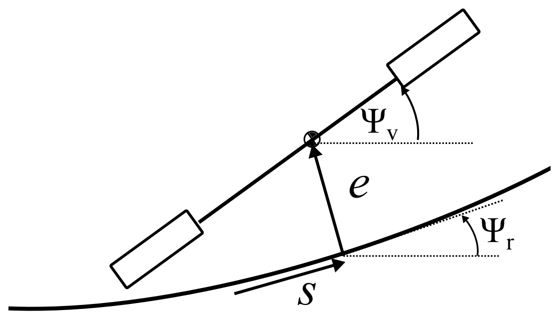
ME 227: Vehicle Dynamics and Control **Spring 2016**

Lecture 6: Path Tracking and Feedforward

April 14 ©JC Gerdes, N Kapania

6.1 Driving on a Curved Path

When the road curves, the equations of motion for the vehicle relative to the lane includes a term related to the curvature of the road.



$$\dot{s} = U_x \cos\Delta\Psi - U_y \sin\Delta\Psi \approx U_x - U_y\Delta\Psi \tag{6.1}$$

$$\dot{e} = U_y \cos\Delta\Psi + U_x \sin\Delta\Psi \approx U_y + U_x\Delta\Psi \tag{6.2}$$

$$\Delta\dot{\Psi} = \dot{\Psi}_\mathrm{v} - \dot{\Psi}_\mathrm{r} \tag{6.3}$$

$$= r - \kappa\dot{s} \approx r - U_x\kappa \tag{6.4}$$

Where $\kappa(s)$ is the instantaneous curvature of the road at distance s along the path. Notice that we now have two heading angles to keep track of: the heading of the vehicle Ψ_v and the heading of the road Ψ_r. The heading error $\Delta\Psi \triangleq \Psi_\mathrm{v} - \Psi_\mathrm{r}$ is now defined to be the difference between the two. The equation of motion for $\Delta\dot{\Psi}$ makes intuitive sense. A heading error can build up because the vehicle is turning or because the road is turning.

6.2 Path Tracking Performance

Earlier this week, we developed a simple *lookahead* feedback control law for driving in a straight line, where $\delta_\mathrm{FB} = -k_\mathrm{p}(e + x_\mathrm{LA}\Delta\Psi)$. What was the steady-state lateral deviation e when lookahead feedback was used to drive on a straight path? Lets use the state-space formulation we developed in the last class:

$$\dot{\tilde{x}} = A\tilde{x} + B\delta \tag{6.5}$$

$$\tilde{x} \triangleq \begin{bmatrix} e \\ \dot{e} \\ \Delta\Psi \\ \Delta\dot{\Psi} \end{bmatrix} \tag{6.6}$$

When we close the loop with the lookahead feedback:

$$\delta = \delta_{\mathrm{FB}} = -K\tilde{x} = -\begin{bmatrix} k_{\mathrm{p}} & 0 & k_{\mathrm{p}} x_{\mathrm{LA}} & 0 \end{bmatrix} \tilde{x} \tag{6.7}$$

and the dynamics of the closed loop system are given by:

$$\dot{\tilde{x}} = A\tilde{x} + B\delta = A\tilde{x} - BK\tilde{x} = (A - BK)\,\tilde{x} \tag{6.8}$$

We discussed in the last class how the eigenvalues of the closed-loop system are the poles of the closed-loop system response. This means that as long as the matrix $(A - BK)$ has eigenvalues with a real negative component, all the states in $\tilde{x}$ will be driven to zero by the feedback controller. As a result, there will be 0 steady-state lateral error and 0 steady-state heading error.

Unfortunately, when the road is curved, this is not the case. Rederiving our equations of motion for $\dot{e}$ and $\Delta\dot{\Psi}$ from earlier this week:

$$\dot{e} = U_y + U_x \Delta\Psi \tag{6.9}$$

$$m\ddot{e} = m\dot{U}_y + mU_x\Delta\dot{\Psi} \tag{6.10}$$

$$= \left(-\frac{c_0}{U_x}U_y - \frac{c_1}{U_x}r - mU_x r + C_f\delta\right) + mU_x\Delta\dot{\Psi} \tag{6.11}$$

$$= -\frac{c_0}{U_x}\left(\dot{e} - U_x\Delta\Psi\right) - \frac{c_1}{U_x}\left(\Delta\dot{\Psi} + U_x\kappa\right) - mU_x\left(\Delta\dot{\Psi} + U_x\kappa\right) + C_f\delta + mU_x\Delta\dot{\Psi} \tag{6.12}$$

$$= -\frac{c_0}{U_x}\dot{e} + c_0\Delta\Psi - \frac{c_1}{U_x}\Delta\dot{\Psi} + C_{\mathrm{f}}\delta - \left(mU_x^2 + c_1\right)\kappa \tag{6.13}$$

$$I_z\Delta\ddot{\Psi} = I_z\dot{r} - I_zU_x\dot{\kappa} \tag{6.14}$$

$$= \left(-\frac{c_1U_y + c_2 r}{U_x} + aC_{\mathrm{f}}\delta\right) - I_zU_x\dot{\kappa} \tag{6.15}$$

$$= -\frac{c_1}{U_x}\left(\dot{e} - U_x\Delta\Psi\right) - \frac{c_2}{U_x}\left(\Delta\dot{\Psi} + U_x\kappa\right) + aC_f\delta - I_zU_x\dot{\kappa} \tag{6.16}$$

$$= -\frac{c_1}{U_x}\dot{e} + c_1\Delta\Psi - \frac{c_2}{U_x}\Delta\dot{\Psi} + aC_{\mathrm{f}}\delta - c_2\kappa - I_zU_x\dot{\kappa} \tag{6.17}$$

$$\tag{6.18}$$

The curvature of the road $\kappa(s)$ is not a state of our system nor is it an input we can control. Instead, we must model it as a *disturbance* input.

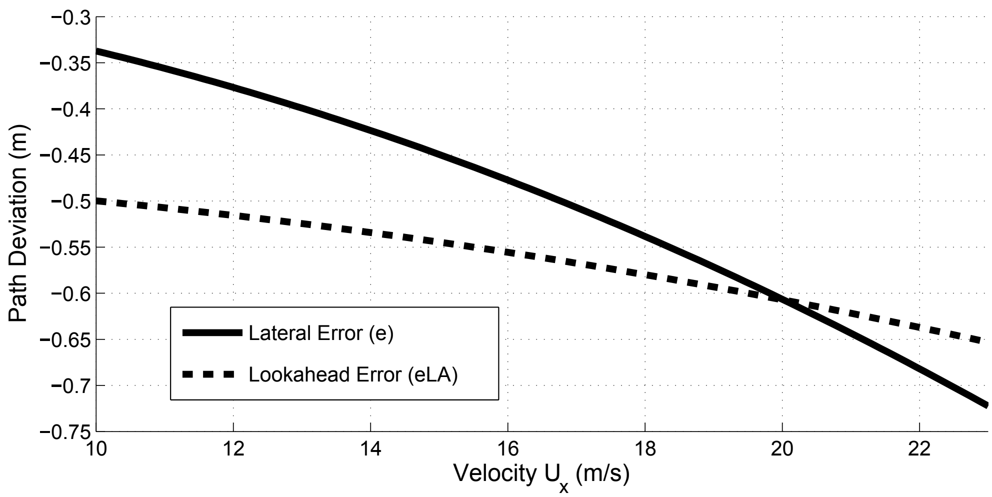
$$\dot{\tilde{x}} = A\tilde{x} + B\delta + d \tag{6.19}$$

$$d = \begin{bmatrix} 0 \\ -\left(U_x^2 + \frac{c_1}{m}\right)\kappa \\ 0 \\ -\frac{c_2}{I_z}\kappa - U_x\dot{\kappa} \end{bmatrix} \tag{6.20}$$

Note that the matrices A and B are given by (5.48) and are unchanged from the previous lecture. Applying the feedback controller gives the following system dynamics:

$$\dot{\tilde{x}} = (A - BK)\,\tilde{x} + d \tag{6.21}$$

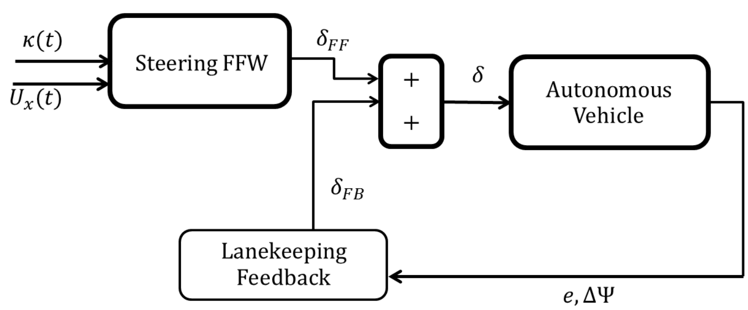
Since d is nonzero and out of our control, the vehicle in general will not have zero tracking error at steady-state. The steady-state values of e and e_{LA} depend on the vehicle speed, curvature, and parameters, but are shown below for a linear vehicle model cornering on a turn with radius $R = 100$ meters
(i.e. $\kappa = .01$ 1/m). As the vehicle goes faster and faster, the vehicle drifts further to the outside of the turn, which makes intuitive sense.



6.3 Adding in Feedforward

While we can't control the curvature of the road, it's not like a typical disturbance considered in E105 because we generally know what the curvature will be at every point on the road. If a human is in the loop, this comes from our eyes, but for an autonomous vehicle, this generally comes from a map of the road.

As a result, we can compensate for the road's curvature by applying a *feedforward* steering input δ_{FF}, which does not depend on the error states e and $\Delta\Psi$ but only on the vehicle's speed and road curvature.



What feedforward steering should we choose? Remember from Lecture 2 that we know the proper steady-state steer angle to drive a circle with radius R (or curvature $1/R$):

$$\delta_{\mathrm{FF}} = \frac{L}{R} + K a_y \tag{6.22}$$

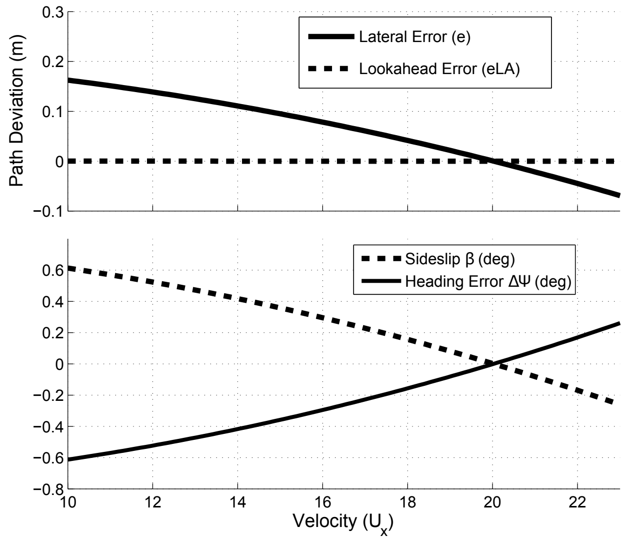
$$= L\kappa + K U_x^2 \kappa \tag{6.23}$$

The dynamics of the vehicle are now given by:

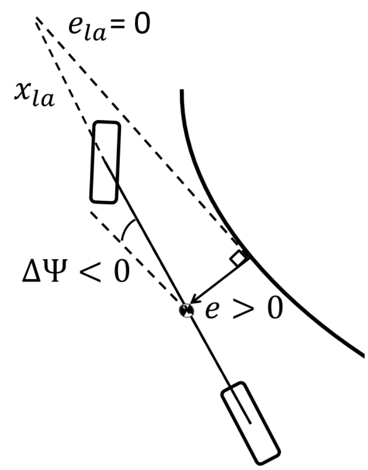
$$\dot{\tilde{x}} = A\tilde{x} + B\left(\delta_{\mathrm{FB}} + \delta_{\mathrm{FF}}\right) + d \tag{6.24}$$

$$= \left(A - BK\right)\tilde{x} + B\delta_{\mathrm{FF}} + d \tag{6.25}$$

Simulating this system, we find that our steady-state errors are a lot closer to zero, but not quite:



What is happening physically here?

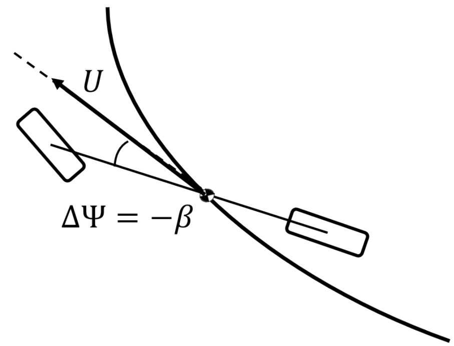


As the diagram indicates, the system settles to a steady-state condition where the lookahead error e_{LA} is zero, but the lateral deviation e and heading deviation $\Delta\Psi$ are nonzero. At steady-state,

$$e_{\text{LA}} = 0 \implies e = -x_{\text{LA}}\Delta\Psi \tag{6.26}$$

This makes sense intuitively. Because we have one control input (δ) but two error states (e and $\Delta\Psi$), it is only possible to control a linear combination of e and $\Delta\Psi$, but not both simultaneously. As you can see from the simulation results, there is a value of U_x where the lateral error and heading error settle to 0. This occurs when the vehicle sideslip is 0.

It turns out that for zero steady-state lateral error, the vehicle sideslip must always be tangent to the desired path. Why?



Another observation is that $\Delta\Psi = -\beta$ at steady-state. In general, we ultimately care about achieving zero steady-state lateral tracking error, so it is OK to have a non-zero heading error if it keeps the vehicle sideslip tangent to the desired path.

Our goal is now to augment our feedforward command so that the steady-state sideslip is always tangent to the desired path. The vehicle sideslip β is related to the rear tire slip:

$$\alpha_\mathrm{r} = \beta - \frac{br}{U_x} \tag{6.27}$$

At steady-state, $r = U_x\kappa$:

$$\alpha_\mathrm{r} = \beta_{ss} - b\kappa \tag{6.28}$$

From the steady state force and moment balance, $F_\mathrm{yr} = \frac{a}{L}mU_x^2\kappa$. From the linear tire model, $F_\mathrm{yr} = -C_\mathrm{r}\alpha_\mathrm{r}$.

$$\alpha_\mathrm{r} = -\frac{F_\mathrm{yr}}{C_\mathrm{r}} = -\frac{amU_x^2}{C_\mathrm{r}L}\kappa = \beta_\mathrm{ss} - b\kappa \tag{6.29}$$

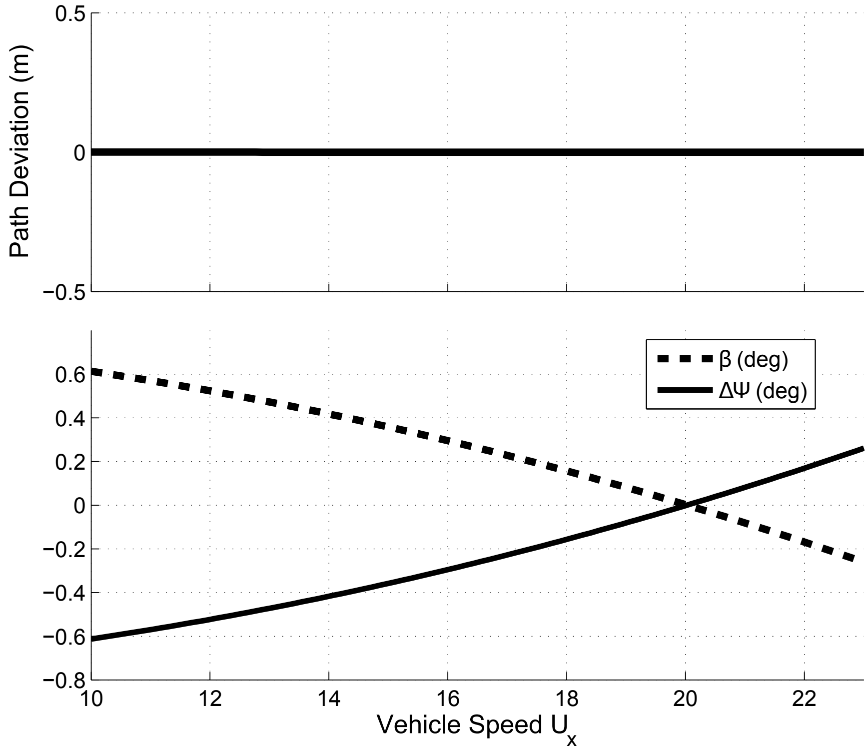
$$\implies \beta_\mathrm{ss} = \left(b - \frac{amU_x^2}{C_\mathrm{r}L}\right)\kappa \tag{6.30}$$

As a sanity check of this equation, $\beta_\mathrm{ss} = 0$ if $U_x = \sqrt{\frac{bLC_\mathrm{r}}{am}}$, a result we derived in Lecture 4. The full steering control algorithm command is now given by the following equation:

$$\delta = -k_\mathrm{p}\left(e + x_\mathrm{LA}\left(\Delta\Psi + \beta_{ss}\right)\right) + \left(L\kappa + KU_x^2\kappa\right) \tag{6.31}$$

$$\beta_{ss} = \left(b - \frac{amU_x^2}{C_\mathrm{r}L}\right)\kappa \tag{6.32}$$

Why do we put the β_{ss} in the lanekeeping feedback term? Can we still write (6.31) in terms of a feedback component δ_FB and δ_FF?



As expected, this final steering controller has zero steady-state lateral error e, but generally non-zero heading error $\Delta\Psi$.

6.4 Mathematical Derivation

The steering controller (6.31) was derived indirectly through physical intuition, but it can also be derived mathematically as well. In state space form, we have:

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\left(\delta_{\mathrm{FB}} + \delta_{\mathrm{FF}}\right) + d & (6.33)\\ &= \left(A - BK\right)\tilde{x} + B\delta_{\mathrm{FF}} + d & (6.34)\\ &= \hat{A}x + \hat{B}\kappa & (6.35) \end{aligned}$$

Where $\hat{A} \triangleq (A - BK)$. Assuming our feedforward command $\delta_{\mathrm{FF}} \triangleq G_{\mathrm{FF}}\kappa$ is linearly dependent on κ:

$$\begin{aligned} \hat{B}\kappa &= B\delta_{\mathrm{FF}} + d & (6.36)\\ &= \begin{bmatrix} 0 \\ \frac{C_f}{m} \\ 0 \\ \frac{aC_f}{I_z} \end{bmatrix} \delta_{\mathrm{FF}} + \begin{bmatrix} 0 \\ -\left(U_x^2 + \frac{c_1}{m}\right) \\ 0 \\ -\frac{c_2}{I_z}\kappa \cancel{-U_x\dot{\kappa}} \end{bmatrix} \kappa & (6.37)\\ &= \begin{bmatrix} 0 \\ \frac{C_f}{m}G_{\mathrm{FF}} - \left(U_x^2 + \frac{c_1}{m}\right) \\ 0 \\ \frac{aC_f}{I_z}G_{\mathrm{FF}} - \frac{c_2}{I_z} \end{bmatrix} \kappa & (6.38) \end{aligned}$$

Notice that we have rewritten our dynamics so that the system input is the curvature. This will let us check the steady-state behavior of our system given a constant step input of $K = 1/R$. The steady-state error of our closed-loop steering system is obtained from the final value theorem.[1]

$$\frac{e_{\text{ss}}}{\kappa} = \lim_{s\to 0} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \left(sI - \hat{A}\right)^{-1} \hat{B} \tag{6.39}$$

$$= \frac{1}{k_{\text{p}}} \left(G_{\text{FF}} - \frac{mU_x^2}{L} \left(\frac{bC_{\text{r}} + aC_f \left(k_{\text{p}} x_{\text{LA}} - 1\right)}{C_{\text{r}} C_f} \right) - L + bk_{\text{p}} x_{\text{LA}} \right) \tag{6.40}$$

The steady-state lateral error can be made zero if the feedforward steering gain G_{FF} is chosen as:

$$G_{\text{FF}} = \frac{mU_x^2}{L} \left(\frac{bC_{\text{r}} + aC_f \left(k_{\text{p}} x_{\text{LA}} - 1\right)}{C_{\text{r}} C_f} \right) + L - bk_{\text{p}} x_{\text{LA}} \tag{6.41}$$

$$= L + KU_x^2 - k_{\text{p}} x_{\text{LA}} \left(b - \frac{amU_x^2}{C_{\text{r}} L} \right) \tag{6.42}$$

Substituting back $\delta_{\text{FF}} = G_{\text{FF}} \kappa$:

$$\delta_{\text{FF}} = L\kappa + KU_x^2 \kappa - k_{\text{p}} x_{\text{LA}} \left(b - \frac{amU_x^2}{C_{\text{r}} L} \right) \kappa \tag{6.43}$$

$$= \left(L\kappa + KU_x^2 \kappa \right) - k_{\text{p}} x_{\text{LA}} \beta_{ss} \tag{6.44}$$

This is the same expression for feedforward steering we obtained from physical intuition in the previous section.

[1]This analysis was done in *Vehicle Dynamics and Control: Second Edition* by Rajesh Rajamani. Available online through Stanford Library.