CS2040 Data Structures and Algorithms
Lecture Note #3

Analysis of Algorithms

Objectives

1

- To introduce the theoretical basis for measuring the efficiency of algorithms

2

- To learn how to use such measure to compare the efficiency of different algorithms

Outline

1. What is an Algorithm?
2. What do we mean by Analysis of Algorithms?
3. Big-O notation – Upper Bound
4. How to find the complexity of a program?

1 What is an algorithm?

1 Algorithm

- A step-by-step procedure for solving a problem.
- Properties of an algorithm:
  - Each step of an algorithm must be exact. unclear/blackbox
  - An algorithm must terminate.
  - An algorithm must be effective.
  - *An algorithm should be general.

2 What do we mean by Analysis of Algorithms?

2.1 What is Analysis of Algorithms?

- Analysis of algorithms
  - Provides tools for comparing the **efficiency** of different methods of solution (rather than programs) time/space
  - **Efficiency** = Complexity of algorithms
- A comparison of algorithms
  - Should focus on significant differences in the efficiency of the algorithms
  - Should not consider reductions in computing costs due to clever coding tricks. Tricks may reduce the readability of an algorithm.

2.2 Determining the Efficiency/Complexity of Algorithms

- To evaluate rigorously the resources (time and space) needed by an algorithm and represent the result of the analysis with a formula
- We will emphasize more on the time requirement rather than space requirement here
- The time requirement of an algorithm is also called its time complexity

2.3 By measuring the run time?

TimeTest.java

```
public class TimeTest {
  public static void main(String[] args) {
    long startTime = System.currentTimeMillis();
    long total = 0;
    for (int i = 0; i < 10000000; i++) {
      total += i;
    }
    long stopTime = System.currentTimeMillis();
    long elapsedTime = stopTime - startTime;
    System.out.println(elapsedTime);
  }
}
```

Note: The run time depends on the compiler, the computer used, and the current work load of the computer.

2.4 Exact run time is not always needed

- Using exact run time is not meaningful when we want to compare two algorithms
  - coded in different languages,
  - running on different computers or
  - using different data sets

2.5 Determining the Efficiency of Algorithms

- Algorithm analysis should be independent of
  - Specific implementations
  - Compilers and their optimizers
  - Computers
  - Data

2.6 Execution Time of Algorithms

- Instead of working out the exact timing, we count the number of some or all of the primitive operations (e.g. +, -, *, /, assignment, …) needed.
- Counting an algorithm's operations is a way to assess its efficiency
  - An algorithm’s execution time is related to the number of operations it requires.

2.7 Counting the number of statements

- To simplify the counting further, we can ignore
  - the different types of operations, and
  - different number of operations in a statement,

  and simply count the number of statements executed.

2.8 Computation cost of an algorithm

- How many operations are required?

```
for (int i=1; i<=n; i++) {
  perform 100 operations;          // A
  for (int j=1; j<=n; j++) {
     perform 2 operations;         // B
  }
}
```

Total Ops = A + B $= \sum_{i=1}^{n} 100 + \sum_{i=1}^{n} (\sum_{j=1}^{n} 2)$

$$= 100n + \sum_{i=1}^{n} 2n \;= 100n + 2n^2 \;= 2n^2 + 100n$$

2.9 Approximation of analysis results

- Very often, we are interested only in using a simple term to indicate how efficient an algorithm is. The exact formula of an algorithm's performance is not really needed.
- Example:

  Given the formula: $2n^2 + 100n$

  - the dominating term $2n^2$ can tell us approximately how the algorithm performs by providing us with a measure of the growth rate (how the number of operations executed grows as n increases in size) of the algorithm
- This is called asymptotic analysis of the algorithm

2.10 Asymptotic analysis

- Asymptotic analysis is an analysis of algorithms that focuses on
  - analyzing the problems of large input size,
  - considering only the leading term of the formula, and
  - ignoring the coefficient of the leading term
- Some notations are needed in asymptotic analysis

2.11 Algorithm Growth Rates (1/2)

- An algorithm's time requirement can be measured as a function of the problem size, say n
- An algorithm's growth rate
  - Enables the comparison of one algorithm with another
  - Examples
    - Algorithm A requires time proportional to n^2
    - Algorithm B requires time proportional to n

2.12 Algorithm Growth Rates (2/2)

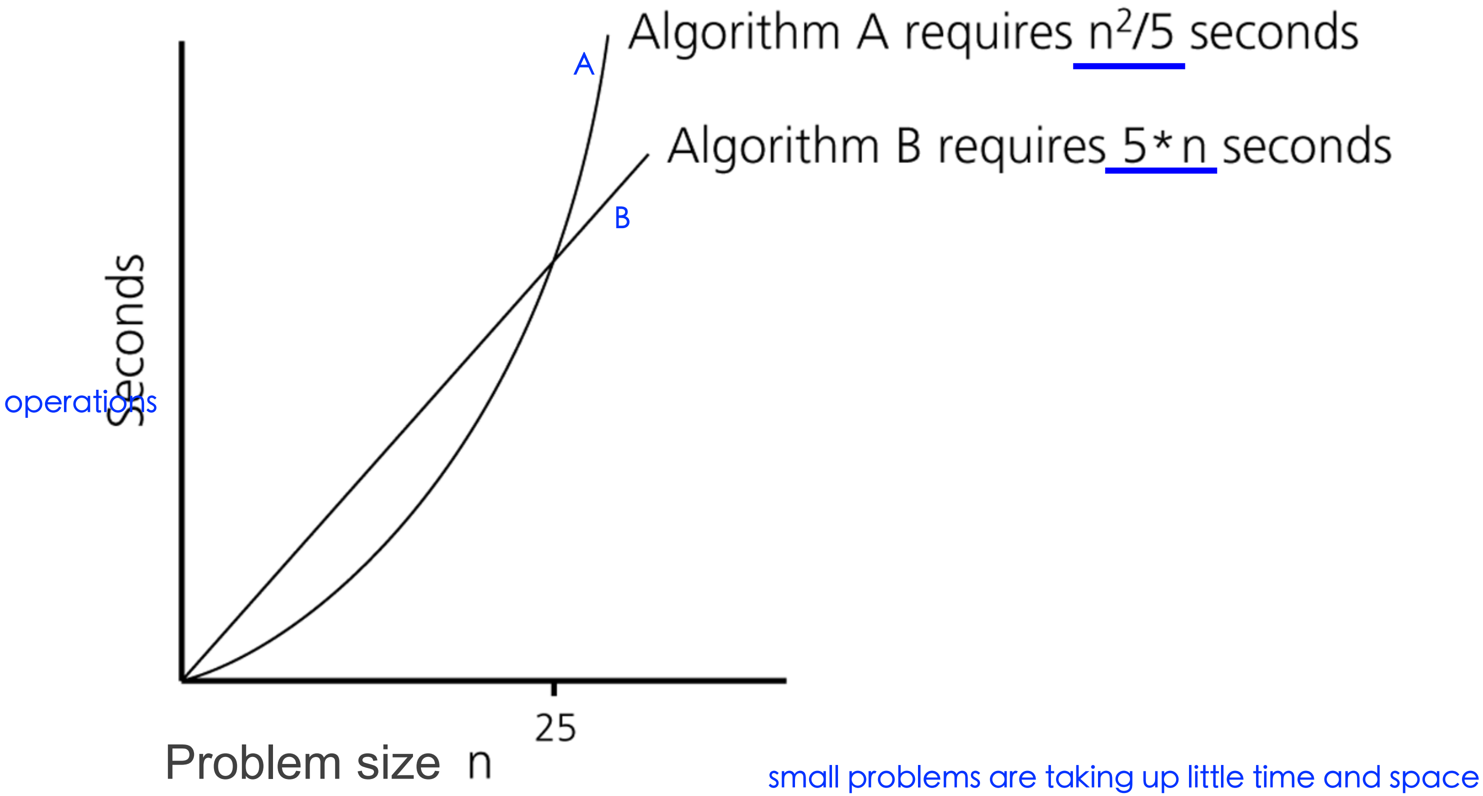


Figure - Time requirements as a function of the problem size n

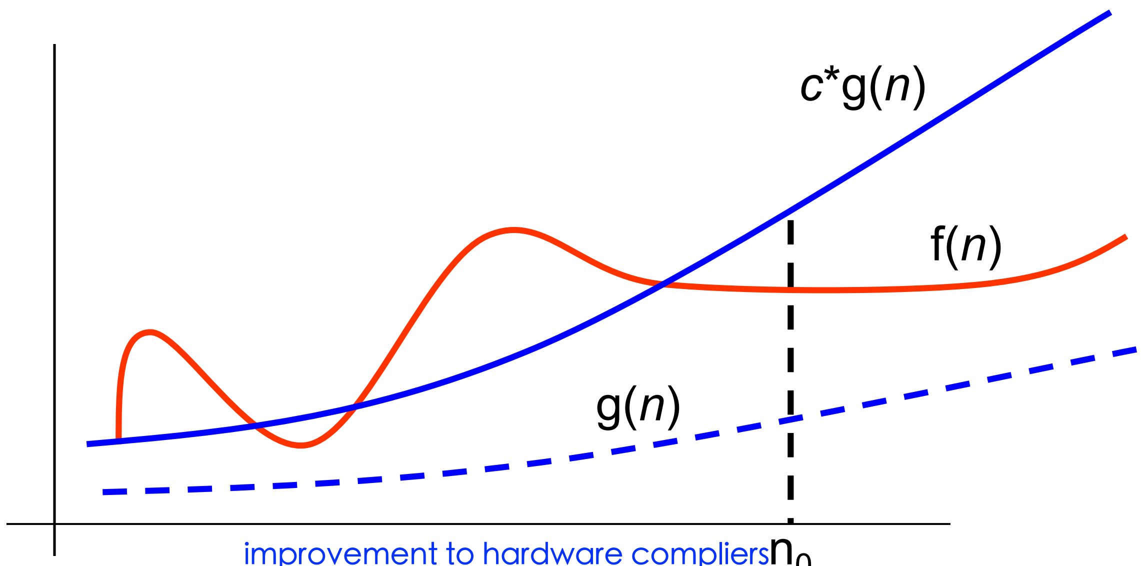
- Algorithm efficiency is typically a concern for large problems only. Why?

3 Big O notation

3.1 Definition – Big O notation

- Given a function f(n), we say g(n) is an (asymptotic) upper bound of f(n), denoted as f(n) = O(g(n)), if there exist a constant $c > 0$, and a positive integer n_0 such that $f(n) \le c*g(n)$ for all $n \ge n_0$.

- f(n) is said to be bounded from above by g(n).
- O() is called the “big O” notation.



improvement to hardware compliers
shift downwards constant improvement

3.2 Ignore the coefficients of all terms

- Based on the definition, $2n^2$ and $30n^2$ have the same upper bound n^2, i.e.,
  - $2n^2 = O(n^2)$

    f1(n) = $2n^2$; g(n) = n^2.
    Let c=3 and n_0=1, since $2n^2 \leq cn^2 \;\; \forall\, n \geq n_0$
    Hence f1(n) = O(g(n))

    Why?

  - $30n^2 = O(n^2)$

    f2(n) = $30n^2$; g(n) = n^2.
    Let c=31 and n_0=1, since $30n^2 \leq cn^2 \;\; \forall\, n \geq n_0$
    Hence f2(n) = O(g(n))

  They differ only in the choice of c.
- Therefore, in big O notation, we can omit the coefficients of all terms in a formula:
  - Example: $f(n) = 2n^2 + 100n = O(n^2) + O(n)$

3.3 Finding the constants c and n_0

- Given $f(n) = 2n^2 + 100n$, prove that $f(n) = O(n^2)$.

Observe that: $2n^2 + 100n \leq 2n^2 + n^2 = 3n^2$ whenever $n \geq 100$.

→ Set the constants to be $c = 3$ and $n_0 = 100$.

By definition, we have $f(n) = O(n^2)$.

Notes:

1. $n^2 \leq 2n^2 + 100n$ for all n, i.e., $g(n) \leq f(n)$, and yet $g(n)$ is an asymptotic upper bound of $f(n)$
2. c and n_0 are not unique.
   For example, we can choose $c = 2 + 100 = 102$, and $n_0 = 1$ (because $f(n) \leq 102n^2 \ \forall\ n \geq 1$)

Q: Can we write $f(n) = O(n^3)$?

Yes

3.4 Is the bound tight?

- The complexity of an algorithm can be bounded by many functions.
- Example:
  - Let $f(n) = 2n^2 + 100n$.
  - $f(n)$ is bounded by n^2, n^3, n^4 and many others according to the definition of big O notation.
  - Hence, the following are all correct:
    - $f(n) = O(n^2)$; $f(n) = O(n^3)$; $f(n) = O(n^4)$
- However, we are more interested in the tightest bound which is n^2 for this case.

3.5 Growth Terms: Order-of-Magnitude

- In asymptotic analysis, a formula can be simplified to a single term with coefficient 1
- Such a term is called a growth term (rate of growth, order of growth, order-of-magnitude)
- The most common growth terms can be ordered as follows: (note: many others are not shown)

polunomial

super exponential/factorial

$$O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n) < O(n!)$$

constant linear loglinear exponential

"fastest" *"slowest"*

Note:

- "log" = log base 2, or $\log_2$; "$\log_{10}$" = log base 10; "ln" = log base e. In big O, all these log functions are the same. (Why?)

3.6 Examples on big O notation

- $f1(n) = \frac{1}{2}n + 4$
  $= O(n)$
- $f2(n) = 240n + 0.001n^2$
  $= O(n^2)$

higher order

- $f3(n) = n \log n + \log n + n \log (\log n)$
  $= O(n \log n)$

3.7 Order-of-Magnitude Analysis and Big O Notation (1/2)

(a)

Function	10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Figure - Comparison of growth-rate functions in tabular form

3.8 Order-of-Magnitude Analysis and Big O Notation (2/2)

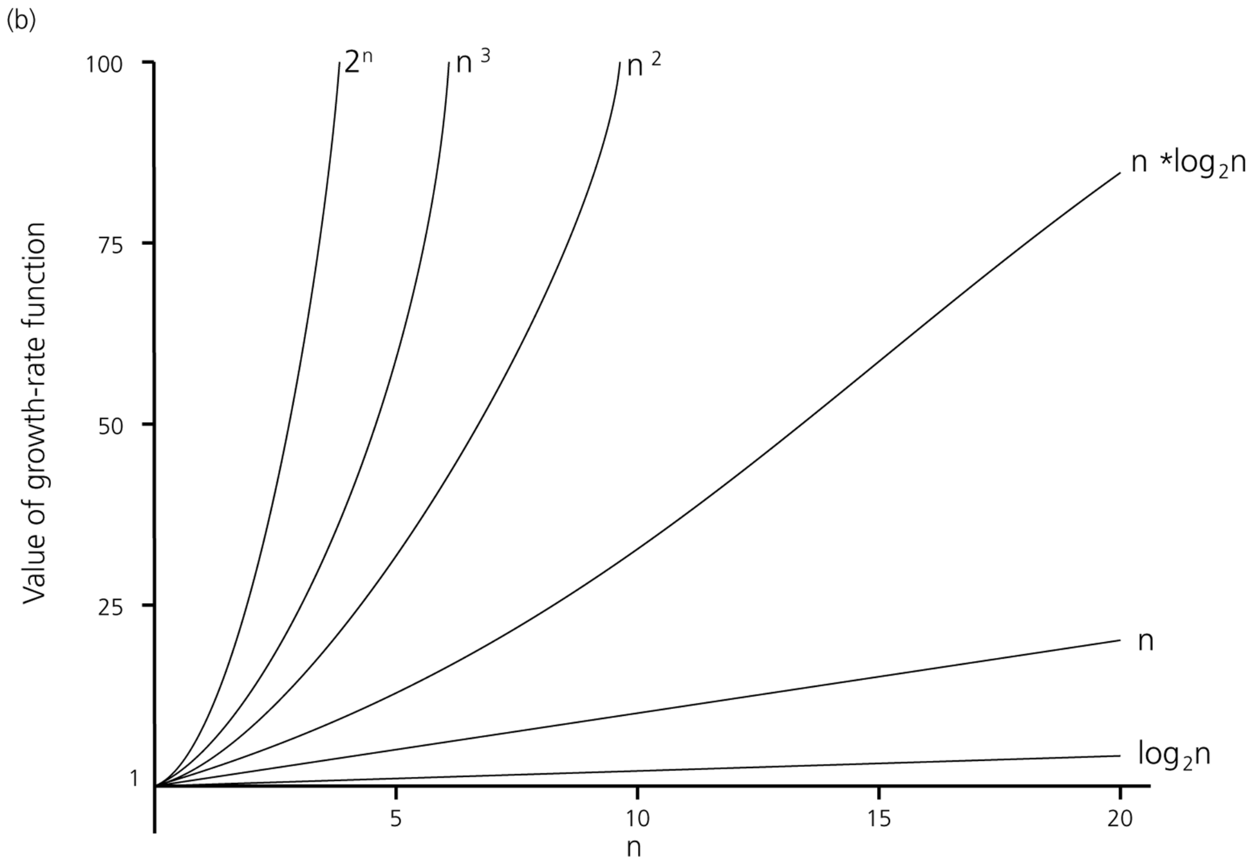


Figure - Comparison of growth-rate functions in graphical form

3.9 Summary: Order-of-Magnitude Analysis and Big O Notation

- Order of growth of some common functions:

  $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n) < O(n!)$

- Properties of growth-rate functions
  - You can ignore low-order terms
  - You can ignore a multiplicative constant in the high-order term
  - $O(f(n)) + O(g(n)) = O(f(n) + g(n))$

4 How to find the complexity of a program? algorithm

4.1 Some rules of thumb and examples

- Basically just count the number of statements executed.
- If there are only a small number of simple statements in a program – O(1)
- If there is a ‘for’ loop dictated by a loop index that goes up to n – O(n)
- If there is a nested ‘for’ loop with outer one controlled by n and the inner one controlled by m – O($n*m$)
- For a loop with a range of values n, and each iteration reduces the range by a fixed constant fraction (eg: ½) – O(log n)
- For a recursive method, each call is usually O(1). So
  - if n calls are made – O(n)
  - if n log n calls are made – O(n log n)

4.2 Examples on finding complexity (1/2)

- What is the complexity of the following code fragment?

```
int sum = 0;
for (int i=1; i<n; i=i*2) {
    sum++;
}
```

- It is clear that sum is incremented only when

  $i = 1, 2, 4, 8, \ldots, 2^k$ where $k = \lfloor \log_2 n \rfloor$

  f(n)=(lgn+1)*c1+c2

  There are $k+1$ iterations. So the complexity is $O(k)$ or $O(\log n)$

Note:

- In Computer Science, $\log n$ means $\log_2 n$.
- When 2 is replaced by 10 in the 'for' loop, the complexity is $O(\log_{10} n)$ which is the same as $O(\log_2 n)$. (Why?)
- $\log_{10} n = \log_2 n / \log_2 10$

4.2 Examples on finding complexity (2/2)

- What is the complexity of the following code fragment?
(For simplicity, let's assume that *n* is some power of 3.)

```
int sum = 0;
for (int i=1; i<n; i=i*3) {
   for (j=1; j<=i; j++) {
      sum++;
   }
}
```

- $f(n) = 1 + 3 + 9 + 27 + \ldots + 3^{(\log_3 n)}$
$= 1 + 3 + \ldots + n/9 + n/3 + n$
$= n + n/3 + n/9 + \ldots + 3 + 1$ (reversing the terms in previous step)
$= n * (1 + 1/3 + 1/9 + \ldots)$
$\leq n * (3/2)$
$= 3n/2$
$= O(n)$

Why is $(1 + 1/3 + 1/9 + \ldots) \leq 3/2$?
This is sum of infinite geometric series see word file "analysis_of_algorithms_useful_equalities.docx"

4.3 Non-recursive Binary Search Algorithm (1)

- Requires array to be sorted in ascending order
- Maintain subarray where x (the search key) might be located
- Repeatedly compare x with m, the middle element of current subarray
  - If x = m, found it!
  - If x > m, continue search in subarray after m
  - If x < m, continue search in subarray before m

4.3 Non-recursive Binary Search Algorithm (2)

- Data in the array a[] are sorted in ascending order

```
public static int binSearch(int[] a, int len, int x)
{
   int mid, low = 0;
   int high = len - 1;
   while (low <= high) {
      mid = (low + high) / 2;
      if (x == a[mid]) return mid;
      else if (x > a[mid]) low = mid + 1;
      else high = mid - 1;
   }
   return -1;
}
```

4.3 Non-recursive Binary Search Algorithm (3)

- At any point during binary search, part of array is "*alive*" (might contain the point x)
- Each iteration of loop eliminates at least half of previously "*alive*" elements
- At the beginning, all n elements are "*alive*", and after
  - After 1 iteration, at most $n/2$ elements are left, or alive
  - After 2 iterations, at most $(n/2)/2 = n/4 = n/2^2$ are left
  - After 3 iterations, at most $(n/4)/2 = n/8 = n/2^3$ are left

    :
  - After i iterations, at most $n/2^i$ are left
  - At the final iteration, at most 1 element is left

4.3 Non-recursive Binary Search Algorithm (4)

In the worst case, we have to search all the way up to the last iteration k with only one element left.

We have:

$$n/2^k = 1$$

$$2^k = n$$

$$k = \log n$$

Hence, the binary search algorithm takes O(f(n)) , or O(log n) times

4.4 Time complexity of recursion: Fibonacci numbers

- Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, …
  - The first two Fibonacci numbers are both 1 (arbitrary numbers)
  - The rest are obtained by adding the previous two together.
- Calculating the n^{th} Fibonacci number recursively:

$$
\begin{aligned}
Fib(n) &= 1 && \text{for } n=1, 2 \\
&= Fib(n-1) + Fib(n-2) && \text{for } n > 2
\end{aligned}
$$

```java
// Precond: n > 0
public static int fib(int n) {
  if (n <= 2)
    return 1;
  else
    return fib(n-1) + fib(n-2);
}
```

4.4 Time complexity of recursion: Fibonacci numbers

- Function for total number of operations is almost exactly like the recursive case of Fibonacci
  - $\boxed{T(n) = T(n-1)+T(n-2)+4}$ (1 comparison, 1 addition, 2 subtraction)

    2nd order linear recurrence → Hard !

$\Rightarrow$ $\boxed{T(n) < 2T(n-1)+c}$ → 1st order linear recurrence → Easier !

$\Rightarrow$ $< 2(2T(n-2)+c)+c$ (expand T(n-1) to be 2*T(n-2)+c)

$\Rightarrow$ $< 4T(n-2)+2c+c$

$\Rightarrow$ $< 8T(n-3)+4c+2c+c$ (cont. the expansion)

$\Rightarrow$ … (will stop when n is 1)

$\Rightarrow$ $< 2^n(T(1))+2^{n-1}c+ 2^{n-2}c+\ldots+c$ (T(1) is c)

$\Rightarrow$ $< 2^n*(1+1/2+1/4+1/8+\ldots)*c$

$\Rightarrow$ $= O(2^n)$

Iterative Method

4.5 Analysis of Different Cases (1)

Worst-Case Analysis

- Interested in the worst-case behaviour.
- A determination of the maximum amount of time that an algorithm requires to solve problem/input of size *n*
- This is the one we will be mostly looking at for the rest of the course

Best-Case Analysis

- Interested in the best-case behaviour
- Not useful

Average-Case Analysis

- A determination of the amount of time that an algorithm requires to solve an “average” input of size *n*
- Have to know the probability distribution of the inputs to determine what is an “average input”
- Not covered in this module (except for some simple examples)

4.5 Analysis of Different Cases (2)

Expected-Case Analysis

- Analysis performed on randomized algorithms – i.e algorithms that employ randomness in their logic
- Often confused with average-case analysis (although they are related)
- Not covered in this module

Amortized Analysis look for the worst case

- Sometimes worst case behavior cannot be possible for every run of the algorithm, meaning that across several runs, only some can induce worst case behavior while others don't
- Amortized analysis determines the total time complexity required for a sequence of runs and thus the "amortized" cost per run
- Need more advanced techniques which will be covered in CS3230, so will only cover some very simple examples in CS2040

4.6 The Efficiency of Searching Algorithms

- Example: Efficiency of Sequential Search (data not sorted)
  - Worst case: O(n)

    Which case?
  - Average case: O(n)
  - Best case: O(1)

    Why? Which case?
  - Unsuccessful search?
- Q: What is the best case complexity of Binary Search (data sorted)?
  - Best case complexity is not interesting. Why?

4.7 Keeping Your Perspective

- If the problem size is always small, you can probably ignore an algorithm's efficiency
- Weigh the trade-offs between an algorithm's time requirements and its memory requirements
- Order-of-magnitude analysis focuses on large problems
- There are other measures, such as big Omega (Ω), big theta (Θ), little oh (o), and little omega (ω). These may be covered in more advanced module.

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Question Time:

- What is the complexity of the following code fragment?

```
String someStr = "A";

for (int i=1; i<n; i++) {
    someStr = someStr + "A";
}
```

1. O(n)
2. O(nlogn)
3. O(logn)
4. $O(n^2)$

String not mutable
String concatenation keeps on create a new object everytime

$f(n)=1+2+3+\cdots+n$
$=n(n+1)/2$