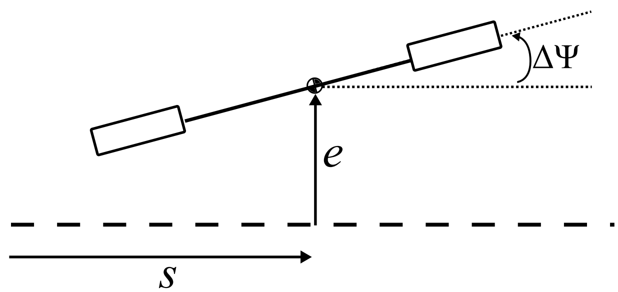
ME 227: Vehicle Dynamics and Control **Spring 2016**

Lecture 5: Lanekeeping

April 12

5.1 Path Tracking Equations of Motion

The bicycle model we developed last week captures many of the essential dynamics of an automobile in terms of the velocity states β (or U_y) and r. However, driving is generally performed on a road, so additional states are required to capture the vehicle's motion relative to a lane. For this lecture, we will look at a straight road or path.



In this diagram, s is defined as the distance traveled along the path. The lateral error e is defined as the lateral distance from the vehicle center of gravity to the path. The heading error $\Delta\Psi$ is the angle between the path and the vehicle centerline. To derive equations of motion for these new states, we will apply the linear tire model and start with the bicycle model equations of motion from last week.

$$m\left(\dot{U}_y + rU_x\right) = F_{\text{yr}} + F_{\text{yf}} \tag{5.1}$$
$$= -C_{\text{r}}\alpha_{\text{r}} - C_{\text{f}}\alpha_{\text{f}} \tag{5.2}$$
$$= -C_{\text{r}}\left(\frac{U_y - br}{U_x}\right) - C_{\text{f}}\left(\frac{U_y + ar}{U_x}\right) - mrU_x + C_{\text{f}}\delta \tag{5.3}$$
$$\implies m\dot{U}_y = -\frac{c_0}{U_x}U_y - \frac{c_1}{U_x}r - mrU_x + C_{\text{f}}\delta \tag{5.4}$$

$$I_z\dot{r} = aF_{\text{yf}} - bF_{\text{yr}} \tag{5.5}$$
$$= -aC_{\text{f}}\alpha_{\text{f}} + bC_{\text{r}}\alpha_{\text{r}} \tag{5.6}$$
$$= -aC_{\text{f}}\left(\frac{U_y + ar}{U_x} - \delta\right) + bC_{\text{r}}\left(\frac{U_y - br}{U_x}\right) \tag{5.7}$$
$$\implies I_z\dot{r} = -\frac{c_1U_y + c_2r}{U_x} + aC_{\text{f}}\delta \tag{5.8}$$

The constants c_0, c_1, c_2 were all defined in Lecture 3:

$$c_0 \triangleq C_\text{f} + C_\text{r} \tag{5.9}$$
$$c_1 \triangleq aC_\text{f} - bC_\text{r} \tag{5.10}$$
$$c_2 \triangleq a^2C_\text{f} + b^2C_\text{r} \tag{5.11}$$

The values of s, e, and $\Delta\Psi$ can be updated by relating their rates of change to the vehicle states:

$$\dot{s} = U_x \cos\Delta\Psi - U_y \sin\Delta\Psi \approx U_x - U_y\Delta\Psi \tag{5.12}$$
$$\dot{e} = U_y \cos\Delta\Psi + U_x \sin\Delta\Psi \approx U_y + U_x\Delta\Psi \tag{5.13}$$
$$\Delta\dot{\Psi} = r \tag{5.14}$$

5.2 Transfer Function Analysis

To analyze the lanekeeping dynamics of the vehicle, we need to get transfer functions for e and $\Delta\Psi$ in terms of the steering input δ. To do that, we need to take second derivatives. Taking the second derivative allows us to eliminate the state variable r and U_y by substitution from (5.13) and (5.14).

$$m\ddot{e} = m\dot{U}_y + mU_x\Delta\dot{\Psi} \tag{5.15}$$
$$= -\frac{c_0}{U_x}U_y - \frac{c_1}{U_x}r - \cancel{mrU_x} + C_\text{f}\delta + \cancel{mrU_x} \tag{5.16}$$
$$= -\frac{c_0}{U_x}\left(\dot{e} - U_x\Delta\Psi\right) - \frac{c_1}{U_x}\Delta\dot{\Psi} + C_\text{f}\delta \tag{5.17}$$
$$= -\frac{c_0}{U_x}\dot{e} + c_0\Delta\Psi - \frac{c_1}{U_x}\Delta\dot{\Psi} + C_\text{f}\delta \tag{5.18}$$

$$I_z\Delta\ddot{\Psi} = I_z\dot{r} = -\frac{c_1U_y + c_2r}{U_x} + aC_\text{f}\delta \tag{5.19}$$
$$= -\frac{c_1\left(\dot{e} - U_x\Delta\Psi\right) + c_2r}{U_x} + aC_\text{f}\delta \tag{5.20}$$
$$= -\frac{c_1}{U_x}\dot{e} + c_1\Delta\Psi - \frac{c_2}{U_x}\Delta\dot{\Psi} + aC_\text{f}\delta \tag{5.21}$$

In matrix form, the equations of motion are given by:

$$\begin{bmatrix} m & 0 \\ 0 & I_z \end{bmatrix}\begin{bmatrix} \ddot{e} \\ \Delta\ddot{\Psi} \end{bmatrix} = \frac{1}{U_x}\begin{bmatrix} -c_0 & -c_1 \\ -c_1 & -c_2 \end{bmatrix}\begin{bmatrix} \dot{e} \\ \Delta\dot{\Psi} \end{bmatrix} + \begin{bmatrix} 0 & c_0 \\ 0 & c_1 \end{bmatrix}\begin{bmatrix} e \\ \Delta\Psi \end{bmatrix} + \begin{bmatrix} C_\text{f} \\ aC_\text{f} \end{bmatrix}\delta \tag{5.22}$$

Just like last week, we can get to the transfer function by taking the Laplace transform of both equations.[1]

Let's define the following Laplace transforms:

$$\mathcal{L}\{e\left(t\right)\} \triangleq E\left(s\right) \tag{5.23}$$
$$\mathcal{L}\{\Delta\Psi\left(t\right)\} \triangleq D\left(s\right) \tag{5.24}$$
$$\mathcal{L}\{\delta\left(t\right)\} \triangleq \Delta\left(s\right) \tag{5.25}$$

[1] Again, those of you who have taken E205 will know that you can put these equations in state-space form and analyze poles/system response with eigenvalues

Applying the Laplace transforms will again result in coupled algebraic equations, just like last week:

$$ms^2E = -\frac{c_0}{U_x}sE + c_0D - \frac{c_1}{U_x}sD + C_f\Delta \tag{5.26}$$

$$I_zs^2D = -\frac{c_1}{U_x}sE + c_1D - \frac{c_2}{U_x}sD + aC_f\Delta \tag{5.27}$$

Again, we will need to algebraically decouple these equations, which will be extremely nasty. I used the symbolic algebra toolbox in MATLAB and got the following transfer function for the lateral error:

$$\frac{E}{\Delta}(s) = \frac{n_2s^2 + n_1s + n_0}{s^2\left(s^2 + d_1s + d_0\right)} \tag{5.28}$$

Where the coefficients are:

$$n_2 = mC_fI_zU_x^2 \tag{5.29}$$

$$n_1 = C_fU_xc_2m - C_fU_xI_zac_1 \tag{5.30}$$

$$n_0 = C_fI_zU_x^2ac_0 - C_fU_x^2c_1m \tag{5.31}$$

$$d_1 = \frac{c_0I_z + c_2m}{I_zmU_x} \tag{5.32}$$

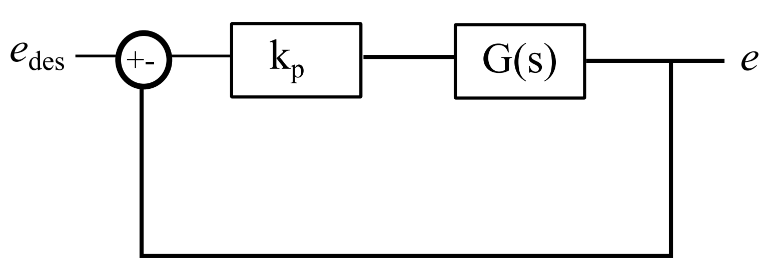
$$d_0 = \frac{C_fC_rL^2 - c_1mU_x^2}{I_zmU_x^2} \tag{5.33}$$

What's important to look at are the poles. We have four poles now - why? Two of the poles correspond to the poles of the bicyle model, which makes sense, because we have mathematically removed the yaw rate and sideslip states, but the vehicle dynamics will still affect the lanekeeping respone. The other two poles are at the origin. What does this mean about system stability?

5.3 Steering Feedback

Since the system is open-loop unstable, it's interesting to think about using the steering angle to keep the vehicle in the lane. The simplest scheme is a proportional feedback control law. Under this control law, we produce a steer angle proportional to the lateral error at the center of gravity:

$$\delta = -k_pe \tag{5.34}$$



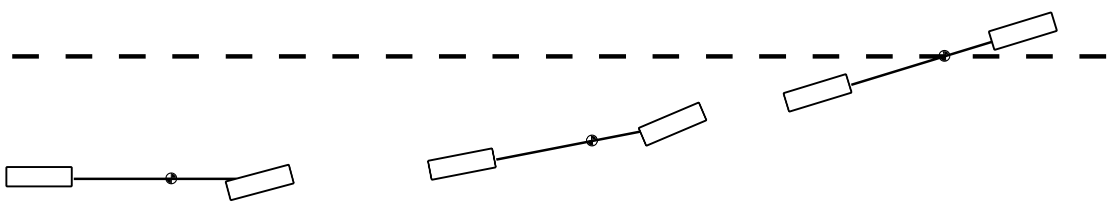
To get the poles of the closed loop transfer function, remember that $\frac{E}{E_{des}}(s) = \frac{k_p G(s)}{1+k_p G(s)}$. The denominator coefficients of the closed loop transfer function are given by:

$$d_1 = \frac{c_0 I_z + c_2 m}{I_z m U_x} \tag{5.35}$$
$$d_2 = \frac{C_\mathrm{f} C_\mathrm{r} L^2 - c_1 m U_x^2 + k_p I_z U_x^2}{I_z m U_x^2} \tag{5.36}$$
$$d_3 = \frac{k_p b L C_\mathrm{r}}{m I_z U_x} \tag{5.37}$$
$$d_4 = \frac{k_p L C_\mathrm{r}}{I_z m} \tag{5.38}$$
$$\tag{5.39}$$

Notice that we no longer have two poles at the origin, meaning our response may be closed-loop stable. For a four-pole system, however, showing that the coefficients are all positive is no longer sufficient to prove stability. In fact, two other stability conditions are also required.[2]

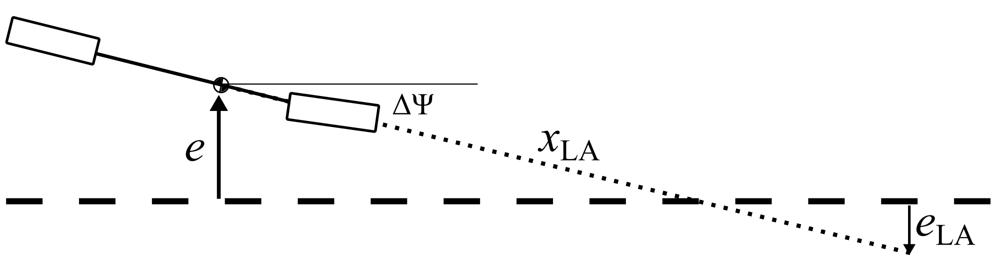
$$d_1 d_2 > d_3 \tag{5.40}$$
$$d_1 d_2 d_3 > d_3^2 + d_1^2 d_4 \tag{5.41}$$

It turns out these two other conditions are not always satisfied and that mathematically we can show a lack of stability, particularly at higher speeds. What is the physical interpretation of this instability?



Suppose a vehicle has a nonzero lateral error. It will yaw as it moves to the desired lateral position, so that when it drives lateral error to zero, there will still be a heading error. This will result in the vehicle overshooting the center of its lane.

The solution to this problem is actually surprisingly straightforward. We need to provide some sort of preview information. The simplest way is to incorporate the heading error into the feedback through the use of lookahead:



The steering command is now:

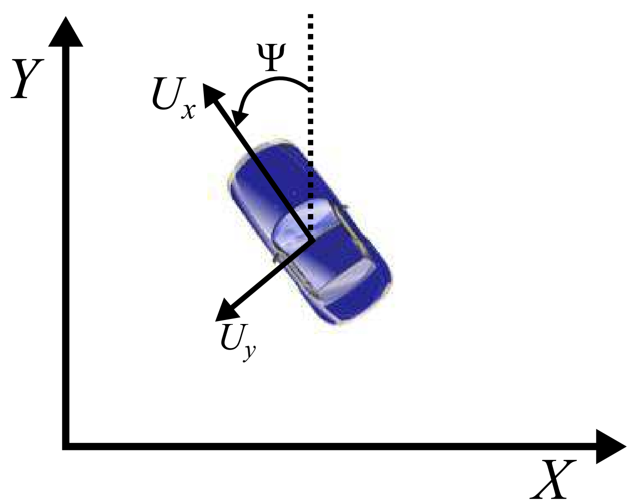
$$\delta = -k_p e_\mathrm{LA} = -k_p \left(e + x_\mathrm{LA} \Delta\Psi\right) \tag{5.42}$$

[2]These conditions come from applying Routh's method, which I believe is covered in E205

Note that this is equivalent to adding feedback on heading error with a gain of $k_p x_{\text{LA}}$ to our original proportional feedback on lateral error. Another way to think about this is a more physically intuitive (and more effective) method of PD control on lateral error e, since the derivative $\dot{e} = U_y + U_x \Delta\Psi$ is linearly dependent on heading error $\Delta\Psi$. Why would lookahead control be easier to implement in practice compared to PD control? Going back to E105, what is the benefit of adding in feedback on the heading error?

5.4 Global Position States

Sometimes when plotting the vehicle's trajectory, we want to know where the car is in the world (i.e. in a static reference frame). This is possible by adding in three more states - X, Y, and Ψ:



Equations of motion relating the rigid body velocities U_x, U_y and r to the global states are as follows:

$$\dot{Y} = U_x \cos\Psi - U_y \sin\Psi \tag{5.43}$$

$$\dot{X} = -U_y \cos\Psi - U_x \sin\Psi \tag{5.44}$$

$$\dot{\Psi} = r \tag{5.45}$$

5.5 State-Space Formulation

As you can probably see, with anything more than two poles, using transfer functions for system analysis becomes an algebraic nightmare. Fortunately, there is a method known as state-space form for elegantly writing and analyzing linear systems. This is covered in complete detail in E205, but I will cover what you need to know for HW3. Define a state vector $\tilde{x}$:

$$\tilde{x} \triangleq \begin{bmatrix} e \\ \dot{e} \\ \Delta\Psi \\ \Delta\dot{\Psi} \end{bmatrix} \tag{5.46}$$

We can then write the equations of motion in the following *state-space* form:

$$\dot{\tilde{x}} = A\tilde{x} + Bu \tag{5.47}$$

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \\ \Delta\dot{\Psi} \\ \Delta\ddot{\Psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{c_0}{mU_x} & \frac{c_0}{m} & -\frac{c_1}{mU_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{c_1}{I_zU_x} & \frac{c_1}{I_z} & -\frac{c_2}{I_zU_x} \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \Delta\Psi \\ \Delta\dot{\Psi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{C_f}{m} \\ 0 \\ \frac{aC_f}{I_z} \end{bmatrix} \delta \tag{5.48}$$

The output of interest in state-space form is given by y:

$$y = C\tilde{x} + \cancel{Du} \tag{5.49}$$

$$e = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tilde{x} \tag{5.50}$$

Several interesting things to note. Finding poles in state-space form is very easy - the poles are simply the eigenvalues of the A matrix. Furthermore, you can easily switch from state-space form to transfer function form in MATLAB using the command `[num,den] = ss2tf(A,B,C,0)`. For the case of steering feedback proportional to lateral error:

$$\delta = -k_p e = -K\tilde{x} = -\begin{bmatrix} k_p & 0 & 0 & 0 \end{bmatrix} \tilde{x} \tag{5.51}$$

$$\tag{5.52}$$

The dynamics of the closed loop system are now given by:

$$\dot{\tilde{x}} = A\tilde{x} + B\delta = A\tilde{x} - BK\tilde{x} = (A - BK)\,\tilde{x} \tag{5.53}$$

What are the poles of the closed-loop system?