scheduled at Wednesday, Nov 4th, 2009, 4:00pm[1] in Ross N605

Short: (On CIC 04) Where is combinatorics?

1. Patterns. This section is a copy-paste of Chapter I of Aigner [14] and, in particular, of Section 4 of that chapter. Let n, r be finite ordinals (numbers). Let N, R be sets of size n, r, respectively; usually $n > 0$, $r > 1$ and N, R are totally ordered as $N = \{1 < \ldots < n\}$, $R = \{1 < \ldots < r\}$. We consider the set $\mathrm{Map}(N, R)$ of all mappings $f : N \to R$. Let G and H be permutation groups on sets N and R, respectively. There is the following permutation group $H^G = \{h^g : g \in G, h \in H\}$ on $\mathrm{Map}(N, R)$, called *power group* and defined as:

$$h^g(f) := hfg \quad \text{for all} f \in \mathrm{Map}(N, R) \ .$$

Moreover, H^G clearly is a permutation group on each of $\mathrm{Inj}(N, R)$, $\mathrm{Sur}(N, R)$, and $\mathrm{Bij}(N, R)$. The H^G-orbits are called *G, H-patterns.*

We now make precise the statement "a set M of *indistinguishable* elements". Let $\pi = A_1|A_2|\cdots|A_k$ be an unordered k-set-partition of M; the understanding is that any two elements are indistinguishable precisely when they are in the same block. We consider the subgroup G of $S(M)$ consisting of all those permutations of M which permute the elements in each block but fix each block as a whole. Then G is such that the G-orbits are precisely the given blocks of M, and G is the maximal such subgroup of $S(M)$. Also, clearly, the inclusions $(S(A_i) \hookrightarrow G)_{1 \le i \le k}$ constitute a coproduct, $G \cong \sum_{i=1}^{k} S(A_i)$. We denote G by $\mathrm{Indist}(\pi)$. Note that if π is the single block M, then $\mathrm{Indist}(\pi) = S(M)$; whereas if all blocks of π are singletons, then $\mathrm{Indist}(\pi) = E(R)$, the identity group on R.

Finally, we arrive at the following two tables. Table 1.1 shows some primitive classes of maps: first $\mathrm{Map}(N, R)$, $\mathrm{Inj}(N, R)$, then $\mathrm{Map}(N, R) \cap \mathrm{Mon}(N, R)$, $\mathrm{Inj}(N, R) \cap \mathrm{Mon}(N, R)$ which use monotonicity. Table 1.2 shows, for each G, H, the representatives of G, H-patterns that are in $\mathrm{Map}(N, R)$, in $\mathrm{Inj}(N, R)$, in $\mathrm{Sur}(N, R)$, and in $\mathrm{Bij}(N, R)$. In particular, Table 1.2 includes the primitives of Table 1.1, thus showing that monotonicity can be obtained through patterns by setting $G = S(N)$, $H = E(R)$. Also in Table 1.2: "ordered" really means "R-indexed", "strict" means "injective", "$\Diamond$" means viewed as "word", "$\triangle$" means viewed as "sortings", "trivial" means "the identity map" or "the inclusion map". Now for some science-fiction:

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Table 1: 1.1 Table

	all	injective
all	n-permutations of r with repetition	n-permutations of r
monotone	n-combinations of r with repetition	n-combinations of r

2. Where is Combinatorics? We probed these textbooks: [1], [2], [3], [4], [5, Volume 1], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Ultimately, Aigner [14, Preface] writes:

> As to the scope of the field, there appears to be a growing consensus that combinatorics should be divided intro three large parts:
> (a) *Enumeration*, including generating functions, inversion, and calculus of finite differences,
> (b) *Order Theory*, including finite posets, and lattices, matroids, and existence results such as Hall's and Ramsey's;
> (c) *Configurations*, including designs, permutation groups, and coding theory.

All of the previous books cover (a), most cover (c). Only Stanley [5, Volume 1] in Chapter 3, Cameron [6] in Chapter 2, Bogard [11] in Chapter 7, van Lint and Wilson [8] in Chapter 27 really do cover (b). Now Vilenkin [13, Preface] writes:

> The branch of mathematics concerned with the number of different arrangements of given objects subject to various restrictions is called *combinatorics*.

Whatever "combinatorics" is, the expectation is that, given a theory, the combinatorics of the "whole" theory *is already exhausted* in the basic parts of the theory; in other words: combinatorics *is sufficient* to allow the "dreamer" to carry out its "whole" agenda. Merging Vilenkin's quote with Aigner's quote, we extract the softer definition:

> "combinatorics" has as ingredients: "data", "coherence", "classification"; we expect as byproduct: the "classification" of "coherent combinations" of "data".

To compare with Vilenkin's quote: instead of having "objects" (a collection) we have abstract "data", instead of outputing a "number" we output a "classification" (possibly an existence answer, or a number, or an order, or a category, ...), instead of a "restriction" we say a "coherence", instead of "arrangement subject

Table 2: 1.2 Table

	all	injective	surjective	bijective
$E(R)^{E(N)}$	$\Diamond$ ordered generalized r-set-partitions of N	$\triangle$ strict words in R of length n	$\Diamond$ ordered r-set-partitions of N	permutations
$E(R)^{S(N)}$ = monotonicity	$\triangle$ monotone nonstrict words in R of length n $\Diamond = n$-multisets of R	$\triangle$ monotone strict words in R of length n $\Diamond = n$-subsets of R	$\triangle$ monotone surjective words in R of length n $\Diamond = n$-multisets of R comprising all of R	trivial
$S(R)^{E(N)}$	unordered generalized r-set-partitions of N	trivial	unordered r-set-partitions of N	trivial
$S(R)^{S(N)}$	unordered generalized r-number-partitions of n	trivial	unordered r-number-partitions of n	trivial
$\mathrm{Indist}(\pi)^{\mathrm{Indist}(\sigma)}$	$\Diamond$ π, r-distributions of N into R			

to restriction" we say "coherent combination". Note that the word "combination" cannot stand by itself since the data is abstract, rather we use "coherent combination". Thus we conclude that the specific definition in Vilenkin's quote really is what is termed "e*num*erative combinatorics".

For example, given G, H permutation groups on N, R respectively, we can set "data" as "Map(N, R)", set "coherent combination" as "G, H-pattern", set "classify" as "count". Actually, since Map(N, R) can be given an *inclusion-as-multisets* order or a *refinement-as-partitions* order, we can set "classify" to "order". Thus there seems to appear an hierarchy:

"pattern" - or zeste of "symmetry"
↓
"cohesion" - or zeste of "agreement"
↓
"coherence" - or zeste of "communication (or telepathy!) is enabled"
↓
"consistency" - or the "minimum"

Orthogonally of "combinatorics" (to combine), there is "recognition" (to recognize) of various manifestations of the *same* "coherence": "algebra" tautologies. Thus we are living concurrent moments: combine, recognize, combine, recognize, combine, ... In other words: combinatorics, algebra, combinatorics, algebra, combinatorics, ...

"En passant" we must mention a short note cdombeu [15], written a long time ago, concerning order, entropy, and combinatorics.

3. Oracles – or Grothendieck's "rêve éveillé". We know of a coherence where the data is the category of proofs of a deductive system; and it is possible to *reduce* that coherence to a coherence where the data is an ordered set. Up to now, we have been referring to an even more abstract notion of coherence ... freely using it as some sort of oracle.

4. On CIC. In the next CIC we really will stick to what we announced earlier: try to extract the "combinatorial content" of Chapters 1–3 of Borceux [16]. Also, the magnet names of the previous announces:

(On CIC 02) infohash D59E746E74CB6AA3B10EF422E51205FCC381E042 or magnet:?xt=urn:btih:2WPHI3TUZNVKHMIO6QROKEQF7TBYDYCC

(On CIC 03) infohash 84F584FFB406C7ABE8E113A5236D425AE404A5AB or magnet:?xt=urn:btih:QT2YJ75UA3D2X2HBCOSSG3KCLLSAJJNL

(defunct "sross514-1.math.yorku.ca")
infohash BC07FE9882D068F7C3185C3E33E683831B308402 or
magnet:?xt=urn:btih:XQD75GEC2BUPPQYYLQ7DHZUDQMNTBBAC

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[13] *N. Ya. Vilenkin* _Combinatorics_ In Academic Press, (1971), English translation by A. Shenitzer, S. Shenitzer.

[14] ***Martin Aigner*** _*Combinatorial Theory*_ In Springer-Verlag, (1979), A Series of Comprehensive Studies in Mathematics 234.

[15] ***cdombeu*** _*Ordre et désordre: le problème de boules*_ In AGRAAF, (2003), English translation "Order and disorder: the problem of having balls" available.

[16] ***Francis Borceux*** _*Handbook of Categorical Algebra 1*_ In Cambridge University Press, (1994), Encyclopedia Of Mathematics And Its Applications 50.