emdash — Functorial programming for strict/lax ω-categories in Lambdapi

https://github.com/hotdocx/emdash

Abstract

We report on **emdash** https://github.com/hotdocx/emdash an ongoing experiment whose goal is a new *type-theoretical* account of strict/lax ω-categories that is both *internal* (expressed inside dependent type theory) and *computational* (amenable to normalization by rewriting). The current implementation target is the Lambdapi logical framework, and the guiding methodological stance is proof-theoretic: many categorical equalities are best presented as *normalization* ("cut-elimination") steps rather than as external propositions.

The central construction is a dependent comma/arrow ("dependent hom") operation that directly organizes "cells over a base arrow" in a simplicial manner. Concretely, let B be a category and let E be a dependent category over B (informally a fibration $E : B \to \mathbf{Cat}$). Fix a base object $b_0 \in B$ and a fibre object $e_0 \in E(b_0)$. We construct a Cat-valued functor that assigns to a base arrow $b_{01} : b_0 \to b_1$ and a fibre object $e_1 \in E(b_1)$ the category of morphisms in the fibre over b_1 from the transport of e_0 along b_{01} to e_1. In slogan form, this is a dependent arrow/comma object

$$\mathrm{Homd}_E(e_0, (-,-)) : E \times_B \left(\mathrm{Hom}_B(b_0, -)\right)^{\mathrm{op}} \to \mathbf{Cat}.$$

In the current kernel snapshot, this construction is computational in the Grothendieck/Grothendieck probe case (via a definitional rule for `homd_`), while full general normalization is still ongoing. The intended iteration yields a simplicial presentation of higher cells (triangles, surfaces, higher simplices), where "stacking" of 2-cells along a 1-cell is expressed *over a chosen base edge*.

As a complementary application, we outline a computational formulation of adjunctions in which unit and counit are first-class 2-cell data and the triangle identities are oriented as definitional reductions on composites (e.g. $\varepsilon_f \circ L(\eta_g) \rightsquigarrow f \circ L(g)$). This showcases the broader emdash theme: coherence is enforced by computation, via stable rewrite heads for functoriality and "off-diagonal" components of transformations. The development is diagram-first: commutative diagrams are specified in a strict JSON format (Arrowgram) and rendered/checked as part of a reproducible paper artifact.

From an engineering perspective, this fits a "MathOps" workflow: a long-running feedback loop between an LLM assistant and a proof-checker/type-checker, where commutative diagrams are first-class artifacts. In emdash we use Arrowgram (a strict JSON diagram format) to make diagrams AI-editable, renderable (e.g. to SVG), and checkable alongside the kernel and the paper.

1. Introduction (what problem are we solving?)

Higher category theory is hard to formalize for two intertwined reasons:

1. *Size of coherence*: weak n-categories and especially weak ω-categories come with an explosion of coherence data.

2. *Where equalities live*: in proof assistants, equalities are propositions to be proved, while in category theory many equalities are "structural" and ought to compute away.

The emdash project explores a kernel design in which:

- "Category theory" is internal: we work *inside* a dependent type theory that has a classifier `Cat` of categories and a classifier `Grpd` of (∞-)groupoids.
- Many categorical laws are definitional: they are enforced by rewrite rules and unification hints, so that normalization *is* diagram chasing.

This paper is a narrative tour of the current kernel snapshot (a Lambdapi specification). It aims to be readable to a non-specialist, while staying faithful to the code. When we mention a kernel identifier (e.g. `homd_int_base`) we also provide a traditional reading.

1.1 Contributions (what is new here?)

This paper's contributions are primarily expository (the kernel is ongoing work), but the kernel already embodies several concrete design/engineering commitments:

1. **A rewrite-head discipline for categorical computation.** Most "structural equalities" are oriented as normalization steps on stable head symbols (e.g. `comp_fapp0`, `fapp1_fapp0`, `tapp0_fapp0`, `tapp1_fapp0`) rather than proved externally.
2. **A simplicial triangle classifier for higher cells over base arrows.** The dependent arrow/comma construction `homd_` (and the internal pipeline `homd_int_base`) provides a computational home for "triangles/surfaces" in the Grothendieck case, and a compositional target for iteration.
3. **An explicit off-diagonal interface for transfors (ordinary and displayed).** Instead of encoding transfors as records with a naturality law, we expose diagonal components (`tapp0_*`, `tdapp0_*`) and off-diagonal components over arrows (`tapp1_*`, `tdapp1_*`) as first-class stable heads.
4. **Executable feasibility evidence.** An earlier executable prototype (bidirectional elaboration with holes + normalization-driven definitional equality) demonstrates that the "kernel spec → elaborating proof assistant" pipeline is realistic.
5. **Continuity with prior warm-ups.** Earlier large-scale rewrite-centric developments already validate the *style* of emdash: categorical interfaces presented with computational rules (universal properties / adjunction transposes; Grothendieck-style geometry interfaces). A key claim of the v2 design is that these interfaces are largely portable into the v2 stable-head discipline, and therefore count as prior progress rather than speculative future work.
6. **New bridge rules in the current snapshot.** The kernel now includes an explicit Grothendieck morphism-action bridge (`Fibration_cov_func`, `Fibration_cov_fapp1_func`) and phase-2 draft strict naturality/exchange rewrites for arrow-indexed components (`tapp1_fapp0`), together with sanity assertions. These bridges also make explicit a key laxness signal: under simplicial iteration, canonical/cartesian source triangles may be sent to non-cartesian target triangles.

2. Technical Overview and Design Principles

emdash is designed around a small set of kernel principles:

1. **Internalization.** Categories, functors, and (higher) transformations are first-class terms, not meta-level structures.
2. **Normalization-first.** Many categorical equalities are oriented as rewrite rules on canonical head symbols.
3. **Stable heads.** Large composite expressions are folded to small "rewrite heads" so normalization is predictable and performant.
4. **Variance by binders.** Functorial/contravariant/object-only dependencies are carried by binder notation at the surface level.

Lambdapi is a natural implementation target because it provides user-defined rewrite rules and higher-order unification. This matters because emdash wants two things simultaneously:

- **Internal definitions**: a functor is not a meta-level function; it is an *object* of a functor category.
- **Computation**: categorical equalities are implemented as *rewrites on stable head symbols*, so they are available during typechecking.

In standard proof assistants you typically have:

- definitional equality: β/δ/ι/ζ reductions;
- propositional equality: theorems you rewrite with.

In emdash we deliberately push a large part of the “categorical equational theory” into definitional equality, using Lambdapi’s rewriting and (carefully!) its unification rules.

2.1 Background: emdash v1 (TypeScript kernel) and “functorial elaboration”

Before the current v2 kernel, the project developed a 1-categorical prototype together with an executable kernel in TypeScript (a bidirectional elaborator with holes, unification-based hole solving, and normalization-driven definitional equality).

The project also maintains a reference codebase for a surface language and its tooling (parsing + elaboration + rendering): `https://github.com/hotdocx/emdash`. This is where the binder-annotated surface reading is meant to become executable.

Two takeaways from v1 matter for the v2 story:

1. **Feasibility of an implementation pipeline.** The kernel concepts can be implemented as an interactive system with holes, bidirectional checking, and a normalization-driven definitional equality. This strongly suggests that the v2 kernel can likewise be turned into a usable assistant once the surface syntax and elaboration layer is designed.
2. **Coherence as computation (“functorial elaboration”).** In the v1 TypeScript kernel, the constructor of a structured object (e.g. a functor) can *compute-check* its laws during elaboration: the system normalizes both sides of functoriality equations in a generic context, and throws a dedicated `CoherenceError` if they do not match definitionally. This is the same philosophical stance as v2’s rewrite-head style: coherence is enforced by computation, not by a separately managed library of lemmas.

The present paper focuses on the v2 Lambdapi kernel itself, but we treat the v1 implementation as evidence that the “specification-first → executable kernel” path is realistic.

Warm-up developments (not discussed in detail here) already stress-test the rewrite-centric style at scale, including interfaces for computation by universal properties (products/exponentials/adjunction transposes) and for Grothendieck-style geometry (sieves/sites/sheafification and related constructions). These serve as a backlog of definitions and computation laws to port into the v2 stable-head discipline.

2.2 How to read kernel vs surface syntax

The v2 kernel is “kernel-first”: it chooses canonical internal heads and rewrite rules. The intended *surface language* (the syntax a user would actually type) is designed so that **variance is tracked by binder notation**, not by explicit “naturality proof terms”.

We will use the following surface-style conventions:

- **Functorial index** `x : A`: a variable intended to vary functorially (e.g. when you have a functor $F : A \to B$, you write $x : A \vdash F[x] : B$).
- **Contravariant index** `x :^- A`: used for arrow-indexed components like $\epsilon_{(f)}$ (it makes "accumulation" rules orient correctly).
- **Object-only index** `x :^o A`: the default for a generic displayed category $E : \mathrm{Catd}(A)$; you may substitute along *paths* in $\mathrm{Obj}(A)$, but you do not assume transport along base arrows unless the structure provides it.

Several kernel projections are intended to be *silent* in surface syntax:

- `τ` (decoding a `Grpd`-code to a type),
- `F[x]` for `fapp0 F x`,
- `E[x]` for `Fibre_cat E x`,
- diagonal components `ϵ[x]` / `ϵ[e]` for transfors and displayed transfors.

The explicit, computationally important data is typically **off-diagonal**: $\epsilon_{(f)}$ (ordinary) and $\epsilon_{(\sigma)}$ (displayed) for "over-a-base-arrow" components. Those correspond to stable heads like `tapp1_fapp0` and `tdapp1_*`.

Conceptually, the kernel can be read as a computational internalization of a categorical semantics for a programming language (a categories-with-families–like stance). The point of the stable-head and binder-mode discipline is that this internal logic is suitable for a bidirectional HOAS-style elaboration engine: coherence is enforced by normalization on kernel primitives, rather than discharged as external proof obligations.

2.3 Technical overview (kernel ↔ surface ↔ mathematics)

Kernel head	Surface reading (intended)	Standard meaning
`Cat`	`⊢ C : Cat`	category / ω-category classifier
`Obj : Cat → Grpd`	`⊢ x : C`	object groupoid of a category
`Hom_cat C x y`	`x :^- C, y : C ⊢ f : x → y`	hom-category (so 1-cells are its objects)
`Functor_cat A B`	`⊢ F : A → B`	functor category
`fapp0 F x`	`F[x]`	object action F_0
`fapp1_fapp0 F f`	`F[f]` (silent)	arrow action $F_1(f)$ (as a 1-cell)
`Transf_cat F G`	`⊢ ϵ : Transf(F,G)`	transformations / transfors
`tapp0_fapp0 Y ϵ`	`ϵ[Y]` (silent)	component ϵ_Y
`tapp1_fapp0 … ϵ f`	`ϵ_(f)`	off-diagonal component "over f"
`Catd Z`	`⊢ E : Catd Z`	displayed category / (iso)fibration over Z
`Fibre_cat E z`	`E[z]`	fibre category over z
`Functord_cat E D`	`⊢ FF : E →_Z D`	displayed functors over fixed base

`fdapp0 … FF z e`	`FF[e]`	fibrewise object action of displayed functor
`Fibration_cov_catd M`	(silent if `M: Z → Cat`)	Grothendieck construction $\int M$
`Total_cat E`	`∫E` (informal)	total category of a displayed category
`fib_cov_tapp0_fapp0 M f u`	$f_!(u)$	Grothendieck transport on fibre objects (strict today)
`homd_`	`Homd_E(w,–)` (informal)	dependent arrow/comma category (triangle classifier)
`Transfd_cat FF GG`	`⊢ ϵ : Transfd(FF,GG)`	displayed transfors
`tdapp0_fapp0 … ϵ`	`ϵ[e]` (silent)	displayed component in a fibre
`tdapp1_*`	`ϵ_(σ)`	displayed off-diagonal component over $\sigma : e \to_f e'$

The kernel also defines stable aliases for "objects of a category" (since `Obj` is not injective in this development): `Hom`, `Functor`, `Transf`, and their displayed analogues (`Fibre`, `Functord`, `Transfd`). We use these aliases in code snippets below.

Notation convention. In surface typing examples, we write `⊢ x : C` as shorthand for `⊢ x : τ (Obj C)` (and similarly `f : x → y` abbreviates `f : τ (Hom C x y)`).

3. Core Type Theory: `Grpd`, `Cat`, and homs-as-categories

3.1 Two classifiers: groupoids vs directed structure

The kernel separates:

- `Grpd : TYPE` — codes for (∞-)groupoids (types with paths);
- `Cat : TYPE` — codes for (strict/lax) ω-categories.

Equality is valued in `Grpd` (so "paths form a groupoid"):

```
 constant symbol = : Π [a: Grpd], τ a → τ a → Grpd;
constant symbol eq_refl : Π [a: Grpd], Π x: τ a, τ (x = x);
symbol ind_eq : Π [a: Grpd], Π [x: τ a], Π [y: τ a], τ (x = y) → Π p: (τ a → Grpd), τ (p y)
→ τ (p x);
```

3.2 Objects are groupoidal

Every category has an object classifier:

```
symbol Obj : Cat → Grpd;
```

This is an explicit design choice: object "equality" in a category is *not* a primitive directed notion; it is a path in the object groupoid.

3.3 Hom is recursive: homs are categories

Instead of `Hom_C(x,y)` being a set/type, in emdash it is a category:

```
 injective symbol Hom_cat : Π (A : Cat) (X_A Y_A : τ (Obj A)), Cat;
symbol Hom (A : Cat) (X_A Y_A : τ (Obj A)) : Grpd ≔ Obj (Hom_cat A X_A Y_A);
```

So a "1-cell" $f : x \to y$ is an *object* of `Hom_cat C x y`. A "2-cell" between parallel 1-cells is then a 1-cell *in that hom-category*, etc. This is the standard globular iteration, but the rest of the paper emphasizes a *simplicial* reorganization of this data.

3.4 Opposites and pointwise composition

The kernel treats the opposite category as a primitive constructor `Op_cat : Cat → Cat` with definitional computation rules:

- objects are unchanged: `Obj (Op_cat A) ↪ Obj A`;
- homs are reversed: `Hom_cat (Op_cat A) X Y ↪ Hom_cat A Y X`;
- identities and composition compute by "forgetting" `Op_cat` and reversing the order.

This is a small example of a general engineering pattern in emdash: if an operation is *structural* (opposite, projections, products), we try to make it compute definitionally so that later constructions can rely on normalization rather than on lemmas.

Similarly, composition in a category A is exposed in two complementary ways:

- a pointwise constructor `comp_fapp0` for composing 1-cells $g : y \to z$ and $f : x \to y$;
- an internal packaging `comp_func` (not shown here) that treats composition itself as a functor out of a product of hom-categories, so that it can carry higher structure when needed.

The pointwise head is what most rewrite rules target: it is the stable normal form for "$g \circ f$".

3.5 Paths as morphisms: `Path_cat` and `Core_cat`

The universe `Obj C : Grpd` means objects come with a path/equality structure. To relate that path structure to actual *directed* morphisms, emdash introduces:

- `Path_cat : Grpd → Cat`, the category whose objects are elements of a groupoid and whose morphisms are paths;
- `Core_cat C := Path_cat (Obj C)`, the "core" (groupoidal) category of C built from object paths.

This provides a clean place to talk about univalence later: one can add bridges between paths in `Obj C` and equivalences/isomorphisms in C without collapsing the whole directed structure into paths.

Concretely, `path_to_hom_func` and its stable-head application `path_to_hom_fapp0` give the direction "path ⇒ morphism":

```
 constant symbol path_to_hom_func : Π [C : Cat], Π (x y : τ (Obj C)),
  τ (Functor (Path_cat (x = y)) (Hom_cat C x y));
symbol path_to_hom_fapp0 : Π [C : Cat], Π (x y : τ (Obj C)), Π (p : τ (x = y)),
  τ (Hom C x y);
```

This is intentionally one-way at the definitional level: the reverse direction (morphisms ⇒ paths) is the dangerous one that can create rewrite loops unless handled by a carefully controlled unification rule (as in the draft univalence bridge present in the current kernel development).

3.6 The "universe categories" `Grpd_cat` and `Cat_cat`

emdash internalizes "the category of groupoids" and "the category of categories" as actual categories:

- `Grpd_cat : Cat` with `τ (Obj Grpd_cat) ≡ Grpd`,
- `Cat_cat : Cat` with `τ (Obj Cat_cat) ≡ Cat`,
- and `Hom_cat Cat_cat A B` computing to `Functor_cat A B`.

This lets us write constructions like "postcompose by opposite" or "compose functors" as ordinary morphisms in `Cat_cat`, and then reuse the same functorial action machinery (`fapp0`, `fapp1_func`) uniformly.

4. Functors and Transfors (ordinary)

4.1 Functors are objects of a functor category

For categories $A, B : \texttt{Cat}$, the category of functors is `Functor_cat A B : Cat`.

```
 constant symbol Functor_cat : Π (A B : Cat), Cat;
symbol Functor (A B : Cat) : Grpd ≔ Obj (Functor_cat A B);
symbol fapp0 : Π [A B : Cat], Π (F_AB : τ (Functor A B)), τ (Obj A) → τ (Obj B);
symbol fapp1_func : Π [A B : Cat], Π (F_AB : τ (Functor A B)), Π [X_A Y_A : τ (Obj A)],
  τ (Functor (Hom_cat A X_A Y_A) (Hom_cat B (fapp0 F_AB X_A) (fapp0 F_AB Y_A)));
```

Here `fapp0` is the object action F_0, and `fapp1_func` is the induced functor on hom-categories F_1 (so it already "knows" about higher cells).

4.2 Stability by "rewrite heads"

In the kernel, many operations are *declared* as symbols (not definitional abbreviations) so that rewrite rules can canonically fold large expressions into small stable heads. For example, applying `fapp1_func` and then `fapp0` is folded into a stable head `fapp1_fapp0`:

```
 symbol fapp1_fapp0 : Π [A B : Cat], Π (F_AB : τ (Functor A B)),
  Π [X_A Y_A : τ (Obj A)],
  Π (f : τ (Hom A X_A Y_A)),
  τ (Hom B (fapp0 F_AB X_A) (fapp0 F_AB Y_A));
```

This “stable head discipline” is crucial for performance and for avoiding brittle matching against huge expanded terms.

4.3 Transfors (transformations) and components

Morphisms in a functor category are transfors (natural transformations / higher analogues). In emdash they live in a category `Transf_cat F G`. A key operation is component extraction: given $\epsilon : F \Rightarrow G$ and $Y \in A$, we want a 1-cell

$$\epsilon_Y : \mathrm{Hom}_B(FY, GY).$$

In the kernel this is a *primitive head* `tapp0_fapp0` with rewrite rules connecting it to the internal packaging:

```
symbol tapp0_fapp0 : Π [A B : Cat], Π [F_AB G_AB : τ (Functor A B)],
 Π (Y_A : τ (Obj A)), Π (ϵ : τ (Transf F_AB G_AB)),
 τ (Hom B (fapp0 F_AB Y_A) (fapp0 G_AB Y_A));
```

The philosophy is: *do not bake “components” into a record definition of transfors*. Instead, compute components by normalization of projection heads.

4.4 Off-diagonal components: `tapp1_fapp0` and “lax naturality data”

In ordinary category theory, a natural transformation $\epsilon : F \Rightarrow G$ is often presented by its *diagonal* components $\epsilon_X : F(X) \to G(X)$, plus a *naturality equation* for every $f : X \to Y$:

$$G(f) \circ \epsilon_X = \epsilon_Y \circ F(f).$$

emdash aims at a more ω-friendly view: the “naturality square” is not necessarily an equality; it can carry higher cell data. Operationally, the kernel therefore exposes an explicit **off-diagonal / arrow-indexed component**

$$\epsilon_{(f)} : F(X) \to G(Y)$$

as a stable-head operation `tapp1_fapp0`. The intended surface syntax is `ϵ_(f)` and it uses the contravariant binder discipline `x :^- A`.

Informally, `tapp1_fapp0` is a way to *name* “the part of a transfor that lives over the base arrow f”. This is exactly the kind of interface we need for cut-elimination style rewrites: we can orient coherence laws as reductions on expressions built from `comp_fapp0`, `fapp1_fapp0`, and `tapp1_fapp0`, without committing to any record encoding of transfors.

Figure 1: an off-diagonal component as a triangle

$$\begin{tikzcd}
F(X) \arrow[dr, "F(f)"'] \arrow[drr, bend left, "\epsilon_{(f)}"] & & \\
& F(Y) \arrow[r, "\epsilon_Y"'] & G(Y)
\end{tikzcd}$$

In a strict 1-category, you may take $\epsilon_{(f)} := \epsilon_Y \circ F(f)$ (or equivalently $G(f) \circ \epsilon_X$) and prove the two definitions equal. In emdash, we keep $\epsilon_{(f)}$ as explicit data because it is the correct home for higher "laxness" witnesses.

Figure 2: the composite case and "accumulation" over base arrows

The same idea becomes more informative when we look at a composite base arrow. If we have $f : X \to Y$ and $g : Y \to Z$, then an "off-diagonal" component can be presented directly over $g \circ f$:

$$\begin{tikzcd}
F(X) \arrow[dr, "F(f)"'] \arrow[drr, bend left, "\epsilon_{(g \circ f)}"] & & \\
& F(Y) \arrow[r, "\epsilon_{(g)}"'] & G(Z)
\end{tikzcd}$$

Kernel-wise, this is exactly where the "contravariant binder / accumulation" discipline pays off: one can orient a coherence law as a rewrite of the form

$$(\epsilon_{(g)}) \circ F(f) \;\rightsquigarrow\; \epsilon_{(g \circ f)},$$

so that normalization *accumulates* the base-arrow index instead of repeatedly expanding/contracting naturality squares.

Exchange law sanity: postcomposition and pasting

Once we expose off-diagonal components, we can also express the familiar 2-categorical **exchange**/pasting phenomenon as an *internal equality between normal forms*.

Fix a category B, an object $M \in B$, and objects $N, L \in B$. Consider the covariant representable `hom_ B M B (id_func B) : B → Cat_cat`. For $f : N \to L$ it yields the postcomposition functor

$$\mathrm{postcomp}(f) : \mathrm{Hom}_B(M, N) \to \mathrm{Hom}_B(M, L),$$

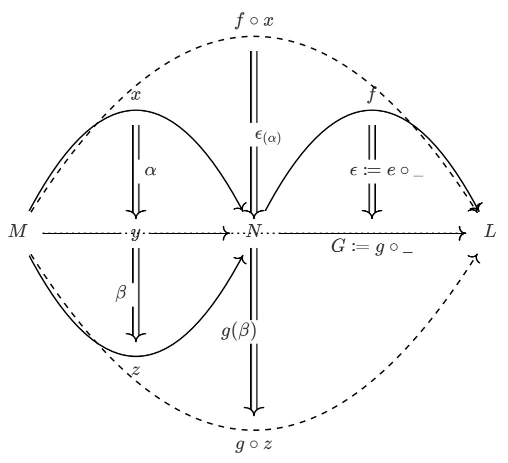
and for a 2-cell $e : f \Rightarrow g$ it yields a transfor $\epsilon : \mathrm{postcomp}(f) \Rightarrow \mathrm{postcomp}(g)$.

Now take composable 2-cells $\alpha : x \Rightarrow y$ and $\beta : y \Rightarrow z$ in $\mathrm{Hom}_B(M, N)$. The exchange law instance we sanity-check in the kernel is the equation

$$\mathrm{postcomp}(g)(\beta) \circ \epsilon_{(\alpha)} \;=\; \epsilon_{(\beta\circ\alpha)}.$$

In kernel heads, this is exactly the shape `comp_fapp0 (fapp1_fapp0 postcomp_g β) (tapp1_fapp0 ϵ α) ≡ tapp1_fapp0 ϵ (comp_fapp0 β α)`.

To emphasize that the **pasting diagram is not pre-composed**, Figure 3 draws the generating 1-cells $x, y, z : M \to N$ and $f, g : N \to L$ together with the 2-cells α, β (vertical) and ϵ (horizontal). The dashed arrows are the *composites* $f \circ x$, $g \circ y$, $g \circ z$ in $\mathrm{Hom}_B(M, L)$; the exchange law says that the same pasted 2-cell from $f \circ x$ to $g \circ z$ is obtained whether we first “whisker/horizontally compose” to form $\epsilon_{(\alpha)}$ and then postcompose by $g(\beta)$, or whether we first vertically compose $\beta \circ \alpha$ and then take the off-diagonal component $\epsilon_{(\beta\circ\alpha)}$.



For reference, the stable head has the following kernel type (here `@` just disables implicit arguments):

```
symbol tapp1_fapp0 : Π [A B : Cat], Π [F_AB G_AB : τ (Functor A B)],
  Π [X_A Y_A : τ (Obj A)],
  Π (ϵ : τ (Transf F_AB G_AB)),
  Π (f : τ (Hom A X_A Y_A)),
  τ (Hom B (fapp0 F_AB X_A) (fapp0 G_AB Y_A));
```

4.5 Representables and the Yoneda-style heads `hom_` / `hom_con`

emdash provides covariant and contravariant representables in a Cat-valued setting:

- `hom_` models $\mathrm{Hom}_A(W, F(-))$ (covariant in the variable, with postcomposition behavior);

- `hom_con` models $\mathrm{Hom}_A(F(-), W)$, implemented by a definitional reduction to `hom_` in the opposite category.

These are not just for "doing Yoneda"; they are also the glue behind the internal packaging of `tapp*` and `homd_int_base`. A recurring kernel tactic is: *encode a textbook naturality statement as a rewrite rule about postcomposition by functors*, so that "naturality" becomes normalization.

4.6 Strictness as optional structure: `StrictFunctor_cat`

While the global goal includes lax/weak structure, the kernel often begins with strict computation rules and relaxes them later. For example, `StrictFunctor_cat A B` classifies functors equipped with rewrite rules stating that identities and composition are preserved *on the nose* at the level of 1-cells:

```
 constant symbol StrictFunctor_cat : Π (A B : Cat), Cat;
symbol StrictFunctor (A B : Cat) : Grpd ≔ Obj (StrictFunctor_cat A B);
injective symbol sfunc_func : Π [A B : Cat], τ (StrictFunctor A B) → τ (Functor A B);
```

This is a pragmatic move: strictness gives stable computation rules, and later "laxness witnesses reduce to identities" can be recovered as special cases once the simplicial machinery is strong enough.

5. Dependent Categories (`Catd`) and Grothendieck Constructions

5.1 Displayed categories as isofibrations

The kernel has a classifier `Catd Z` of dependent categories over a base category Z (intended to model isofibrations / displayed categories).

```
constant symbol Catd : Π (Z : Cat), TYPE;
```

In the paper we write $E : \texttt{Catd}\ Z$ and read it as a fibration-like structure $p : \int E \to Z$ whose fibre over z is a category $E[z]$.

5.2 The Grothendieck constructor as computation

For an honest Cat-valued functor $M : Z \to \mathbf{Cat}$, emdash provides a displayed category `Fibration_cov_catd M : Catd Z` (think "the Grothendieck construction" $\int M$), and in this special case fibres compute definitionally:

```
constant symbol Fibration_cov_catd : Π [Z : Cat], τ (Functor Z Cat_cat) → Catd Z;
```

This is the main place where the kernel commits to concrete computations for `Catd`: general displayed categories are abstract, but Grothendieck ones compute.

Figure 3: Grothendieck morphisms lie over base arrows

$$\begin{tikzcd}
b \arrow[r, "f"] & b' \\
(b,e) \arrow[u, dashed, "p"'] \arrow[r, "(f,\sigma)"] & (b',x) \arrow[u, dashed, "p"]
\end{tikzcd}$$

Here (f,σ) is the usual Grothendieck idea: a morphism in the total category contains a base arrow f plus a fibre morphism σ with the correct transported source.

5.3 Slice-style displayed functors: `Functord_cat`

There are (at least) two common ways to formalize displayed functors:

1. **General base map**: a displayed functor can live "over" an arbitrary base functor $F : X \to Y$.
2. **Slice-style**: fix a base Z, and consider only functors over id_Z between objects of the slice $\mathbf{Cat}/Z$.

emdash2 chooses the slice-style presentation because it makes composition and normalization stable:

- displayed categories over Z are terms of `Catd Z`,
- displayed functors over Z are objects of `Functord_cat E D`,
- composition stays in the same base automatically.

The kernel still supports the "general base map" intuition via pullback: a functor over F becomes an ordinary slice-style functor into a pullback `Pullback_catd D F`.

5.4 Totals, projections, and "internalized" totalization

Given $M : Z \to \mathbf{Cat}$, the Grothendieck total category $\int M$ is represented as `Total_cat (Fibration_cov_catd M)`. In the current kernel, the object layer of totals is now definitionally Σ-shaped for arbitrary displayed categories:

$$\tau\big(\mathrm{Obj}(\mathrm{Total\_cat}(E))\big) \;\rightsquigarrow\; \sum_{z \in \mathrm{Ob}(Z)} \mathrm{Obj}(E[z]).$$

In the Grothendieck case this specializes to $\mathrm{Ob}(\int M) \;\simeq\; \sum_{z \in \mathrm{Ob}(Z)} \mathrm{Ob}(M(z))$.

Morphisms/homs of `Total_cat E` for general `E : Catd Z` remain mostly abstract; Grothendieck-specific rules provide the main computational hom-level behavior.

For composing constructions inside `Cat_cat`, emdash also packages “totalization” as a functor object:

```
symbol Total_func [Z : Cat] : τ (Functor (Functor_cat Z Cat_cat) Cat_cat);
```

with a β-rule on objects `fapp0 (Total_func[Z]) M ↪ Total_cat (Fibration_cov_catd M)`. This is the same stable-head pattern as elsewhere: we want *a small head symbol* we can compose with, rather than unfolding a large definition every time.

5.5 Strict Grothendieck transport on fibre objects: `fib_cov_tapp0_fapp0`

If $M : Z \to \mathbf{Cat}$ is Cat-valued, then for a base arrow $f : z \to z'$ we can “transport” a fibre object $u \in M(z)$ to $f_!(u) := M(f)(u) \in M(z')$. emdash exposes this as a stable head `fib_cov_tapp0_fapp0` with strict functoriality on objects:

- $(\mathrm{id})_!(u) \rightsquigarrow u$,
- $g_!(f_!(u)) \rightsquigarrow (g \circ f)_!(u)$.

The orientation is cut-elimination style: nested transports fold to one transport along the composite. This strictness is explicitly temporary; it is a placeholder for the later lax/weak story where “functoriality of transport” comes with higher cells rather than with definitional equalities.

5.6 Fibrewise products of displayed categories: `Product_catd` and `prodFib`

Displayed categories over the same base admit a fibrewise product `Product_catd U A : Catd Z`, with fibres

$$(U \times A)[z] \equiv U[z] \times A[z].$$

For Grothendieck displayed categories $\int E$ and $\int D$, emdash includes a definitional rule

$$(\textstyle\int E) \times_Z (\textstyle\int D) \;\rightsquigarrow\; \textstyle\int (E \times D),$$

implemented via a stable-head functor `prodFib` for pointwise product of Cat-valued functors. This is a representative example of “rewrite hygiene”: we introduce a small head so later rules can match without normalizing a huge composed expression in `Cat_cat`.

6. Dependent Arrow/Comma Categories: `homd_` and `homd_int`

This section is the technical center of the paper. It explains the kernel constructions that we use to model “cells over a base arrow” without starting from a raw globular presentation.

6.1 Classical picture: dependent hom / comma object

Let:

- Z be a category,
- E be a dependent category over Z (morally $E : Z \to \mathbf{Cat}$),
- $W \in Z$ be a chosen base object, and $w \in E(W)$ be a chosen fibre object.

Given a base arrow $f : W \to z$ and an object $x \in E(z)$, we want the category of "arrows from the transport of w along f to x". In textbook fibration language this is:

$$\mathrm{Hom}_{E(z)}(f_! w, x).$$

This is a *dependent arrow category*: its objects are "triangles" with base edge f and a displayed edge in the fibre.

6.2 What `homd_` computes (in the Grothendieck case)

The kernel symbol `homd_` is declared abstractly, but it has a key computation rule when the displayed categories involved are Grothendieck constructions. In words:

> if $E = \int E_0$ and $D = \int D_0$ are Grothendieck displayed categories, then `homd_` evaluated at a triple (z, d, f) reduces to the hom-category in the fibre $E_0(z)$ from the transported w to the image of d under the displayed functor.

This is precisely the "dependent arrow/comma" intuition.

At the definitional level, the key pointwise computation rule is (Grothendieck–Grothendieck case):

```
rule fapp0 (@homd_ $Z
              (@Fibration_cov_catd $Z $E0)
              (@Fibration_cov_catd $Z $D0)
              $FF
              $W_Z $W)
            (Struct_sigma $z (Struct_sigma $d $f))
  ↪ Hom_cat (fapp0 $E0 $z)
      (fapp0 (fapp1_fapp0 $E0 $f) $W)
      (@fdapp0 _ _ _ $FF $z $d);
```

Figure 4: a displayed arrow over a base arrow

$$\begin{tikzcd}
e \in E(b) \arrow[r, bend left, "\sigma : f_! e \to x"] \arrow[d, dashed, "p"'] & x \in E(b') \arrow[d, dashed, "p"] \\
b \arrow[r, "f"] & b'
\end{tikzcd}$$

The slogan is: *a 2-cell is a 1-cell in a dependent arrow category*.

6.3 The "more internal" pipeline: `homd_int_base`

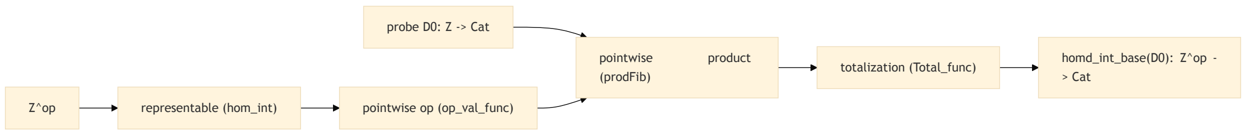
The kernel contains a more internal version of the dependent hom construction, designed to be composed with other internal operations without exploding rewriting. It is built from stable-head building blocks (pointwise opposite, pointwise product, totalization, etc.). The key helper is:

```
symbol homd_int_base [Z : Cat] (D0 : τ (Functor Z Cat_cat))
  : τ (Functor (Op_cat Z) Cat_cat);
```

Conceptually, `homd_int_base D0` is a Cat-valued functor on Z^{op} whose value at z packages:

- a representable $\mathrm{Hom}_Z(-, z)$ (built via `hom_int` and then dualized),
- paired with the probe family $D0[z]$ (via `prodFib`),
- then "totalized" (via `Total_func`) so that later constructions can range over *base arrows* explicitly.

  The point is not the exact combinatorics but the *shape*: we are building a simplicial indexing category (objects z together with base edges into z) in a way that remains computationally stable.


probe D0: Z -> Cat
Z^op
representable (hom_int)
pointwise op (op_val_func)
pointwise product (prodFib)
totalization (Total_func)
homd_int_base(D0): Z^op -> Cat

6.4 `homd_int` as a displayed functor (current status)

The kernel exposes `homd_int` as a displayed functor object (currently without computation rules). Informally it packages:

- a probe displayed category $\int D0$,
- a displayed functor $FF : \int D0 \to E$,

- and returns a displayed functor over E^{op} whose fibre over (z, e) classifies arrows "from e to $FF(d)$ over a base arrow".

```
constant symbol homd_int : Π [Z : Cat], Π [E : Catd Z],
  Π (D0 : τ (Functor Z Cat_cat)),
  Π (FF : τ (Functord (Fibration_cov_catd D0) E)),
  τ (Functord (Op_catd E)
              (Op_catd (Fibration_con_catd
                (comp_cat_fapp0 Fib_func (homd_int_base D0)))));
```

The important message for the reader is: *emdash organizes higher cells by iterating dependent arrow categories*; `homd_int` is the internalized, compositional version of the triangle classifier. In parallel, the current kernel also uses a `homd_curry` / `Homd_func` pipeline as a direct computational bridge for total-category homs in Grothendieck-shaped cases.

7. Displayed Transfors and Simplicial Iteration

7.1 Displayed transfors: pointwise (`tdapp0_*`) and off-diagonal (`tdapp1_*`)

In addition to ordinary transfors between ordinary functors, the kernel includes **displayed transfors**: transformations between displayed functors over the same base Z.

If $FF, GG : E \to_Z D$ are displayed functors (objects of `Functord_cat E D`), then `Transfd_cat FF GG` is the category of displayed transfors. As with ordinary transfors, emdash provides stable-head projections rather than a record encoding:

- `tdapp0_fapp0` extracts the **diagonal component** at a base point (Y, V), i.e. a morphism in the fibre $D[Y]$:

```
symbol tdapp0_fapp0 : Π [Z : Cat], Π [E D : Catd Z],
  Π [FF GG : τ (Functord E D)],
  Π (Y_Z : τ (Obj Z)),
  Π (V : τ (Fibre E Y_Z)),
  Π (ϵ : τ (Transfd FF GG)),
  τ (Hom (Fibre_cat D Y_Z) (fdapp0 FF Y_Z V) (fdapp0 GG Y_Z V));
```

- `tdapp1_fapp0_funcd` (and internal variants like `tdapp1_int_*`) package the **off-diagonal component** over a displayed arrow $\sigma : e \to_f e'$ (surface syntax `ϵ_(σ)`), mirroring the ordinary `tapp1_*` story.

This mirrors the surface discipline: diagonal components are silent (`ϵ[e]`), while off-diagonal components are explicit computational interfaces.

One particularly important normalization rule connects identity displayed transfors to the *action on dependent homs*. Informally:

> the identity transfor 1_{FF} induces exactly the "functorial action of FF on homd-data".

In the kernel this is a fold from `tdapp1_*` at an identity to the stable head `fdapp1_funcd`. This is the displayed analogue of the philosophy "coherence is computation": functorial action on higher structure is obtained by normalizing a canonical expression rather than by proving a lemma.

7.2 From globes to simplices: triangles, surfaces, and “stacking”

Classically, ω-categories are presented globularly: 0-cells, 1-cells, 2-cells between 1-cells, etc. emdash keeps the globular core (homs are categories), but it tries to *use simplicial indexing for computation*:

- a 2-cell is a triangle over a base edge;
- a 3-cell is a tetrahedron over a base triangle;
- and so on.

The kernel contains the beginnings of this “simplicial engine”:

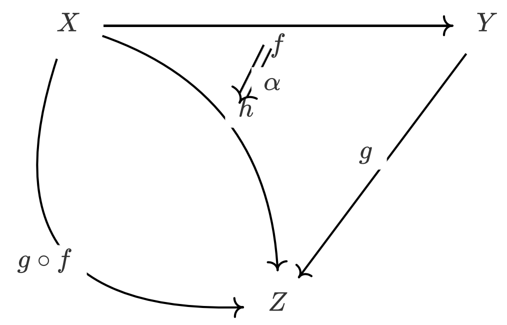
- `homd_` (triangle/surface classifier) with a Grothendieck/Grothendieck computation rule,
- draft operations like `fdapp1_funcd` / `fdapp1_int_transfd` for iterating the construction,
- and phase-2 draft strict accumulation rules on `tapp1_fapp0`, including a concrete exchange-law sanity assertion for representable postcomposition.

A useful way to read the current `fapp1_funcd` / `fdapp1_funcd` comments is as follows. For a lax functor action $F_1 : \mathrm{Hom}_C(x,-) \to \mathrm{Hom}_D(F_0x, F-)$, the iterated action is fibred and can be non-cartesian:

$$((F_1)_1)_0 : \mathrm{Homd}_{\mathrm{Hom}_C(x,-)}\big(f, (g \circ f, g)\big) \to \mathrm{Homd}_{\mathrm{Hom}_D(F_0x,F-)}\big((F_1)_0 f, \big((F_1)_0(g \circ f), (F_1)_0 g\big)\big).$$

Conceptually, an identity-like (cartesian) source triangle over a base edge may normalize to a non-identity (non-cartesian) target triangle; that non-identity image is the laxness witness. In the strict case, the intended deferred rule would collapse this witness back to identity.

Figure 5: a 2-cell between parallel composites (Arrowgram arrow-to-arrow)



$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[ddr, bend left, "h"] \arrow[ddr, bend right=60, "g \circ f"'] \arrow[r, phantom, shift right=3, "\Downarrow \alpha"] & & Y \arrow[dl, "g"] \\
 & & \\
 & Z &
\end{tikzcd}$$

This is the common 2-categorical picture (a 2-cell between parallel composites). The kernel aim is to recover such cells as objects of a dependent hom construction, so that “stacking” and exchange laws can be derived from functoriality of the indexing.

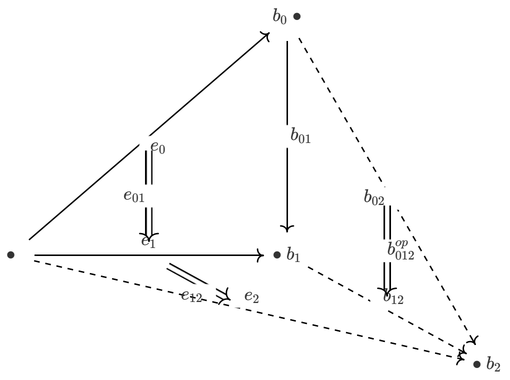
Figure 6: stacking 2-cells along a 1-cell (a tetrahedral “over-a-base-edge” picture)

The “stacking” operation (horizontal composition of 2-cells along a 1-cell rather than along a 0-cell) is easiest to see as a tetrahedron of base edges, with 2-cells comparing *direct* edges with *composite* edges.

Recall from §6 that the dependent arrow/comma classifier organizes “2-cells over a chosen base edge” via a functor

$$\mathrm{Homd}_E(e_0, --) : E \times_B \left(\mathrm{Hom}_B(b_0, -)\right)^{\mathrm{op}} \to \mathbf{Cat}.$$

Stacking then corresponds to composing such base edges and asking for a *computational* interface where normalization re-associates and “exchanges” these simplicial indices.



In emdash terms, this is the kind of geometry the kernel is aiming to make *computational*: rather than proving a separate “exchange law”, one wants an interface where these higher comparisons arise by functoriality/transport in a suitably internalized indexing. The current snapshot already includes a representable-instance exchange sanity assertion; a fully generic stacking interface is still in progress.

In slogan form, this matches a cut-accumulation reading of naturality on off-diagonal components:

$$(G(b)) \cdot \epsilon_{(a)} \rightsquigarrow \epsilon_{(b \cdot a)}, \qquad \epsilon_{(b)} \cdot (F(a)) \rightsquigarrow \epsilon_{(b \cdot a)},$$

and a familiar exchange-law instance can be summarized as a normalization:

$$(g \circ \beta) \cdot (e \circ \alpha) \rightsquigarrow e \circ (\beta \cdot \alpha).$$

8. Computational Adjunctions (cut-elimination rules)

The kernel contains a draft interface for adjunctions inspired by Došen–Petrić’s proof-theoretic view of categories.

8.1 Adjunction data as first-class terms

An adjunction is represented by:

- categories R, L,
- functors $LAdj : R \to L$ and $RAdj : L \to R$,
- transfors (unit and counit)

$$\eta : \mathrm{Id}_R \Rightarrow RAdj \circ LAdj, \qquad \epsilon : LAdj \circ RAdj \Rightarrow \mathrm{Id}_L.$$

In the kernel this is a type former:

```
constant symbol adj :
 Π [R L : Cat],
 Π (LAdj : τ (Functor R L)),
 Π (RAdj : τ (Functor L R)),
 Π (η : τ (Transf (@id_func R) (comp_cat_fapp0 RAdj LAdj))),
 Π (ϵ : τ (Transf (comp_cat_fapp0 LAdj RAdj) (@id_func L))),
 TYPE;
```

8.2 The triangle law as a rewrite rule

One of the most concrete successes of this approach is that a triangle identity can be oriented as a rewrite rule (“cut-elimination”). Very roughly (omitting indices), the rule has the shape:

$$\epsilon_f \circ L(\eta_g) \ \rightsquigarrow\ f \circ L(g).$$

This matches the abstract’s goal: make the triangle identity a *definitional computation step* rather than a lemma.

Figure 7: the familiar triangle identity (as reduction)

$$\begin{tikzcd}
& L(R(L(A))) \arrow[dr, "\epsilon_{L(A)}"] & \\
L(A) \arrow[ur, "L(\eta_A)"] \arrow[rr, dashed, "\mathrm{id}"] & & L(A)
\end{tikzcd}$$

The rule is implemented at the level of the stable heads for composition (`comp_fapp0`), functor action on morphisms (`fapp1_fapp0`), and arrow-indexed transfor components (`tapp1_fapp0`). The result is that normalizing a composite term performs the triangle reduction. In the current snapshot this remains draft-level: the rewrite is present and typechecks, while a fully closed regression term is still marked TODO in the kernel.

This computational adjunction story is best viewed as an application of the *off-diagonal component infrastructure for transfors* (`tapp1_*` / `tdapp1_*` and their “functional” packaging), rather than as a special application of the dependent-hom layer. The dependent-hom constructions (`homd_` , `homd_int`) target a different goal: simplicial organization of higher cells over chosen base arrows, so that stacking/exchange phenomena can later be obtained uniformly.

9. Metatheory and Normalization Discipline

emdash relies on Lambdapi’s core metatheoretic contract: typechecking is done modulo a conversion relation generated by β-reduction plus user-supplied rewrite rules (and some built-in definitional equalities). Lambdapi checks subject-reduction for each rewrite rule; global termination/confluence are left to the user.

The practical question, therefore, is not “can we state the equations?” but “can we orient a useful subset as a terminating, robust normalization strategy?” The kernel comments call this **rewrite hygiene**.

9.1 Stable heads and canonicalization

The most frequent move is to introduce a *stable-head* symbol for a conceptually important operation and add a canonicalization rule that folds a large redex into that head. Examples:

- `fapp1_fapp0` folds “apply `fapp1_func` then `fapp0`” into a single head.
- `tapp0_fapp0` is the head for “component at Y”, rather than unfolding it as an evaluation of a packaged functor.
- `fib_cov_tapp0_fapp0` is the head for Grothendieck-style object transport.

This makes rewriting predictable (matching sees the head) and keeps later rewrite rules small.

9.2 Rewriting vs unification rules

emdash uses both:

- `rule` for *computation* (normalization steps that should fire during reduction),
- `unif_rule` for *inference hints* (guiding elaboration/conversion without changing normal forms).

The slogan is: if a law should not change canonical normal forms (e.g. “the component at an identity arrow agrees with the diagonal component”), prefer `unif_rule` over `rule`.

9.3 Avoiding rewrite explosions

Two common failure modes for rewrite-based kernels are:

- **conversion blowups** caused by matching against too much inferred structure in a rule LHS;
- **non-termination** caused by loops between “expanding” and “folding” rules.

The kernel mitigates this by (i) keeping inferred arguments as `_` on LHS patterns unless essential, and (ii) orienting many laws in a cut-elimination direction (nested structure folds to a single constructor).

9.4 Timeouts as a diagnostic tool

Because a hung typecheck often indicates rewrite/unification pathology, the repo’s workflow treats a short timeout as a diagnostic: a timeout is a bug signal, not a reason to increase the timeout.

10. Implementation and Evaluation (February 9, 2026 snapshot)

The kernel is intentionally “small but sharp”. Some parts compute definitionally today; others are interfaces intended to be refined.

- **Computational today** (examples): opposites (`Op_cat`), products (`Product_cat` , `Product catd` with Groth bridges via `prodFib`), total-object Σ-computation for arbitrary displayed categories (`τ (Obj (Total_cat E))`), Grothendieck fibres and transport (`Fibration_cov_catd` , `fib_cov_tapp0_fapp0`), Grothendieck morphism-action bridge (`Fibration_cov_func` , `Fibration_cov_fapp1_func`), direct total-hom bridges (`homd_curry` , `Homd_func`) in Grothendieck-shaped cases, pointwise computation for `homd_` in the Grothendieck–Grothendieck case, pointwise component extraction for transfors/displayed transfors (`tapp0_fapp0` , `tdapp0_fapp0`), phase-2 draft strict naturality/exchange rewrites for `tapp1_fapp0` , and a draft triangle cut-elimination rule for adjunctions.
- **Abstract / TODO** (examples): full computation rules for `homd_int` , full hom-level computation for general displayed categories `E : Catd Z` , full displayed Groth morphism-action bridge needed to derive external heads like `fdapp1_funcd` from internal `tdapp1_int_*` / `fdapp1_int_*` pipelines, replacing temporary strict phase-2 laws by lax higher-cell data (including explicit cartesian vs non-cartesian triangle behavior as rewrite-level infrastructure), and the user-facing surface syntax/elaboration layer (variance-aware binders, implicit coercions).

This division is deliberate: the kernel tries to avoid committing to heavy encodings (Σ-records for functors/transfors) until the rewrite story is stable.

10.2 Typechecking the kernel

The Lambdapi development is designed to typecheck quickly; long typechecks are treated as a symptom of rewrite/unification pathologies. The repo provides a timeout-protected check (`EMDASH_TYPECHECK_TIMEOUT=60s make check`) and a watch mode that logs to `logs/typecheck.log` .

The paper renderer is also exercised as a reproducible artifact:

- `npm run validate:paper -w print` validates all embedded Arrowgram/Vega-Lite JSON blocks.
- `npm run check:render -w print` runs validation + build + a headless browser console check (fails on KaTeX warnings and rendering errors).

11. Limitations, Related Work, and Future Directions

11.1 Limitations (current kernel snapshot)

The current kernel is intentionally incomplete; some gaps are design choices, others are pending infrastructure.

- **No global record encoding for functors/transfors.** Objects of `Functor_cat` and `Transf_cat` are intentionally abstract; the user-facing interface is via projection heads (`fapp0` , `tapp0_*` , `tapp1_*` , etc.).
- **Displayed categories are only partially computational.** Generic `Catd` now has computational object-level structure in several places (e.g. `Fibre_cat` , `τ (Obj (Total_cat E))` , pullback/terminal/opposite rules), but hom-level behavior is still mostly computational only in Grothendieck-shaped cases.

- **Strictness placeholders.** Some computation rules (e.g. Grothendieck object transport) are strict today and are intended to be relaxed to a lax/weak story with higher cells.
- **Adjunction layer is still draft-level.** One triangle cut-elimination rewrite is present, but the surrounding bridge infrastructure and closed regression terms are incomplete.
- **"Metatheory by engineering."** Termination/confluence are managed by rewrite discipline and tests rather than by a formal metatheorem at this stage.

11.2 Related ideas and influences

emdash draws on three complementary threads:

1. **Proof-theoretic categories** (Došen–Petrić): treat categorical equalities as cut-elimination / normalization steps.
2. **Type-theoretic higher categories** (e.g. Finster–Mimram): define weak ω-categories by internal type-theoretic structure.
3. **Parametricity-inspired internalization** (e.g. "bridge types"): avoid explicit interval objects while retaining internal reasoning principles reminiscent of parametricity.

Our distinctive emphasis is *computational organization of higher cells over base arrows* via dependent arrow/comma categories, combined with an explicit off-diagonal component interface for transfors that supports computation-first coherence laws.

11.3 Next steps

- Finish the simplicial iteration story (derive/justify the external heads like `fdapp1_funcd` from internal `tdapp1_int_*` / `fdapp1_int_*` pipelines; extend computation beyond the Grothendieck case).
- Add the missing user-facing layer: a variance-aware elaborator for the intended surface language (as in the v1 TypeScript kernel, but for the v2 primitives).
- Extend the adjunction interface from the first triangle cut-elimination rule to a robust set of whiskering/triangle/exchange normalizations, with closed regression terms.
- Port the warm-up libraries: re-express the universal-property and geometry interfaces within the v2 heads (`Functor_cat`, `Transf_cat`, and eventually a displayed-profunctor layer), so they become stable regression tests for the kernel discipline.

12. Conclusion

emdash is an attempt to make a small computational kernel in which:

- higher categorical structure is *internal* (functors and transformations are first-class objects),
- equalities are *operational* (rewrite rules),
- and higher cells are organized *simplicially* via dependent arrow categories (`homd_`, `homd_int`).

The most concrete result so far is a faithful computational core for Grothendieck-style dependent categories (now including explicit morphism-action bridges) together with draft strict off-diagonal naturality/exchange rules and a first computational adjunction triangle rule. The next step is to replace strict placeholders by lax higher-cell structure and finish the internal-to-external simplicial derivations.

References

1. F. Blanqui et al. *The Lambdapi Logical Framework*.
2. K. Došen and Z. Petrić. *Cut-Elimination in Categories*.
3. E. Finster and S. Mimram. *A Type-Theoretical Definition of Weak ω-Categories*.
4. T. Altenkirch, Y. Chamoun, A. Kaposi, M. Shulman. *Internal Parametricity, without an Interval*.
5. H. Herbelin, R. Ramachandra. *Parametricity-based formalizations of semi-simplicial / semi-cubical structures*.

Appendix A. Reading guide to the code (kernel identifiers → math)

- `Cat` , `Obj` , `Hom_cat` : category classifier, object groupoid, hom-category (iterated for higher cells).
- `Functor_cat` , `fapp0` , `fapp1_func` , `fapp1_fapp0` : functors as objects; object and hom action; stable-head application.
- `Transf_cat` , `tapp0_fapp0` , `tapp1_fapp0` : transfors; object-indexed and arrow-indexed components (lax naturality lives “off-diagonal”).
- `Transfd_cat` , `tdapp0_fapp0` , `tdapp1_fapp0_funcd` , `fdapp1_funcd` : displayed transfors; pointwise components in fibres; packaged off-diagonal components over displayed arrows; action on dependent-hom data.
- `Catd` , `Fibre_cat` , `Functord_cat` : displayed categories (isofibrations); fibres; displayed functors over a fixed base.
- `Fibration_cov_catd` , `Fibration_cov_func` , `Fibration_cov_fapp1_func` : Grothendieck construction and its object-/morphism-action bridge.
- `hom_` , `hom_int` : (co)representables / internalized hom functors.
- `homd_` , `homd_curry` , `Homd_func` : dependent arrow/comma constructions (triangle classifier and internal curry pipeline), with Grothendieck-case computation bridges.
- `homd_int_base` , `homd_int` : internalized pipeline and displayed packaging for the same idea (currently mostly interface-level).
- `fdapp1_int_transfd` , `tdapp1_int_func_transfd` : internal displayed off-diagonal packaging used to derive external heads.
- `adj` and the triangle rewrite: adjunction data and draft cut-elimination rule.