Mutual Exclusion (P1)

$B = (\{s_0, s_1, s_2, s_3, t_0, t_1, t_2, t_3, 0, 1\}, \{=\})$, $V = \{a, b, x, y, t\}$
$M = (T, \Theta)$ over (B, V) as follows, with the usual interpretation I.

T	$a = s_0$	$\longrightarrow$	$(y, t, a) := (1, 1, s_1)$
	$a = s_1 \wedge (x = 0 \vee t = 0)$	$\longrightarrow$	$(a) := (s_2)$
	$a = s_2$	$\longrightarrow$	$(y, a) := (0, s_3)$
	$a = s_3$	$\longrightarrow$	$(y, t, a) := (1, 1, s_1)$
	$b = t_0$	$\longrightarrow$	$(x, t, b) := (1, 0, t_1)$
	$b = t_1 \wedge (y = 0 \vee t = 1)$	$\longrightarrow$	$(b) := (t_2)$
	$b = t_2$	$\longrightarrow$	$(x, b) := (0, t_3)$
	$b = t_3$	$\longrightarrow$	$(x, t, b) := (1, 0, t_1)$
Θ	$(a = s_0 \wedge b = t_0 \wedge x = 0 \wedge y = 0 \wedge t = 0)$		

Prove: $(T, \Theta) \models_I G(a = s_1 \wedge b \neq t_1 \wedge b \neq t_2 \rightarrow (a = s_2 R b \neq t_2))$

Proof Rule

Proof Rule:

$$\frac{\begin{array}{l}\zeta \Rightarrow \varphi' \\ \varphi' \wedge \neg\psi \rightarrow [T]\varphi' \\ \varphi' \Rightarrow \varphi\end{array}}{\zeta \Rightarrow \psi R \varphi}$$

We have:

$$\begin{array}{lcl}\zeta & \equiv & (a = s_1 \wedge b \neq t_1 \wedge b \neq t_2) \\ \psi & \equiv & (a = s_2) \\ \varphi & \equiv & (b \neq t_2) \\ \varphi' & \equiv & ?\end{array}$$

Attempt 1

Proof Rule:

$$\frac{\begin{array}{l}\zeta \Rightarrow \varphi' \\ \varphi' \wedge \neg\psi \rightarrow [T]\varphi' \\ \varphi' \Rightarrow \varphi\end{array}}{\zeta \Rightarrow \psi R \varphi}$$

We have:

$$\begin{array}{lcl}\zeta & \equiv & (a = s_1 \wedge b \neq t_1 \wedge b \neq t_2) \\ \psi & \equiv & (a = s_2) \\ \varphi & \equiv & (b \neq t_2) \\ \varphi' & \equiv & \varphi\end{array}$$

Remain to prove:

$$\varphi' \wedge \neg\psi \rightarrow [T]\varphi'$$

Attempt 1

(1)
t_1: $a = s_0 \longrightarrow (y, t, a) := (1, 1, s_1)$
$[t_1]\varphi' = (a = s0 \rightarrow \varphi'(1/y, 1/t, s_1/a)) = (a = s0 \rightarrow b \neq t_2)$
$\varphi' \wedge \neg\psi \rightarrow [t_1]\varphi'$: $(b \neq t_2) \wedge \neg(a = s_2) \rightarrow [t_1](b \neq t_2)$
OK.
...
(6)
t_6: $b = t_1 \wedge (y = 0 \vee t = 1) \longrightarrow (b) := (t_2)$
$[t_6]\varphi' = (b = t_1 \wedge (y = 0 \vee t = 1) \rightarrow t_2 \neq t_2)$
$\varphi' \wedge \neg\psi \rightarrow [t_6]\varphi'$: $(b \neq t_2) \wedge \neg(a = s_2) \rightarrow [t_1](b \neq t_2)$
FAIL.

NEED to strengthen φ'.

Attempt 2

Proof Rule:

$$\frac{\begin{array}{l}\zeta \Rightarrow \varphi' \\ \varphi' \wedge \neg\psi \rightarrow [T]\varphi' \\ \varphi' \Rightarrow \varphi\end{array}}{\zeta \Rightarrow \psi R \varphi}$$

We have:

$$\begin{array}{lcl}
\zeta & \equiv & (a = s_1 \wedge b \neq t_1 \wedge b \neq t_2) \\
\psi & \equiv & (a = s_2) \\
\varphi & \equiv & (b \neq t_2) \\
\varphi' & \equiv & (a = s_1 \wedge (b = t_0 \vee b = t_3 \vee (b = t_1 \wedge x = 1 \wedge t = 0)))\vee \\
& & (a = s_2 \wedge b \neq t_2)
\end{array}$$

Remain to prove:

$$\varphi' \wedge \neg\psi \rightarrow [T]\varphi'$$

Attempt 2

Try to prove $\varphi' \wedge \neg\psi \rightarrow [t_6]\varphi'$

$$
\begin{aligned}
&((a = s_1 \wedge (b = t_0 \vee b = t_3 \vee (b = t_1 \wedge x = 1 \wedge t = 0)))\vee \\
&(a = s_2 \wedge b \neq t_2)) \wedge (a \neq s_2) \wedge (b = t_1 \wedge (y = 0 \vee t = 1)) \\
&\rightarrow (a = s_1 \wedge (t_2 = t_0 \vee t_2 = t_3 \vee (t_2 = t_1 \wedge x = 1 \wedge t = 0)))\vee \\
&\qquad (a = s_2 \wedge t_2 \neq t_2)
\end{aligned}
$$

$$
\begin{aligned}
&((a = s_1 \wedge (b = t_0 \vee b = t_3 \vee (b = t_1 \wedge x = 1 \wedge t = 0)))\vee \\
&(a = s_2 \wedge b \neq t_2)) \wedge (a \neq s_2) \wedge (b = t_1 \wedge (y = 0 \vee t = 1)) \\
&\rightarrow (a = s_2)
\end{aligned}
$$

$$
\begin{aligned}
&((a = s_1 \wedge ((b = t_1 \wedge x = 1 \wedge t = 0))) \wedge (y = 0) \\
&\rightarrow (a = s_2)
\end{aligned}
$$

FAIL.

Solution

Proof Rule:

$$\frac{\begin{array}{l}\zeta \Rightarrow \varphi' \\ \varphi' \wedge \neg\psi \rightarrow [T]\varphi' \\ \varphi' \Rightarrow \varphi\end{array}}{\zeta \Rightarrow \psi R \varphi}$$

We have:

$$\begin{array}{lcl}\zeta & \equiv & (a = s_1 \wedge b \neq t_1 \wedge b \neq t_2) \\ \psi & \equiv & (a = s_2) \\ \varphi & \equiv & (b \neq t_2) \\ \varphi' & \equiv & (a = s_1 \wedge (b = t_0 \vee b = t_3 \vee (b = t_1 \wedge x = 1 \wedge t = 0 \wedge y = 1))) \\ & & \vee (a = s_2 \wedge b \neq t_2)\end{array}$$

Remain to prove:

$$\varphi' \wedge \neg\psi \rightarrow [T]\varphi'$$

Solution OK

Try to prove $\varphi' \wedge \neg\psi \rightarrow [t_6]\varphi'$

$$\begin{array}{l}((a = s_1 \wedge (b = t_0 \vee b = t_3 \vee (b = t_1 \wedge x = 1 \wedge t = 0 \wedge y = 1)))\vee \\ (a = s_2 \wedge b \neq t_2)) \wedge (a \neq s_2) \wedge (b = t_1 \wedge (y = 0 \vee t = 1)) \\ \rightarrow (a = s_1 \wedge (t_2 = t_0 \vee t_2 = t_3 \vee (t_2 = t_1 \wedge x = 1 \wedge t = 0 \wedge y = 1)))\vee \\ \qquad (a = s_2 \wedge t_2 \neq t_2)\end{array}$$

$$\begin{array}{l}((a = s_1 \wedge (b = t_0 \vee b = t_3 \vee (b = t_1 \wedge x = 1 \wedge t = 0 \wedge y = 1)))\vee \\ (a = s_2 \wedge b \neq t_2)) \wedge (a \neq s_2) \wedge (b = t_1 \wedge (y = 0 \vee t = 1)) \\ \rightarrow \mathit{false}\end{array}$$

$$\begin{array}{l}((a = s_1 \wedge ((b = t_1 \wedge x = 1 \wedge t = 0 \wedge y = 1)))) \wedge (y = 0) \\ \rightarrow \mathit{false}\end{array}$$

OK.

Further Thinking

$$a = s_1 \wedge b \neq t_1 \wedge b \neq t_2 \Rightarrow (a = s_2 R b \neq t_2)$$

$$a = s_1 \wedge (b = t_0 \vee b = t_3) \Rightarrow (a = s_2 R b \neq t_2)$$

$a = s_1 \wedge (b = t_0) \Rightarrow (a = s_2 R b \neq t_2)$ and
$a = s_1 \wedge (b = t_3) \Rightarrow (a = s_2 R b \neq t_2)$

May be easier to prove the last two properties.

$$\frac{\zeta_0 \Rightarrow \psi R \varphi \qquad \zeta_1 \Rightarrow \psi R \varphi}{\zeta_0 \vee \zeta_1 \Rightarrow \psi R \varphi}$$

Integer Square Root (P1)

Given $M = (T, \Theta)$, and the usual interpretation I over integers.

T	$a = s_0 \longrightarrow (y_1, y_2, y_3, a) := (0, 1, 1, s_1)$ $a = s_1 \wedge (y_3 \leq x) \longrightarrow (a) := (s_2)$ $a = s_1 \wedge \neg(y_3 \leq x) \longrightarrow (a) := (s_4)$ $a = s_2 \longrightarrow (y_1, y_2, a) := (y_1 + 1, y_2 + 2, s_3)$ $a = s_3 \longrightarrow (y_3, a) := (y_3 + y_2, s_1)$
Θ	$(a = s_0)$

Prove $(T, \Theta) \models_I x > 0 \rightarrow G(a = s_4 \rightarrow y_1 = \sqrt{x})$

Preparation

Proof Rule:

$$\frac{\begin{array}{l}\zeta \Rightarrow \varphi' \\ \varphi' \rightarrow [T]\varphi' \\ \varphi' \Rightarrow \varphi\end{array}}{\zeta \Rightarrow G\varphi}$$

Suppose that we have
$(T, \Theta) \models_I G(a = s_0 \wedge x > 0 \rightarrow G(a = s_4 \rightarrow y_1 = \sqrt{x}))$

Then $(T, \Theta) \models_I (a = s_0 \wedge x > 0 \rightarrow G(a = s_4 \rightarrow y_1 = \sqrt{x}))$

In addition, we have $(T, \Theta) \models_I a = s_0$.

Therefore $(T, \Theta) \models_I (x > 0 \rightarrow G(a = s_4 \rightarrow y_1 = \sqrt{x}))$

Proof Rule

Proof Rule:

$$\frac{\begin{array}{l}\zeta \Rightarrow \varphi' \\ \varphi' \rightarrow [T]\varphi' \\ \varphi' \Rightarrow \varphi\end{array}}{\zeta \Rightarrow G\varphi}$$

We have:

$$\begin{array}{lcl}\zeta & \equiv & (x > 0 \wedge a = s_0) \\ \varphi & \equiv & (a = s_4 \rightarrow y_1 = \sqrt{x}) \\ \varphi' & \equiv & ?\end{array}$$

Solution

Let

$$
\begin{aligned}
\zeta &\equiv (x > 0 \wedge a = s_0) \\
\varphi &\equiv (a = s_4 \rightarrow y_1 = \sqrt{x}) \\
\varphi' &\equiv (a = s_0 \wedge \varphi_0) \vee (a = s_1 \wedge \varphi_1) \vee (a = s_2 \wedge \varphi_2) \vee (a = s_3 \wedge \varphi_3) \vee \\
&\quad\ (a = s_4 \wedge \varphi_4)
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_0 &\equiv (x > 0) \\
\varphi_1 &\equiv (y_1^2 \leq x \wedge y_2 = 2 * y_1 + 1 \wedge y_3 = (y_1 + 1)^2) \\
\varphi_2 &\equiv ((y_1 + 1)^2 \leq x \wedge y_2 = 2 * y_1 + 1 \wedge y_3 = (y_1 + 1)^2) \\
\varphi_3 &\equiv (y_1^2 \leq x \wedge y_2 = 2 * y_1 + 1 \wedge y_3 = y_1^2) \\
\varphi_4 &\equiv (y_1 = \sqrt{x}) \\
&\equiv (y_1^2 \leq x \wedge x < (y_1 + 1)^2
\end{aligned}
$$

Remain to prove:

$$\varphi' \rightarrow [T]\varphi'$$

Mutual Exclusion (P2)

Given $M = (T, \Theta)$, and the usual interpretation I.

T	$a = s_0$	$\longrightarrow$	$(y, t, a) := (1, 1, s_1)$
	$a = s_1 \wedge (x = 0 \vee t = 0)$	$\longrightarrow$	$(a) := (s_2)$
	$a = s_2$	$\longrightarrow$	$(y, a) := (0, s_3)$
	$a = s_3$	$\longrightarrow$	$(y, t, a) := (1, 1, s_1)$
	$b = t_0$	$\longrightarrow$	$(x, t, b) := (1, 0, t_1)$
	$b = t_1 \wedge (y = 0 \vee t = 1)$	$\longrightarrow$	$(b) := (t_2)$
	$b = t_2$	$\longrightarrow$	$(x, b) := (0, t_3)$
	$b = t_3$	$\longrightarrow$	$(x, t, b) := (1, 0, t_1)$
Θ	$(a = s_0 \wedge b = t_0 \wedge x = 0 \wedge y = 0 \wedge t = 0)$		

Prove: $(T, \Theta) \models_I G(a = s_1 \rightarrow F(a = s_2))$

Proof Rule

Proof Rule:

$$\frac{\begin{array}{l} \varphi \Rightarrow (\psi \vee \zeta) \\ \zeta \Rightarrow (w_x^e \wedge (\psi \vee E(T))) \\ \zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e \sqsubset v)) \end{array}}{\varphi \Rightarrow F\psi}$$

We have:

$$\begin{array}{lcl} \varphi & \equiv & (a = s_1) \\ \psi & \equiv & (a = s_2) \\ \zeta & \equiv & ? \\ w & \equiv & ? \\ e & \equiv & ? \end{array}$$

We may assume $(T, \Theta) \models_I G(E(T))$

Attempt 1

Define f such that:

$$
\begin{array}{llll}
I(f(t_0,0)) = 1 & I(f(t_1,0)) = 0 & I(f(t_2,0)) = 2 & I(f(t_3,0)) = 1 \\
I(f(t_0,1)) = 1 & I(f(t_1,1)) = 3 & I(f(t_2,1)) = 2 & I(f(t_3,1)) = 1
\end{array}
$$

Let

$$
\begin{array}{lcl}
W & = & (\{0,1,2,3\}, \leq) \\
w & = & (0 \leq x \leq 3) \\
e & = & f(b,t) \\
\zeta & = & (a = s_1)
\end{array}
$$

Need

$$
\begin{array}{l}
\varphi \Rightarrow (\psi \vee \zeta) \\
\zeta \Rightarrow w_x^e \wedge (\psi \vee E(T)) \\
\zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e < v))
\end{array}
$$

Remain to prove:

$$\zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e < v))$$

Attempt 1

t_6: $b = t_1 \wedge (y = 0 \vee t = 1) \longrightarrow (b) := (t_2)$

$$((a = s_1 \wedge f(b,t) = v) \wedge b = t_1 \wedge (y = 0 \vee t = 1))$$
$$\rightarrow (a = s_2 \vee (a = s_1 \wedge f(t_2,t) < v))$$

$$((a = s_1 \wedge f(t_1,t) = v) \wedge b = t_1 \wedge (y = 0 \vee t = 1))$$
$$\rightarrow (a = s_2 \vee (a = s_1 \wedge f(t_2,t) < v))$$

$$((a = s_1) \wedge b = t_1 \wedge (y = 0 \vee t = 1))$$
$$\rightarrow (a = s_2 \vee (a = s_1 \wedge f(t_2,t) < f(t_1,t)))$$

FAIL.

Need to strengthen $\zeta = (a = s_1)$ with $\zeta = (a = s_1 \wedge y = 1)$.
Then it is ok.

Solution

$$
\begin{aligned}
W &= (\{0,1,2,3\}, \leq) \\
w &= (0 \leq x \leq 3) \\
e &= f(b,t) \\
\varphi &= (a = s_1) \\
\psi &= (a = s_2) \\
\zeta &= (a = s_1 \wedge y = 1)
\end{aligned}
$$

Ok.

Need:
$\varphi \Rightarrow (\psi \vee \zeta)$, i.e., $a = s_1 \Rightarrow (a = s_2 \vee (a = s_1 \wedge y = 1))$.
It is ok, since we have the following (can be proved separately).
$(T, \Theta) \models G(a = s_1 \rightarrow y = 1)$.

Integer Square Root (P2)

Given $M = (T, \Theta)$, and the usual interpretation I over natural numbers (!).

T	$a = s_0 \longrightarrow (y_1, y_2, y_3, a) := (0, 1, 1, s_1)$ $a = s_1 \wedge (y_3 \leq x) \longrightarrow (a) := (s_2)$ $a = s_1 \wedge \neg(y_3 \leq x) \longrightarrow (a) := (s_4)$ $a = s_2 \longrightarrow (y_1, y_2, a) := (y_1 + 1, y_2 + 2, s_3)$ $a = s_3 \longrightarrow (y_3, a) := (y_3 + y_2, s_1)$
Θ	$(a = s_0)$

Prove $(T, \Theta) \models_I x > 0 \rightarrow F(a = s_4)$

Preparation

Proof Rule:

$$\frac{\begin{array}{l}\varphi \Rightarrow (\psi \vee \zeta) \\ \zeta \Rightarrow (w_x^e \wedge (\psi \vee E(T)) \\ \zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e \sqsubset v))\end{array}}{\varphi \Rightarrow F\psi}$$

Suppose that we have $(T, \Theta) \models_I G(a = s_0 \wedge x > 0 \rightarrow F(a = s_4))$

Then $(T, \Theta) \models_I (x > 0 \rightarrow F(a = s_4))$

Proof Rule

Proof Rule:

$$\frac{\begin{array}{l} \varphi \Rightarrow (\psi \vee \zeta) \\ \zeta \Rightarrow (w_x^e \wedge (\psi \vee E(T))) \\ \zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e \sqsubset v)) \end{array}}{\varphi \Rightarrow F\psi}$$

We have:

$$\begin{array}{rcl} \varphi & \equiv & (a = s_0 \wedge x > 0) \\ \psi & \equiv & (a = s_4) \\ \zeta & \equiv & ? \\ w & \equiv & ? \\ e & \equiv & ? \end{array}$$

We may assume $(T, \Theta) \models_I G(\psi \vee E(T))$

Solution

Define f such that:

$$
\begin{aligned}
&I(f(s_0, x, y_3)) = 3x + 1 \\
&I(f(s_i, x, y_3)) = 3(x + 1 - y_3) + 1 - i \quad (i = 1, 2, 3) \\
&I(f(s_4, x, y_3)) = 0
\end{aligned}
$$

Let

$$
\begin{array}{lclclcl}
W & = & (NAT, \leq) & \qquad & \varphi & = & (x > 0 \wedge a = s_0) \\
w & = & true & & \psi & = & (a = s_4) \\
e & = & f(a, x, y_3) & & \zeta & = & \varphi'
\end{array}
$$

Need

$$
\begin{aligned}
&\varphi \Rightarrow (\psi \vee \zeta) \\
&\zeta \Rightarrow w_x^e \wedge (\psi \vee E(T)) \\
&\zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e < v))
\end{aligned}
$$

Remain to prove:

$$
\zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e < v))
$$

Solution, Ok with some Modification

$(\zeta \wedge e = v) \equiv (\zeta \wedge f(a, x, y_3) = v)$

t_1: $a = s_0 \rightarrow f(s_1, x, 1) < v$,
i.e., $a = s_0 \rightarrow 3(x + 1 - 1) < 3x + 1$.

t_2: $a = s_1 \wedge y_3 \leq x \rightarrow f(s_2, x, y_3) < v$,
i.e., $a = s_1 \rightarrow 3(x + 1 - y_3) - 1 < 3(x + 1 - y_3)$. [need $y_3 \leq x$, ok]

t_3: $a = s_1 \wedge \neg(y_3 \leq x) \rightarrow (f(s_4, x, y_3) < v) \vee (s_4 = s_4)$,
ok.

t_4: $a = s_2 \rightarrow f(s_3, x, y_3) < v$,
i.e., $a = s_2 \rightarrow 3(x + 1 - y_3) - 2 < 3(x + 1 - y_3) - 1$.
[need $y_3 \leq x$, also ok]

t_5: $a = s_3 \rightarrow f(s_1, x, y_3 + y_2) < v$,
i.e., $a = s_3 \rightarrow 3(x + 1 - (y_3 + y_2)) < 3(x + 1 - y_3) - 2$.
(need $y_2 \geq 1$ and $y_3 \leq x$, we need to add $y_2 \geq 1$ to ζ, ok)

Integer Square Root (P2a)

Given $M = (T, \Theta)$, and the usual interpretation I over integers.

T	$a = s_0$	$\longrightarrow$	$(y_1, y_2, y_3, a) := (0, 1, 1, s_1)$
	$a = s_1 \wedge (y_3 \leq x)$	$\longrightarrow$	$(a) := (s_2)$
	$a = s_1 \wedge \neg(y_3 \leq x)$	$\longrightarrow$	$(a) := (s_4)$
	$a = s_2$	$\longrightarrow$	$(y_1, y_2, a) := (y_1 + 1, y_2 + 2, s_3)$
	$a = s_3$	$\longrightarrow$	$(y_3, a) := (y_3 + y_2, s_1)$
Θ	$(a = s_0)$		

Prove $(T, \Theta) \models_I x > 0 \rightarrow F(a = s_4)$

Preparation

Proof Rule:

$$\frac{\begin{array}{l}\varphi \Rightarrow (\psi \vee \zeta) \\ \zeta \Rightarrow (w_x^e \wedge (\psi \vee E(T)) \\ \zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e \sqsubset v))\end{array}}{\varphi \Rightarrow F\psi}$$

Suppose that we have $(T, \Theta) \models_I G(a = s_0 \wedge x > 0 \rightarrow F(a = s_4))$

Then $(T, \Theta) \models_I (x > 0 \rightarrow F(a = s_4))$

Proof Rule

Proof Rule:

$$\frac{\begin{array}{l}\varphi \Rightarrow (\psi \vee \zeta) \\ \zeta \Rightarrow (w_x^e \wedge (\psi \vee E(T))) \\ \zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e \sqsubset v))\end{array}}{\varphi \Rightarrow F\psi}$$

We have:

$$\begin{array}{lcl} \varphi & \equiv & (a = s_0 \wedge x > 0) \\ \psi & \equiv & (a = s_4) \\ \zeta & \equiv & ? \\ w & \equiv & ? \\ e & \equiv & ? \end{array}$$

We may assume $(T, \Theta) \models_I G(\psi \vee E(T))$

Attempt 1

Define f such that:

$$I(f(s_0, x, y_3)) = 3x + 1$$
$$I(f(s_i, x, y_3)) = 3(x + 1 - y_3) + 1 - i \quad (i = 1, 2, 3)$$
$$I(f(s_4, x, y_3)) = 0$$

Let

$$\begin{array}{lcl} W & = & (NAT, \leq) \\ w & = & x \geq 0 \\ e & = & f(a, x, y_3) \end{array} \qquad \begin{array}{lcl} \varphi & = & (x > 0 \wedge a = s_0) \\ \psi & = & (a = s_4) \\ \zeta & = & \varphi' \end{array}$$

Need

$$\varphi \Rightarrow (\psi \vee \zeta)$$
$$\zeta \Rightarrow w_x^e \wedge (\psi \vee E(T)) \qquad ???$$
$$\zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e < v))$$

Solution

Define f such that:

$$\begin{array}{lll} I(f(s_0, x, y_3)) & = 3x + 1 & \\ I(f(s_i, x, y_3)) & = 3(x + 1 - y_3) + 1 - i \quad (i = 1, 2, 3) & y_3 \leq x \\ & = 0 & \neg(y_3 \leq x) \\ I(f(s_4, x, y_3)) & = 0 & \end{array}$$

Let

$$\begin{array}{lllllll} W & = & (NAT, \leq) & \quad & \varphi & = & (x > 0 \wedge a = s_0) \\ w & = & x \geq 0 & & \psi & = & (a = s_4) \\ e & = & f(a, x, y_3) & & \zeta & = & \varphi' \end{array}$$

Need

$$\begin{array}{l} \varphi \Rightarrow (\psi \vee \zeta) \\ \zeta \Rightarrow w_x^e \wedge (\psi \vee E(T)) \\ \zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e < v)) \end{array}$$

Remain to prove:

$$\zeta \wedge e = v \rightarrow [T](\psi \vee (\zeta \wedge e < v))$$

Solution, Ok with some Modification

$(\zeta \wedge e = v) \equiv (\zeta \wedge f(a, x, y_3) = v)$

t_1: $a = s_0 \rightarrow f(s_1, x, 1) < v$,
i.e., $a = s_0 \rightarrow 3(x + 1 - 1) < 3x + 1$, or $a = s_0 \rightarrow 0 < 3x + 1$.

t_2: $a = s_1 \wedge y_3 \leq x \rightarrow f(s_2, x, y_3) < v$,
i.e., $a = s_1 \rightarrow 3(x + 1 - y_3) - 1 < 3(x + 1 - y_3)$.

t_3: $a = s_1 \wedge \neg(y_3 \leq x) \rightarrow (f(s_4, x, y_3) < v) \vee (s_4 = s_4)$. [ok]

t_4: $a = s_2 \rightarrow f(s_3, x, y_3) < v$,
i.e., $a = s_2 \rightarrow 3(x + 1 - y_3) - 2 < 3(x + 1 - y_3) - 1$, or $-2 < -1$.

t_5: $a = s_3 \rightarrow f(s_1, x, y_3 + y_2) < v$,
i.e., $a = s_3 \rightarrow f(s_1, x, y_3 + y_2) < 3(x + 1 - y_3) - 2$. $[y_3 \leq x]$
Either $f(s_1, x, y_3 + y_2) = 0$,
or $f(s_1, x, y_3 + y_2) = 3(x + 1 - (y_3 + y_2))$ and we add $y_2 \geq 1$ to ζ.