第一次作业参考答案 2017 年 10 月 19 日

1．证明：

S 非空，对$\forall \mathrm{Ax_1}, \mathrm{Ax_2} \in \mathrm{S} \to \mathrm{Ax_1} + \mathrm{Ax_2} = \mathrm{A}(\mathrm{x_1} + \mathrm{x_2}) \in \mathrm{S}$, 故对加法运算封闭；

$\forall \mathrm{Ax} \in \mathrm{S}, \mathrm{c} \in \mathrm{R} \to \mathrm{A}(\mathrm{cx}) \in \mathrm{S}$, 故对数乘运算封闭。S 为线性子空间。

设 V 是$\mathrm{B}x$的解空间，定义线性变换$\varphi: \mathrm{V} \to \mathrm{S}$，则

$$dim\mathrm{S} = dim\mathrm{V} - ker\,\varphi = dim\{x|\mathrm{B}x = 0, x \in \mathrm{R}^m\} - \dim\left\{x \middle| \begin{pmatrix} A \\ B \end{pmatrix} x = 0, x \in \mathrm{R}^m\right\}$$

$$= \left(m\text{-}rank(B)\right) - \left(m\text{-}rank\begin{pmatrix} A \\ B \end{pmatrix}\right) = rank\begin{pmatrix} A \\ B \end{pmatrix} \text{-} rank(B)$$

2．解：因为 $y_i = \mu_i + e_{1i}, i = 1, 2 \ldots n_1$，$z_j = \mu_j + e_{2j}, j = 1.2 \ldots n_2$

且 $\begin{Bmatrix} y_1 \\ \vdots \\ y_{n1} \\ z_1 \\ \vdots \\ z_{n_2} \end{Bmatrix} = \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{Bmatrix} e_{11} \\ \vdots \\ e_{1n_1} \\ e_{21} \\ \vdots \\ e_{2n_2} \end{Bmatrix}$ $Ee = 0, \mathrm{cov}(e) = \sigma^2 I_{n_1+n_2}$，则可知 $\begin{pmatrix} 1 & 0 \\ \ldots & \ldots \\ 1 & 0 \\ 0 & 1 \\ \ldots & \ldots \\ 0 & 1 \end{pmatrix}$ 为设计矩阵

3、对 A 进行初等变换，得到：

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 1/2 & -3/2 & 1 & 0 \\ -1/2 & -1/2 & 0 & 1 \end{pmatrix} \times A \times \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

令 P=$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 1/2 & -3/2 & 1 & 0 \\ -1/2 & -1/2 & 0 & 1 \end{pmatrix}$，Q=$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$，则$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P \times A \times Q$

则 $A = P^{-1} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times Q^{-1}$，所以 $A^{-} = Q \times \begin{pmatrix} I_2 & B \\ C & D \end{pmatrix} \times P$，其中 B 为 2×2，C 为 1×2，

D 为 1×1 阶任意矩阵。

4、 解：将 $Q(x_1,x_2,x_3)$ 化为二次型形式：

$$Q(x_1,x_2,x_3)=X'\begin{pmatrix}2&1&1\\1&2&-1\\1&-1&3\end{pmatrix}X-\begin{pmatrix}6\\12\\-8\end{pmatrix}X+19,\ X\sim N_3(\mu,\Sigma)$$

由多维正态分布的密度形式可知:

$$\Sigma^{-1}=\begin{pmatrix}2&1&1\\1&2&-1\\1&-1&3\end{pmatrix}$$

故 $\Sigma=\begin{pmatrix}5/3&-4/3&-1\\-4/3&5/3&1\\-1&1&1\end{pmatrix}$，$|\Sigma|=1/3$，所以 $C=(2\pi)^{3/2}(1/3)^{1/2}$，

根据 $-2\Sigma^{-1}\mu=-\begin{pmatrix}6\\12\\-8\end{pmatrix}$, 解出 $\mu=\begin{pmatrix}1\\2\\-1\end{pmatrix}$

5. 证明：特征函数

$$\varphi(t_1,t_2,t_3,t_4)=\exp\left(-\frac{1}{2}t'\Sigma t\right)=\exp\left(-\frac{1}{2}\sum_{k=1}^{4}\sum_{l=1}^{4}\sigma_{kl}t_kt_l\right),$$

$$\frac{\partial\varphi}{\partial t_1}=-\varphi\sum_{l=1}^{4}\sigma_{1l}t_l\ ,$$

$$\frac{\partial^2\varphi}{\partial t_1\partial t_2}=-\frac{\partial\varphi}{\partial t_2}\sum_{l=1}^{4}\sigma_{1l}t_l-\varphi\sigma_{12}\ ,$$

$$\begin{aligned}\frac{\partial^3\varphi}{\partial t_1\partial t_2\partial t_3}&=-\frac{\partial^2\varphi}{\partial t_2\partial t_3}\sum_{l=1}^{4}\sigma_{1l}t_l-\frac{\partial\varphi}{\partial t_2}\sigma_{13}-\frac{\partial\varphi}{\partial t_3}\sigma_{12}\\&=\frac{\partial\varphi}{\partial t_3}\sum_{l=1}^{4}\sigma_{2l}t_l\sum_{l=1}^{4}\sigma_{1l}t_l+\varphi\sum_{l=1}^{4}\sigma_{1l}t_l\sigma_{23}+\varphi\sum_{l=1}^{4}\sigma_{2l}t_l\sigma_{13}+\varphi\sum_{l=1}^{4}\sigma_{3l}t_l\sigma_{12}\quad,\\&=\frac{\partial\varphi}{\partial t_3}\sum_{l=1}^{4}\sigma_{2l}t_l\sum_{l=1}^{4}\sigma_{1l}t_l+\varphi\left(\sum_{l=1}^{4}\sigma_{1l}t_l\sigma_{23}+\varphi\sum_{l=1}^{4}\sigma_{2l}t_l\sigma_{13}+\varphi\sum_{l=1}^{4}\sigma_{3l}t_l\sigma_{12}\right)\end{aligned}$$

$$Ex_1x_2x_3x_4=\left.\frac{\partial\varphi(t)}{\partial t_4\partial t_3\partial t_2\partial t_1}\right|_{t=0}=\sigma_{14}\sigma_{23}+\sigma_{24}\sigma_{13}+\sigma_{34}\sigma_{12}\text{。}$$

而

$$Ex_k x_l = -\left.\frac{\partial \varphi(t)}{\partial t_k \partial t_l}\right|_{t=0} = -\sigma_{kl}，\quad 1 \le k, l \le 4 \text{。}$$

故 $Ex_1x_2x_3x_4 = Ex_1x_2Ex_3x_4 + Ex_1x_3Ex_2x_4 + Ex_2x_3Ex_1x_4$ 得证。

6. 证明：

$X'\Sigma^{-1}X - \frac{x_1^2}{\sigma_{11}^2} = X'\left[\Sigma^{-1} - \begin{pmatrix} 1/\sigma_{11}^2 & 0 \\ 0 & 0 \end{pmatrix}\right]X$ ，记 $A = \Sigma^{-1} - \begin{pmatrix} 1/\sigma_{11}^2 & 0 \\ 0 & 0 \end{pmatrix}$ ，则

$A\Sigma = \begin{pmatrix} 1 & \sigma_{12}/\sigma_{11} \\ 0 & 0 \end{pmatrix}$，且 $A\Sigma A\Sigma = A\Sigma$，即 $A\Sigma$ 幂等，且由于 Σ^{-1} 存在，所以

$rank(A) = rank(A\Sigma) = 1$，并且非中心参数为 0，故 $X'\Sigma^{-1}X - \frac{x_1^2}{\sigma_{11}^2} \sim \chi_1^2$ 得证

7. 证明：

$$f_V(v) = \sum_{k=0}^{\infty} f_{V,W}(\mathrm{v}, W=k) = \sum_{k=0}^{\infty} f_{V|W}(\mathrm{v} \mid W=k)\,\mathrm{P}(W=k)$$

$$= \sum_{k=0}^{\infty}\left[\frac{\left(\frac{1}{2}\right)^{\frac{n+2k}{2}}}{\Gamma\left(\frac{n+2k}{2}\right)} v^{\frac{n+2k}{2}-1} \mathrm{e}^{-\frac{v}{2}}\right] \cdot \left[\frac{\left(\frac{\lambda}{2}\right)^k}{k!} \mathrm{e}^{-\frac{\lambda}{2}}\right]$$

$$= e^{-\frac{v+\lambda}{2}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^{\frac{n}{2}+2k}}{k!\,\Gamma(\frac{n}{2}+k)} v^{\frac{n}{2}+k-1} \lambda^k$$

此即 $\chi_{n,\lambda}^2$ 的密度函数，因此 $V \sim \chi_{n,\lambda}^2$。

8. (1)证明：令 $Y = X - \mu \Rightarrow X = Y + \mu$ ，则 $Y \sim N_p(0, \Sigma)$ 。

$$\begin{aligned} &Cov(X, X'AX) = Cov(Y+\mu, (Y+\mu)'A(Y+\mu)) = Cov(Y, Y'AY + 2Y'A\mu) \\ &= E(YY'AY) + 2E(YY'A\mu) = E(YY'AY) + 2\sum A\mu \end{aligned}$$

$E(YY'AY)$ 为奇数阶矩且 $EY = 0$，则 $E(YY'AY) = 0$ ，故 $Cov(X, X'AX) = 2A\Sigma\mu$ 。

(2)证明：$Var(X'AX) = E(X'AX)^2 - \left[E(X'AX)\right]^2$，

因为 $\left[E(X'AX)\right]^2 = \left[\mu'A\mu + tr(A\Sigma)\right]^2$，

$$E(X'AX)^2 = E\left[(Y+\mu)'A(Y+\mu)\right]^2$$
$$= E\left[(Y'AY)^2 + 4(\mu'AY)^2 + 4\mu'AYY'AY + 2Y'AY\mu'A\mu + (\mu'A\mu)^2\right],$$

$$\begin{aligned} E(Y'AY)^2 &= E(\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} a_{ij}a_{kl}Y_iY_jY_kY_l) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} a_{ij}a_{kl}\left(EY_iY_jEY_kY_l + EY_iY_kEY_jY_l + EY_iY_lEY_jY_k\right), \\ &= \left[tr(A\Sigma)\right]^2 + 2tr(A\Sigma)^2 \end{aligned}$$

$$E(\mu'AY)^2 = \mu'A\cdot E(YY'A\mu) = \mu'A\Sigma A\mu\ ,$$

$$E(\mu'AYY'AY) = 0\ 。$$

$$\begin{aligned} Var(X'AX) &= \left[tr(A\Sigma)\right]^2 + 2tr(A\Sigma)^2 + 4\mu'A\Sigma A\mu + 2tr(A\Sigma)\mu'A\mu + (\mu'A\mu)^2 \\ &\quad - \left[\mu'A\mu + tr(A\Sigma)\right]^2 \\ &= 2\left[tr(A\Sigma)\right]^2 + 4\mu'A\Sigma A\mu \end{aligned}\ 。$$