{v}_1)\sigma^{1,2}_\mu(\vec{x}, \vec{v}_1, t) \frac{\rho_1(\vec{x}, \vec{v}_1, t)}{\langle\rho_1\rangle} d^2\vec{v}_1\,, \\
q_\mu(\vec{x}, \vec{v}_\mu, t) &:= \int \hat{q}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^{\,0}_\mu, t) d^2\vec{v}_\mu\, d^2\vec{v}^{\,0}_\mu\,, && (16) \\
Q_\mu\left(\frac{\psi_\mu(\vec{v}_\mu)}{m_\mu}\right) &:= \int \frac{\psi_\mu(\vec{v}_\mu)}{m_\mu} m_\mu q_\mu(\vec{x}, \vec{v}_\mu, t) d^2\vec{v}_\mu\,.
\end{aligned}$$

Here, $\psi_\mu(\vec{v}_\mu, \vec{v}^{\,0}_\mu)$ is some arbitrary function of $\vec{v}_\mu$ and $\vec{v}^{\,0}_\mu$.

As far as pedestrians of type $\mu$ are concerned, we are mostly interested in their density $\langle\rho_\mu\rangle$, their mean velocity $\langle\vec{v}_\mu\rangle$ and the variance $\langle(\delta v_{\mu,i})^2\rangle$ of their velocity components $v_{\mu,i}$

(at a given place $\vec{x}$ and time $t$). Since it is formally equivalent and more comparable to fluid dynamics, we shall instead search equations for the *mass density*

$$\langle \varrho_\mu \rangle := m_\mu \langle \rho_\mu \rangle \, ,$$

the mean *momentum density*

$$\langle \rho_\mu \rangle \langle m_\mu \vec{v}_\mu \rangle = \langle \varrho_\mu \rangle \langle \vec{v}_\mu \rangle$$

and the mean *energy density* (in direction $i$)

$$\langle \epsilon_{\mu,i} \rangle := \langle \rho_\mu \rangle \left\langle \frac{m_\mu}{2} v_{\mu,i}^2 \right\rangle = \langle \varrho_\mu \rangle \frac{\langle v_{\mu,i} \rangle^2}{2} + \langle \varrho_\mu \rangle \left\langle \frac{(\delta v_{\mu,i})^2}{2} \right\rangle .$$

By multiplication of (3) with $\psi_\mu(\vec{v}_\mu) = m_\mu$, $m_\mu \vec{v}_\mu$ or $m_\mu v_{\mu,i}^2/2$ and integration over $\vec{u}_\mu$ one obtains the following equations (keeping in mind that the GAUSSIAN surface integrals vanish) (see chapter 2.10 of Ref. [18]):

$$\begin{aligned} \frac{\partial \langle \varrho_\mu \rangle}{\partial t} &= -\frac{\partial}{\partial x_{\mu,\alpha}} \left( \langle \varrho_\mu \rangle \langle v_{\mu,\alpha} \rangle \right) + Q_\mu(1) && \text{(17a)} \\ &+ \sum_\nu \left[ \frac{m_\mu}{m_\nu} \langle \varrho_\nu \rangle \chi_\mu^{\nu\mu}(1) - \langle \varrho_\mu \rangle \chi_\mu^{\mu\nu}(1) \right] && \text{(17b)} \end{aligned}$$

for the mass density,

$$\begin{aligned} \frac{\partial \left( \langle \varrho_\mu \rangle \langle v_{\mu,\beta} \rangle \right)}{\partial t} &= -\frac{\partial}{\partial x_{\mu,\alpha}} \left( \langle \varrho_\mu \rangle \langle v_{\mu,\alpha} \rangle \langle v_{\mu,\beta} \rangle + p_{\mu,\alpha\beta} \right) + \langle \varrho_\mu \rangle \frac{f_{\mu,\beta}}{m_\mu} + Q_\mu(v_{\mu,\beta}) && \text{(18a)} \\ &+ \langle \varrho_\mu \rangle \frac{1}{\tau_\mu} \left( \langle v_{\mu,\beta}^0 \rangle - \langle v_{\mu,\beta} \rangle \right) && \text{(18b)} \\ &+ \sum_\nu \langle \varrho_\mu \rangle \langle \varrho_\nu \rangle \frac{1}{m_\nu} \left[ \chi_{\mu\nu}(v_{\mu,\beta}^*) - \chi_{\mu\nu}(v_{\mu,\beta}) \right] && \text{(18c)} \\ &+ \sum_\nu \left[ \frac{m_\mu}{m_\nu} \langle \varrho_\nu \rangle \chi_\mu^{\nu\mu} \left( v_{\nu,\beta} \right) - \langle \varrho_\mu \rangle \chi_\mu^{\mu\nu} \left( v_{\mu,\beta} \right) \right] && \text{(18d)} \end{aligned}$$

for the momentum density, and

$$\begin{aligned} \frac{\partial \langle \epsilon_{\mu,i} \rangle}{\partial t} &= -\frac{\partial}{\partial x_{\mu,\alpha}} \left( \langle v_{\mu,\alpha} \rangle \langle \epsilon_{\mu,i} \rangle + p_{\mu,\alpha i} \langle v_{\mu,i} \rangle + j_{\mu,\alpha,i} \right) + \langle \varrho_\mu \rangle \langle v_{\mu,i} \rangle \frac{f_{\mu,i}}{m_\mu} + Q_\mu \left( \frac{v_{\mu,i}^2}{2} \right) && \text{(19a)} \\ &+ \langle \varrho_\mu \rangle \frac{1}{\tau_\mu} \left( \langle v_{\mu,i}^0 \rangle^2 - \langle v_{\mu,i} \rangle^2 \right) && \text{(19b)} \\ &+ \langle \varrho_\mu \rangle \frac{1}{\tau_\mu} \left( \langle (\delta v_{\mu,i}^0)^2 \rangle - \langle (\delta v_{\mu,i})^2 \rangle \right) && \text{(19c)} \\ &+ \sum_\nu \langle \varrho_\mu \rangle \langle \varrho_\nu \rangle \frac{1}{m_\nu} \left[ \chi_{\mu\nu} \left( \frac{v_{\mu,i}^{*\,2}}{2} \right) - \chi_{\mu\nu} \left( \frac{v_{\mu,i}^2}{2} \right) \right] && \text{(19d)} \\ &+ \sum_\nu \left[ \frac{m_\mu}{m_\nu} \langle \varrho_\nu \rangle \chi_\mu^{\nu\mu} \left( \frac{v_{\nu,i}^2}{2} \right) - \langle \varrho_\mu \rangle \chi_\mu^{\mu\nu} \left( \frac{v_{\mu,i}^2}{2} \right) \right] && \text{(19e)} \end{aligned}$$

for the energy density. Here, we have used the EINSTEINian summation convention to sum over terms in which the Greek indices $\alpha$, $\beta$ or $\gamma$ occur twice.

## 3.1 Interpretation

(17a), (18a) and (19a) are the well known hydrodynamic equations (see chapters 2.4 and 2.10 of Ref. [18]). (18c), (19d) describe the effects of interactions between two individuals of type $\mu$ and $\nu$ (for details see section 6). These terms do not vanish as they would do, if a conservation of momentum and energy would be fulfilled in a *strict* sense (see chapter 2.10 of Ref. [18]). However, since the individuals try to approach the intended velocity $\vec{v}_\mu^0$, there is a *tendency* to restore momentum and energy that is described by (18b), (19b), (19c).

(17b), (18d), (19e) are additional terms due to individuals who change their type of motion. In the following we will assume the special case that these terms as well as the terms $Q_\mu(\psi_\mu(\vec{v}_\mu)/m_\mu)$ due to individuals entering or leaving the system $\mathcal{M}$ vanish (by compensation of inflow into $\mu$ and outflow from $\mu$). For concrete situations the quantities $\chi_{\mu}^{\cdot\cdot}$ and $Q_\mu(.)$ have to be obtained empirically.

$p_{\mu,\alpha\beta}$ is, in thermodynamics, termed the tensor of *pressure*. $p_{\mu,\alpha\beta}n_\beta$ has the meaning of the force which is used by the individuals of type $\mu$ to change their movement when crossing a line of unit length $l$ (or, more exactly, the meaning of the component of this force in the direction $\vec{n}$ perpendicular to the line). $\vec{j}_{\mu,i}$ is, in thermodynamics, called the *heat flow*. For pedestrians it describes the tendency of the velocity variance $\langle(\delta v_{\mu,i})^2\rangle$ to equalize with time (see (31)). The variance

$$\theta_{\mu,i} := \langle(\delta v_{\mu,i})^2\rangle \equiv k_B T_{\mu,i}/m_\mu$$

is the thermodynamic equivalent to the *absolute temperature* $T_{\mu,i}$ in direction $i$. Approximate expressions for $p_{\mu,\alpha\beta}$ and $\vec{j}_{\mu,i}$ are derived in sections 4 and 5.

## 3.2 Problems of small densities

For pedestrian crowds the densities $\hat{\rho}_\mu$ are usually very small. As a consequence, equation (3) won't be fulfilled very well and a discrete formulation would be more appealing (see [11]). However, we could equivalently start from the continuity equation (20), which holds better, because the densities $\rho_\mu$ are only moderately small. The macroscopic equations will be even better fulfilled, because they are only equations for the mean values $\langle\varrho_\mu\rangle$, $\langle\vec{v}_\mu\rangle$, $\langle\epsilon_{\mu,i}\rangle$ and could be also set up by plausibility considerations.

In order to have small fluctuations of the variables $\langle\varrho_\mu\rangle$, $\langle\vec{v}_\mu\rangle$, $\langle\epsilon_{\mu,i}\rangle$ with time, $\hat{\rho}_\mu$ in equation (1) has to be averaged over a finite area $A$ and a finite volume $V$, which should be taken sufficiently large. If $T$ denotes (appart from fluctuations) the time scale for the temporal change of $\langle\varrho_\mu\rangle$, $\langle\vec{v}_\mu\rangle$ and $\langle\epsilon_{\mu,i}\rangle$, these variables can be also averaged over time intervalls $\Delta t \ll T$:

$$\overline{\langle\varrho_\mu(\vec{x},t)\rangle\langle\psi_\mu(\vec{x},t)\rangle} := \frac{1}{\Delta t}\int\limits_{t-\Delta t/2}^{t+\Delta t/2} \langle\varrho_\mu(\vec{x},t')\rangle\langle\psi_\mu(\vec{x},t')\rangle\, dt' \, .$$

Then, (17) to (19) will be proper approximations for the movement of pedestrians.

Another complication by low densities is, that KNUDSEN *corrections* have to be taken into account [12]: According to these corrections, the "temperature" $\theta_{\mu,i}$ and the tangential velocity $\langle v_{\mu,\|}\rangle$ change discontinuously at a boundary $\partial\mathcal{M}$, which therefore seems to be shifted by a small distance $\xi$ that is comparable to the mean interaction free path (see §14 of Ref. [19]).

# 4 Pedestrians in equilibrium

In order to calculate $p_{\mu,\alpha\beta}$, $\vec{j}_{\mu,i}$ and $\chi_{\mu\nu}$ we need the explicit form of $\rho_\mu$ (see (10), and (12) to (15)). $\rho_\mu$ is the density of individuals of type $\mu$ at place $\vec{x}$ and time $t$ having the actual velocity $\vec{v}_\mu$ but arbitrary intended velocity $\vec{v}_\mu^0$ (see (10)). It is directly measurable in pedestrian crowds. By integration of (3) over $\vec{v}_\mu^0$ we obtain the theoretical dependence

$$
\begin{aligned}
\frac{d\rho_\mu}{dt} &\equiv \frac{\partial\rho_\mu}{\partial t} + \nabla_{\vec{x}}\left(\rho_\mu\vec{v}_\mu\right) + \nabla_{\vec{v}_\mu}\left(\rho_\mu\frac{\vec{f}_\mu}{m_\mu}\right) \\
&= \frac{\rho_\mu^0-\rho_\mu}{\tau_\mu} + \sum_\nu S_{\mu\nu} \\
&+ \sum_\nu C_{\mu\nu} + q_\mu
\end{aligned} \tag{20}
$$

(compare to [27, 26]). So, the temporal development of the density $\rho_\mu$ is given by a tendency to walk with the intended velocity $\vec{v}_\mu^0$ (see (4)), by the effects $S_{\mu\nu}$ of pair interactions (see (7)), by the effects $C_{\mu\nu}$ of pedestrians changing their type of motion (see (9)) and by the effect $q_\mu$ of pedestrians entering or leaving the system $\mathcal{M}$ (see (11), (16)). The last two effects shall be be neglected in the following (see the comment in section 3.1).

Equation (20) can be solved in a suitable approximation by the recursive method of Chapman and Enskog [4, 5]. The lowest order approximation presupposes the condition $d\rho_\mu^e/dt = 0$ of *local equilibrium*, which is approximately fulfilled by the Gaussian distribution

$$
\rho_\mu^e(\vec{x},\vec{v}_\mu,t) = \langle\rho_\mu\rangle\cdot\frac{1}{2\pi b_\mu\theta_{\mu,\|}}\mathrm{e}^{-[(v_{\mu,\|}-\langle v_{\mu,\|}\rangle)^2/(2\theta_{\mu,\|})+(v_{\mu,\perp}-\langle v_{\mu,\perp}\rangle)^2/(2\theta_{\mu,\perp})]} \tag{21}
$$

according to empirical data [13, 14, 23].

$$
(b_\mu)^2 := \frac{\theta_{\mu,\perp}}{\theta_{\mu,\|}} \le 1
$$

describes the fact that the velocity variance $\theta_{\mu,i}$ perpendicular ($\perp$) to the mean intended direction of movement $\langle\vec{v}_\mu^0\rangle$ is normally less than parallel ($\|$) to it [23].

For each type $\mu$ of motion let us perform a particular transformation

$$
\vec{x} \longrightarrow \vec{X}_\mu := \begin{pmatrix} x_{\mu,\|} \\ x_{\mu,\perp}/b_\mu \end{pmatrix},
$$

$$
\vec{v}_\mu \longrightarrow \vec{V}_\mu := \begin{pmatrix} v_{\mu,\|} \\ v_{\mu,\perp}/b_\mu \end{pmatrix},
$$

$$
\vec{f}_\mu \longrightarrow \vec{F}_\mu := \begin{pmatrix} f_{\mu,\|} \\ f_{\mu,\perp}/b_\mu \end{pmatrix},
$$

$$\langle \varrho_{\mu,\perp}(\vec{x})\rangle := \frac{1}{\Delta x_\|}\int\limits_{\Delta x_\|} \langle \varrho_\mu\rangle dx_{\mu,\|} \longrightarrow b_\mu \langle \varrho_{\mu,\perp}(\vec{X}_\mu)\rangle \,,$$

$$p_{\mu,\alpha\beta} =: \epsilon_{\mu,\alpha\gamma} P_{\mu,\gamma\beta} \longrightarrow P_{\mu,\alpha\beta}\,,$$

$$j_{\mu,\alpha,i} =: \epsilon_{\mu,\alpha\beta} J_{\mu,\beta,i} \longrightarrow J_{\mu,\alpha,i}$$

with

$$\underline{\epsilon}_\mu \equiv (\epsilon_{\mu,\alpha\beta}) := \begin{pmatrix} 1 & 0 \\ 0 & b_\mu^2 \end{pmatrix} .$$

This transformation stretches the direction perpendicular to $\langle \vec{v}^0_\mu\rangle$ by the factor $1/b_\mu$ and simplifies equations (17) to (19) to *isotropic* ones (that means to equations with local rotational symmetry). With

$$\theta_\mu := \theta_{\mu,\|}$$

we get

$$P^e_{\mu,\alpha\beta} = \langle \varrho_\mu\rangle \theta_\mu \delta_{\alpha\beta} =: P^e_\mu \delta_{\alpha\beta}\,, \tag{22}$$

for the pressure and

$$J^e_{\mu,\alpha,i} = 0 \tag{23}$$

for the "heat flow" (see chapter 2.10 of Ref. [18]). In addition, the EULER equations

$$\frac{d\langle \varrho_\mu\rangle}{dt} := \frac{\partial \langle \varrho_\mu\rangle}{\partial t} + \langle V_{\mu,\alpha}\rangle \frac{\partial \langle \varrho_\mu\rangle}{\partial X_{\mu,\alpha}} = -\langle \varrho_\mu\rangle \frac{\partial \langle V_{\mu,\alpha}\rangle}{\partial X_{\mu,\alpha}}\,, \tag{24}$$

$$\frac{d\langle V_{\mu,\beta}\rangle}{dt} := \frac{\partial \langle V_{\mu,\beta}\rangle}{\partial t} + \langle V_{\mu,\alpha}\rangle \frac{\partial \langle V_{\mu,\beta}\rangle}{\partial X_{\mu,\alpha}} = -\frac{1}{\langle \varrho_\mu\rangle}\frac{\partial P^e_{\mu,\alpha\beta}}{\partial X_{\mu,\alpha}} + \frac{F_{\mu,\beta}}{m_\mu}\,, \tag{25}$$

$$\frac{d\theta_\mu}{dt} := \frac{\partial \theta_\mu}{\partial t} + \langle V_{\mu,\alpha}\rangle \frac{\partial \theta_\mu}{\partial X_{\mu,\alpha}} = -\theta_\mu \frac{\partial \langle V_{\mu,\alpha}\rangle}{\partial X_{\mu,\alpha}} \tag{26}$$

can be derived from (3) (see chapter 16.2 of Ref. [31]).

## 4.1 Behavior on a dance floor

On a dance floor like that of a discotheque, two types of motion can be found: Type 1 describing dancing individuals, type 2 describing individuals standing around and looking on. We can assume to have an isotropic case, that means $b_\mu = 1$ and $\theta_{\mu,\|} = \theta_{\mu,\perp}$. According to (19c), the variance $\langle (\delta v^0_{\mu,i})^2\rangle$ of the *intended* velocities $v^0_{\mu,i}$ is *causal* for the temperature $\theta_{\mu,i}$, that means for the variance $\langle (\delta v_{\mu,i})^2\rangle$ of the actual velocities $v_{\mu,i}$. So for the temperatures $\theta_1$, $\theta_2$ of individuals dancing and individuals standing around it holds

$$\theta_1 > \theta_2\,,$$

since the dancing individuals intend to move with higher variance $\langle (\delta v^0_{1,i})^2\rangle > \langle (\delta v^0_{2,i})^2\rangle$. (This is even so in the case of equilibrium, because we have to take the effect of the KNUDSEN corrections into account, see section 3.2.) The equilibrium condition of equal pressure

$$P^e_1 = P^e_2$$

now leads to

$$\langle \varrho_1\rangle = \frac{\theta_2}{\theta_1}\langle \varrho_2\rangle < \langle \varrho_2\rangle$$

(see (22)). Therefore, the dancing individuals will have less density than the individuals standing around (see figure 1). This effect can actually be observed.

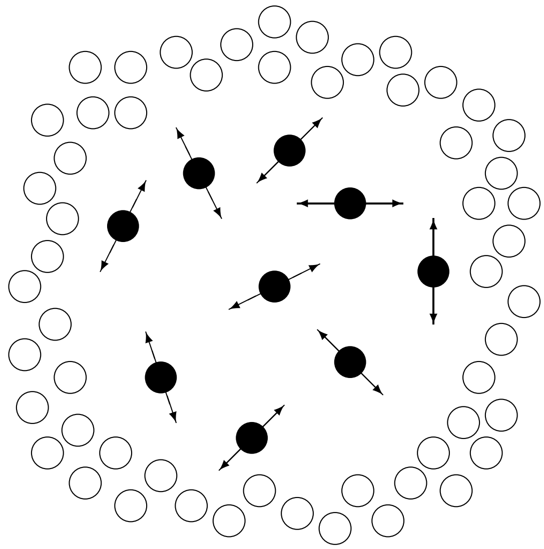

Figure 1: Behavior on a dance floor: Dancing individuals (filled circles) show a lower density than the individuals standing around (empty circles), since they intend to move with a greater velocity variance.

## 4.2 Propagation of density waves

In a nearly homogenuous pedestrian crowd with small density variations one can assume

$$\vec{V}_\mu \cdot \nabla_{\vec{X}_\mu} \langle \varrho_\mu \rangle \approx 0\,, \quad \langle \vec{V}_\mu \rangle \nabla_{\vec{X}_\mu} \langle \vec{V}_\mu \rangle \approx \vec{0}\,, \quad \langle \vec{V}_\mu \rangle \nabla_{\vec{X}_\mu} \theta_\mu \approx 0\,, \quad F_{\mu,\beta} \approx 0\,.$$

From the Euler equations (24) to (26) the equation

$$\frac{\partial^2 \langle \varrho_\mu \rangle}{\partial t^2} - \Delta_{\vec{X}_\mu} P^e_\mu = 0$$

can be derived, then (see chapter 16.2 of Ref. [31]). Subtracting $\langle \varrho_\mu \rangle / \theta_\mu \times (26)$ from (24) and making use of (22) and (23), the *adiabatic law*

$$\frac{P^e_\mu}{\langle \varrho_\mu \rangle^2} = const.$$

can be shown, which leads to the *linear wave equation*

$$\langle \varrho_\mu \rangle \kappa^S_\mu \frac{\partial^2}{\partial t^2} \langle \varrho_\mu \rangle - \Delta_{\vec{X}_\mu} \langle \varrho_\mu \rangle = 0 \tag{27}$$

with the *adiabatic compressibility*

$$\kappa^S_\mu := \frac{1}{\langle \varrho_\mu \rangle} \left( \frac{\partial P^e_\mu}{\partial \langle \varrho_\mu \rangle} \right)_S = \frac{1}{2} \frac{\langle \varrho_\mu \rangle}{P^e_\mu} = \frac{1}{2 \langle \varrho_\mu \rangle \theta_\mu}$$

(see chapter 16.2 of Ref. [31]). (For the description of nonlinear waves see [20] and chapter 2.1 of Ref. [35].) Equation (27) describes the propagation of density waves with velocity

$$c_\mu = \frac{1}{\sqrt{\langle \varrho_\mu \rangle \kappa^S_\mu}}\,. \tag{28}$$

On the other hand, the velocity of propagation is given by the mean distance $d_\mu = 1/\sqrt{\langle\rho_\mu\rangle}$ of the succeeding individual devided by its mean reaction time $\zeta_\mu$:

$$c_\mu = \frac{1}{\sqrt{\langle\rho_\mu\rangle}\zeta_\mu} .$$

Inserting this into (28), it follows that for small densities $\langle\rho_\mu\rangle$ the adiabatic compressibility grows with the mean *reaction time* $\zeta_\mu$ of individuals according to

$$\kappa_\mu^S = \frac{(\zeta_\mu)^2}{m_\mu} .$$

# 5 Nonequilibrium equations

In cases of deviations

$$\delta\rho_\mu(\vec{X}_\mu, \vec{V}_\mu, t) := \rho_\mu(\vec{X}_\mu, \vec{V}_\mu, t) - \rho_\mu^e(\vec{X}_\mu, \vec{V}_\mu, t) \tag{29}$$

from local equilibrium $\rho_\mu^e(\vec{X}_\mu, \vec{V}_\mu, t) := \rho_\mu^e(\langle\rho_\mu\rangle, \langle\vec{V}_\mu\rangle, \theta_\mu)$, we have to find a higher order approximation of equation (20) than in section 4. If the deviations $\delta\rho_\mu$ remain small compared with $\rho_\mu$, we can linearize equation (20) around $\rho_\mu^e$ and get

$$\frac{d\rho_\mu^e}{dt} \approx \frac{d\rho_\mu^e}{dt} + \frac{d\delta\rho_\mu}{dt} = \frac{d\rho_\mu}{dt} \approx -\frac{\delta\rho_\mu}{\tau_\mu} + \sum_\nu \frac{\delta\rho_\nu}{\tau_{\mu\nu}}$$

(see chapter 15 of Ref. [17]). Here, $\tau_{\mu\nu}$ is the mean interaction free time between an individual of type $\mu$ and individuals of type $\nu$ (see (35) and chapter 16.2 of Ref. [31]). From (21) and (22) to (26) one finds [31]

$$\begin{aligned}
\frac{d\rho_\mu^e}{dt} &= \frac{\partial\rho_\mu^e}{\partial\langle\rho_\mu\rangle}\frac{d\langle\rho_\mu\rangle}{dt} + \left(\nabla_{\langle\vec{V}_\mu\rangle}\rho_\mu^e\right)\cdot\frac{d\langle\vec{V}_\mu\rangle}{dt} + \frac{\partial\rho_\mu^e}{\partial\theta_\mu}\frac{d\theta_\mu}{dt} \\
&= \rho_\mu^e\left[\frac{\delta\vec{V}_\mu}{\theta_\mu}\cdot\nabla_{\vec{X}_\mu}\theta_\mu\left(\frac{(\delta\vec{V}_\mu)^2}{2\theta_\mu} - 2\right)\right] \\
&+ \rho_\mu^e\left[\frac{1}{\theta_\mu}\left(\delta V_{\mu,\alpha}\frac{\partial\langle V_{\mu,\beta}\rangle}{\partial X_{\mu,\alpha}}\delta V_{\mu,\beta} - \frac{(\delta\vec{V}_\mu)^2}{2}\nabla_{\vec{X}_\mu}\langle\vec{V}_\mu\rangle\right)\right] .
\end{aligned}$$

If

$$(\tau_\mu\delta_{\mu\nu} + \overline{\tau}_{\mu\nu})$$

denotes the inverse matrix of

$$\left(\frac{1}{\tau_\mu}\delta_{\mu\nu} - \frac{1}{\tau_{\mu\nu}}\right) ,$$

the relation

$$\delta\rho_\mu = -\tau_\mu\frac{d\rho_\mu^e}{dt} - \sum_\nu \overline{\tau}_{\mu\nu}\frac{d\rho_\nu^e}{dt} \tag{30}$$

leads, because of $\rho_\mu = \rho_\mu^e + \delta\rho_\mu$, to the corrected tensor of pressure

$$P_{\mu,\alpha\beta} = P_\mu^e \delta_{\alpha\beta} - \eta_\mu \Lambda_{\mu,\alpha\beta} - \sum_\nu \eta_{\mu\nu} \Lambda_{\nu,\alpha\beta}$$

(see (13)) and the corrected heat flow

$$J_{\mu,\alpha} = -\kappa_\mu \frac{\partial \theta_\mu}{\partial X_{\mu,\alpha}} - \sum_\nu \kappa_{\mu\nu} \frac{\partial \theta_\nu}{\partial X_{\nu,\alpha}} \tag{31}$$

(see (14)). Here,

$$\Lambda_{\mu,\alpha\beta} := \left( \frac{\partial \langle V_{\mu,\alpha} \rangle}{\partial X_{\mu,\beta}} + \frac{\partial \langle V_{\mu,\beta} \rangle}{\partial X_{\mu,\alpha}} - \frac{\partial \langle V_{\mu,\alpha} \rangle}{\partial X_{\mu,\alpha}} \delta_{\alpha\beta} \right)$$

is the *tensor of strain*,

$$\eta_\mu = \tau_\mu \theta_\mu \langle \varrho_\mu \rangle \, , \qquad \eta_{\mu\nu} = \overline{\tau}_{\mu\nu} \theta_\nu \langle \varrho_\nu \rangle$$

are coefficients of the *shear viscosity*, and

$$\kappa_\mu = 2\tau_\mu \theta_\mu \langle \varrho_\mu \rangle \, , \qquad \kappa_{\mu\nu} = 2\overline{\tau}_{\mu\nu} \theta_\nu \langle \varrho_\nu \rangle$$

are coefficients of the *thermal conductivity*.

Note, that the effect of restoring the local equilibrium distribution $\rho_\mu^e$ results from the tendency to approach the intended velocity distribution $\rho_\mu^0(\vec{X}_\mu, \vec{V}_\mu, t)$ with a time constant $\tau_\mu$, but not from interaction processes as usual (see chapter 13.3 of Ref. [30]). Therefore the viscosity $\eta_\mu$ is dependent on the density $\langle \rho_\mu \rangle$ in contrast to ordinary fluids (see pages 323 and 327 in Ref. [31]). For vanishing densities $\langle \rho_\nu \rangle \longrightarrow 0$ the interaction rates $1/\tau_{\mu\nu}$ become negligible (see (35)) and $\overline{\tau}_{\mu\nu}$, $\eta_{\mu\nu}$, $\kappa_{\mu\nu}$ vanish in comparison with $\tau_\mu$, $\eta_\mu$, $\kappa_\mu$, respectively. According to (30), the deviation $\delta\rho_\mu$ from the local equilibrium distribution $\rho_\mu^e$ and, therefore, the viscosity and the thermal conductivity are consequences of finite relaxation times $\tau_\mu$, $\tau_{\mu\nu}$.

## 5.1 Effect of viscosity

For pedestrians the effect of viscosity is not compensated by a gradient of pressure as in ordinary fluids, but instead by the tendency of pedestrians to reach their intended velocity described by (18b). In the case of a stationary flow in one direction (that means of one type of motion) parallel to the boundaries $\partial\mathcal{M}$ we have essentially the equation

$$0 = \frac{\partial \langle \varrho_\mu \rangle \langle V_{\mu,\parallel} \rangle}{\partial t} = \eta_\mu \frac{\partial^2}{\partial X_{\mu,\perp}^2} \langle V_{\mu,\parallel} \rangle + \langle \varrho_\mu \rangle \frac{1}{\tau_\mu} \left( \langle V_{\mu,\parallel}^0 \rangle - \langle V_{\mu,\parallel} \rangle \right) \, , \tag{32}$$

if $\eta_\mu \gg \eta_{\mu\mu}$ (see (18a), (18b)). For a lane of effective width $2W$ (with the origin $X_{\mu,\perp} = 0$ in the middle) equation (32) has the *hyperbolic* solution

$$\langle V_{\mu,\parallel} \rangle = \langle V_{\mu,\parallel}^0 \rangle \left[ 1 - \frac{\cosh(X_{\mu,\perp}/D_\mu)}{\cosh(W/D_\mu)} \right] \tag{33}$$

with a *boundary layer* of width

$$D_\mu = \sqrt{\frac{\eta_\mu \tau_\mu}{\langle \varrho_\mu \rangle}} = \tau_\mu \sqrt{\theta_\mu} \, .$$

In comparison with this, a pressure gradient

$$\frac{\partial P^e_{\mu,\|\|}}{\partial X_{\mu,\|}} := \frac{\Delta P^e_\mu}{L}$$

generating the driving force would lead instead to the *parabolic* solution

$$\langle V_{\mu,\|} \rangle = \frac{\Delta P^e_\mu}{\eta_\mu L}(W^2 - X^2_{\mu,\perp}) \,, \tag{34}$$

and the mean tangential velocity $\langle V_{\mu,\|} \rangle$ would depend on the length $L$ of the lane (see chapter 3.3 of Ref. [21]). The hyperbolic solution (33) and the parabolic solution (34) are depicted in figure 2.

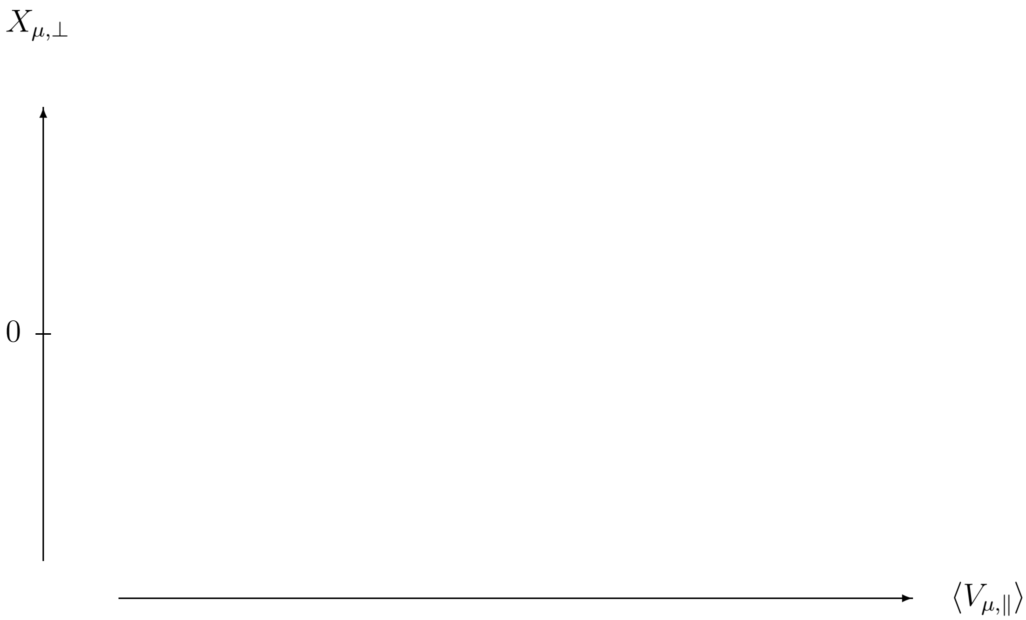


Figure 2: Effect of viscosity (internal friction): Ordinary fluids show a parabolic velocity profile (broken line). In contrast to this, a hyperbolic velocity profile is expected for pedestrian crowds (solid line). Whereas in ordinary fluids the internal friction is compensated by a pressure gradient, for pedestrian crowds this role is played by the accelerating effect of the intended velocity. Due to the KNUDSEN corrections the fluid slips at the boundary, that means the *effective* width is greater than the *actual* width.

# 6 Effects of interactions

The *scattering rates* $\sigma^*_{\mu\nu}$ of interactions (see (7), (8)) are proportional to the relative velocity $\|\vec{v}_\mu - \vec{v}_\nu\|$ of the interacting pedestrians and to the *scattering cross section* $l_{\mu\nu}$ (which is a length here and of order of a pedestrian's stride) [18]:

$$\sigma^*_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu) = l_{\mu\nu} \|\vec{v}_\mu - \vec{v}_\nu\| P_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu) \,.$$

It is

$$\frac{1}{\tau_{\mu\nu}} := \frac{1}{\langle\rho_\mu\rangle}\iint \rho_\mu(\vec{x},\vec{v}_\mu,t)\rho_\nu(\vec{x},\vec{v}_\nu,t)l_{\mu\nu}\|\vec{v}_\mu-\vec{v}_\nu\|d^2\vec{v}_\mu\, d^2\vec{v}_\nu$$
$$= \langle\rho_\nu\rangle l_{\mu\nu}\langle\|\vec{v}_\mu-\vec{v}_\nu\|\rangle \qquad (35)$$

the mean rate of interactions of an individual of type $\mu$ with individuals of type $\nu$, and $\tau_{\mu\nu}$ the corresponding mean interaction free time [18]. For the mean relative velocity $\langle\|\vec{v}_\mu-\vec{v}_\nu\|\rangle$ (see figure 3) the following limits can be calculated by making use of (21), (29) and neglecting terms of order $O(\delta\rho_\mu)$:

$$\langle\|\vec{v}_\mu-\vec{v}_\nu\|\rangle \approx \begin{cases} \sqrt{\pi\theta_\mu} & \text{if } \langle\vec{v}_\mu\rangle\approx\langle\vec{v}_\nu\rangle,\ \theta_\mu\approx\theta_\nu \\ \|\langle\vec{v}_\mu\rangle-\langle\vec{v}_\nu\rangle\| & \text{if } \|\langle\vec{v}_\mu\rangle-\langle\vec{v}_\nu\rangle\|\gg\theta_\mu,\theta_\nu\,. \end{cases} \qquad (36)$$

$\langle\|\vec{v}_\mu-\vec{v}_\nu\|\rangle$

$\|\langle\vec{v}_\mu\rangle-\langle\vec{v}_\nu\rangle\|$

Figure 3: The mean relative velocity $\langle\|\vec{v}_\mu-\vec{v}_\nu\|\rangle$ in dependence of $\|\langle\vec{v}_\mu\rangle-\langle\vec{v}_\nu\rangle\|$ for the special case $\theta_{\mu,i}=1=\theta_{\nu,i}$.

Let us introduce

$$\tau^*_{\mu\nu}=\tau^*_{\mu\nu}(\langle\vec{v}_\mu\rangle,\langle\vec{v}_\nu\rangle,\theta_\mu,\theta_\nu):=\tau_{\mu\nu}\langle\varrho_\nu\rangle=\tau_{\mu\nu}m_\nu\langle\rho_\nu\rangle\,, \qquad (37)$$

and the *total rate of interactions*

$$\frac{1}{\hat{\tau}_\mu}:=\sum_\nu\frac{1}{\tau_{\mu\nu}}\,. \qquad (38)$$

Then,

$$r_\mu = \mathrm{e}^{-\Delta t_\mu / \hat{\tau}_\mu} \tag{39}$$

is the probability for the possibility to pass an individual to the right or left, if this would need an interaction free time of at least $\Delta t_\mu$ (see chapter 12.1 of Ref. [30]).

$$P_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}_\mu^*) = \sum_k P_{\mu\nu}^k(\vec{v}_\mu, \vec{v}_\nu; \vec{v}_\mu^*)$$

is the probability, that two individuals of types $\mu$ and $\nu$ have velocities $\vec{v}_\mu$ and $\vec{v}_\nu$ before their interaction, and the individual of type $\mu$ has the velocity $\vec{v}_\mu^*$ thereafter. We shall distinguish three types $k$ of interaction:

If an individual of type $\mu$ is hindered by another individual of type $\nu$, it tries to pass the other to the right with probability $p_{\mu\nu}$ or to the left with probability $1 - p_{\mu\nu}$:

$$P_{\mu\nu}^1(\vec{v}_\mu, \vec{v}_\nu; \vec{v}_\mu^*) = r_\mu \left[ p_{\mu\nu}\delta(\vec{v}_\mu^* - \underline{S}_{\beta_{\mu\nu}}\vec{v}_\mu) + (1 - p_{\mu\nu})\delta(\vec{v}_\mu^* - \underline{S}_{\beta_{\mu\nu}}^{-1}\vec{v}_\mu) \right] .$$

$\vec{v}_\mu^* = \underline{S}_{\beta_{\mu\nu}}\vec{v}_\mu$ describes a rotation of velocity $\vec{v}_\mu$ to the right side by an angle $\beta_{\mu\nu}$ in order to avoid the hindering pedestrian, $\vec{v}_\mu^* = \underline{S}_{\beta_{\mu\nu}}^{-1}\vec{v}_\mu$ is the inverse rotation to the left side.

In cases, where it is impossible to avoid the individual of type $\nu$ having a velocity $\vec{v}_\nu$, the individual of type $\mu$ tries to walk with velocity $\vec{v}_\mu^* = \vec{v}_\nu$, if $\vec{v}_\nu$ has a positive component $\vec{v}_\nu \cdot \vec{e}_\mu > 0$ in the intended direction $\vec{e}_\mu := \vec{v}_\mu^0 / \|\vec{v}_\mu^0\|$ of motion:

$$P_{\mu\nu}^2(\vec{v}_\mu, \vec{v}_\nu; \vec{v}_\mu^*) = (1 - r_\mu)\delta(\vec{v}_\mu^* - \vec{v}_\nu)\Theta(\vec{v}_\nu \cdot \vec{e}_\mu > 0) .$$

This corresponds to situations, where one moves for a short time within a gap behind a pedestrian being in the way (or sometimes, for different directions $\vec{e}_\mu \neq \vec{e}_\nu$, in front of it). The *decision function* $\Theta$ is defined by

$$\Theta(x) := \begin{cases} 1 & \text{if } x \text{ is fulfilled} \\ 0 & \text{else.} \end{cases}$$

If $\vec{v}_\nu \cdot \vec{e}_\mu < 0$ (that means in the case of a negative component of the hindering pedestrian's velocity $\vec{v}_\nu$ with respect to the intended direction $\vec{e}_\mu$ of movement) it is better for the individual(s) to stop ($\vec{v}_\mu^* = \vec{0}$):

$$P_{\mu\nu}^3(\vec{v}_\mu, \vec{v}_\nu; \vec{v}_\mu^*) = (1 - r_\mu)\delta(\vec{v}_\mu^* - \vec{0})\Theta(\vec{v}_\nu \cdot \vec{e}_\mu \leq 0) .$$

This results in

$$\begin{aligned} & \frac{1}{m_\nu}\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle \left[ \chi_{\mu\nu}^k(\vec{v}_\mu^*) - \chi_{\mu\nu}^k(\vec{v}_\mu) \right] \\ \approx \quad & \frac{\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle}{\tau_{\mu\nu}^*} \cdot \begin{cases} r_\mu[p_{\mu\nu}\langle\underline{S}_{\beta_{\mu\nu}}\vec{v}_\mu\rangle + (1 - p_{\mu\nu})\langle\underline{S}_{\beta_{\mu\nu}}^{-1}\vec{v}_\mu\rangle - \langle\vec{v}_\mu\rangle] \,, & k = 1 \qquad (40.1) \\ (1 - r_\mu)\langle\Theta_{\mu\nu}\rangle[\langle\vec{v}_\nu\rangle - \langle\vec{v}_\mu\rangle] \,, & k = 2 \qquad (40.2) \\ -(1 - r_\mu)(1 - \langle\Theta_{\mu\nu}\rangle)\langle\vec{v}_\mu\rangle \,, & k = 3 \qquad (40.3) \end{cases} \end{aligned}$$

and

$$\frac{1}{m_\nu}\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle\left[\chi^k_{\mu\nu}\left(\frac{(v^*_{\mu,i})^2}{2}\right)-\chi^k_{\mu\nu}\left(\frac{(v_{\mu,i})^2}{2}\right)\right]$$

$$\approx \frac{\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle}{\tau^*_{\mu\nu}}\cdot\begin{cases} r_\mu[p_{\mu\nu}\langle(\underline{S}_{\beta_{\mu\nu}}\vec{v}_\mu)^2_i\rangle+(1-p_{\mu\nu})\langle(\underline{S}^{-1}_{\beta_{\mu\nu}}\vec{v}_\mu)^2_i\rangle-\langle v^2_{\mu,i}\rangle]\,, & k=1 \quad (41.1)\\ (1-r_\mu)\langle\Theta_{\mu\nu}\rangle[\langle v^2_{\nu,i}\rangle-\langle v^2_{\mu,i}\rangle]\,, & k=2 \quad (41.2)\\ -(1-r_\mu)(1-\langle\Theta_{\mu\nu}\rangle)\langle v^2_{\mu,i}\rangle\,, & k=3\,, \quad (41.3)\end{cases}$$

which allows to calculate (18c) and (19d) explicitly. We have used the abbreviation

$$\begin{aligned}\langle\Theta_{\mu\nu}\rangle &= \langle\Theta_{\mu\nu}\rangle(\vec{e}_\mu,\langle\vec{v}_\nu\rangle,\theta_\nu):=\langle\Theta(\vec{v}_\nu\cdot\vec{e}_\mu>0)\rangle\\ &\approx \begin{cases} 1-\frac{1}{2}\mathrm{e}^{-y^2_{\mu\nu}/(2\theta_\nu)}+\frac{y_{\mu\nu}}{\sqrt{2\pi\theta_\nu}}\left[1-\Phi\left(\frac{y_{\mu\nu}}{\sqrt{2\theta_\nu}}\right)\right] & \text{if } y_{\mu\nu}\ge 0\\ \frac{1}{2}\mathrm{e}^{-y^2_{\mu\nu}/(2\theta_\nu)}-\frac{|y_{\mu\nu}|}{\sqrt{2\pi\theta_\nu}}\left[1-\Phi\left(\frac{|y_{\mu\nu}|}{\sqrt{2\theta_\nu}}\right)\right] & \text{if } y_{\mu\nu}<0\end{cases}\end{aligned}$$

(see figure 4) with

$$y_{\mu\nu}:=\langle\vec{v}_\nu\rangle\cdot\vec{e}_\mu$$

and the GAUSSian *error function*

$$\Phi(z):=\int_0^z\frac{2}{\sqrt{\pi}}\mathrm{e}^{-x^2}dx\,.$$

$(1-r_\mu)(1-\langle\Theta_{\mu\nu}\rangle)$ is the relative frequency of stopping processes and $r_\mu$ the relative frequency of avoiding processes to the left or right due to interactions.

## 6.1 Interpretation

(a) **Development of lanes**
According to (40.1), an asymmetrical avoiding probability $p_{\mu\nu}\neq 1-p_{\mu\nu}$ (see [9]) leads to a momentum density that tends to the right (for $p_{\mu\nu}>1/2$) or to the left (for $p_{\mu\nu}<1/2$). This momentum density vanishes when the products $\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle$ of the densities $\langle\varrho_\mu\rangle$, $\langle\varrho_\nu\rangle$ have become zero. Therefore, it causes a separation of different types $\mu\neq\nu$ of motion into several lanes (see figure 5). This effect can be observed at least for high densities $\langle\varrho_\mu\rangle$, $\langle\varrho_\nu\rangle$ [9, 23, 24, 25] and has the advantage as well as the purpose to reduce the total rate $1/\hat{\tau}_\mu$ of interactions.

The width of the lanes of two opposing directions 1 and 2 can be calculated from the equilibrium condition of equal pressure:

$$P^e_1=P^e_2\,,\qquad\text{that means}\qquad\langle\varrho_1\rangle\theta_1=\langle\varrho_2\rangle\theta_2\,.$$

Since in the case of a lane of width $W_\mu$ and length $L$ consisting of $N_\mu$ individuals the relation

$$\langle\varrho_\mu\rangle=m_\mu\langle\rho_\mu\rangle=m_\mu\frac{N_\mu}{W_\mu L}$$

holds for the mass density $\langle\varrho_\mu\rangle$, we get

$$\frac{N_1}{N_2}\approx\frac{B_1}{B_2}\,,$$

$\langle \Theta_{\mu\nu} \rangle$

$|y_{\mu\nu}|$

Figure 4: The function $\langle \Theta_{\mu\nu} \rangle$ in dependence of $|y_{\mu\nu}| = |\langle \vec{v}_\nu \rangle \cdot \vec{e}_\mu|$ for the special case $\theta_{\nu,i} = 1$.

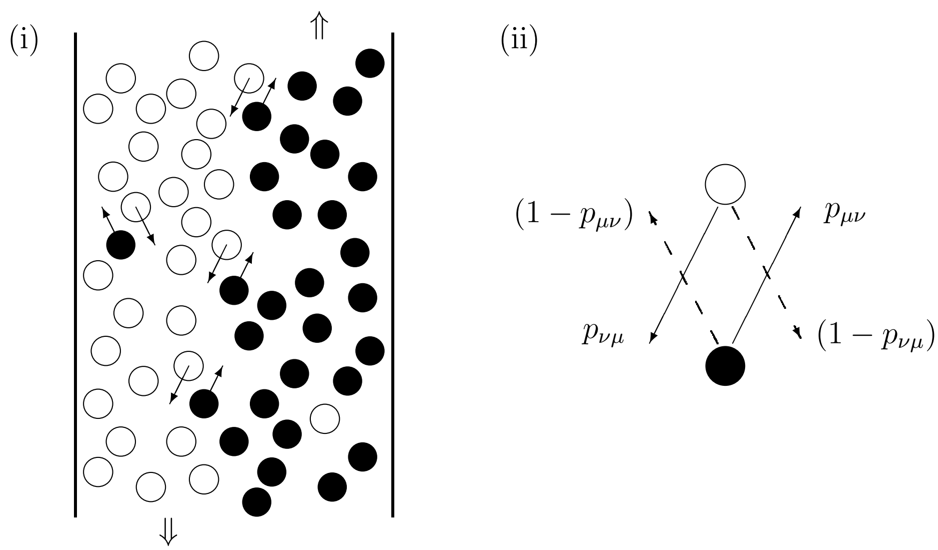


Figure 5: (i) Opposite directions of motion normally use separate lanes. Avoiding maneuvers are indicated by arrows. (ii) For pedestrians with an opposite direction of motion it is advantageous, if both prefer either the right hand side or the left hand side when trying to pass each other. Otherwise, they would have to stop in order to avoid a collision. The probability $p_{\mu\nu}$ for choosing the right hand side is usually greater than the probability $(1 - p_{\mu\nu})$ for choosing the left hand side.

if $m_1\theta_1 \approx m_2\theta_2$. That means, the lane width $B_\mu$ will be proportional to the number $N_\mu$ of individuals (see [23], taking into account the KNUDSEN corrections described in section 3.2.)

(b) **Crossings**
If the direction $\mu$ of motion is crossed by the direction $\nu$ of motion, it suffers a momentum density (40.2), which causes the individuals of type $\mu$ to be "pushed" partly in direction $\langle\vec{v}_\nu\rangle$ of type $\nu$. (For the *delay effect* of crossings see [22].)

(c) **Pedestrian jams**
In order to find out the consequences of (40.3) (that means of stopping processes), we can consider the equation

$$\frac{\partial\langle\varrho_\mu\rangle\langle\vec{v}_\mu\rangle}{\partial t} = \frac{1}{\tau_\mu}\langle\varrho_\mu\rangle(\langle\vec{v}^0_\mu\rangle - \langle\vec{v}_\mu\rangle) - \sum_\nu \langle\varrho_\mu\rangle\langle\varrho_\nu\rangle s^3_{\mu\nu}\langle\vec{v}_\mu\rangle \tag{42}$$

describing the tendency to walk with the intended velocity $\vec{v}^0_\mu$ as well as stopping processes (see (18b), (18c)).

$$s^3_{\mu\nu} := \frac{1-r_\mu}{\tau^*_{\mu\nu}}(1-\langle\Theta_{\mu\nu}\rangle)$$

has been introduced for shortness. The stationary solution of equation (42) is given by

$$\langle\vec{v}_\mu\rangle = \frac{1/\tau_\mu}{1/\tau_\mu + \sum_\nu \langle\varrho_\nu\rangle s^3_{\mu\nu}}\langle\vec{v}^0_\mu\rangle =: k_\mu\langle\vec{v}^0_\mu\rangle$$

(compare to [25]). According to (35) to (39) we find

$$\frac{\partial s^3_{\mu\nu}}{\partial\langle\rho_\nu\rangle} > 0$$

and, using the abbreviation $\langle\delta v_{\mu\nu}\rangle := \langle\|\vec{v}_\mu - \vec{v}_\nu\|\rangle$,

$$\frac{\partial s^3_{\mu\nu}}{\partial\langle\delta v_{\mu\nu}\rangle} > 0\,, \qquad s^3_{\mu\nu}(\langle\delta v_{\mu\nu}\rangle = 0) = 0\,. \tag{43}$$

Due to (43) a development of pedestrian jams (that means $k_\mu < 1$) is *caused by the variation* $\delta\vec{v}_{\mu\nu} := \vec{v}_\mu - \vec{v}_\nu$ *of the velocities.* This is even the case for a lane consisting of individuals of one type $\mu$ only (where $\langle\varrho_\nu\rangle = 0$ for $\nu \neq \mu$, see (a)), since $s^3_{\mu\mu}$ is growing with the velocity variance $\theta_\mu$:

$$\frac{\partial s^3_{\mu\mu}}{\partial\theta_\mu} > 0\,, \qquad s^3_{\mu\mu}(\theta_\mu = 0) = 0\,.$$

(d) (41.3), describes a loss of variance (a loss of "temperature") by stopping processes.

# 7 Applications

## 7.1 Optimal motion

From sections 5.1 and 6.1 we can conclude that the motion of pedestrians can be optimized by

- avoiding crossings of different directions $\mu$ of motion, for example by bridges or round-about traffic,
- separation of opposite directions of movement, for example by different lanes for each direction (the right line being preferred [23, 24, 25]), or walking at a narrow passage *by turns*,
- avoiding great velocity variances $\theta_\mu$, for example by walking in formation (as done by the military) [23],
- avoidance of obstacles, narrow passages and great densities.

These rules are applicable to town- and traffic-planning.

## 7.2 Maximal diversity of perceptions

In a museum or super market the individuals will perceive more details (and probably buy more goods) if they walk slowly. So the opposite of the rules in 7.1 could be applied for planning museums and markets.

## 7.3 Critical situations

In critical situations pedestrians may panic. If the mean total interaction free time $\hat{\tau}_\mu$ (see (38)) is less than the mean reaction time $\zeta_\mu$, the danger of falling and getting injured is great.

$$\hat{\tau}_\mu > \zeta_\mu$$

gives a condition for the critical density $\langle\rho_\mu\rangle^{crit}$ of pedestrians that should not be exceeded (see (35) to (38)).

# 8 Conclusions

Starting from the microscopic view of explicit gaskinetic equations we have derived some fluid dynamic equations for the movement of pedestrians. These equations are anisotropic (that means without local rotational symmetry). They look similar to the equations for ordinary fluids, but they are coupled equations for *several* fluids, that means for several types of motion $\mu$, each consisting of individuals having approximately the same intended velocity $\vec{v}_\mu^0$. They also contain some additional terms that are characteristic for *pedestrian* fluids. These terms are due to the tendency of pedestrians to walk with an intended velocity, due to pedestrians changing their type (direction) $\mu$ of motion, and due to interactions between pedestrians (that means due to avoiding maneuvers).

For high densities $\langle\rho_\mu\rangle$ the interactions between pedestrians are very important. As a consequence, a development of pedestrian jams and of separate lanes for different directions of motion is expected. Pedestrian jams can be understood as an deceleration effect due to avoiding maneuvers, and become the worse the greater the velocity variance is. The seperation into several lanes is caused by asymmetrical probabilities for avoiding a pedestrian to the right or to the left. This asymmetry effect has the advantage to reduce the situations, where hindering avoiding behavior is necessary.

For pedestrian crowds the mechanism of approaching equilibrium is essentially given by the tendency to walk with the intended velocity, not by interaction processes as in ordinary fluids. As a consequence, the viscosity $\eta_\mu$ (that means the coefficient of internal friction) is growing with the pedestrian density. In addition, we have seen that variations within pedestrian density will show wave-like propagation with a velocity $c_\mu$ which depends on the mean reaction time.

Last but not least quantities like "temperature" and "pressure" play another role as in ordinary fluids. It can be shown, that the "temperature" (that means the velocity variance) $\theta_\mu$ is produced by the variance of the *intended* velocities, the individuals *want* to move with. As a consequence, two contacting groups of individuals belonging to different types of motion can show different "temperatures". This is the case, for example, on a dance floor. On the other hand, whereas a pressure gradient is compensating the effect of internal friction in ordinary fluids, for pedestrian crowds this role is played by the accelerating effect of the intended velocity. Therefore a hyperbolic stationary velocity profile is found instead of a parabolic one.

# 9 Outlook

Current investigations of pedestrian movement deal with the problem how to specify the forces $\vec{f}_\mu$, and the rates $\chi_\mu^{\cdot\cdot}$ of pedestrians changing their type of motion. These questions call for a detailled model for the intentions of pedestrians.

Pedestrian intentions can be modelled by stochastic laws. They are functions

- of a pedestrian's demand for certain commodities,
- of the city center entry points (parking lots, metro stations, bus stops, and so forth),
- of the location of stores offering the required commodieites,
- of the expenditures (for example prices, ways), and
- of unexpected attractions (shop windows, entertainment, and so forth).

Models of this kind have been developed and empirically tested by BORGERS and TIMMERMANS [2, 3]. A model, which takes into account pedestrian intentions as well as gaskinetic aspects will be presented in a forthcoming paper [11]. It can be formulated in a way that is also suitable for Monte Carlo simulations of pedestrian dynamics with a computer. Computer simulations of this kind are an ideal tool for town- and traffic-planning. Their results can be directly compared with films of pedestrian crowds (see [11]).

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# References

[1] E. Alberti and G. Belli, "Contributions to the Boltzmann-like Approach for Traffic Flow—A Model for Concentration Dependent Driving Programs," *Transportation Research*, **12** (1978) 33-42.

[2] A. Borgers and H. J. P. Timmermans, "City Centre Entry Points, Store Location Patterns and Pedestrian Route Choice Behaviour: A Microlevel Simulation Model," *Socio-Economic Planning Science*, **20** (1986) 25-31.

[3] A. Borgers and H. Timmermans, "A Model of Pedestrian Route Choice and Demand for Retail Facilities within Inner-City Shopping Areas," *Geographical Analysis*, **18** (1986) 115-128.

[4] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University, Cambridge, 1970).

[5] D. Enskog, *PhD* (Almqvist and Wiksell, Upsala, 1917).

[6] D. C. Gazis, *Traffic Science* (Wiley, New York, 1974).

[7] X. van der Hagen, H. Timmermans and A. Borgers, "Modeling Pedestrian Movement: A Brief State-of-the-Art Review," *Metar*, **20** (1989) 1-7.

[8] F. A. Haight, *Mathematical Theories of Traffic Flow* (Academic Press, New York, 1963).

[9] D. Helbing, "A Mathematical Model for the Behavior of Pedestrians," *Behavioral Science*, **36** (1991) 298-310.

[10] D. Helbing, "Interrelations between Stochastic Equations for Systems with Pair Interactions," *Physica A*, **181** (1992) 29-52.

[11] D. Helbing, "Mathematical Models of Pedestrian Behavior", unpublished notes (1992) (forthcoming in *Transportation Research*).

[12] L. F. Henderson, "On the Fluid Mechanics of Human Crowd Motion," *Transportation Research*, **8** (1974) 509-515.

[13] L. F. Henderson, "The Statistics of Crowd Fluids," *Nature*, **229** (1971) 381-383.

[14] L. F. Henderson, "Sexual Differences in Human Crowd Motion," *Nature*, **240** (1972) 353-355.

[15] L. F. Henderson and D. M. Jenkins, "Response of Pedestrians to Traffic Challenge," *Transportation Research*, **8** (1973) 71-74.

[16] "Improving How a Street Works for all Users. Pedestrian Movement Analysis" (1986) (Department of City Planning, Transportation Division, New York City).

[17] J. Jäckle, *Einführung in die Transporttheorie* (Vieweg, Braunschweig, 1978).

[18] J. Keizer, *Statistical Thermodynamics of Nonequilibrium Processes* (Springer, New York, 1987).

[19] L. D. Landau and E. M. Lifschitz, *Lehrbuch der Theoretischen Physik X: Physikalische Kinetik* (Akademie-Verlag, Berlin, 1983).

[20] M. J. Lighthill and G. B. Whitham, "On Kinematic Waves: II. A Theory of Traffic on Long Crowded Roads," *Proceedings of the Royal Society*, **A229** (1955) 317-345.

[21] R. Lüst, *Hydrodynamik* (Bibliographisches Institut, Mannheim, 1978).

[22] A. J. Mayne, "Some Further Results in the Theory of Pedestrians and Road Traffic," *Biometrica*, **41** (1954) 375-389.

[23] P. D. Navin and R. J. Wheeler, "Pedestrian Flow Characteristics," *Traffic Engineering*, **39** (1969) 31-36.

[24] D. Oeding, "Verkehrsbelastung und Dimensionierung von Gehwegen und anderen Anlagen des Fußgängerverkehrs," Straßenbau und Straßenverkehrstechnik, 22 (1963) (Bonn).

[25] S. J. Older, "Movement of Pedestrians on Footways in Shopping Streets," *Traffic Engineering and Control*, **10** (1968) 160-163.

[26] S. L. Paveri-Fontana, "On Boltzmann-like Treatments for Traffic Flow. A Critical Review of the Basic Model and an Alternative Proposal for Dilute Traffic Analysis," *Transportation Research*, **9** (1975) 225-235.

[27] I. Prigogine, "A Boltzmann-Like Approach to the Statistical Theory of Traffic Flow," in *Theory of Traffic Flow*, edited by R. Herman (Elsevier, Amsterdam, 1961).

[28] I. Prigogine and R. Herman, *Kinetic Theory for Vehicular Traffic* (Elsevier, New York, 1971).

[29] B. Pushkarev and J. M. Zupan, "Pedestrian Travel Demand," Highway Research Record, 355 (1971) (Highway Research Board, Washington, D. C.)

[30] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, Auckland, 1985).

[31] A. Rieckers and H. Stumpf, *Thermodynamik, Band2*, (Vieweg, Braunschweig, 1977).

[32] J. Sandahl and M. Percivall, "A Pedestrian Traffic Model for Town Centers," *Traffic Quarterly*, **26** (1972) 359-372.

[33] A. J. Scott, "A Theoretical Model of Pedestrian Flow," *Socio-Economic Planning Science*, **8** (1974) 317-322.

[34] I. B. Stilitz, "The Role of Static Pedestrian Groups in Crowded Spaces," *Ergonomics*, **12** (1969) 821-839.

[35] G. B. Whitham, *Lectures on Wave Propagation* (Springer, Berlin, 1979).