课程名称：高等代数(2) 学号：1131190111 姓名：[redacted] 班级：信计1901

一、

(1) 证：$\alpha_1, \alpha_2 \in V$

$$\alpha_1 = \begin{pmatrix} x_1 & x_2+ix_3 \\ x_2-ix_3 & -x_1 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} y_1 & y_2+iy_3 \\ y_2-iy_3 & -y_1 \end{pmatrix}$$

$$\alpha_1+\alpha_2 = \begin{pmatrix} x_1+y_1 & (x_2+y_2)+i(x_3+y_3) \\ (y_2+x_2)-i(x_3+y_3) & -(x_1+y_1) \end{pmatrix} \in V$$

$$k\alpha_1 = \begin{pmatrix} kx_1 & kx_2+ikx_3 \\ kx_2-ikx_3 & -kx_1 \end{pmatrix} \in V$$

故是一线性空间

(2) 维$(V)=3$

其中 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 为一组基.

$$\forall A = \begin{pmatrix} x_1 & x_2+ix_3 \\ x_2-ix_3 & -x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + x_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_3\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

且 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$、$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$、$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 线性无关

故为一组基，且维$(V)=3$

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二、

解：① $L(\alpha_1,\alpha_2,\alpha_3)+L(\beta_1,\beta_2,\beta_3)=L(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)$

$$A=(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)=\begin{pmatrix}1&1&1&1&1&2\\1&1&2&2&-2&3\\0&-1&1&0&2&1\\2&3&-2&-6&4&-5\end{pmatrix}$$

$$A\to\begin{pmatrix}1&0&0&0&10&2\\0&1&0&0&-6&-1\\0&0&1&0&-4&0\\0&0&0&1&1&1\end{pmatrix}$$

维$(L(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3))=$ 秩$(A)=4$

$\alpha_1,\alpha_2,\alpha_3,\beta_1$ 为一组基

② $B=(\alpha_1,\alpha_2,\alpha_3)$ $C=(\beta_1,\beta_2,\beta_3)$

维$(V_1)=$ 秩$(B)=3$，维$(V_2)=$ 秩$(C)=3$

维$(V_1\cap V_2)=$ 维$(V_1)+$ 维$(V_2)-$ 维$(V_1+V_2)=2$

设 $\xi\in V_1\cap V_2$

$\xi=x_1\alpha_1+x_2\alpha_2+x_3\alpha_3=y_1\beta_1+y_2\beta_2+y_3\beta_3$

$$\begin{pmatrix}1&1&1&-1&-1&-2\\1&1&2&-2&2&-3\\0&-1&1&0&-2&-1\\2&3&-2&6&-4&5\end{pmatrix}\to\begin{pmatrix}1&0&0&0&-10&-2\\0&1&0&0&6&1\\0&0&1&0&4&0\\0&0&0&1&1&1\end{pmatrix}$$

基础解系为：$(10,-6,-4,-1,1,0),(2,-1,0,-1,0,1)$

$\xi_1=(10\alpha_1-6\alpha_2-4\alpha_3)^T=(10,-4,2,10)^T$

$\xi_2=(2\alpha_1-\alpha_2)^T=(1,1,1,1)^T$

ξ_1,ξ_2 为一组基

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三、(1)证：取 $P[x]_3$ 的一组基 $1, x, x^2$

$$(f_1(x), f_2(x), f_3(x)) = (1, x, x^2)\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = (1, x, x^2)A$$

$$(g_1(x), g_2(x), g_3(x)) = (1, x, x^2)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = (1, x, x^2)B$$

$r(A) = r(B) = 3$

故 $f_1(x), f_2(x), f_3(x)$ 为基，$g_1(x), g_2(x), g_3(x)$ 为基

(2) $(g_1(x), g_2(x), g_3(x)) = (f_1(x), f_2(x), f_3(x))A^{-1}B$

$$A^{-1}B = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

(3) $f(x) = 2 + 5x + 3x^2 = (1, x, x^2)\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$

$$B^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= (g_1(x), g_2(x), g_3(x))B^{-1}\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

$$= (g_1(x), g_2(x), g_3(x))\begin{pmatrix} 1 \\ -\frac{3}{2} \\ 2 \end{pmatrix}$$

四、(1)解：$(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3)\begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix} = (\alpha_1, \alpha_2, \alpha_3) B$

$$\mathcal{A}(\beta_1, \beta_2, \beta_3) = B^{-1}A = \begin{pmatrix} -\frac{1}{7} & -\frac{3}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{6}{7} & \frac{1}{7} \\ \frac{2}{7} & -\frac{1}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 4 & 11 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & -2 \\ 2 & 3 & -1 \end{pmatrix}$$

(2)解：$\alpha = (\alpha_1, \alpha_2, \alpha_3)\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$\mathcal{A}(\alpha) = (\beta_1, \beta_2, \beta_3) B^{-1}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$= (\beta_1, \beta_2, \beta_3)\begin{pmatrix} -\frac{8}{7} \\ \frac{23}{7} \\ \frac{2}{7} \end{pmatrix}$$

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五、(1) 解：$f(x_1, x_2, x_3) \to \begin{pmatrix} 1 & -2 & b \\ -2 & a & 4 \\ b & 4 & -2 \end{pmatrix} = A$

$f(y_1, y_2, y_3) \to \begin{pmatrix} -7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = B$

$|A| = |B| \Rightarrow -8 - 2a + 16b - ab^2 = -28$

$Tr(A) = Tr(B)$ 故 $\begin{cases} a = -2 \\ b = 2 \end{cases}$

且 $b < 2$

(2) 解：① 当 $\lambda = -7$ 时 $|\lambda E - A| = 0$

解得其基础解系：$\alpha_1 = (-1, -2, 2)$

$\eta_1 = \frac{\alpha_1}{|\alpha_1|} = \frac{1}{3}(-1, -2, 2)$

② 当 $\lambda = 2$ 时，$|\lambda E - A| = 0$

基础解系：$\alpha_2 = (2, 0, 1)$，$\alpha_3 = (-2, 1, 0)$

$\beta_2 = (2, 0, 1)$，$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_2$

$= (-\frac{2}{5}, 1, \frac{4}{5})$

$\eta_2 = \frac{1}{\sqrt{5}}(2, 0, 1)$ $\eta_3 = (-\frac{2}{3\sqrt{5}}, \frac{\sqrt{5}}{3}, \frac{4}{3\sqrt{5}})$

取 $X = \begin{pmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \end{pmatrix} Y$

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六、证：由 $(f(x),g(x))=1$ 知 $\exists u(x),v(x)$ 使 $u(x)f(x)+v(x)g(x)=1$

故 $u(\mathscr{A})f(\mathscr{A})+v(\mathscr{A})g(\mathscr{A})=E$

对 $\forall \xi\in V$ 有 $\xi=u(\mathscr{A})f(\mathscr{A})(\xi)+v(\mathscr{A})g(\mathscr{A})(\xi)$

由 $u(\mathscr{A})f(\mathscr{A})+v(\mathscr{A})g(\mathscr{A})=E$，知 $u(\mathscr{A})f(\mathscr{A})(\xi)=\{0\}=v(\mathscr{A})g(\mathscr{A})(\xi)$

所以 $u(\mathscr{A})f(\mathscr{A})(\xi)\in \ker(g(\mathscr{A}))$

$v(\mathscr{A})g(\mathscr{A})(\xi)\in \ker(f(\mathscr{A}))$

$\forall \xi\in \ker(f(\mathscr{A}))+\ker(g(\mathscr{A}))$

$f(\mathscr{A})(\xi)=0 \quad g(\mathscr{A})(\xi)=0$

则 $\ker(f(\mathscr{A}))\cap\ker(g(\mathscr{A}))=\{0\}$

则 $\ker(f(\mathscr{A})g(\mathscr{A}))=\ker(f(\mathscr{A}))\oplus\ker(g(\mathscr{A}))$

七、证：因为 A 正定，所以存在可逆矩阵 X 使 $X^TAX=E$

对实对称矩阵 X^TBX，存在正交矩阵 Y

有 $Y^T(X^TBX)Y$ 为对角

$Y^T(X^TBX)Y=(XY)^TB(XY)=Y^TY=E$

故令 $P=XY$ 即满足