第二章　命题逻辑: 形式推演

上次内容

- 计算机科学中的逻辑
- 命题
- 命题逻辑的语言，语法：表达式，公式，辖域
- 命题逻辑的语义：模型，可满足公式

上次内容

命题逻辑的被形式化对象

- 命题
- 复合命题的真假值

命题逻辑的形式对象:

◇ 命题逻辑的语言

◇ 命题逻辑的表达式

◇ 命题逻辑的公式

表达式是否是公式的判定方法

上次内容

命题逻辑的(形式)语义
赋值$v : P \to \{0, 1\}$
公式在一个赋值下的真假值
永真公式/永假公式
逻辑推论: $\Sigma \models A$ (读作Σ满足A)
永真公式与逻辑推论之间的关系:

$$\models A \text{当且仅当} \emptyset \models A$$

上次内容

系统之外: 被形式化的对象
系统之内: 逻辑语言, 合适公式
逻辑的语法: 公式, 公理, 推理规则
逻辑的语义: 解释, 可满足性, 模型

今天内容

- 推理的必要性
- 形式推演规则
- 形式证明

逻辑推论判定的复杂性

$$\emptyset \models [A \to (B \to C)] \to [(A \to B) \to (A \to C)].$$

逻辑推论判定的复杂性

$$\emptyset \models [A \to (B \to C)] \to [(A \to B) \to (A \to C)].$$

A^v	B^v	C^v	$(B \to C)^v$	$(A \to (B \to C))^v$	$((A \to B) \to (A \to C))^v$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	0	1	1
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

其中$\varphi = [A \to (B \to C)] \to [(A \to B) \to (A \to C)]$.

推理

如果天下雨则地是湿的
天下雨

所以地是湿的.

推理

$$\begin{array}{l}\text{如果天下雨则地是湿的}\\ \text{天下雨}\\ \hline \text{所以地是湿的.}\end{array}$$

$$\begin{array}{l}A \to B\\ A\\ \hline B\end{array}$$

推理

$$\frac{\begin{array}{l} A \to B \\ A \end{array}}{B}$$

$$A \to B, A \models B$$

推理

$$\frac{\begin{array}{l} A \to B \\ A \end{array}}{B}$$

$$\frac{\begin{array}{l} \models A \to B \\ \models A \end{array}}{\models B}$$

推理

$$\frac{\models A \to B \qquad \models A}{\models B}$$

$$\frac{\emptyset \models A \to B \qquad \emptyset \models A}{\emptyset \models B}$$

推理

$$\frac{\models A \to B \qquad \models A}{\models B}$$

$$\frac{\Sigma \models A \to B \qquad \Sigma \models A}{\Sigma \models B}$$

推理

$$\frac{\begin{array}{l}\Sigma \models A \to B \\ \Sigma \models A\end{array}}{\Sigma \models B}$$

如果$\Sigma \models A \to B$,
并且$\Sigma \models A$,
则$\Sigma \models B$.

推理

如果$\Sigma \models A \to B$,
并且$\Sigma \models A$,
则$\Sigma \models B$.

如果$\Sigma \vdash A \to B$,
并且$\Sigma \vdash A$,
则$\Sigma \vdash B$.

形式推理

希望找到一个形式推理的方法使得$\Sigma \vdash A$表示(前提)Σ可以推理出(结论)A, 满足下列条件: 该方法

(1) 不是基于赋值的, 即不是语义的, 而是语法的;

(2) 保真的(truth-preserving), 即在任何赋值v下, 当前提在赋值v下为真, 那么结论在赋值v下也为真;

(3) 完备的(complete), 每个逻辑推论在这个形式推理下可以推导出.

形式推理

给定公式集合Σ和公式A, 如果A可以由Σ形式推理得出, 记为

$$\Sigma \vdash A.$$

形式推理

形式推理是由一些推演规则定义的.

形式推演规则: 自反

$$(Ref) \text{ 自反} \qquad A \vdash A.$$

形式推演规则: 单调性

$$\boxed{(+)\qquad \begin{array}{l}\text{如果}\Sigma \vdash A,\\ \text{则}\Sigma, \Sigma' \vdash A.\end{array}}$$

其中Σ'是任何公式集合.

形式推演规则: $\neg$

$$\boxed{\begin{array}{lll} (\neg^-) & \text{如果} & \Sigma, \neg A \vdash B, \\ & & \Sigma, \neg A \vdash \neg B, \\ & \text{则} & \Sigma \vdash A. \end{array}}$$

这是反证法的形式推演形式;
可以验证相应的逻辑推理形式是正确的. 即如果$\Sigma, \neg A \models B$并且$\Sigma, \neg A \models \neg B$ 则$\Sigma \models A$.

形式推演规则: $\neg$

$$
\boxed{
\begin{array}{lll}
(\neg^-) \qquad & \text{如果} & \Sigma, \neg A \vdash B, \\
& & \Sigma, \neg A \vdash \neg B, \\
& \text{则} & \Sigma \vdash A.
\end{array}
}
$$

这个形式推演规则是保真的, 即如果$\Sigma, \neg A \models B$为真并且$\Sigma, \neg A \models \neg B$为真则$\Sigma \models A$为真.

形式推演规则: $\to$

$(\to^-)$	如果	$\Sigma \vdash A \to B$,
		$\Sigma \vdash A$,
	则	$\Sigma \vdash B$.
$(\to^+)$	如果	$\Sigma, A \vdash B$,
	则	$\Sigma \vdash A \to B$.

形式推演规则: $\to$

$$\boxed{\begin{array}{lrl} (\to^-) & \text{如果} & \Sigma \vdash A \to B, \\ & & \Sigma \vdash A, \\ & \text{则} & \Sigma \vdash B. \end{array}}$$

$$\begin{array}{l} \text{如果天下雨则地是湿的} \\ \text{天下雨} \\ \hline \text{所以地是湿的.} \end{array}$$

形式推演规则: $\to$

$$
\begin{array}{llr l}
(\to^-) & & \text{如果} & \Sigma \models A \to B, \\
 & & & \Sigma \models A, \\
 & & \text{则} & \Sigma \models B.
\end{array}
$$

形式推演规则: $\wedge$

$$
\begin{array}{lrl}
(\wedge^-) & 如果 & \Sigma \vdash A \wedge B, \\
 & 则 & \Sigma \vdash A, \\
 & & \Sigma \vdash B. \\
 & & \\
(\wedge^+) & 如果 & \Sigma \vdash A, \\
 & & \Sigma \vdash B, \\
 & 则 & \Sigma \vdash A \wedge B.
\end{array}
$$

形式推演规则: $\vee$

$$
\begin{array}{llll}
(\vee^-) & 如果 & \Sigma, A \vdash C, \\
 & & \Sigma, B \vdash C, \\
 & 则 & \Sigma, A \vee B \vdash C. \\
 & & \\
(\vee^+) & 如果 & \Sigma \vdash A, \\
 & 则 & \Sigma \vdash A \vee B, \\
 & & \Sigma \vdash B \vee A.
\end{array}
$$

形式推演规则: $\leftrightarrow$

$$
\begin{array}{lrl}
(\leftrightarrow^-) & \text{如果} & \Sigma \vdash A \leftrightarrow B, \\
 & & \Sigma \vdash A, \\
 & \text{则} & \Sigma \vdash B; \\
 & & \\
 & \text{如果} & \Sigma \vdash A \leftrightarrow B, \\
 & & \Sigma \vdash B, \\
 & \text{则} & \Sigma \vdash A; \\
 & & \\
(\leftrightarrow^+) & \text{如果} & \Sigma, A \vdash B, \\
 & & \Sigma, B \vdash A, \\
 & \text{则} & \Sigma \vdash A \leftrightarrow B.
\end{array}
$$

形式证明的例子1

证明: $(\in)$ 如果$A \in \Sigma$ 则$\Sigma \vdash A$.

$$
\begin{array}{llll}
(1) & A & \vdash A & (Ref) \\
(2) & A, \Sigma' & \vdash A & (+, (1)) \\
(3) & \Sigma & \vdash A & (\text{约定})
\end{array}
$$

其中$\Sigma = \{A\} \cup \Sigma' = A, \Sigma'$.

$\square$

注意: 作为集合$A, \Sigma' \neq \Sigma$. 我们约定$A, \Sigma' \vdash A$就是$\{A\} \cup \Sigma' \vdash A$.

形式证明的例子2

证明: $\neg A \to B \vdash \neg B \to A$.
问题?

形式证明的例子2

证明: $\neg A \to B \vdash \neg B \to A$.

(1) $\quad \neg A \to B, \neg B, \neg A \quad (\in)$

形式证明的例子2

证明: $\neg A \to B \vdash \neg B \to A$.

$$(1) \qquad \neg A \to B, \neg B, \neg A \vdash \neg A \to B \qquad (\in)$$

形式证明的例子2

证明: $\neg A \to B \vdash \neg B \to A$.

(1)	$\neg A \to B, \neg B, \neg A$	$\vdash \neg A \to B$	$(\in)$
(2)	$\neg A \to B, \neg B, \neg A$	$\vdash \neg A$	$(\in)$

形式证明的例子2

证明: $\neg A \to B \vdash \neg B \to A$.

$$
\begin{array}{llll}
(1) & \neg A \to B, \neg B, \neg A & \vdash \neg A \to B & (\in) \\
(2) & \neg A \to B, \neg B, \neg A & \vdash \neg A & (\in) \\
(3) & \neg A \to B, \neg B, \neg A & \vdash B & (\to^-, (1), (2))
\end{array}
$$

形式证明的例子2

证明: $\neg A \to B \vdash \neg B \to A$.

(1)	$\neg A \to B, \neg B, \neg A$	$\vdash \neg A \to B$	$(\in)$
(2)	$\neg A \to B, \neg B, \neg A$	$\vdash \neg A$	$(\in)$
(3)	$\neg A \to B, \neg B, \neg A$	$\vdash B$	$(\to^{-}, (1), (2))$
(4)	$\neg A \to B, \neg B, \neg A$	$\vdash \neg B$	$(\in)$

形式证明的例子2

证明: $\neg A \to B \vdash \neg B \to A$.

$$
\begin{array}{lrll}
(1) & \neg A \to B, \neg B, \neg A & \vdash \neg A \to B & (\in) \\
(2) & \neg A \to B, \neg B, \neg A & \vdash \neg A & (\in) \\
(3) & \neg A \to B, \neg B, \neg A & \vdash B & (\to^-, (1), (2)) \\
(4) & \neg A \to B, \neg B, \neg A & \vdash \neg B & (\in) \\
(5) & \neg A \to B, \neg B & \vdash A & (\neg^-, (3), (4))
\end{array}
$$

形式证明的例子2

证明: $\neg A \to B \vdash \neg B \to A$.

(1)	$\neg A \to B, \neg B, \neg A$	$\vdash \neg A \to B$	$(\in)$
(2)	$\neg A \to B, \neg B, \neg A$	$\vdash \neg A$	$(\in)$
(3)	$\neg A \to B, \neg B, \neg A$	$\vdash B$	$(\to^-, (1), (2))$
(4)	$\neg A \to B, \neg B, \neg A$	$\vdash \neg B$	$(\in)$
(5)	$\neg A \to B, \neg B$	$\vdash A$	$(\neg^-, (3), (4))$
(6)	$\neg A \to B$	$\vdash \neg B \to A$	$(\to^+, (5))$

$\square$

形式证明的例子-2

证明: $\neg A \to B \vdash \neg B \to A$.

(6) $\quad \neg A \to B \quad \vdash \neg B \to A$

形式证明的例子-2

证明: $\neg A \to B \vdash \neg B \to A$.

$$\begin{array}{lrll}
(5) & \neg A \to B, \neg B & \vdash A & \\
(6) & \neg A \to B & \vdash \neg B \to A & (\to^{+}, (5))
\end{array}$$

形式证明的例子-2

证明: $\neg A \to B \vdash \neg B \to A$.

$$
\begin{array}{lrll}
(4) & \neg A \to B, \neg B, \neg A & \vdash \neg B & \\
(5) & \neg A \to B, \neg B & \vdash A & (\neg^{-}, (3), (4)) \\
(6) & \neg A \to B & \vdash \neg B \to A & (\to^{+}, (5))
\end{array}
$$

形式证明的例子-2

证明: $\neg A \to B \vdash \neg B \to A$.

$$
\begin{array}{lrll}
(3) & \neg A \to B, \neg B, \neg A & \vdash B & \\
(4) & \neg A \to B, \neg B, \neg A & \vdash \neg B & (\in) \\
(5) & \neg A \to B, \neg B & \vdash A & (\neg^-, (3), (4)) \\
(6) & \neg A \to B & \vdash \neg B \to A & (\to^+, (5))
\end{array}
$$

形式证明的例子-2

证明: $\neg A \to B \vdash \neg B \to A$.

$$
\begin{array}{llll}
(2) & \neg A \to B, \neg B, \neg A & \vdash \neg A & \\
(3) & \neg A \to B, \neg B, \neg A & \vdash B & (\to^-, (1), (2)) \\
(4) & \neg A \to B, \neg B, \neg A & \vdash \neg B & (\in) \\
(5) & \neg A \to B, \neg B & \vdash A & (\neg^-, (3), (4)) \\
(6) & \neg A \to B & \vdash \neg B \to A & (\to^+, (5))
\end{array}
$$

形式证明的例子-2

证明: $\neg A \to B \vdash \neg B \to A$.

$$
\begin{array}{llll}
(1) & \neg A \to B, \neg B, \neg A & \vdash \neg A \to B & (\in) \\
(2) & \neg A \to B, \neg B, \neg A & \vdash \neg A & (\in) \\
(3) & \neg A \to B, \neg B, \neg A & \vdash B & (\to^{-}, (1), (2)) \\
(4) & \neg A \to B, \neg B, \neg A & \vdash \neg B & (\in) \\
(5) & \neg A \to B, \neg B & \vdash A & (\neg^{-}, (3), (4)) \\
(6) & \neg A \to B & \vdash \neg B \to A & (\to^{+}, (5))
\end{array}
$$

$\square$

形式证明的定义

定义. A是在命题逻辑中由Σ形式可推演(或形式可证明)的, 记为$\Sigma \vdash A$, 如果存在一个形式可推演性模式序列

$$\Sigma_1 \vdash A_1, \ldots, \Sigma_n \vdash A_n$$

使得
(1) $\Sigma_n \vdash A_n = \Sigma \vdash A$;
(2) 对每个$k \leq n$, $\Sigma_k \vdash A_k$可由前面的形式可推演性模式和形式推演规则得到.

形式证明的例子-2

证明: $\neg A \to B \vdash \neg B \to A$.

$$
\begin{array}{c|c rl|c|r}
\Sigma_1 & (1) & \neg A \to B, \neg B, \neg A & \vdash \neg A \to B & A_1 & (\in) \\
\Sigma_2 & (2) & \neg A \to B, \neg B, \neg A & \vdash \neg A & A_2 & (\in) \\
\Sigma_3 & (3) & \neg A \to B, \neg B, \neg A & \vdash B & A_3 & (\to^-, (1), (2)) \\
\Sigma_4 & (4) & \neg A \to B, \neg B, \neg A & \vdash \neg B & A_4 & (\in) \\
\Sigma_5 & (5) & \neg A \to B, \neg B & \vdash A & A_5 & (\neg^-, (3), (4)) \\
\Sigma_6 & (6) & \neg A \to B & \vdash \neg B \to A & A_6 & (\to^+, (5))
\end{array}
$$

形式证明的定义

$$\Sigma_1 \vdash A_1, ..., \Sigma_n \vdash A_n$$

(2) 对每个$k \leq n$, $\Sigma_k \vdash A_k$可由前面的形式可推演性模式和形式推演规则得到.
比如, 如果$\Sigma_k \vdash A_k$是由$(\neg^-)$得到的，则存在$i, j < k$使得

$$\begin{aligned} \Sigma_i \vdash A_i &= \Sigma_k, \neg A_k \vdash B; \\ \Sigma_j \vdash A_j &= \Sigma_k, \neg A_k \vdash \neg B. \end{aligned}$$

其中=是符号串的相等.

形式证明的定义

$$\Sigma_1 \vdash A_1, ..., \Sigma_n \vdash A_n$$

(2) 对每个$k \leq n$, $\Sigma_k \vdash A_k$可由前面的形式可推演性模式和形式推演规则得到.

比如, 如果$\Sigma_k \vdash A_k = \Sigma', B \vee C \vdash A_k$是由$(\vee^-)$得到的，则存在$i, j < k$使得

$$\begin{aligned} \Sigma_i \vdash A_i &= \Sigma', B \vdash A_k; \\ \Sigma_j \vdash A_j &= \Sigma', C \vdash A_k; \\ \Sigma_k &= \Sigma' \cup \{B \vee C\}. \end{aligned}$$

形式证明的例子3

定理2.6.4.

$$
\begin{array}{ll}
(i) & A \to B, A \vdash B. \\
(ii) & A \vdash B \to A. \\
(iii) & A \to B, B \to C \vdash A \to C. \\
(iv) & A \to (B \to C), A \to B \vdash A \to C.
\end{array}
$$

形式证明的例子3

证明: $A \to B, B \to C \vdash A \to C$.

形式证明的例子3

证明: $A \to B, B \to C \vdash A \to C$.

$$(1) \qquad A \to B, B \to C, A \ \vdash A \to B \qquad (\in)$$

形式证明的例子3

证明: $A \to B, B \to C \vdash A \to C$.

(1)	$A \to B, B \to C, A \ \vdash A \to B$	$(\in)$
(2)	$A \to B, B \to C, A \ \vdash A$	$(\in)$

形式证明的例子3

证明: $A \to B, B \to C \vdash A \to C$.

(1)	$A \to B, B \to C, A \vdash A \to B$	$(\in)$
(2)	$A \to B, B \to C, A \vdash A$	$(\in)$
(3)	$A \to B, B \to C, A \vdash B$	$(\to^-, (1), (2))$

形式证明的例子3

证明: $A \to B, B \to C \vdash A \to C$.

$$
\begin{array}{lll}
(1) & A \to B, B \to C, A \vdash A \to B & (\in) \\
(2) & A \to B, B \to C, A \vdash A & (\in) \\
(3) & A \to B, B \to C, A \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A \vdash B \to C & (\in)
\end{array}
$$

形式证明的例子3

证明: $A \to B, B \to C \vdash A \to C$.

$$
\begin{array}{lllr}
(1) & A \to B, B \to C, A & \vdash A \to B & (\in) \\
(2) & A \to B, B \to C, A & \vdash A & (\in) \\
(3) & A \to B, B \to C, A & \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A & \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A & \vdash C & (\to^-, (3), (4))
\end{array}
$$

形式证明的例子3

证明: $A \to B, B \to C \vdash A \to C$.

(1)	$A \to B, B \to C, A \vdash A \to B$	$(\in)$
(2)	$A \to B, B \to C, A \vdash A$	$(\in)$
(3)	$A \to B, B \to C, A \vdash B$	$(\to^-, (1), (2))$
(4)	$A \to B, B \to C, A \vdash B \to C$	$(\in)$
(5)	$A \to B, B \to C, A \vdash C$	$(\to^-, (3), (4))$
(6)	$A \to B, B \to C \vdash A \to C$	$(\to^+, (5))$

$\square$

形式证明的例子-3

证明: $A \to B, B \to C \vdash A \to C$.

(6) $A \to B, B \to C \quad \vdash A \to C$

形式证明的例子-3

证明: $A \to B, B \to C \vdash A \to C$.

$$
\begin{array}{lll}
(5) & A \to B, B \to C, A & \vdash C \\
(6) & A \to B, B \to C & \vdash A \to C \quad (\to^+, (5))
\end{array}
$$

形式证明的例子-3

证明: $A \to B, B \to C \vdash A \to C$.

$$
\begin{array}{lrll}
(4) & A \to B, B \to C, A & \vdash B \to C & \\
(5) & A \to B, B \to C, A & \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C & \vdash A \to C & (\to^+, (5))
\end{array}
$$

形式证明的例子-3

证明: $A \to B, B \to C \vdash A \to C$.

$$
\begin{array}{llll}
(3) & A \to B, B \to C, A & \vdash B & \\
(4) & A \to B, B \to C, A & \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A & \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C & \vdash A \to C & (\to^+, (5))
\end{array}
$$

形式证明的例子-3

证明: $A \to B, B \to C \vdash A \to C$.

$$
\begin{array}{lll}
(2) & A \to B, B \to C, A \vdash A & \\
(3) & A \to B, B \to C, A \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C \vdash A \to C & (\to^+, (5))
\end{array}
$$

形式证明的例子-3

证明: $A \to B, B \to C \vdash A \to C$.

$$
\begin{array}{llr}
(1) & A \to B, B \to C, A \vdash A \to B & (\in) \\
(2) & A \to B, B \to C, A \vdash A & (\in) \\
(3) & A \to B, B \to C, A \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C \vdash A \to C & (\to^+, (5))
\end{array}
$$

$\square$

形式证明的例子4

证明: $\neg\neg A \vdash A$.

形式证明的例子4

证明: $\neg\neg A \vdash A$.

$$
\begin{array}{llll}
(1) & \neg\neg A, \neg A & \vdash \neg A & (\in) \\
(2) & \neg\neg A, \neg A & \vdash \neg\neg A & (\in) \\
(3) & \neg\neg A & \vdash A & (\neg^{-}, (1), (2))
\end{array}
$$

$\square$

因为$\neg\neg A = \neg(\neg A)$

形式证明的例子5

证明:

$$\begin{array}{rl} \text{如果} & \Sigma, A \vdash B, \\ & \Sigma, A \vdash \neg B, \\ \text{则} & \Sigma \vdash \neg A. \end{array}$$

形式证明的例子5

证明:

$$
\begin{array}{rl}
\text{如果} & \Sigma, A \vdash B, \\
 & \Sigma, A \vdash \neg B, \\
\text{则} & \Sigma \vdash \neg A.
\end{array}
$$

$$
\boxed{
(\neg^{-}) \qquad
\begin{array}{rl}
\text{如果} & \Sigma, \neg A \vdash B, \\
 & \Sigma, \neg A \vdash \neg B, \\
\text{则} & \Sigma \vdash A.
\end{array}
}
$$

形式证明的例子5

如果$\Sigma, A \vdash B$并且$\Sigma, A \vdash \neg B$, 则$\Sigma \vdash \neg A$.

$$(1) \qquad \Sigma, \neg\neg A \ \vdash \Sigma \qquad\qquad (\in)$$

形式证明的例子5

如果$\Sigma, A \vdash B$并且$\Sigma, A \vdash \neg B$, 则$\Sigma \vdash \neg A$.

(1)	$\Sigma, \neg\neg A$	$\vdash \Sigma$	$(\in)$
(2)	$\neg\neg A$	$\vdash A$	$((i))$

形式证明的例子5

如果$\Sigma, A \vdash B$并且$\Sigma, A \vdash \neg B$, 则$\Sigma \vdash \neg A$.

$$
\begin{array}{llll}
(1) & \Sigma, \neg\neg A & \vdash \Sigma & (\in) \\
(2) & \neg\neg A & \vdash A & ((i)) \\
(3) & \Sigma, \neg\neg A & \vdash A & (+, (2))
\end{array}
$$

形式证明的例子5

如果$\Sigma, A \vdash B$并且$\Sigma, A \vdash \neg B$, 则$\Sigma \vdash \neg A$.

$$
\begin{array}{llll}
(1) & \Sigma, \neg\neg A & \vdash \Sigma & (\in) \\
(2) & \neg\neg A & \vdash A & ((i)) \\
(3) & \Sigma, \neg\neg A & \vdash A & (+, (2)) \\
(4) & \Sigma, A & \vdash B & (assumption)
\end{array}
$$

形式证明的例子5

如果$\Sigma, A \vdash B$并且$\Sigma, A \vdash \neg B$, 则$\Sigma \vdash \neg A$.

$$
\begin{array}{llll}
(1) & \Sigma, \neg\neg A & \vdash \Sigma & (\in) \\
(2) & \neg\neg A & \vdash A & ((i)) \\
(3) & \Sigma, \neg\neg A & \vdash A & (+, (2)) \\
(4) & \Sigma, A & \vdash B & (assumption) \\
(5) & \Sigma, \neg\neg A & \vdash B & (Tr, (3), (4))
\end{array}
$$

形式证明的例子5

如果$\Sigma, A \vdash B$并且$\Sigma, A \vdash \neg B$, 则$\Sigma \vdash \neg A$.

$$
\begin{array}{llll}
(1) & \Sigma, \neg\neg A & \vdash \Sigma & (\in) \\
(2) & \neg\neg A & \vdash A & ((i)) \\
(3) & \Sigma, \neg\neg A & \vdash A & (+, (2)) \\
(4) & \Sigma, A & \vdash B & (\mathit{assumption}) \\
(5) & \Sigma, \neg\neg A & \vdash B & (\mathit{Tr}, (3), (4)) \\
(6) & \Sigma, \neg\neg A & \vdash \neg B & (\mathit{ibid})
\end{array}
$$

形式证明的例子5

如果$\Sigma, A \vdash B$并且$\Sigma, A \vdash \neg B$, 则$\Sigma \vdash \neg A$.

$$
\begin{array}{llll}
(1) & \Sigma, \neg\neg A & \vdash \Sigma & (\in) \\
(2) & \neg\neg A & \vdash A & ((i)) \\
(3) & \Sigma, \neg\neg A & \vdash A & (+, (2)) \\
(4) & \Sigma, A & \vdash B & (\mathit{assumption}) \\
(5) & \Sigma, \neg\neg A & \vdash B & (Tr, (3), (4)) \\
(6) & \Sigma, \neg\neg A & \vdash \neg B & (\mathit{ibid}) \\
(7) & \Sigma & \vdash \neg A & (\neg^{-}, (5), (6))
\end{array}
$$

$\square$

形式证明

定理. $A \to \neg B \vdash B \to \neg A$.

形式证明

定理. $A \to \neg B \vdash B \to \neg A$.

证明.

$$
\begin{array}{lrll}
(1) & A \to \neg B, B, A & \vdash B & (\in) \\
(2) & A \to \neg B, A & \vdash \neg B & (2.6.4(i)) \\
(3) & A \to \neg B, B, A & \vdash \neg B & (+, (2)) \\
(4) & A \to \neg B, B & \vdash \neg A & (\neg^+, (1), (3)) \\
(5) & A \to \neg B & \vdash B \to \neg A & (\to^+, (4))
\end{array}
$$

$\square$

形式证明

证明: 由(自反), (+), ($\rightarrow^+$)以及下列规则:

(1) 如果$\Sigma \vdash \neg\neg A$,则$\Sigma \vdash A$;

(2) 如果$\Sigma \vdash A \rightarrow \neg B, A \rightarrow B^{\text{注意}}$ 则$\Sigma \vdash \neg A$,

推出($\neg^-$).

注意: 书上表示为$\Sigma \vdash A \rightarrow \neg B, B$, 意思为$\Sigma \vdash A \rightarrow \neg B$并且$\Sigma \vdash A \rightarrow B$.

我们这里: $\Sigma \vdash A, B$意思为$\Sigma \vdash A$并且$\Sigma \vdash B$.

形式证明

证明: 由(自反), $(+)$, $(\to^+)$以及下列规则:

(1) 如果$\Sigma \vdash \neg\neg A$,则$\Sigma \vdash A$;

(2) 如果$\Sigma \vdash A \to \neg B, A \to B$ 则$\Sigma \vdash \neg A$,

推出$(\neg^-)$.

$$
\begin{array}{lll}
(\neg^-) \qquad & \text{如果} & \Sigma, \neg A \vdash B, \\
& & \Sigma, \neg A \vdash \neg B, \\
& \text{则} & \Sigma \vdash A.
\end{array}
$$

形式证明

证明: 由(自反), $(+)$, $(\to^+)$以及下列规则:
(1) 如果$\Sigma \vdash \neg\neg A$,则$\Sigma \vdash A$;
(2) 如果$\Sigma \vdash A \to \neg B, A \to B$ 则$\Sigma \vdash \neg A$,
推出$(\neg^-)$.
推理规则是原子的, 结论$\Sigma \vdash A$也是一个规则.

形式证明

证明: 由(自反), $(+)$, $(\to^+)$以及下列规则: (1) 如果$\Sigma \vdash \neg\neg A$,则$\Sigma \vdash A$; (2) 如果$\Sigma \vdash A \to \neg B, A \to B$ 则$\Sigma \vdash \neg A$, 推出$(\neg^-)$.

证明. 假设$\Sigma, \neg A \vdash B$和$\Sigma, \neg A \vdash \neg B$.

$$
\begin{array}{rll}
\Sigma, \neg A \vdash & B & (assumption) \\
\Sigma \vdash & \neg A \to B & (\to^+) \\
\Sigma, \neg A \vdash & \neg B & (assumption) \\
\Sigma \vdash & \neg A \to \neg B & (\to^+) \\
\Sigma \vdash & \neg\neg A & (2) \\
\Sigma \vdash & A & (1)
\end{array}
$$

形式证明

证明: $\neg A \vee \neg B \vdash \neg(A \wedge B)$.

形式证明(第一种证明)

$$\neg A, A \wedge B \vdash \quad A \wedge B \qquad (\in)$$

形式证明(第一种证明)

$$
\begin{aligned}
\neg A, A \wedge B \vdash \quad & A \wedge B && (\in) \\
A \wedge B \vdash \quad & A && (\wedge^-)
\end{aligned}
$$

形式证明(第一种证明)

$$\begin{aligned}
\neg A, A \wedge B \vdash &\quad A \wedge B && (\in) \\
A \wedge B \vdash &\quad A && (\wedge^{-}) \\
\neg A, A \wedge B \vdash &\quad A && (trans)
\end{aligned}$$

形式证明(第一种证明)

$$\begin{aligned}
\neg A, A \wedge B &\vdash A \wedge B && (\in) \\
A \wedge B &\vdash A && (\wedge^{-}) \\
\neg A, A \wedge B &\vdash A && (trans) \\
\neg A, A \wedge B &\vdash \neg A && (\in)
\end{aligned}$$

形式证明(第一种证明)

$$
\begin{aligned}
\neg A, A \wedge B \vdash \quad & A \wedge B && (\in) \\
A \wedge B \vdash \quad & A && (\wedge^-) \\
\neg A, A \wedge B \vdash \quad & A && (trans) \\
\neg A, A \wedge B \vdash \quad & \neg A && (\in) \\
\neg A \vdash \quad & \neg(A \wedge B) && (\neg^-)
\end{aligned}
$$

形式证明(第一种证明)

$$
\begin{array}{rll}
\neg A, A \wedge B \vdash & A \wedge B & (\in) \\
A \wedge B \vdash & A & (\wedge^-) \\
\neg A, A \wedge B \vdash & A & (trans) \\
\neg A, A \wedge B \vdash & \neg A & (\in) \\
\neg A \vdash & \neg(A \wedge B) & (\neg^-) \\
\neg B, A \wedge B \vdash & A \wedge B & (\in) \\
A \wedge B \vdash & B & (\wedge^-) \\
\neg B, A \wedge B \vdash & B & (trans) \\
\neg B, A \wedge B \vdash & \neg B & (\in) \\
\neg B \vdash & \neg(A \wedge B) & (\neg^-)
\end{array}
$$

形式证明(第一种证明)

$$
\begin{array}{rll}
\neg A, A \wedge B \vdash & A \wedge B & (\in) \\
A \wedge B \vdash & A & (\wedge^-) \\
\neg A, A \wedge B \vdash & A & (trans) \\
\neg A, A \wedge B \vdash & \neg A & (\in) \\
\neg A \vdash & \neg(A \wedge B) & (\neg^-) \\
\neg B, A \wedge B \vdash & A \wedge B & (\in) \\
A \wedge B \vdash & B & (\wedge^-) \\
\neg B, A \wedge B \vdash & B & (trans) \\
\neg B, A \wedge B \vdash & \neg B & (\in) \\
\neg B \vdash & \neg(A \wedge B) & (\neg^-) \\
\neg A \vee \neg B \vdash & \neg(A \wedge B) & (\vee^-).
\end{array}
$$

形式证明(第二种证明)

$$
\begin{array}{rll}
A \wedge B \vdash & A & (\wedge^-) \\
\neg A, A \wedge B \vdash & A & (+) \\
\neg A, A \wedge B \vdash & \neg A & (\in) \\
\neg A \vdash & \neg(A \wedge B) & (\neg^-) \\
A \wedge B \vdash & B & (\wedge^-) \\
\neg B, A \wedge B \vdash & B & (+) \\
\neg B, A \wedge B \vdash & \neg B & (\in) \\
\neg B \vdash & \neg(A \wedge B) & (\neg^-) \\
\neg A \vee \neg B \vdash & \neg(A \wedge B) & (\vee^-).
\end{array}
$$

形式证明(第三种证明)

$$\neg A, A, \neg\neg B \vdash \quad A \qquad\qquad (\in)$$

形式证明(第三种证明)

$$\neg A, A, \neg\neg B \vdash \quad A \qquad (\in)$$
$$\neg A, A, \neg\neg B \vdash \quad \neg A \qquad (\in)$$

形式证明(第三种证明)

$$\begin{aligned}
\neg A, A, \neg\neg B \vdash \quad & A && (\in) \\
\neg A, A, \neg\neg B \vdash \quad & \neg A && (\in) \\
\neg A, A \vdash \quad & \neg B && (\neg^-)
\end{aligned}$$

形式证明(第三种证明)

$$
\begin{aligned}
\neg A, A, \neg\neg B \vdash \quad & A && (\in) \\
\neg A, A, \neg\neg B \vdash \quad & \neg A && (\in) \\
\neg A, A \vdash \quad & \neg B && (\neg^{-}) \\
\neg A \vdash \quad & A \to \neg B && (\to^{+})
\end{aligned}
$$

形式证明(第三种证明)

$$
\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^{-}) \\
\neg A \vdash & A \to \neg B & (\to^{+}) \\
\neg B, A \vdash & \neg B & (\in)
\end{array}
$$

形式证明(第三种证明)

$$
\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^-) \\
\neg A \vdash & A \to \neg B & (\to^+) \\
\neg B, A \vdash & \neg B & (\in) \\
\neg B \vdash & A \to \neg B & (\to^+)
\end{array}
$$

形式证明(第三种证明)

$$
\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^-) \\
\neg A \vdash & A \to \neg B & (\to^+) \\
\neg B, A \vdash & \neg B & (\in) \\
\neg B \vdash & A \to \neg B & (\to^+) \\
\neg A \vee \neg B \vdash & A \to \neg B & (\vee^-)
\end{array}
$$

形式证明(第三种证明)

$$
\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^-) \\
\neg A \vdash & A \to \neg B & (\to^+) \\
\neg B, A \vdash & \neg B & (\in) \\
\neg B \vdash & A \to \neg B & (\to^+) \\
\neg A \vee \neg B \vdash & A \to \neg B & (\vee^-) \\
\neg A \vee \neg B, A \wedge B \vdash & A \to \neg B & (+)
\end{array}
$$

形式证明(第三种证明)

$$
\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^-) \\
\neg A \vdash & A \to \neg B & (\to^+) \\
\neg B, A \vdash & \neg B & (\in) \\
\neg B \vdash & A \to \neg B & (\to^+) \\
\neg A \vee \neg B \vdash & A \to \neg B & (\vee^-) \\
\neg A \vee \neg B, A \wedge B \vdash & A \to \neg B & (+) \\
\neg A \vee \neg B, A \wedge B \vdash & A \wedge B & (\in)
\end{array}
$$

形式证明(第三种证明)

$$\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^-) \\
\neg A \vdash & A \rightarrow \neg B & (\rightarrow^+) \\
\neg B, A \vdash & \neg B & (\in) \\
\neg B \vdash & A \rightarrow \neg B & (\rightarrow^+) \\
\neg A \vee \neg B \vdash & A \rightarrow \neg B & (\vee^-) \\
\neg A \vee \neg B, A \wedge B \vdash & A \rightarrow \neg B & (+) \\
\neg A \vee \neg B, A \wedge B \vdash & A \wedge B & (\in) \\
\neg A \vee \neg B, A \wedge B \vdash & A & (\wedge^-)
\end{array}$$

形式证明(第三种证明)

$$
\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^{-}) \\
\neg A \vdash & A \to \neg B & (\to^{+}) \\
\neg B, A \vdash & \neg B & (\in) \\
\neg B \vdash & A \to \neg B & (\to^{+}) \\
\neg A \vee \neg B \vdash & A \to \neg B & (\vee^{-}) \\
\neg A \vee \neg B, A \wedge B \vdash & A \to \neg B & (+) \\
\neg A \vee \neg B, A \wedge B \vdash & A \wedge B & (\in) \\
\neg A \vee \neg B, A \wedge B \vdash & A & (\wedge^{-}) \\
\neg A \vee \neg B, A \wedge B \vdash & B & (\wedge^{-})
\end{array}
$$

形式证明(第三种证明)

$$
\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^-) \\
\neg A \vdash & A \to \neg B & (\to^+) \\
\neg B, A \vdash & \neg B & (\in) \\
\neg B \vdash & A \to \neg B & (\to^+) \\
\neg A \vee \neg B \vdash & A \to \neg B & (\vee^-) \\
\neg A \vee \neg B, A \wedge B \vdash & A \to \neg B & (+) \\
\neg A \vee \neg B, A \wedge B \vdash & A \wedge B & (\in) \\
\neg A \vee \neg B, A \wedge B \vdash & A & (\wedge^-) \\
\neg A \vee \neg B, A \wedge B \vdash & B & (\wedge^-) \\
\neg A \vee \neg B, A \wedge B \vdash & \neg B & (\to^-)
\end{array}
$$

形式证明(第三种证明)

$$
\begin{array}{rll}
\neg A, A, \neg\neg B \vdash & A & (\in) \\
\neg A, A, \neg\neg B \vdash & \neg A & (\in) \\
\neg A, A \vdash & \neg B & (\neg^{-}) \\
\neg A \vdash & A \to \neg B & (\to^{+}) \\
\neg B, A \vdash & \neg B & (\in) \\
\neg B \vdash & A \to \neg B & (\to^{+}) \\
\neg A \vee \neg B \vdash & A \to \neg B & (\vee^{-}) \\
\neg A \vee \neg B, A \wedge B \vdash & A \to \neg B & (+) \\
\neg A \vee \neg B, A \wedge B \vdash & A \wedge B & (\in) \\
\neg A \vee \neg B, A \wedge B \vdash & A & (\wedge^{-}) \\
\neg A \vee \neg B, A \wedge B \vdash & B & (\wedge^{-}) \\
\neg A \vee \neg B, A \wedge B \vdash & \neg B & (\to^{-}) \\
\neg A \vee \neg B \vdash & \neg(A \wedge B). &
\end{array}
$$

推理树

一个证明可以看作是一棵树.
例子: $A \to B, B \to C \vdash A \to C$.

$$
\begin{array}{lrlr}
(1) & A \to B, B \to C, A & \vdash A \to B & (\in) \\
(2) & A \to B, B \to C, A & \vdash A & (\in) \\
(3) & A \to B, B \to C, A & \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A & \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A & \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C & \vdash A \to C & (\to^+, (5))
\end{array}
$$

推理树

$\boxed{A \to B, B \to C, A \vdash A \to B}$ $\boxed{A \to B, B \to C, A \vdash A}$

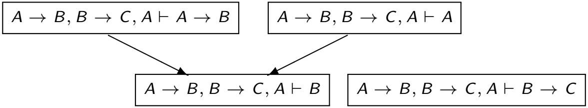
$$
\begin{array}{llll}
(1) & A \to B, B \to C, A & \vdash A \to B & (\in) \\
(2) & A \to B, B \to C, A & \vdash A & (\in) \\
(3) & A \to B, B \to C, A & \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A & \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A & \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C & \vdash A \to C & (\to^+, (5))
\end{array}
$$

推理树

$$
\begin{tikzcd}
\boxed{A \to B, B \to C, A \vdash A \to B} \arrow[dr] & & \boxed{A \to B, B \to C, A \vdash A} \arrow[dl] \\
& \boxed{A \to B, B \to C, A \vdash B} &
\end{tikzcd}
$$

$$
\begin{array}{lrll r}
(1) & A \to B, B \to C, A & \vdash A \to B & & (\in) \\
(2) & A \to B, B \to C, A & \vdash A & & (\in) \\
(3) & A \to B, B \to C, A & \vdash B & & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A & \vdash B \to C & & (\in) \\
(5) & A \to B, B \to C, A & \vdash C & & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C & \vdash A \to C & & (\to^+, (5))
\end{array}
$$

推理树



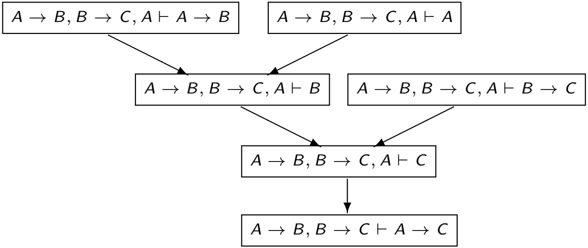
$$
\begin{array}{llll}
(1) & A \to B, B \to C, A & \vdash A \to B & (\in) \\
(2) & A \to B, B \to C, A & \vdash A & (\in) \\
(3) & A \to B, B \to C, A & \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A & \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A & \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C & \vdash A \to C & (\to^+, (5))
\end{array}
$$

推理树

$$\begin{tikzcd}
\boxed{A \to B, B \to C, A \vdash A \to B} \arrow[dr] & & \boxed{A \to B, B \to C, A \vdash A} \arrow[dl] & \\
& \boxed{A \to B, B \to C, A \vdash B} \arrow[dr] & & \boxed{A \to B, B \to C, A \vdash B \to C} \arrow[dl] \\
& & \boxed{A \to B, B \to C, A \vdash C} &
\end{tikzcd}$$

$$\begin{array}{lrll}
(1) & A \to B, B \to C, A & \vdash A \to B & (\in) \\
(2) & A \to B, B \to C, A & \vdash A & (\in) \\
(3) & A \to B, B \to C, A & \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A & \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A & \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C & \vdash A \to C & (\to^+, (5))
\end{array}$$

推理树



$$
\begin{array}{llr}
(1) & A \to B, B \to C, A \vdash A \to B & (\in) \\
(2) & A \to B, B \to C, A \vdash A & (\in) \\
(3) & A \to B, B \to C, A \vdash B & (\to^-, (1), (2)) \\
(4) & A \to B, B \to C, A \vdash B \to C & (\in) \\
(5) & A \to B, B \to C, A \vdash C & (\to^-, (3), (4)) \\
(6) & A \to B, B \to C \vdash A \to C & (\to^+, (5))
\end{array}
$$

推理树

$$
\begin{tikzcd}
\boxed{\in} \arrow[dr] & & \boxed{\in} \arrow[dl] & \\
& \boxed{\to^{-}} \arrow[dr] & & \boxed{\in} \arrow[dl] \\
& & \boxed{\to^{-}} \arrow[d] & \\
& & \boxed{\to^{+}} &
\end{tikzcd}
$$

推理树

定义. A是在命题逻辑中由Σ形式可推演(或形式可证明)的, 记为$\Sigma \vdash A$, 如果存在一个形式可推演性模式序列

$$\Sigma_1 \vdash A_1, ..., \Sigma_n \vdash A_n$$

使得
(1) $\Sigma_n \vdash A_n = \Sigma \vdash A$;
(2) 对每个$k \leq n$, $\Sigma_k \vdash A_k$可由前面的形式可推演性模式和形式推演规则得到.

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(Ref) 自反 $A \vdash A$.
($\in$) 如果$A \in \Sigma$则$\Sigma \vdash A$.

结论

推理是简单的.
命题逻辑的推理是可以算法化的.
为什么?

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作业2可以做了.