Problems and examples (Physics 2021-22 CST)

1. **(Scaling).** The reasoning of a single cell must be in microscopic size to provide a rigid structure that would seem to contradict the fact that human nerve cells in the spinal cord can be as much as a meter long, although their widths are still very small. Why is this possible?

   Solution. We may let r and L be the radius and length of a tube for the calculation of the area-volume ratio:
   $$\frac{\text{Area}}{\text{Volume}} = \frac{2\pi r L}{\pi r^2 L} = \frac{2}{r}.$$
   The ratio is a constant since r is supposed to be a constant. Therefore, no matter how long a nerve cell is, the area-volume ratio is always the same. It implies that the length of a nerve cell does not affect the strength of the cell body.

2. **(Dimensional analysis).** Please use dimensional analysis to find the expression for centripetal force F. Suppose that the force F of a particle with mass m moving with uniform speed v in a circle of radius r is proportional to some powers of m, say m^a; r as r^b, and the v as v^c.

   Solution. We expression is simply
   $$F \propto m^a r^b v^c$$
   Using the method of dimensional analysis, we get
   $$\begin{aligned} MLT^{-2} &= M^a L^b (LT^{-1})^c \\ &= M^a L^{b+c} T^{-c}. \end{aligned}$$
   Thus, $a = 1$, $b = -1$, $c = 2$. The expression is
   $$F \propto \frac{mv^2}{r}.$$

3. **(Vector).** Find the angle between the vectors $\mathbf{F}_1 = 10\hat{\mathbf{i}} - 20.4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{F}_2 = -15\hat{\mathbf{i}} - 6.2\hat{\mathbf{k}}$.

   Solution. Rearranging the terms of the dot product formula we get
   $$\cos\theta = \frac{\mathbf{F}_1 \cdot \mathbf{F}_2}{|\mathbf{F}_1||\mathbf{F}_2|} = \frac{10(-15) + (-20.4)(0) + 2(-6.2)}{\sqrt{10^2 + 20.4^2 + 2^2}\sqrt{15^2 + 6.2^2}} = -0.439$$
   Thus $\theta = 116°$.

4. **(Vector).** What is the magnitude of the vector $\mathbf{A} = 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ in the direction of the vector $\mathbf{B} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$?

   Solution. That is, to find the dot product of $\mathbf{A} \cdot \hat{\mathbf{B}}$ where $\hat{\mathbf{B}}$ is the unit vector of $\mathbf{B}$. It is
   $$(2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) \cdot \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{\sqrt{1^2 + (-2)^2 + 3^2}} = -\frac{8}{\sqrt{14}}.$$

5. Please prove the cosine rule
   $$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$
   where R is the resultant of the vectors A and B with and acute angle θ between them.

Solution. Using the parallelogram rule and the Pythagorean theorem, the resultant R is followed by

$$\begin{aligned} R &= \sqrt{(A\cos\theta + B)^2 + A^2\sin^2\theta} \\ &= \sqrt{A^2\cos^2\theta + 2AB\cos\theta + B^2 + A^2\sin^2\theta} \\ &= \sqrt{A^2 + B^2 + 2AB\cos\theta} \end{aligned}$$

6. **(Velocity).** A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by the equation

$$y = 10000 - 50\left(t + 5\mathrm{e}^{-t/5}\right)$$

metre. Find her velocity at $t = 7.0$ s.

Solution. Differentiating y with respect to t gives

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 - 50\frac{\mathrm{d}}{\mathrm{d}t}\left(t + 5\mathrm{e}^{-t/5}\right) \\ &= -50(1 - \mathrm{e}^{-t/5}) \end{aligned}$$

When $t = 7$, her velocity is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=7} = -50(1 - \mathrm{e}^{-7/5}) = -37.67\mathrm{ms}^{-1}$$

The negative sign indicates her moving downward.

7. **(Acceleration).** A child throws a marble into the air with an initial speed v_t. Another child drops a ball at the same instant. Compare the accelerations of the two objects while they are in flight.

Solution. We only need to consider the vertical components of the both cases since the horizontal acceleration of them are zero.

For the marble, after it is released from the hand there is no other external force (acceleration) to act on it except gravity, no matter how the boy throws the marble to any direction. The second object—the ball obviously falls by gravity. Therefore, the accelerations of the two objects in air are the same—gravity.

8. **(Acceleration).** Alice has a position as a function of time given by $x = a/(b + t^2)$ where a, b are positive constants. Find her maximum speed.

Solution. Differentiate the function once and twice with respect to time for the velocity and acceleration:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{2at}{(b + t^2)^2} \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= a\frac{(b + t^2)^2(-2) - (-2t)[2(b + t^2)2t]}{(b + t^2)^4} \end{aligned}$$

Put $\mathrm{d}^2x/\mathrm{d}t^2 = 0$ for the maximum velocity, we obtain

$$b + t^2 = 0 \quad \text{or} \quad b - 3t^2 = 0$$

(Actually we need to differentiate the acceleration again evaluated at that time to verify if it is concave upward or download for the minimum or maximum.) Rejecting the first solution, since b is supposed to be positive the first equation will give imaginary time (unphysical). Thus, at

$$t = \sqrt{\frac{b}{3}}$$

gives her maximum velocity:

$$\frac{dx}{dt} = \frac{-2a\sqrt{b/3}}{(b + b/3)^2} = -a\left(\frac{4b}{3}\right)^{-3/2}.$$

9. **(Velocity).** In good weather the drive from City A to City B on the highway takes 2 hours 47 minutes at an average speed of 115 km/h. In a rainy day, however, it is not unusual to average only 75 km/h. How long would the trip take at this average speed?

   Solution. Let t be the time taken for the trip in a rainy day drive. Remember to change the minute to normal decimal, thus

   $$t = \frac{115}{75}\left(2 + \frac{47}{60}\right) = 4.27 \text{ hours}$$

   In terms of minutes it is

   $$4 \text{ hr } + 27 \times \frac{60}{100} \text{ min} = 4 \text{ hours } 16 \text{ minutes.}$$

10. **(Kinematics).** (a) Calculate the height of a cliff if it takes 2.35 s for a rock to hit the ground when it is thrown straight up from the cliff with an initial velocity of 8.00 m/s. (b) How long would it take to reach the ground if it is thrown straight down with the same speed?

    Solution. (a) As usual, positive for upward, negative for downward we have the height of the cliff

    $$s = 8 \times 2.35 + \frac{1}{2}(-9.8) \times 2.35^2 = -8.26 \text{ m.}$$

    (b) The time taken is

    $$-8.26 = -8t + \frac{1}{2}(-9.8)t^2 \quad \Rightarrow \quad t = 0.42 \text{ s.}$$

11. **(Kinematics).** Show that throwing an object straight up spends as much time rising as it does falling back to its starting point. *(Hint: Set up two equations for the upward direction, and one equation for the downward.)*

    Solution. Let positive s be the displacement the ground to the highest position, positive u the initial velocity, t_1, t_2 are the time taken for the upward and downward intervals. There are two upward suvat-equations

    $$\begin{aligned} s &= ut_1 + \frac{1}{2}gt_1^2, \\ 0 &= u + gt_1, \end{aligned}$$

and one downward equation

$$-s = 0 + \frac{1}{2}gt_2^2.$$

Summing the first and the third equations to eliminate s, then inserting into the second to eliminate u gives

$$-gt_1^2 + \frac{1}{2}g(t_1^2 + t_2^2) = 0$$

It follows $t_1 = t_2$.

12. **(Projectile).** A stone is thrown from the top of a building upward at an angle of 30° to the horizontal and with an initial speed of 20 m/s, as shown in Figure 1. If the height of the building is 45 m,

    (a) how long is it before the stone hits the ground?

    (b) What is the speed of the stone just before it strikes the ground?

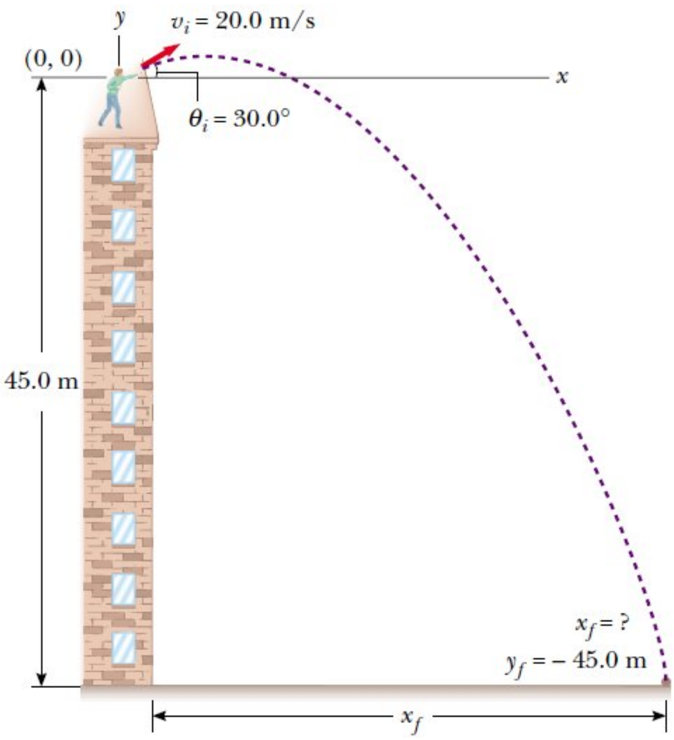


Figure 1:

Solution. (a) The initial x and y components of the stone's velocity are

$$v_{xi} = v_i \cos\theta_i = 20 \times \cos 30 = 17.3 \text{ m/s},$$
$$v_{yi} = v_i \sin\theta_i = 20 \times \sin 30 = 10 \text{ m/s}.$$

To find the time taken t, we can directly use $y_f = v_{yi}t + \frac{1}{2}a_y t^2$ with $y_f = -45$ m, $a_y = -g$, and $v_{yi} = 10$ m/s:

$$-45 = 10t - \frac{1}{2} \times 9.8 \times t^2$$

Solving the quadratic equation for t gives, for the positive root, $t = 4.22$ second.

(b) We can use $v_{yf} = v_{yi} + a_y t$, with $t = 4.22$ s to obtain the y component of the velocity just before the stone strikes the ground:

$$v_{yf} = 10 - 9.8 \times 4.22 = -31.4 \text{ m/s}.$$

Since $v_{xf} = v_{xi} = 17.3$ m/s, the required speed is

$$v_f = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(17.3)^2 + (-31.4)^2} = 35.9 \text{ m/s}.$$

13. **(Projectile).** A football player punts the ball at a 45.0° angle. Without an effect from the wind, the ball would travel 60.0 m horizontally. (a) What is the initial speed of the ball? (b) When the ball is near its maximum height it experiences a brief gust of wind that reduces its horizontal velocity by 1.50 m/s. What distance does the ball travel horizontally?

    Solution. (a) Using the equation $R_{\max} = v^2 \sin 2\theta/g$, the launching speed is $\sqrt{60 \times 9.8} = 24.25 \text{ ms}^{-1}$ with 45° launching angle. (b) The horizontal velocity is supposed being $24.25 \cos 45 = 17.15 \text{ ms}^{-1}$, it becomes $17.15 - 1.50 = 15.65 \text{ ms}^{-1}$ in its second half journey. The actual distance travel is

    $$\frac{60}{2} + \frac{15.65}{17.15} \times \frac{60}{2} = 57.4 \text{ m}.$$

    We have used the simple ratio for the distance, as the inertial horizontal motion is linear.

14. **(Terminal velocity).** Calculate the radius of a parachute that will slow a 70-kg parachutist to a terminal velocity of 14 m/sec.

    Solution. Assuming the parachute has no mass. Let R be the radius of the parachute. The terminal velocity is given by

    $$v_t = \sqrt{\frac{W}{CA}} = \sqrt{\frac{W}{CR^2\pi}}$$

    where W, C, A are the weight of the parachutist, coefficient of air resistance, and the normal surface area against the falling direction. Rearranging the terms of the equation we have

    $$R = \sqrt{\frac{W}{C\pi v_t^2}}.$$

    Substituting the data into the equation we obtain

    $$R = \sqrt{\frac{70 \times 10}{0.88\pi \times 14^2}} \approx 1.1 \text{ metre}.$$

15. **(Projectile).** In the standing broad jump, one squats and then pushes off with the legs to see how far one can jump. Suppose the extension of the legs from the crouch position is 0.6 m and the acceleration achieved from this position is 1.3 times the acceleration due to gravity, g. How far can she/he jump?

Solution. You have got the detail explanation of this type of problem in the lecture notes and examples. The launching velocity is $\sqrt{2(1.3 \times 9.8) \times 0.6} = 3.91 \text{ ms}^{-1}$. Using the formula $R = \dfrac{v^2 \sin 2\theta}{g}$ with setting $\theta = 45°$ for the maximum range $R_{\max}$, we get

$$R_{\max} = \frac{3.91^2}{9.8} = 1.56 \text{ m}.$$

16. **(Projectile).** A long-jumper leaves the ground at an angle of 20° above the horizontal and at a speed of 11 ms^{-1}. (a) How far does he jump in the horizontal direction? (b) What is the maximum height reached?

Solution. (a) As discussed in the lecture we write down the horizontal $R = 11 \cos 20\, t$ and the vertical $y = 11 \sin 20\, t + (-9.8)t^2/2$ component equations. We obtain the range

$$R = \frac{11^2 \sin(2 \times 20)}{9.8} = 7.94 \text{ m}.$$

The time interval for the maximum height is just half of time of the whole journey by symmetry. We may directly solve for t from the above equations again, or by differentiating the vertical equation with respect to time then setting it to zero:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(11 \sin 20\, t + \frac{1}{2}(-9.8)t^2\right) = 0 \quad \text{gives} \quad t = \frac{11 \sin 20}{9.8} = 0.384 \text{ s}.$$

The maximum height is

$$11 \sin 20(0.384) + \frac{1}{2}(-9.8)(0.384)^2 = 0.722 \text{ m}.$$

17. **(Force).** A helicopter of mass m is taking off vertically. The only forces acting on it are the earth's gravitational force and the force, F, of the air pushing up on the propeller blades.

(a) If the helicopter lifts off at $t = 0$, what is its vertical speed at time t?

(b) Plug numbers into your equation from part (a), using $m = 2300$ kg, $F = 27000$ N, and $t = 4.0$ s.

Solution. (a) The net force of the system is

$$F - W = ma \quad \Rightarrow \quad a = \frac{F - W}{m}$$

where W, a are the weight and the acceleration of the helicopter. Setting $u = 0$ we have

$$v = u + at \quad \Rightarrow \quad v = \frac{(F - W)t}{m}.$$

(b)

$$v = \frac{(27000 - 2300 \times 9.81) \times 4}{2300} = 7.7 \text{ ms}^{-1}.$$

18. **(Force).** Two blocks of masses m_1 and m_2 are placed in contact with each other on a frictionless horizontal surface. A constant horizontal force $\mathbf{F}$ is applied to the block of mass m_1. (a) Determine the magnitude of the acceleration of the two-block system. (b) Determine the magnitude of the contact force between the two blocks.

Solution. (a) Since $\mathbf{F}$ is horizontal, from Newton's second law we have

$$\mathbf{F} = (m_1 + m_2)\mathbf{a} \quad \Rightarrow \quad \mathbf{a} = \frac{\mathbf{F}}{m_1 + m_2}.$$

(b) Let the forces between the blocks are $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively. The resultant force of the whole system is

$$\mathbf{F} + \mathbf{F}_1 + \mathbf{F}_2 = (m_1 + m_2)\mathbf{a}.$$

But in (a) we know that $\mathbf{F} = (m_1 + m_2)\mathbf{a}$. Therefore

$$\mathbf{F}_2 = -\mathbf{F}_1.$$

The force $\mathbf{F}_2$ acting on the second block m_2 is just

$$\mathbf{F}_2 = m_2\mathbf{a} = -\mathbf{F}_1.$$

The interaction between the blocks only depends on the mass in front of, namely the m_2. Written in terms of the applying force, the interaction is

$$\mathbf{F}_2 = \frac{m_2}{m_1 + m_2}\mathbf{F}.$$

19. **(Friction).** A block of mass m_1 on a rough, horizontal surface is connected to a ball of mass m_2 by a lightweight cord over a lightweight, frictionless pulley, as shown in Figure 2. A force of magnitude F at an angle θ with the horizontal is applied to the block as shown. The coefficient of kinetic friction between the block and surface is μ_k. Determine the magnitude of the acceleration of the two objects.

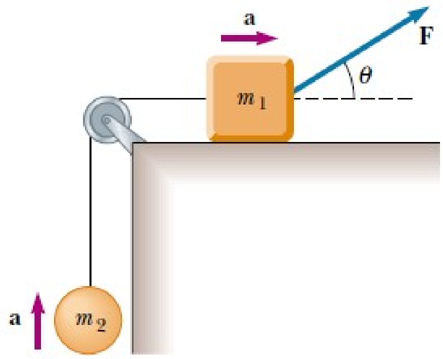


Figure 2:

Solution. The applied force $\mathbf{F}$ has x and y components $F\cos\theta$ and $F\sin\theta$, respectively. Applying Newton's second law to both objects and assuming the motion of the block is to the right, we obtain the equations of motion of the block:

$$\sum F_x = F\cos\theta - f_k - T = m_1 a_x = m_1 a, \tag{1}$$
$$\sum F_y = n + F\sin\theta - m_1 g = m_1 a_y = 0 \tag{2}$$

where f_k and n are the kinetic friction and the normal reaction forces. The equations of motion of the ball:

$$\sum F_x = m_2 a_x = 0,$$
$$\sum F_y = T - m_2 g = m_2 a_y = m_2 a. \tag{3}$$

Because the two objects are connected, we can equate the magnitudes of the x component of the acceleration of the block and the y component of the acceleration of the ball. Since $f_k = \mu_k n$, and $n = m_1 g - F\sin\theta$, we obtain

$$f_k = \mu_k(m_1 g - F\sin\theta). \tag{4}$$

That is, the frictional force is reduced because of the positive y component of $\mathbf{F}$. Substituting (4) and the value of T from (3) into (1) gives

$$F\cos\theta - \mu_k(m_1 g - F\sin\theta) - m_2(a+g) = m_1 a.$$

Solving for a, we obtain

$$a = \frac{F(\cos\theta + \mu_k\sin\theta) - g(m_2 + \mu_k m_1)}{m_1 + m_2}.$$

20. **(Centripetal acceleration).** Calculate the centripetal acceleration of a point 7 cm from the axis of an ultracentrifuge spinning at 7.5×10^4 rev/min. Determine the ratio of this acceleration to that due to gravity.

Solution. The centripetal acceleration is

$$a = r\dot{\theta}^2 = 7\times10^{-2}\times\left(\frac{7.5\times10^4\times2\pi}{60}\right) = 4.318\times10^6\ \mathrm{ms}^{-2}.$$

The ratio of it to the gravitational acceleration by Earth is

$$\frac{4.318\times10^6}{9.8} = 4.4\times10^5.$$

21. **(Centripetal force).** A pilot of mass m in a jet aircraft executes a loop-thee-loop, as shown in Figure 3. In this manoeuvre, the aircraft moves in a vertical circle of radius 2.7 km at a constant speed of 225 m/s. Determine the force exerted by the seat on the pilot (a) at the bottom of the loop and (b) at the top of the loop. Express your answers in terms of the weight of the pilot mg.

Solution. (a) The net force is the product of the mass and the centripetal acceleration:

$$n_{\mathrm{bot}} - mg = m\frac{v^2}{r}$$

Substituting the values given for the speed and radius gives

$$n_{\mathrm{bot}} = mg\left(1 + \frac{v^2}{rg}\right) = mg\left(1 + \frac{225}{2.7\times10^3\times9.8}\right) = 2.91\,mg.$$

Hence, the magnitude of the force $\mathbf{n}_{\mathrm{bot}}$ exerted by the seat on the pilot is greater than the weight of the pilot by a factor of 2.91, i.e., he experiences an apparent weight that is greater than his true weight by a factor of 2.91. (b) On the other hand, the net force equation for the top case is

$$n_{\mathrm{top}} + mg = m\frac{v^2}{r}.$$

Thus,

$$n_{\mathrm{top}} = mg\left(\frac{v^2}{rg} - 1\right) = mg\left(\frac{225}{2.7\times10^3\times9.8} - 1\right) = 0.913\,mg.$$

The magnitude of the force exerted by the seat on the pilot is less than his true weight by a factor of 0.913, and the pilot feels lighter.

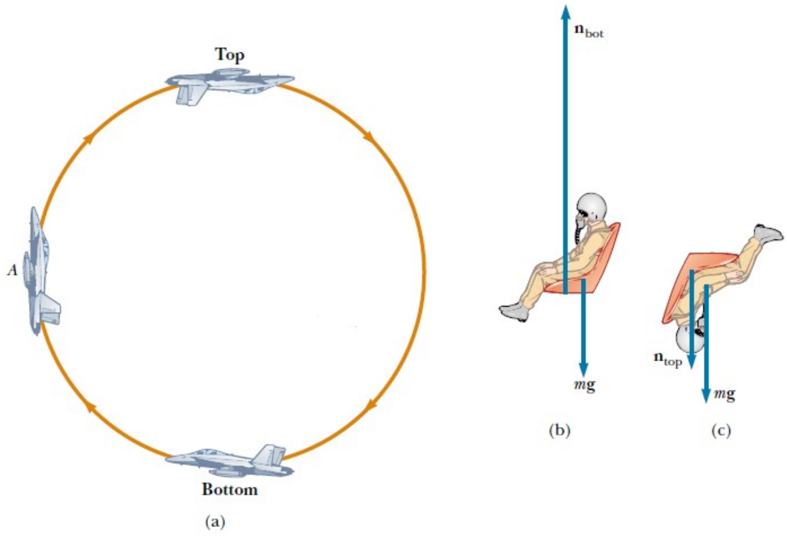


Figure 3:

22. **(Centripetal force).** A ball of mass 0.50 kg is attached to the end of a cord 1.50 m long. The ball is whirled in a horizontal circle . If the cord can withstand maximum tension of 50.0 N, what is the maximum speed the ball can attain before the cord breaks? Assume that the string remains horizontal during the motion.

   Solution. The equation for uniform horizontal circular motion is

$$\frac{mv^2}{r} = T.$$

   Now putting $T = 50$ N for the maximum speed, we get

$$\frac{0.5v_{\max}^2}{1.5} = 50 \quad \Rightarrow \quad v_{\max} = 12.25 \text{ ms}^{-1}$$

23. **(Centripetal force).**

   (a) Consider a satellite of mass m moving in a circular orbit round the Earth at a constant speed v and at an altitude h above the Earth's surface, as illustrated in Figure 4. Determine the speed of the satellite in terms of G, h, R_E (the radius of the Earth), and M_E (the mass of the Earth).

   (b) The satellite is in a circular orbit around the Earth at an altitude of 1000 km. The radius of the Earth is equal to 6.37×10^6 m, and its mass is 5.98×10^{24} kg. Find the speed of the satellite, and then find the period. ($G = 6.673 \times 10^{-11}$ Nm^2/kg^2)

   Solution. (a) The only external force acting on the satellite is gravity, which acts toward the centre of the Earth and keeps the satellite in its circular orbit. Therefore,

$$\frac{GM_Em}{r^2} = \frac{mv^2}{r},$$

   gravity is the source of the centripetal force, otherwise, the satellite would move to a straight line and leave away the Earth. Solving for v and because $r = R_E + h$, we obtain

$$v = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{GM_E}{R_E + h}}.$$

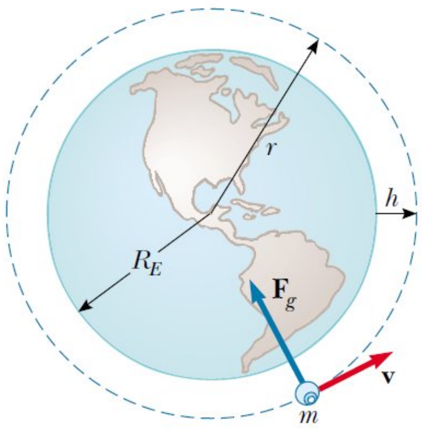


Figure 4:

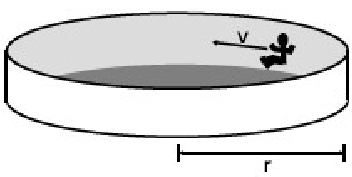
(b) Directly substituting the data into above equation, it gives

$$v = \sqrt{\frac{6.673 \times 10^{-11} \times 5.98 \times 10^{24}}{6.37 \times 10^{6} + 10^{6}}} = 7.36 \times 10^{3} \text{ m/s}.$$

The time taken for one complete revolution (period) is given by

$$\begin{aligned} T &= \frac{2\pi(R_E + h)}{v} = 2\pi\sqrt{\frac{(R_E + h)^3}{GM_E}} \\ &= 2\pi\sqrt{\frac{(6.37 \times 10^{6} + 10^{6})^3}{6.673 \times 10^{-11} \times 5.98 \times 10^{24}}} \\ &= 6.29 \times 10^{3} \text{ sec.} \end{aligned}$$

24. **(Centripetal force).** Consider the carnival ride in which the riders stand against the wall inside a large cylinder. As the cylinder rotates, the floor of the cylinder drops and the passengers are pressed against the wall by the centrifugal force. Assuming that the coefficient of friction between a rider and the cylinder wall is 0.6 and that the radius of the cylinder is 5 m, what is the minimum angular velocity of the cylinder that will hold the rider firmly against the wall?



Solution. Let N be the normal reaction force perpendicular to the wall toward the centre of the cylinder. The vertical and horizontal equations for equilibrium (the rider does not fall) are:

$$mr\dot{\theta}^2 = N, \quad mg = \mu N$$

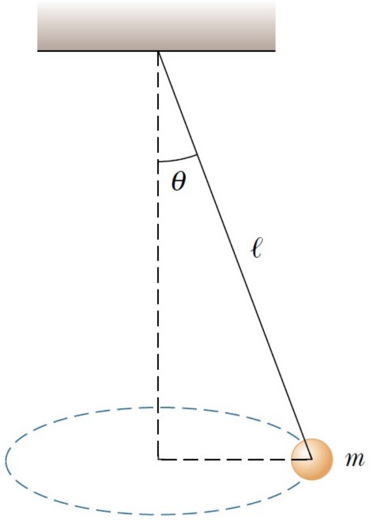
where $\dot{\theta}$ is the angular velocity, μ the coefficient of static friction between the rider and the wall. Solving for $\dot{\theta}$ we have

$$\dot{\theta} = \sqrt{\frac{g}{r\mu}}.$$

Putting $g = 9.8$, $r = 5$ and $\mu = 0.6$ into the equation we obtain

$$\dot{\theta} = \sqrt{\frac{9.8}{5 \times 0.6}} = 1.8 \text{ rad/sec.}$$

25. **(Centripetal force).** A conical pendulum consists of a bob of mass m in motion in a circular path in a horizontal plane. During the motion, the supporting wire of length l maintains the constant angle θ with the vertical. Show that the magnitude of the



angualr momentum of the mass about the centre of the circle is

$$L = \sqrt{\frac{m^2 g l^3 \sin^4\theta}{\cos\theta}}.$$

Solution. Let T be the tension along the cord, the equation for the plane circular motion is

$$T \sin\theta = \frac{mv^2}{r}.$$

Since the angular momentum $L = mvr$ and $r = l\sin\theta$, above equation may simply be rewritten as

$$T = \frac{L^2}{ml^3\sin^4\theta}.$$

Using the vertical equation $T\cos\theta = mg$, eliminating T for L we get

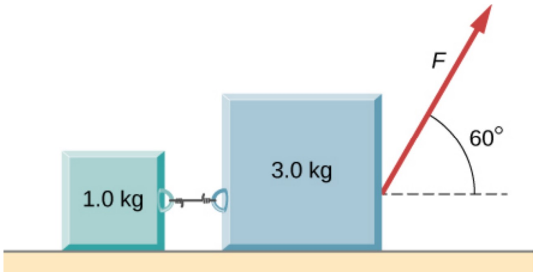
$$\frac{mg}{\cos\theta} = \frac{L^2}{ml^3\sin^4\theta} \quad\Rightarrow\quad L = \sqrt{\frac{m^2 g l^3 \sin^4\theta}{\cos\theta}}$$

26. **(Friction).** Two blocks connected by a string are pulled across a horizontal surface by a force applied to one of the blocks, as shown below. The coefficient of kinetic friction between the blocks and the surface is 0.25. If each blcok has an acceleration of 2.0 ms^{-2} to the right, what is the magnitude F of the applied force?

Solution. The equation of motion of the system is

$$F\cos 60 - \mu(3g - \mu F\sin 60) - \mu g = (1+3)a$$

Inserting the given data the force is $F = 24.8$ N.



27. **(Energy).** Compare the work required to accelerate a car of mass 2000 kg from 30.0 to 40.0 km/h with that required for an acceleration from 50.0 to 60.0 km/h.

Solution. Although in both cases the velocity differences are the same, the kinetic energies required to bring them to the desired velocities are different owing to the nonlinear proportionality of increment. The explicit calculation is

$$\frac{1}{2}2000(40^2-30^2)\left(\frac{10^3}{60^2}\right)^2 = 54012 \approx 54 \text{ kJ},$$
$$\frac{1}{2}2000(60^2-50^2)\left(\frac{10^3}{60^2}\right)^2 = 84876 \approx 85 \text{ kJ}$$

28. **(Gravitational potential).** A particle of mass m is displaced through a small vertical distance h near the earth's surface. Show that in this situation the general expression for the change in gravitational potential energy reduces to the familiar relationship $U = mgh$.

Solution. Since
$$U = -GMm\left(\frac{1}{r_f}-\frac{1}{r_i}\right) = GMm\left(\frac{r_f-r_i}{r_ir_f}\right)$$
where r_i, r_f are the initial and final altitudes from the centre of the earth. Because both the initial and final positions of the particle are close to the earth's surface, then

$$r_f - r_i = h \quad \text{and} \quad r_ir_f \approx R_E^2$$

where R_E is the radius of the earth. Since $g = GMm/R_E^2$, we obtain

$$U \approx \frac{GMmh}{R_E^2} = mgh.$$

29. **(Energy).** Engineers desire to model the magnitude of the elastic force of a bungee cord using the equation

$$F(x) = a\left[\frac{x+9}{9} - \left(\frac{9}{x+9}\right)^2\right],$$

where x is the stretch of the cord along its length and a is a constant. If it takes 22.0 kJ of work to stretch the cord by 16.7 m, determine the value of the constant a.

Solution. Integrating the given force function over the stretched length gives the energy

$$
\begin{aligned}
22 \times 10^3 &= \int_0^{16.7} a \left[\frac{x+9}{9} - \left(\frac{9}{x+9} \right)^2 \right] \mathrm{d}x \\
&= a \left[\frac{x^2}{18} + x + \frac{81}{x+9} \right]_0^{16.7} \\
&= a \left(\frac{16.7^2}{18} + 16.7 + \frac{81}{25.7} - 9 \right)
\end{aligned}
$$

which gives $a = 835$ N.

30. **(Gravitational acceleration).** Assuming the density of Earth is constant, show that the gravitational acceleration being a function of the radial distance by

$$g(r) = \frac{GMr}{R^3}$$

where G, M, R and r $(r < R)$ are the gravitational constant, mass of Earth, radius of Earth and the radial distance from the centre.

Solution. (a) Let $m(r)$ be the mass of the interior sphere with radius r, which is written as

$$m(r) = M \frac{r^3}{R^3}$$

Thus, the gravitational acceleration at that level with radius r is

$$g(r) = \frac{Gm(r)}{r^2} = \frac{G}{r^2} M \frac{r^3}{R^3} = \frac{GMr}{R^3}.$$

31. **(Momentum).** A very massive object with velocity v collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity $2v$. (Hint: Use the centre of mass frame of reference.)

Solution. In one-dimensional elastic collision, we have the expressions of conservation of momentum and conservation of kinetic energy:

$$
\begin{aligned}
m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2, \\
\frac{m_1 u_1^2}{2} + \frac{m_2 u_2^2}{2} &= \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2}
\end{aligned}
$$

where m_1, m_2 are the masses, and u_1, u_2 are the initial velocities before the collision, v_1, v_2 are the final velocities after the collision. Solving these simultaneous equations for the v_1, v_2 we get

$$
\begin{aligned}
v_1 &= \frac{u_1(m_1 - m_2) + 2m_2 u_2}{m_1 + m_2}, \\
v_2 &= \frac{u_2(m_2 - m_1) + 2m_1 u_1}{m_1 + m_2}.
\end{aligned}
$$

In our case we set $m_1 \gg m_2$, and $u_1 = v$, $u_2 = 0$. Using the second formula for the v_2,

$$v_2 = \frac{2m_1 v}{m_1 + m_2} \simeq 2v.$$

32. **(Impulse).** A car traveling at 27 m/s collides with a building. The collision with the building causes the car to come to a stop in approximately 1 second. The driver, who weighs 860 N, is protected by a combination of a variable-tension seat-belt and an airbag. The airbag and seat-belt slow his velocity, such that he comes to a stop in approximately 2.5 s.

   (a) What average force does the driver experience during the collision?

   (b) Without the seat-belt and airbag, his collision time would have been approximately 0.20 s. What force would he experience in this case?

   Solution. (a) The average force or the impulsive force is

$$F_{\mathrm{av}} = \frac{860}{9.8} \times (0 - 27) \times \frac{1}{2.5} = -948 \text{ N}.$$

   (b) Without the airbag and the seatbelt his collision time becomes $1 + 0.2 = 1.2$ s. The impulsive force is

$$F_{\mathrm{av}} = \frac{860}{9.8} \times (0 - 27) \times \frac{1}{1.2} = -1975 \text{ N}$$

33. **(Moment of inertia).** A circular hoop of mass m and radius r spins like a wheel while its centre remains at rest. Its period is T. Find the moment of inertia of the hoop. Show that its kinetic energy equals $2\pi^2 mr^2/T^2$.

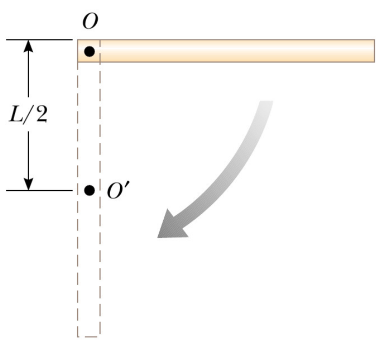
   Solution. Let ρ be the mass per unit length of the hoop. The moment of inertia of it is

$$I = \int_0^{2\pi} r^2 \cdot \rho r d\theta = 2\pi\rho r^3 = mr^2.$$

   The rotational kinetic energy is thus

$$\mathrm{KE}_R = \frac{1}{2} I\dot{\theta}^2 = \frac{1}{2} mr^2 \left(\frac{2\pi}{T}\right)^2 = \frac{2\pi^2 mr^2}{T^2}.$$

34. **(Rigid body motion).** A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end as shown. The rod is released from rest in the horizontal position. (a) What is its angular speed when it reaches its lowest position? (b) Determine the linear speed of the centre of mass v_{cm} and the linear speed of the lowest point on the rod when it is in the vertical position.



Solution. (a) At the moment when the rod rotates to the shown position the kinetic energy due to rotation is equal to the potential energy of the rod before it released;

$$\frac{1}{2}MgL = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2$$

where we have used the fact $I = ML^2/3$. Therefore the angular speed is

$$\omega = \sqrt{\frac{3g}{L}}.$$

(b) The linear speed of the centre of mass is

$$v_{\text{cm}} = \frac{L}{2}\omega = \frac{1}{2}\sqrt{3gL}.$$

Because the lowest point has the double linear speed of the point CM, it is

$$2v_{\text{cm}} = \sqrt{3gL}.$$

35. **(Angular momentum).** A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley, as shown in Figure 5. The radius of the pulley is R, and the moment of inertia about its axle is I. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

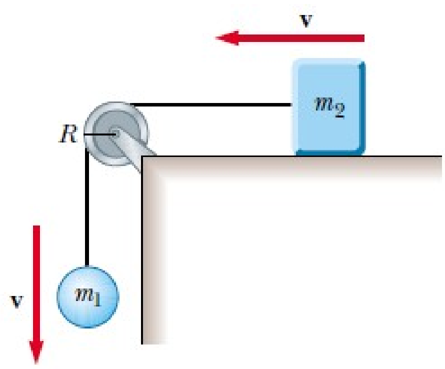


Figure 5:

Solution. At the instant the sphere and block have a common speed v, the angular momentum of the sphere is m_1vR, and that of the block is m_2vR. At the same instant, the angular momentum of the pulley is $I\omega = Iv/R$. Hence, the total angular momentum of the system is

$$L = m_1vR + m_2vR + \frac{Iv}{R}. \tag{5}$$

Now let us evaluate the total external torque acting on the system about the pulley axle. Because it has a moment arm of zero, the force exerted by the axle on the pulley does not contribute to the torque. Furthermore, the normal force acting on the block is balanced by the force of gravity m_2g, and so these forces do not contribute to the torque. The force of gravity m_1g acting on the sphere produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. This is the total external torque about the pulley axle; that is, $\sum \tau = m_1gR$. Using this result with (5), we find

$$\begin{aligned}\sum \tau &= \frac{\mathrm{d}L}{\mathrm{d}t} \\ m_1gR &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(m_1+m_2)Rv + \frac{Iv}{R}\right] \\ &= (m_1+m_2)R\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{I}{R}\frac{\mathrm{d}v}{\mathrm{d}t}.\end{aligned}$$

Because $\mathrm{d}v/\mathrm{d}t = a$, thus

$$a = \frac{m_1g}{(m_1+m_2) + I/R^2}.$$

36. **(Angular momentum).** A disk has radius r and mass M at rest mounted with its axis vertical. A bullet of mass m and velocity v which is fired horizontally and tangential to the disk lodges in the perimeter of the disk. What is the angular velocity of the disk?

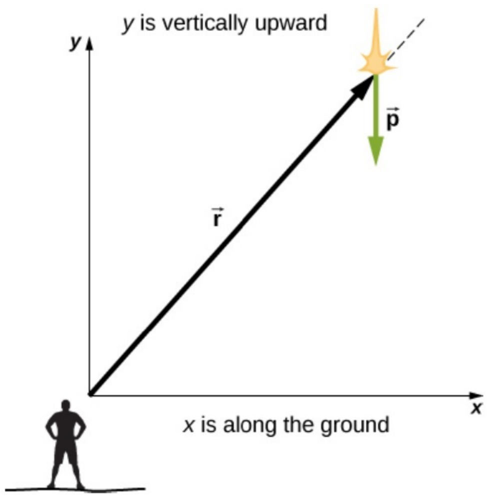
Solution. The problem is similar to some linear inelastic collision problems, here we shall use angular momentum conservation instead. The angular momentum of the bullet relative to the axis of the disk is mvr. After the collision we have

$$mvr = mR^2\dot{\theta} + I\dot{\theta}.$$

Putting $I = Mr^2/2$ for a disk we get

$$\dot{\theta} = \left(\frac{2m}{2m+M}\right)\frac{v}{r}.$$

37. **(Angular momentum).** A meteor enters Earth's atmosphere and is observed by someone on the ground by the position vector $\mathbf{r} = 25\hat{\mathbf{i}} + 28\hat{\mathbf{j}}$ km before it burns up in the atmosphere. At the instant the observer sees the meteror, it has linear momentum $\mathbf{p} = -30\hat{\mathbf{j}}$ kg km s^{-1}, with a mass 16 kg and a constant acceleration $-2\hat{\mathbf{j}}$ ms^{-2} along its path. (a) What is the angular momentum of the meteor observed by the person? (c) What is the torque on the meteor about the origin (observer)?



Solution. (a) The angular momentum about the observer is the cross product of position vector by the linear momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = (25\hat{\mathbf{i}} + 28\hat{\mathbf{j}})(10^3) \times (-30\hat{\mathbf{j}})(10^3) = -7.5 \times 10^8\hat{\mathbf{k}} \text{ kg ms}^{-1}$$

(b) The torque about the observer is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = (25\hat{\mathbf{i}} + 28\hat{\mathbf{j}})(10^3) \times 16(-2\hat{\mathbf{j}}) = 8 \times 10^5\hat{\mathbf{k}} \text{ Nm}$$

Note. This problem shows that a linear motion may also be viewed as a rotation about some point of reference.

38. **(Rotation).** A wheel 1 m in diameter rotates with an angular acceleration of 4 rad s^{-2}. (a) If the wheel's initial angular velocity is 2.0 rad s^{-1}, what is its angular velocity after 10 s? (b) Through what angle does it rotate in the 10 s interval? (c) What are the tangential speed and acceleration of a point on the rim of the wheel at the end of the 10 s interval?

Solution. (a) Using the formula $\dot{\theta}_f = \dot{\theta}_i + \ddot{\theta}t$, we get

$$\dot{\theta}_f = 2 + 4(10) = 42 \text{ rad s}^{-1}.$$

(b) The rotated angle is

$$\Delta\theta = \dot{\theta}_i t + \frac{1}{2}\ddot{\theta}t^2 = 2 \times 10 + \frac{1}{2} \times 4 \times 10^2 = 220 \text{ rad}.$$

(c) The tangential speed is

$$r\dot{\theta}_f = 0.5 \times 42 = 21 \text{ ms}^{-1},$$

and the tangential acceleration is

$$r\ddot{\theta} = 0.5 \times 4 = 2 \text{ ms}^{-2}.$$

39. **(Pressure).** In about 1657 Otto von Guericke, inventor of the air pump, evacuated a sphere made of two brass hemispheres (Figure 6). Two teams of eight horses each could pull the hemispheres apart only on some trials, and then "with greatest difficultly," with the resulting sound likened to a cannon firing. (a) Show that the force F required to pull the evacuated hemispheres apart is $\pi R^2(P_0 - P)$, where R is the radius of the hemispheres and P is the pressure inside the hemispheres, which is much less than P_0. (b) Determine the force if $P = 0.1P_0$ and $R = 0.3$m.

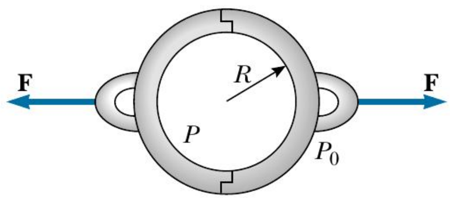


Figure 6:

Solution. The half of the surface area of the interior of the sphere is $2\pi R^2$. The total pressure produced by the forces equals to the difference of atmospheric pressure P_0 and the interior pressure P

$$\frac{F}{2\pi R^2} + \frac{F}{2\pi R^2} = P_0 - P \quad \Rightarrow \quad F = \pi R^2(P_0 - P)$$

Putting all the numbers into the formula one should obtain 2.564×10^4 N.

40. **(Pressure).** You can chew through very tough objects with your incisors because they exert a large force on the small area of a pointed tooth. What pressure can you create by exerting a force of 500 N with your tooth on an area of 1 mm^2?

    Solution. The pressure is $500/(1 \times 10^{-6}) = 5 \times 10^8\ \text{Nm}^{-2}$.

41. **(Pressure).** A U-tube originally containing mercury has water added to one arm to a depth of 20 cm. What is the pressure at the water-mercury interface? What is the height of the mercury column as measured from the water mercury level? (density of mercury: 14,000 kg/m^3)

    Solution. Since we would like to find the gauge value of pressure only, we may simply ignore the atmospheric pressure in our calculation. (Atmospheric pressure P_0 will be canceled out on both sides of the tube.) The gauge pressure at the water-mercury interface is

    $$1000 \times 9.8 \times 0.2 = 1960 \text{ Pa}.$$

    This pressure is balanced by the pressure due to mercury, i.e., $1960 = \rho_m g h_m$. Therefore,

    $$h_m = \frac{1960}{14000 \times 9.8} = 0.14 \text{ m}.$$

42. **(Viscosity).** A small spherical particle falls in a liquid against the buoyant force and the drag force which is assumed to be given by the Stokes law, $F_s = 6\pi r\eta v$. Show that the terminal speed is given by $v = \dfrac{2R^2 g}{9\eta}(\rho_s - \rho_l)$, where R is the radius of the sphere, ρ_s is its density, and ρ_l is the density of the fluid and η the coefficient of viscosity.

    Solution. The equation of motion of the particle is $W - B - F_s = ma$ where W, B, F_s are the weight of the particle, the buoyancy and the drag force against it. The equation becomes

    $$\rho_s(4\pi R^3/3)g - \rho_l(4\pi R^3/3)g - 6\pi R\eta v = \rho_s(4\pi R^3/3)a$$

    The particle attains the terminal velocity when $a = 0$, thus

    $$\frac{2}{9}R^2(\rho_s - \rho_l)g - \eta v = 0 \quad \Rightarrow \quad v = \frac{2R^2 g}{9\eta}(\rho_s - \rho_l)$$

43. **(Buoyancy).** A solid sphere has a diameter of 1.2 cm. It floats in water with 0.4 cm of its diameter above water level (Figure 7). Determine the density of the sphere.

    Solution. We have to find the general formula for a portion of a sphere first. Suppose a circle $x^2 + y^2 = a^2$ with radius a, the volume of a portion of it is

    $$\int_{-a}^{x'} y^2 \pi \mathrm{d}x = \int_{-a}^{x'} \pi(a^2 - x^2)\mathrm{d}x = \frac{\pi}{3}(3a^2 x' + 2a^3 - x'^3)$$

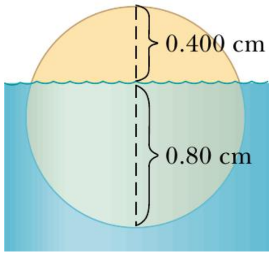


Figure 7:

where x' is the length from the centre along the x-axis. A simple checking putting $x' = a$ gives $4\pi a^3/3$ for the volume of a sphere. For the volume shown in the diagram, putting $a = 0.6$ and $x' = 0.2$ into the formula gives the volume of the immersed portion of the sphere being 0.67 m^3. Since the whole volume is $(0.6)^3\pi \times 4/3 = 0.9$, the density of the sphere is

$$\frac{0.67}{0.9} \times 1000 = 744 \text{ kgm}^{-3}.$$

44. **(Flow).** How many cubic metre of blood does the heart pump in a 75-year lifetime, assuming the average flow rate is 5 L/min? (*Ans:* 2.0×10^5 m^3)

    Solution. The product of flow rate and time is the total volume of blood over that interval. Therefore,

$$(5 \times 10^{-3} \times 60^{-1}) \times (75 \times 365.25 \times 24 \times 60^2) = 197235 \approx 2 \times 10^5 \text{ m}^3.$$

45. **(Bernoulli's equation.)** When a wind blows between two large buildings, a significant drop in pressure can be created. The air pressure is normally 1 atm inside the building, so the drop in pressure just outside can cause a plate glass window to pop out of the building and crash to the street below. What pressure difference would result from a 27 m/s wind? What force would be exerted on a $2 \times 3\,\text{m}^2$ plate glass window? The density of air is 1.29 kg/m^3 at 27° and 1 atm.

    Solution. The pressures inside and outside of the building are supposed to be the same if no wind blowing at outside. When wind blows, air pressure will drop according to Bernoulli's equation. The net pressure pressing on the glass is

$$\Delta P = \frac{1}{2} \times 1.29 \times 27^2 = 470 \text{ Pa}.$$

    And it is equivalent to a force $470 \times 2 \times 3 = 2820$ N acting on the glass.

46. **(Bernoulli's equation).** Fluid is flowing through the horizontal pipe as shown in Figure 8. Please show that

$$v_2 = A_1\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}.$$

    Solution. From the equation of continuity, $A_1 v_1 = A_2 v_2$, we find that

$$v_1 = \frac{A_2}{A_1} v_2.$$

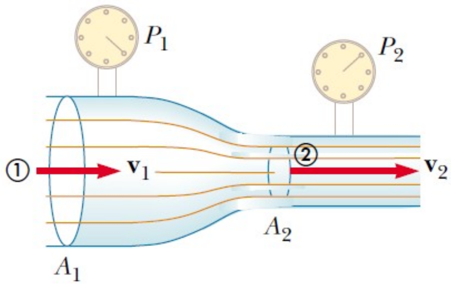


Figure 8: Fluid flows through the pipe.

Substituting this expression into the Bernoulli's equation with the same altitude, i.e., $h = 0$, gives

$$P_1 + \frac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^2 = P_2 + \frac{1}{2}\rho v_2^2.$$

Therefore,

$$v_2 = A_1\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}.$$

47. **(Bernoulli's equation).** A very large diameter no lid tank placing on the ground filled with water to a height h. The tank is punctured with a small hole at a height 0.5 m above the bottom. What is the water level h being able to eject a stream of water through the hole landing 3 m from the tank?

    Solution. Pressures are supposed to be the same at the height h and at 0.5 m height, the Bernoulli equation gives

    $$\frac{1}{2}\rho v_t^2 + \rho g h = \frac{1}{2}\rho v^2 + 0.5\rho g$$

    where v_t and v are the velocities of water dropping at the top and ejecting at the hole, respectively. Since $v \gg v_t$, we neglect v_t and put it to zero. Thus, the velocity of water ejecting from the hole is $v = \sqrt{2g(h - 0.5)}$. In the vertical direction, the time taken for landing

    $$0.5 = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{1}{g}}.$$

    As the horizontal range is 3 m, $3 = \sqrt{2g(h-0.5)} \cdot \sqrt{\frac{1}{g}} = \sqrt{2(h-0.5)}$ gives $h = 5$ m.

48. **(Heat capacity).** A 20 g piece of aluminium at 90 °C is dropped into a cavity in a large block of ice at 0 °C. How much ice does the aluminium melt? (Given specific heat of aluminium is 0.215 cal/g-°C, latent heat of fusion of water 3.33×10^5 J/kg)

    Solution. Assume all the energy going to melt the ice from the piece of aluminium. We have the conservation equation for the energy:

    $$20 \times 0.215 \times 90 \times 4.184 = 3.33 \times 10^5 \times m$$

    where m is the mass of ice being melted in kilogram, and we have put the conversion 1 cal = 4.184 J. Thus,

    $$m = 0.00486 \text{ kg} = 4.86 \text{ g}$$

    of ice being melted.

49. **(Heat capacity).** A person fires a silver bullet with a mass of 2 g and with a muzzle speed of 200 m/s into the pine wall of a saloon. Assume that all the internal energy generated by the impact remains with the bullet. What is the temperature change of the bullet (Given: specific heat capacity of silver is 234 J/kg °C).

Solution. The kinetic energy of the bullet is $mv^2/2$. The temperature of the bullet increases because the kinetic energy becomes the extra internal energy. The temperature change is the same as that which would take place if the kinetic energy were transferred by heat from a stove to the bullet. Using 234 J/kg °C as the specific heat of silver, we obtain

$$\Delta T = \frac{Q}{mc} = \frac{\frac{1}{2}mv^2}{mc} = \frac{v^2}{2c} = \frac{200^2}{2 \times 234} = 85.5\,^{\circ}\mathrm{C}.$$

50. **(Latent heat).** Liquid helium has a very low boiling point, 4.2 K, and a very low latent heat of vaporization, 2.09×10^4 J/kg. If energy is transferred to a container of boiling liquid helium from an immersed electric heater at a rate of 10.0 W, how long does it take to boil away 1.00 kg of the liquid?

Solution. We must supply 2.09×10^4 J of energy to boil away 1.00 kg. Because 10.0 W = 10.0 J/s, 10.0 J of energy is transferred to the helium each second. Therefore, the time it takes to transfer 2.09×10^4 J of energy is

$$t = \frac{2.09 \times 10^4}{10} = 2.09 \times 10^3 \text{ s} \approx 35 \text{ min}.$$

51. **(Heat flows).** The average thermal conductivity of the walls (including the windows) and roof of the house depicted in Figure 9 is 0.480 W/m °C, and their average thickness is 21.0 cm. The house is heated with naturnal gas having a heat of combustion (that is, the energy provided per cubic metre of gas burned) of 9,300 kcal/m^3. How many cubic meters of gas must be burned each day to maintain an inside temperature of 25.0°C if the outside temperature is 0.0°C? Disregard radiation and the energy lost by heat through the ground.

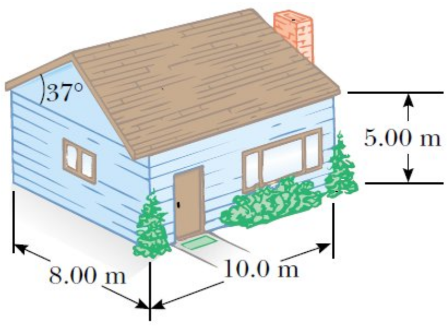


Figure 9:

Solution. The energy flow rate P of conduction is given by

$$P = kA\frac{\Delta T}{L} \tag{6}$$

where k is the thermal conductivity, A the area of the normal direction of heat flow, L the permeable length and ΔT the temperature difference. We have all the numerical data except the area of the house and flow rate that we have got to calculate:

$$A = 2(10 \times 5) + 2(8 \times 5) + \frac{2}{2} \times 8 \times 4 \tan 37 + 2 \times 10 \times \frac{4}{\cos 37} = 304.28 \text{ m}^2$$

and

$$P = \frac{9300 \times 4184 \times V}{24 \times 60 \times 60}$$

where V is the volume of the natural gas required for one day we want to calculate. We have changed the unit from kcal to Joule, and have expressed the 24 hours in term of second. Substituting all these values and $\Delta T = 25$ K, $L = 0.21$ m and $k = 0.48$ into the flow rate equation (6), i.e.,

$$\frac{9300 \times 4184 \times V}{24 \times 60 \times 60} = 0.48 \times 304.28 \times \frac{25}{0.21}$$

we obtain $V = 38.6$ m^3.

52. **(Heat flow).** Two slabs of thickness L_1 and L_2 and thermal conductivities k_1 and k_2 are in thermal contact with each other, as shown in Figure 10. The temperature of their outer surface are T_1 and T_2, respectively, and $T_2 > T_1$. Determine the temperature at the interface and the rate of energy transfer per unit area by conduction through the slabs in the steady-state condition.

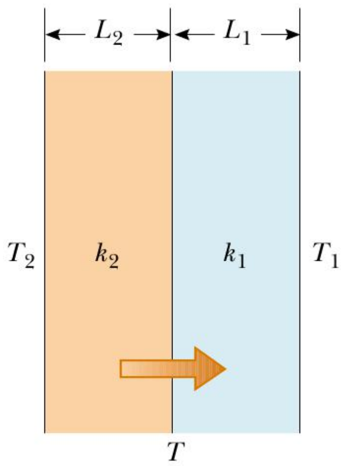


Figure 10:

Solution. If T is the temperature at the interface, then the rates at which energies are transferred through the first and second slabs are

$$H_1 = \frac{k_1 A (T - T_1)}{L_1}, \quad H_2 = \frac{k_2 A (T_2 - T)}{L_2}$$

where A is the area of each interface. When a steady state is reached, these two rates must be equal; hence,

$$\frac{k_1 A (T - T_1)}{L_1} = \frac{k_2 A (T_2 - T)}{L_2}$$

Solving for T gives

$$T = \frac{k_1 L_2 T_1 + k_2 L_1 T_2}{k_1 L_2 + k_2 L_1}.$$

Substituting it into the previous one of the equations we obtain the heat flow per unit area

$$H = \frac{A(T_2 - T_1)}{L_1/k_1 + L_2/k_2}.$$

53. **(Pressure).** A person takes about 20 breaths per minute with 0.5 litre of air in each breath. How much heat is removed per hour by the moisture in the exhaled breath if the incoming air is dry and the exhaled breath is fully saturated? Assume that the water vapour pressure in the saturated exhaled air is 24 torr.

Solution. We change all the units to SI. The exhaled air pressure in each breath is $1.33 \times 10^2 \times 24$ Pa. The volume of the exhaled air is 0.5×10^{-3} m^3. Because

$$\Delta PV = \text{Change of thermal energy}$$

and it is assumed that the exhaled air is fully saturated of water vapour. Therefore, the heat content of the exhaled air in an hour is

$$(1.33 \times 10^2 \times 24) \times (0.5 \times 10^{-3}) \times (20 \times 60) = 191.52 \text{ J}.$$

54. **(Ideal gas).** How many molecules are in 1 cm^3 of helium gas at 20°C? (*Ans:* 2.47×10^{19})

Solution. Assuming the gas obeys ideal gas law under 1 atmospheric pressure, we have

$$\begin{aligned} PV &= Nk_BT \\ N &= \frac{PV}{k_BT} \\ &= \frac{10^5 \times 10^{-6}}{1.38 \times 10^{-23} \times (273 + 20)} \\ &= 2.47 \times 10^{19} \end{aligned}$$

number of helium molecules.

55. **(Ideal gas).** (a) What is the average kinetic energy of a gas molecule at 24 °C? (b) Find the rms (root mean square) speed of an oxygen molecule at this temperature. (Given the mass of an oxygen molecule: 5.356×10^{-26} kg)

Solution. (a) Since the average kinetic energy of a particle is defined by $\frac{1}{2}mv_{\text{rms}}^2 = \frac{3}{2}k_BT$, it gives

$$\frac{3}{2} \times 1.38 \times 10^{-23} \times (273 + 24) = 6.1479 \times 10^{-21} \text{ J}.$$

(b) Assume that the rms speed of nitrogen and oxygen molecules are similar. Since the mass of an oxygen molecule is about 5.356×10^{-26} kg, its rms speed is

$$v_{\text{rms}} = \sqrt{6.1479 \times 10^{-21} \times 2/(5.356 \times 10^{-26})} = 479.14 \text{ ms}^{-1}.$$

56. **(Ideal gas).** Given that the mean free path of a freely moving particle is

$$L = \frac{1}{\sqrt{2}\pi d^2 n}$$

where d and n are the diameter and the number of particles per unit volume, respectively. Please approximate the air around you as a collection of nitrogen molecules, each of which has a diameter of 2×10^{-10} m. How far does a typical molecule move before it collides with another molecule at room temperature (20 °C)?

Solution. In ideal gas, we have

$$PV = NkT \quad \Rightarrow \quad P = \frac{N}{V}kT = nkT \quad \Rightarrow \quad n = \frac{P}{kT}$$

Putting n into the given mean free path equation we get

$$L = \frac{1}{\sqrt{2}\pi d^2 n} = \frac{kT}{\sqrt{2}\pi d^2 P}$$

Inserting $d = 2 \times 10^{-10}$, $T = 273 + 20 = 293$, the Boltzmann constant and 1 atmospheric pressure into the formula we obtain $L = 2.25 \times 10^{-7}$m.

57. **(Thermodynamics.)** A coal-fired power station uses heat transfer from burning coal to do work to turn turbines for generating electricity. Suppose that 2.2×10^{14} J of heat transfer from coal and 1.35×10^{14} J of heat transfer into the environment. (a) What is the work done by the power station? (b) What is the efficiency of the power station?

    Solution. (a) Assume the ideal situation having no energy going into the internal energy, i.e., $\Delta U = 0$, we have the work output

    $$W = Q_h - Q_c = (2.2 - 1.35) \times 10^{14} = 0.85 \times 10^{14} \text{ J}.$$

    (b) The efficiency of the power station is given by

    $$\frac{0.85 \times 10^{14}}{2.2 \times 10^{14}} \times 100\% = 38.6\%$$

58. **(Entropy).** A large, cold object is at −10°C, and a large, hot object is at 130°C. Show that it is impossible for a small amount of energy, for example, 6 J, to be transferred spontaneously from the cold object to the hot one without a decrease in the entropy of the universe and therefore a violation of the second law.

    Solution. By definition the change of entropy is the change of heat transfer divided by the temperature. In order to transfer the 6 J energy from the cold object to the hot object, the entropy change of the **system** is

    $$\left(\frac{6}{273+130}\right)_{\text{hot object}} + \left(\frac{-6}{273-10}\right)_{\text{cold object}} = -0.007925$$

    The main point of the result is that the negative sign indicates the decreasing of the entropy of the system whilst the energy transfer. The process is prohibited by the second law of thermodynamics.

59. Suppose you toss 5 coins starting with 4 heads and 1 tail, and you get one the most likely results, 2 heads and 3 tails. What is the change in entropy?

    Solution. The change of the entropy is the final entropy minus the initial entropy of the system

    $$\Delta S = S_f - S_i = k \ln \Omega_f - k \ln \Omega_i.$$

    The number of ways to get 4 heads and 1 tail is ${}^5C_1 = 5$, to get 2 heads and 3 tails is ${}^5C_2 = 10$. Thus,

    $$\Delta S = 1.38 \times 10^{-23}(\ln 10 - \ln 5) = 9.57 \times 10^{-24} \text{ JK}^{-1}.$$

60. **(Sinusoidal wave).** A sinusoidal wave traveling in the positive x direction has an amplitude of 15 cm, a wavelength of 40 cm, and a frequency of 8 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15 cm, as shown in Figure 11.

(a) Find the angular wave number k, period T, angular frequency ω, and speed v of the wave.

(b) Determine the phase constant ϕ, and write a general expression for the wave function.

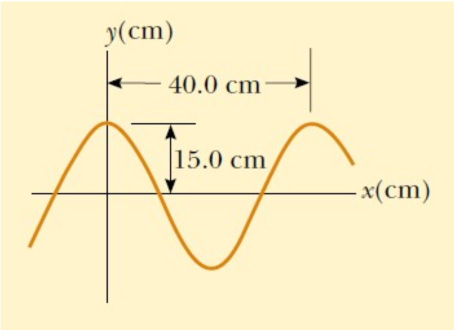


Figure 11: A sinusoidal wave

Solution. (a) The wave number, angular velocity, period, and velocity are:

$$
\begin{aligned}
k &= \frac{2\pi}{\lambda} = \frac{2\pi}{40} = 0.157 \text{ rad/cm},\\
\omega &= 2\pi\nu = 2\pi \times 8 = 50.3 \text{ rad/s},\\
T &= \frac{1}{\nu} = \frac{1}{8} = 0.125 \text{ sec},\\
v &= \lambda\nu = 40 \times 8 = 320 \text{ cm/s}.
\end{aligned}
$$

(b) Since $A = 15$ cm and $y = 15$ cm at $x = 0$ and $t = 0$, thus

$$15 = 15\sin\phi \quad \text{or} \quad \sin\phi = 1.$$

We may take the principal value $\phi = \pi/2$. Hence, the wave function is of the form

$$y = A\sin\left(kx - \omega t + \frac{\pi}{2}\right) = A\cos(kx - \omega t).$$

61. **(Sound velocity).** A uniform cord has a mass of 0.3 kg and a length of 6 m (Figure 12). The cord passes over a pulley and supports a 2 kg object. Find the speed of a pulse travelling along this cord.

Solution. Neglecting the little mass of the vertical potion of the cord, the mass per unit length and tension of the cord are

$$
\begin{aligned}
\mu &= \frac{0.3}{6} = 0.05 \text{ kgm}^{-1},\\
T &= 2g = 2 \times 9.8 = 19.6 \text{ N}.
\end{aligned}
$$

Using the formula for the velocity in a medium, we obtain

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{19.6}{0.05}} = 19.8 \text{ m/s}.$$

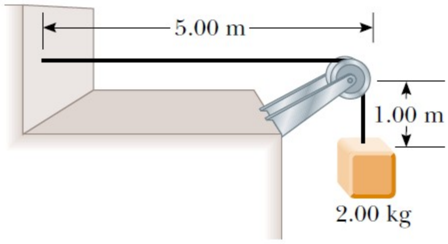


Figure 12: The tension T in the cord is maintained by the suspended object.

62. The G string of a mandolin is 0.34 m long and has a mass per unit length of 0.004 kgm^{-1}. If the string tension of the string is 68.2 N. What is the fundamental frequency of the string?
Solution. Since the two ends of the string are fixed, the length of the string is half of the wavelength of the fundamental frequency. Thus,
$$\begin{aligned}\nu &= \frac{v}{2L} \\ &= \frac{1}{2L}\sqrt{\frac{T}{\mu}} \\ &= \frac{1}{2\times 0.34}\sqrt{\frac{68.2}{0.004}} \\ &= 192 \text{ Hz}\end{aligned}$$

63. A string wave is described by $y = 0.002\sin(0.5x - 628t)$. Determine the amplitude, frequency, period, wavelength, and velocity of the wave.
Solution. Since the amplitude, wave number and angular frequency are
$$y_0 = 0.002 \text{ m}, \quad k = 0.5, \quad \omega = 628 \text{ rad/s}$$
Thus the period, wavelength, and velocity are
$$\begin{aligned}T &= \frac{2\pi}{\omega} = \frac{2\pi}{628} = 0.01 \text{ s}, \\ \lambda &= \frac{2\pi}{k} = \frac{2\pi}{0.5} = 12.6 \text{ m}, \\ v &= \frac{\lambda}{T} = \frac{12.6}{0.01} = 1260 \text{ ms}^{-1}\end{aligned}$$

64. **(Superposition).** If two waves are traveling to the right and have the same frequency, wavelength, and amplitude but differ in phase, what is the resultant wave as a result of superposition? Interpret your result.
Solution. Let the wave amplitudes (functions) of the waves be
$$\psi_1 = \psi_0\sin(kx - \omega t) \quad \text{and} \quad \psi_2 = \psi_0\sin(kx - \omega t + \phi),$$
respectively, where ϕ is the phase difference between the waves. Using the superposition principle, the new wave is just the summation of the waves:
$$\begin{aligned}\psi &= \psi_1 + \psi_2 \\ &= \psi_0[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)] \\ &= 2\psi_0\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right).\end{aligned}$$

The resultant wave function ψ is also sinusoidal and has the same frequency and wavelength described by the sine function term. But the amplitude is varying controlled by the cosine function term if ϕ is not a constant. The variation of the amplitude is also a wave, called envelope.

If $\cos(\phi/2) = 1$, i.e. $\phi = 2n\pi$ where $n = 0, 1, 2, \ldots$, this situation is constructive (in phase), the resultant amplitude is doubled. Similarly, when $\phi = (2n + 1)\pi$ for $n = 0, 1, 2, \ldots$, there is the destructive interference (out of phase).

65. **(Standing waves).** A section of drainage culvert 1.23 m in length makes a howling noise when the wind blows. (a) Determine the frequencies of the first three harmonics of the culvert if it is open at both ends. Take $v = 343$ m/s as the speed of sound in air. (b) What are the three lowest natural frequencies of the culvert if it is blocked at one end?

Solution. (a) The wavelengths of the first three harmonics are

$$\begin{aligned}\lambda_1 &= 2 \times 1.23 = 2.46 \text{ m},\\ \lambda_2 &= 1 \times 1.23 = 1.23 \text{ m},\\ \lambda_3 &= \frac{2}{3} \times 1.23 = 0.82 \text{ m}.\end{aligned}$$

The corresponding frequencies are

$$\begin{aligned}\nu_1 &= \frac{343}{2.46} = 139.4 \text{ Hz},\\ \nu_2 &= \frac{343}{1.23} = 278.9 \text{ Hz},\\ \nu_3 &= \frac{343}{0.82} = 418.3 \text{ Hz}.\end{aligned}$$

(b) The wavelengths of the three lowest frequencies are $4L$, $4L/3$, $4L/5$ where L is the length of the culvert. Using the $\lambda = v/\lambda$ again, we have

$$\begin{aligned}\nu_1 &= \frac{343}{4 \times 1.23} = 69.7 \text{ Hz},\\ \nu_2 &= \frac{343}{4 \times 1.23/3} = 209.1 \text{ Hz},\\ \nu_3 &= \frac{343}{4 \times 1.23/5} = 348.6 \text{ Hz}.\end{aligned}$$

66. **(Resonance).** As shown in Figure 13, water is pumped into a long vertical cylinder at a rate of 18 cm^3/s. The radius of the cylinder is 4 cm, and at the open top of the cylinder is tuning fork vibrating with a frequency of 200 Hz. As the water rises, how much time elapses between successive resonances?

Solution. Resonance occurs when the length of the one-end-closed and one-end-opened pipe provides a length with an integral multiple of $(2n - 1)\lambda/4$, where λ this time is a fixed wavelength according to the frequency of the tuning fork. The height of the water rising for two successive resonances is $\lambda/2$. Let r be the radius of the cylinder, the volume of rised water is

$$\frac{r^2\pi\lambda}{2} = \frac{r^2\pi v}{2\nu} = 18t$$

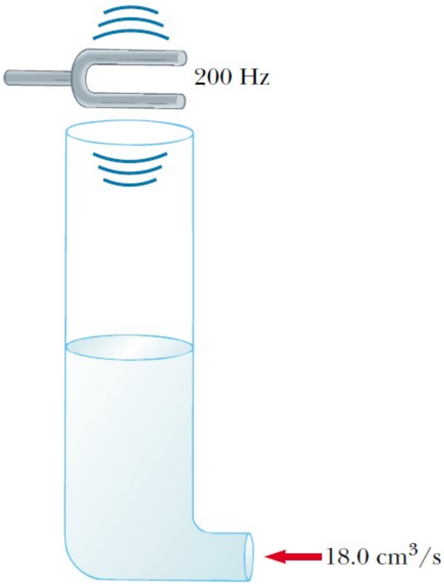


Figure 13:

where t is the time taken between successive resonances. Substituting the numerical values into the equation with speed of sound $v = 330 \times 10^2$ cm/s, we obtain

$$t = \frac{4^2\pi \times 330 \times 10^2}{2 \times 18 \times 200}$$
$$= 230 \text{ sec.}$$

67. **(Intensity).** Two identical machines are positioned the same distance from a worker. The intensity of sound delivered by each machine at the location of the worker is 2×10^{-7} W/m^2. Find the sound level heard by the worker (a) when one machine is operating and (b) when both machines are operating.

Solution. (a) One machine is operating:

$$\text{Logarithmic intensity} = 10 \log \left(\frac{2 \times 10^{-7}}{10^{-12}} \right) = 10 \log(2 \times 10^5) = 53 \text{ dB}.$$

(b) When both machines are operating, the intensity is double to 4×10^{-7} W/m^2, therefore

$$10 \log \left(\frac{4 \times 10^{-7}}{10^{-12}} \right) = 10 \log(4 \times 10^5) = 56 \text{ dB}.$$

From these, we see that when the intensity is doubled, the sound level increases by only 3 dB.

68. **(Intensity).** The intensity of a sound wave at a distance r from the source is given by

$$I = \frac{P_{\text{av}}}{A} = \frac{P_{\text{av}}}{4\pi r^2}$$

where P_{av} is the average power emitted by the source. If a point source emits sound waves with an average power output 80 W (a) Find the intensity 3 m from the source. (b) Find the distance at which the sound level is 40 dB.

Solution. (a) Use the given formula, we obtain

$$I = \frac{80}{4\pi \times 3^2} = 0.707 \text{ W/m}^2.$$

(b) The given intensity is in the unit of dB. We must change it to W/m^2 for the given formula:

$$10^{-12} \log^{-1} \frac{40}{10} = 10^{-8} \text{ W/m}^2.$$

Therefore,

$$\begin{aligned} r &= \sqrt{\frac{P_{\text{av}}}{4\pi I}} \\ &= \sqrt{\frac{80}{4\pi \times 10^{-8}}} \\ &= 25231 \text{ m.} \end{aligned}$$

This unexpected result seems contradicting to our daily experiences. How could a point source of sound only having 80W propagate to over 25 km long distance far away from the source with sound level 40 dB where we are able to hear? The calculation is assumed that there is no ambient air pressure variation, no wind, etc. However in the situation on earth, sound (air pressure) is always diluted by the surroundings.

69. **(Doppler effect).** A commuter train passes a passenger platform at a constant speed of 40 m/s. The train horn is sounded at its characteristic frequency of 320 Hz. (a) What wavelength is detected by a person on the platform as the train approaches? (b) What change in frequency is detected by a person on the platform as the train passes?

Solution. (a) The detected wavelength that is shorter than the wavelength measured in the moving system is

$$vT - 40T = \frac{343 - 40}{320} = 0.95 \text{ m}$$

where T is the period and v the speed of sound in air. (b) The changes of the wavelengths in the cases of approaching and leaving away are $vT - 40T$ and $vT + 40T$, respectively. Thus, the change of frequency while the training passing is

$$\begin{aligned} \Delta\nu &= \frac{v}{vT + 40T} - \frac{v}{vT - 40T} \\ &= \frac{343 \times 320}{343 + 40} - \frac{343 \times 320}{343 - 40} \\ &= -75.7 \text{ Hz.} \end{aligned}$$

The negative sign indicates the decrease of the frequency.

70. **(Doppler effect).** Write down the equations of the observed frequencies for the four cases of Doppler effect in terms of the speed of sound source v_s, the speed of observer v_o and the speed of sound v with the condition $v_s, v_o < v$.

Solution. (a)

$$v_s \longrightarrow \qquad v_o \longrightarrow$$

As O moving away from S (freq. $\downarrow$) and S approaching to O (freq. $\uparrow$) the observed frequency is $\nu' = \dfrac{v - v_o}{v - v_s}\nu$.

(b)

$$\longleftarrow v_s \qquad \longleftarrow v_o$$

As O approaching to S (freq. ↑) and S moving away from O (freq. ↓) the observed frequency is $\nu' = \dfrac{v + v_o}{v + v_s}\nu$.

(c)

$$\longleftarrow v_s \qquad v_o \longrightarrow$$

As O moving away from S (freq. ↓) and S moving away from O (freq. ↓) the observed frequency is $\nu' = \dfrac{v - v_o}{v + v_s}\nu$.

(d)

$$v_s \longrightarrow \qquad \longleftarrow v_o$$

As O approaching to S (freq. ↑) and S approaching to O (freq. ↑) the observed frequency is $\nu' = \dfrac{v + v_o}{v - v_s}\nu$.

Note. It does not matter either the speed v_s or v_o is larger so long as smaller than v.

71. **(Doppler effect).** As an ambulance travels east down a highway at a speed of 33.5 m/s, its siren emits sound at a frequency of 400 Hz. What frequency is heard by a person in a car traveling west at 24.6 m/s (a) as the car approaches the ambulance and (b) as the car moves away from the ambulance? (Given: speed of sound 343 m/s)

Solution. (a) Let v, v_o, v_s be the speed of sound, the speed of the observer and the speed of the ambulance. Because the cars are moving approaching to each other, the new wavelength of the sound in front of the ambulance created by the siren is $vT - v_sT$ where T is the period. That is, $(v - v_s)/\nu$. The frequency of the sound heard by the person is

$$\begin{aligned}\frac{\text{relative velocity of the observer and sound}}{\text{wavelength of sound in front of the ambulance}} &= \frac{v + v_o}{v - v_s}\nu \\ &= \frac{343 + 24.6}{343 - 33.5} \times 400 \\ &= 475 \text{ Hz.}\end{aligned}$$

(b) As the car moves away from the ambulance, the sound wave at the back side of the ambulance is elongated; the wavelength is $vT + v_sT = (v + v_s)/\nu$. The relative velocity of the observer and the sound is also slowed to $v - v_o$. Therefore, the person should hear the new frequency of the sound to be

$$\frac{v - v_o}{v + v_s}\nu = \frac{343 - 24.6}{343 + 33.5} \times 400 = 338 \text{ Hz.}$$

72. **(Optics).** For a spherical convex mirror of 14 cm radius of curvature, describe the image of a 2.5 cm object placed 30 cm out on the principal axis of the mirror.

Solution. We take a negative focal length for the convex mirror and expect the image distance being negative. Hence,

$$\frac{1}{30} + \frac{1}{q} = -\frac{1}{7} \quad \text{gives} \quad q = -5.7 \text{ cm.}$$

The magnification is the ratio of the image and object distances:

$$M = -\frac{-5.7}{30} = 0.19$$

The height of the image is $h = 0.19 \times 2.5 = 0.48$ cm. The minus sign of the image distance indicates that the (virtual) image is behind the mirror. Noted that a negative sign has been added by hand for positive magnification (upright image).

73. **(Optics).** A diverging lens has a focal length of -20 cm. An object 2 cm tall is placed 30 cm in front of the lens. Locate the image.

Solution. Using the thin-less formula with $p = 30$ cm and $f = -20$ cm, we obtain

$$\frac{1}{30} + \frac{1}{q} = \frac{1}{-20}$$
$$q = -12 \text{ cm.}$$

74. **(Optics).** What is the maximum magnification that is possible with a lens having a focal length of 10 cm, and what is the magnification of this lens when the eye is relaxed? (Given the near point distance of the person is 25 cm.)

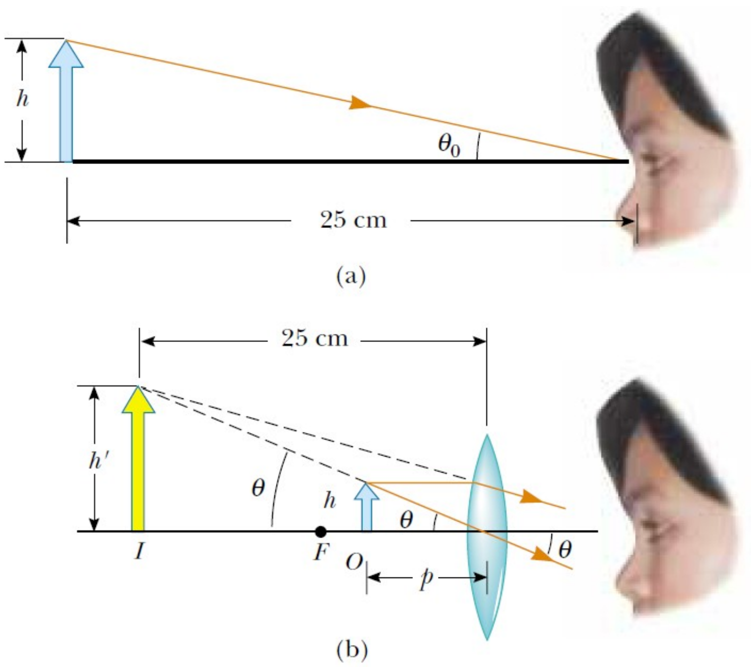


Figure 14: (a) An object placed at the near point of the eye subtends an angle $\theta_0 \approx h/25$ at the eye. (b) An object placed near the focal point of converging lens produces a magnified image that subtends an angle $\theta \approx h'/25$ at the eye.

Solution. In Figure 14, the angular magnification with the object at the near point is

$$m = \frac{\theta}{\theta_0}. \tag{7}$$

The angular magnification is a maximum when the image is at the near point of the eye (i.e., closer to the eye to see bigger). When $q = -25$ cm, the object distance corresponding to this image distance is

$$\frac{1}{p} + \frac{1}{-25} = \frac{1}{f}$$
$$p = \frac{25f}{25+f}$$

where f is the focal length of the magnifier. Since in small angle approximations,

$$\tan\theta_0 \approx \theta_0 \approx \frac{h}{25} \quad \text{and} \quad \tan\theta \approx \theta \approx \frac{h}{p}.$$

Equation (7) becomes

$$m_{\text{max}} = \frac{\theta}{\theta_0} = \frac{h/p}{h/25} = \frac{25}{p} = \frac{25}{25f/(25+f)} = 1 + \frac{25}{f}. \tag{8}$$

When the image is at infinity, the eye is most relaxed (but the image also the smallest as the farthest away). For the image formed by the magnifying lens to appear at infinity, the object has to be at the focal point of the lens. Thus, (7) becomes

$$\theta_0 \approx \frac{h}{25} \quad \text{and} \quad \theta \approx \frac{h}{f}$$

and the magnification is

$$m_{\text{min}} = \frac{\theta}{\theta_0} = \frac{25}{f}. \tag{9}$$

Therefore, substituting $f = 10$ cm into (8) and (9), the maximum and minimum magnifications are

$$m_{\text{max}} = 1 + \frac{25}{10} = 3.5,$$
$$m_{\text{min}} = \frac{25}{10} = 2.5.$$

75. **(Electricity).** Earth and the moon are bound together by gravity. If the force of attraction were the result of each having a charge of the same magnitude but opposite in sign instead, find the quantity of charge that would have to be placed on each to produce the required force.

Solution. Let M, m be the masses of earth and the moon, Q be the mentioned corresponding electric charge. The gravitational force and the Coulomb force equations are

$$F = -\frac{GMm}{r^2} \quad \text{and} \quad F = k\frac{Q(-Q)}{r^2}$$

where r is the distance separation between the centres of earth and the moon. Equating the equation we get

$$Q^2 = \frac{GMm}{k} = \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 7.34 \times 10^{22}}{8.99 \times 10^9}.$$

It gives $Q = 5.702 \times 10^{13}$ C which is the corresponding charge to create the force equivalent to the gravitation.

76. **(Electricity).** Consider a line of point charges, $q_1 = 8\,\mu$C at the origin, $q_2 = -12\,\mu$C at 2 cm, and $q_3 = 10\,\mu$C at 4 cm. What is the force on q_3 due to the other two charges?

Solution. The force on q_3 is just the vector sum of the two forces F_{31} and F_{32}:

$$\frac{9\times10^9\times8\times10\times(10^{-6})^2}{(4\times10^{-2})^2}+\frac{9\times10^9\times(-12)\times10\times(10^{-6})^2}{(2\times10^{-2})^2}=2250\text{ N}$$

toward q_1 and q_2.

77. **(Electricity).** An electron enters the region of a uniform electric field as shown in Figure 15, with $v_i = 3\times10^6$ m/s and $E = 200$ N/C. The horizontal length of the plates is $l = 0.1$ m. (a) Find the acceleration of the electron while it is in the electric field. (b) Find the time it takes the electron to travel through the field. (c) What is the vertical displacement y of the electron while it is in the field?

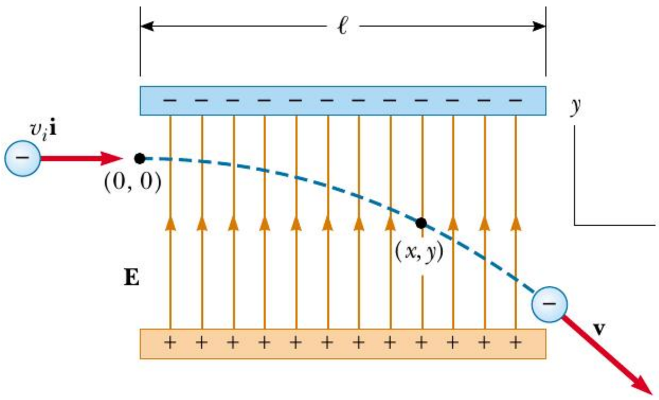


Figure 15:

Solution. (a) The charge on the electron has an value of 1.6×10^{-19} C, and $m = 9.11\times10^{-31}$ kg. Therefore, the magnitude of the acceleration along $y-axis$ is

$$a=\frac{F}{m}=-\frac{eE}{m}=-\frac{1.6\times10^{-19}\times200}{9.11\times10^{-31}}=-3.51\times10^{13}\text{ ms}^{-2}.$$

(b)The time taken is

$$t=\frac{l}{v_i}=\frac{0.1}{3\times10^6}=3.33\times10^{-8}\text{ s}.$$

(c) Like doing in Newtonian mechanics, the vertical displacement is

$$y=\frac{1}{2}at^2=\frac{1}{2}(-3.51\times10^{13})\times(3.33\times10^{-8})^2=-0.0195\text{ m}.$$

78. **(Electricity).** An electron placed between two charged parallel plates separated by 2×10^{-2} m is observed to accelerate at 5×10^{14} ms^{-2}. What is the voltage on the plates?

Solution. The force on the electron is

$$F=ma=9.1\times10^{-31}\times5\times10^{14}=4.6\times10^{-16}\,\text{N}.$$

The electric field required to move this electron is

$$E=\frac{F}{e^-}=\frac{4.6\times10^{-16}}{1.6\times10^{-19}}=2.8\times10^3\,\text{N/C}.$$

The voltage for this separation is

$$V = \frac{U}{e^-} = \frac{Fd}{e^-} = Ed.$$

Thus,

$$V = 2.8 \times 10^3 \times 2 \times 10^{-2} = 57 \text{ V}.$$

79. **(Electricity).** Find the equivalent capacitance between a and b for the combination of capacitors shown in Figure 16. All capacitances are in microfarads.

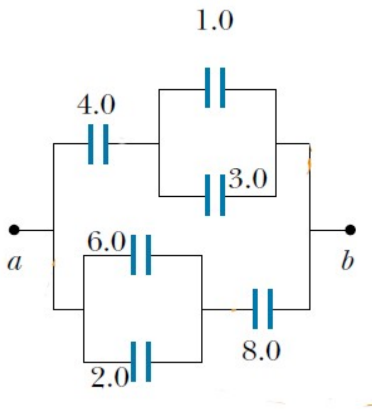


Figure 16:

Solution. We reduce the combination step by step as indicated in Figure 17. The 1 μF and 3 μF capacitors are in parallel and combine according to the expression $C_{eq} = C_1 + C_2 = 4\,\mu$F. The 2 μF and 6 μF capacitors also are in parallel and have an equivalent capacitance of 8 μF. The upper branch in Figure 17b consists of two 4 μF capacitors in series, which combine as follows:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$C_{eq} = \frac{1}{\frac{1}{2}} = 2\,\mu\text{F}.$$

The lower branch in Figure 17b consists of two 8 μF capacitors in series, which

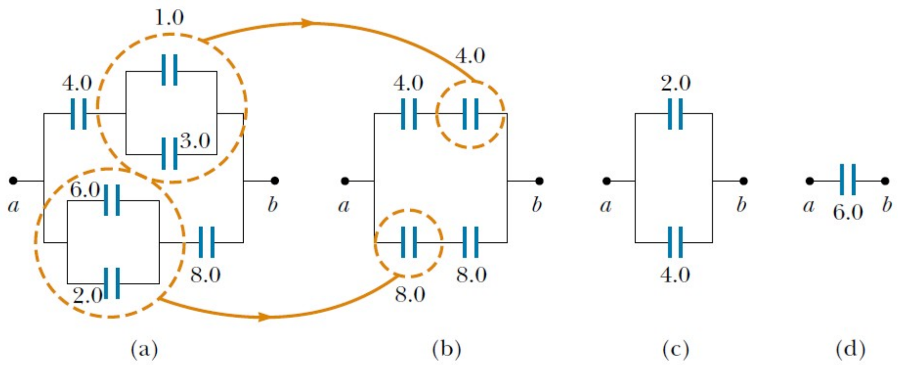


Figure 17:

combine to yield an equivalent capacitance of 4 μF. Finally, the 2 μF and 4 μF capacitors in Figure 17c are in parallel and thus have an equivalent capacitance of 6 μF.

80. Charge a 12 μF capacitor with a 200 V source, then place this capacitor in parallel with a uncharged 7 μF capacitor and calculate the "new" voltage.

Solution. The amount of electric charge stored by a capacitor is given by $Q = CV$. The charged 12 μF capacitor connecting to a 7 μF capacitor in parallel gives

$$12 \times 200 = (12 + 7)V_n$$

where V_n is the new voltage. Thus, $V_n \approx 126$ V.

Remark. We cannot use the law of conservation of energy directly such as

$$\frac{1}{2} \times 12 \times 200^2 = \frac{1}{2}(12 + 7)V_n^2$$

to give $V_n \approx 160$ V which is in fact incorrect that we discussed wrongly. The situation here is quite similar to the inelastic collision in Newtonian mechanics in which energy is lost during collision. In our example, energy is lost during the process of charging the 7 μF capacitor by the charged 12μF one.

81. A 1000 W electric heater operates at 115 V. (a) Calculate the current, resistance, and energy generated in one hour. (b) If the voltage is reduced to 110 V (assume no change in resistance) how does the heat output change?

Solution. (a) The current, resistance, and energy are

$$\begin{aligned} I &= \frac{P}{V} = \frac{1000}{115} = 8.7 \text{ A}, \\ R &= \frac{V}{I} = \frac{V^2}{P} = \frac{115^2}{1000} = 13.2\ \Omega, \\ U &= 1000 \times 60 \times 60 = 3.6 \times 10^6 \text{ J}. \end{aligned}$$

(b) Since $P = V^2/R$, now R is a constant, i.e.,

$$\frac{V_1^2}{P_1} = \frac{V_2^2}{P_2} \quad \Rightarrow \quad P_2^2 = \frac{110^2}{115^2} \times 1000$$

Therefore, the new power is $P_2 = 915$ W.

82. Defibrillator is a clinical device designed for synchronizing the heart by using the tetanizing effect of large currents. A capacitor in this device is charged to about 6000 V and stores about 200 J of energy. Two electrodes connected to the capacitor through a switch are placed on the chest. When the switch is closed, the capacitor rapidly discharges through the body. The current pulse lasts about 5 msec, during which the heart is tetanized.

    (a) Please calculate the capacitance of the capacitor.
    (b) Find the magnitude of the average current flowing during the pulse.

Solution. (a) Since

$$E = \frac{1}{2}CV^2$$

where E, C, V are the energy storage, capacitance and potential difference of the capacitor. Thus, the capacitance is

$$\begin{aligned} 200 &= \frac{1}{2}C(6000)^2 \\ C &= 1.1 \times 10^{-5} \text{ Faraday} \\ &= 11\ \mu F. \end{aligned}$$

(b) The average current I is the change of charge divided by a given interval:

$$I = \frac{Q_f - Q_i}{t_f - t_0}$$

where Q_f, Q_i are the final and initial charges passing through the device, and t_f, t_i are the corresponding moment of time. Since $Q_f = 0$ and we set the initial moment $t_i = 0$, then we have

$$\begin{aligned} I &= \frac{0 - Q_i}{t_f - 0} = \frac{-\Delta Q}{\Delta t} \\ &= \frac{-CV}{\Delta t} \\ &= \frac{-2E}{V\Delta t} \\ &= \frac{-2 \times 200}{6000 \times 5 \times 10^{-3}} \\ &= -13.3 \text{ A.} \end{aligned}$$

The magnitude of the average current flowing is 13.3 A.

83. A battery with voltage of 12 V and an internal resistance of $0.05\,\Omega$. Its terminals are connected to a load resistance of $3.00\,\Omega$. (a) Find the current in the circuit and the terminal voltage of the battery. (b) Calculate the power delivered to the load resistor, the power delivered to the internal resistance of the battery, and the power delivered by the battery.

Solution. (a) The current is

$$I = \frac{12}{3 + 0.05} = 3.93 \text{ A.}$$

Thus, the terminal voltage of the battery is

$$12 - 3.93 \times 0.05 = 11.8 \text{ V.}$$

(b) The power delivered to the load resistor is

$$I^2R = 3.93^2 \times 3 = 46.3 \text{ W.}$$

The power delivered to the internal resistance is

$$I^2r = 3.93^2 \times 0.05 = 0.772 \text{ W.}$$

Hence, the power delivered by the battery is the sum of these quantities,

$$46.3 + 0.772 \approx 47.1 \text{ W.}$$

84. If your car is 4.3 m long when it is parked, how much shorter does it appear to a stationary road-side observer as you drive by at 30 m/s?

Solution. The observer sees the horizontal length of the car to be contracted to a length

$$L = L_p\sqrt{1-\frac{v^2}{c^2}} \approx L_p\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)$$

where we have used the binomial expansion for the factor $\sqrt{1-v^2/c^2}$. The roadside observer sees the car's length as having changed by an amount $L_p - L$:

$$\begin{aligned} L_p - L \approx \frac{L_p}{2}\left(\frac{v^2}{c^2}\right) &= \frac{4.3}{2} \times \left(\frac{3\times 10^1}{3\times 10^8}\right)^2 \\ &= 2.2 \times 10^{-14} \text{ m}. \end{aligned}$$

This is much smaller than the diameter of an atom.

85. Suppose you are driving your car on a business trip and are traveling at 30 m/s. Your boss, who is waiting at your destination, expects the trip to take 5 hours. When you arrive late, your excuse is that your car clock registered the passage of 5 hours but that you were driving fast and so your clock ran more slowly than your boss's clock. If your car clock actually did indicate a 5 hours trip, how much time passed on your boss's clock, which was at rest on the Earth?

Solution. The gamma term is

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(3\times 10^1)^2/(3\times 10^8)^2}} = \frac{1}{\sqrt{1-10^{-14}}}.$$

We use binomial expansion instead of using calculator, because you may probably obtain $\gamma = 1$ in calcualor.

$$\gamma = (1-10^{-14})^{-1/2} \approx 1 + \frac{1}{2}\times 10^{-14} = 1 + 5\times 10^{-15}.$$

The time dilation formula, $\Delta t = \gamma\Delta t'$ where Δt and $\Delta t'$ are the time in rest frame and proper time (time in moving frame). The time interval measured by your boss, to be

$$\begin{aligned} \Delta t = \gamma\Delta t' &= (1 + 5\times 10^{-15}) \times 5 \text{ hours} \\ &= 5 + 2.5\times 10^{-14} \text{ hours} \\ &= 5 \text{ hours } + 0.09 \text{ nano-second}. \end{aligned}$$

That is, your boss' clock would be only 0.09 nano-second ahead of your car clock. You might want to try another excuse!

86. **(Relativistic mechanics).** What is the speed of a particle to produce the kinetic energy equal to one-third of its rest energy?

Solution. The total energy is

$$E = \gamma m_0c^2 = \text{KE} + m_0c^2 = \frac{1}{3}m_0c^2 + m_0c^2 = \frac{4}{3}m_0c^2.$$

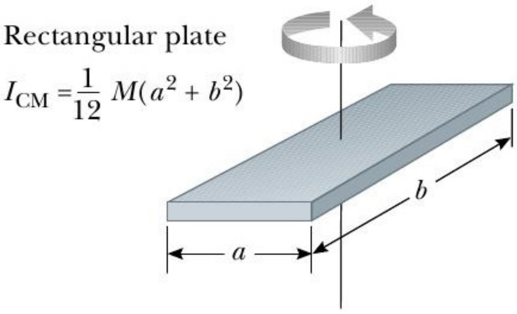
It implies

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{4}{3}.$$

Therefore, $v = \sqrt{\frac{7}{16}}c = 0.66c.$

Supplementary

87. (a) Drive the formula for I_{CM} as shown in the figure. (Hint: You may consider the distance between the rotational axis and a mass element being $r = \sqrt{x^2 + y^2}$.) (b) Find the new moment of inertia if the rotating axis is shifted to one of the corners.



Solution. (a) The formula of moment of inertia is given by $\int r^2 \mathrm{d}m$. Written in terms of Cartesian coordinate system that $r^2 = x^2 + y^2$ and $\mathrm{d}m = \rho(\mathrm{d}x\mathrm{d}y)$ where ρ is the mass per unit area. Thus,

$$
\begin{aligned}
I &= \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \rho(x^2 + y^2)\mathrm{d}x\mathrm{d}y \\
&= \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \rho x^2 \mathrm{d}x\mathrm{d}y + \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \rho y^2 \mathrm{d}x\mathrm{d}y \\
&= \int_{-b/2}^{b/2} \rho \frac{x^3}{3}\bigg|_{-a/2}^{a/2} \mathrm{d}y + \int_{-b/2}^{b/2} \rho x y^2 \bigg|_{-a/2}^{a/2} \mathrm{d}y \\
&= \frac{\rho}{12} a^3 y \bigg|_{-b/2}^{b/2} + \rho a \frac{y^3}{3}\bigg|_{-b/2}^{b/2} \\
&= \frac{\rho}{12} a^3 b + \frac{\rho}{12} a b^3
\end{aligned}
$$

Since $M = \rho ab$, the moment of inertia is $I_{CM} = \dfrac{1}{12} M(a^2 + b^2)$. □

(b) The distance between the centre of mass to one of the corners is $\sqrt{\dfrac{a^2}{4} + \dfrac{b^2}{4}}$. Using the parallel axis theorem, the new moment of inertia is

$$
I = \frac{1}{12} M(a^2 + b^2) + M\left(\frac{a^2}{4} + \frac{b^2}{4}\right) = \frac{1}{3} M(a^2 + b^2).
$$

88. One of the string of a violin is 0.32 m long and has a mass per unit length of 0.003 kgm^{-1}. If the string tension of the string is 76.4 N. What is the frequency of the fifth harmonics of the string?

Solution. Since the two ends of the string are fixed, the length of the string is 5/2 of wavelength of the fifth harmonics. Thus,

$$\nu = \frac{v}{2L/5}$$
$$= \frac{5}{2L}\sqrt{\frac{T}{\mu}}$$
$$= \frac{5}{2 \times 0.32}\sqrt{\frac{76.4}{0.003}}$$
$$= 1247 \text{ Hz}$$

89. What is the intensity at a distance of 3.6 m from a point source emitting sound uniformly in all directions at a power level of 95 W?

Solution. The intensity is the power per unit area. For a point source emitting sound wave, applying the spherical area we have

$$I = \frac{P}{4\pi r^2} = \frac{95}{4\pi \times 3.6^2} = 0.583 \text{ Wm}^{-2}.$$

90. A machine operates at about 68 dB measured from 1.6 m away. What is the intensity measured at 2.5 m away from the source?

Solution. Since

$$68 = 10\log_{10}\frac{I}{10^{-12}},$$

thus, $I = 10^{6.8} \times 10^{-12}$ Wm^{-2}. At 2.5 m away from the source the intensity is

$$10^{6.8} \times 10^{-12} \times \frac{1.6^2}{2.5^2} = 2.58 \times 10^{-6} \text{ Wm}^{-2}.$$

91. At a distance of 4 m from a source the sound level is 88 dB. How far away has the level dropped to 52 dB?

Solution. According to the inverse square law, $\frac{d^2}{4^2} = \frac{I_4}{I_d}$ where d is the distance to be determined, I_d, I_5 are the corresponding intensities measured at d and at 4 metre away from the source, respectively. Thus,

$$52 - 88 = 10\left(\log\frac{I_d}{I_0} - \log\frac{I_4}{I_0}\right)$$
$$= 10\log\frac{I_d}{I_4}$$
$$-3.6 = \log\frac{4^2}{d^2}$$
$$10^{-3.6} = \frac{16}{d^2}$$
$$d = 252.4 \text{ m}$$

92. Singing that is off-pitch by more than about 1.1% sounds bad. How fast would a singer have to be moving relative to the rest of a band to make this much of a change in pitch due to the Doppler effect? (Given the speed of sound 343 ms^{-1} in rest frame. Obviously, assuming there are no microphone and loudspeaker.)

Solution. Using the moving sound source Doppler effect formula, for the higher pitch (approaching to the band) we have

$$\nu = \nu_0 \frac{v}{v - v_s} \quad \Rightarrow \quad \frac{\nu}{\nu_0} = 1.011 = \frac{v}{v - v_s}$$

It gives $v_s = 0.0109v$ where v is the speed of sound. That is, if the singer moves about $3.740\ \mathrm{ms}^{-1}$ towards to the band the sound is off-pitch by 1.1%. Similarly, for the lower off-pitch (moving away from the band)

$$\nu = \nu_0 \frac{v}{v + v_s} \quad \Rightarrow \quad 0.989 = \frac{v}{v + v_s}$$

gives the moving speed

$$v_s = \frac{0.011}{0.089} v = 3.812\ \mathrm{ms}^{-1}.$$

93. An 8-hour exposure to a sound intensity level of 90.0 dB may cause hearing damage. What energy in joules falls on a 0.78 cm diameter eardrum so exposed?

Solution. The intensity at 90 dB is

$$90 = 10 \log \frac{I}{10^{-12}} \quad \Rightarrow \quad 10^9 = \frac{I}{10^{-12}} \quad \Rightarrow \quad I = 10^{-3}\ \mathrm{Wm}^{-2}$$

The total energy contribution on the eardrum is

$$10^{-3} \times 8 \times 60 \times 60 \times (0.0078/2)^2 \pi = 1.376 \times 10^{-3}\ \mathrm{J}$$

94. Two thin lenses 1 (left) and 2 (right) are separated by 8 cm with focal lengths 25 cm and 10 cm along the same principle axes. A bug is positioned 7 cm from the front side of the lens 1. Where is its final image?

Solution. The position of the image formation for the first lens is obtained by

$$\frac{1}{25} = \frac{1}{7} + \frac{1}{q_1} \quad \Rightarrow \quad q_1 = -9.72\ \mathrm{cm}.$$

The image that is at the front side of the first lens becomes the object for the second lens. The object distance is $8 + |-9.72| = 17.72$ cm. The final image is at

$$\frac{1}{10} = \frac{1}{17.72} + \frac{1}{q_2} \quad \Rightarrow \quad q_2 = 22.95 \approx 23\ \mathrm{cm}$$

at the back side of the second lens, and it is real.

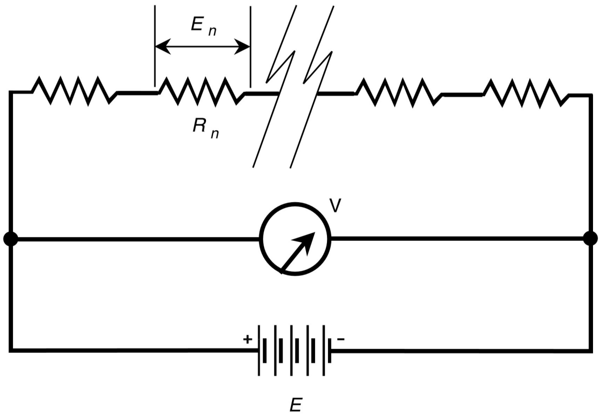
95. Suppose that there are nine resistors in series as shown. Five of them have values of 8 Ω, and the other four have values of 20 Ω. The power source is 24 V DC. What is the voltage across one of the 8-ohm resistors? Across one of the 20-ohm resistors?

Solution. Since in the series configuration the currents passing through the resistors are the same. We may directly use a simple ratio to compute the potential differences over the resistors. The voltage across one of the 8-ohm resistors is

$$24 \times \frac{8}{8 \times 5 + 20 \times 4} = 1.6\ \mathrm{V}.$$

Across one of the 20-ohm resistors is

$$24 \times \frac{20}{8 \times 5 + 20 \times 4} = 4\ \mathrm{V}.$$

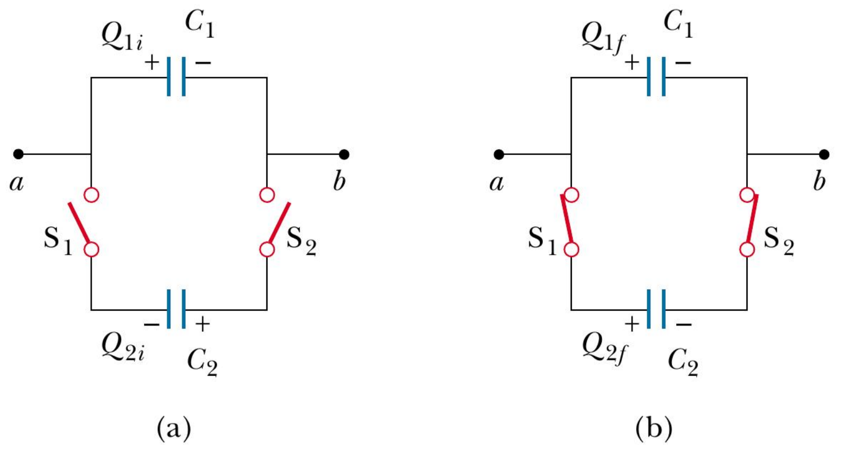


96. A charge $-q$ is placed at one corner of a square of side a, and charges $+q$ are placed at each of the other corners. What is the potential at the centre of the square to the ground?

Solution.

$$\begin{aligned} V &= \text{electric field} \cdot \text{distance} \\ &= \frac{ka}{\sqrt{2}}\left(\frac{q}{(a/\sqrt{2})^2} + \frac{q}{(a/\sqrt{2})^2} + \frac{q}{(a/\sqrt{2})^2} - \frac{q}{(a/\sqrt{2})^2}\right) \\ &= \frac{2\sqrt{2}q}{ka} \end{aligned}$$

97. Two capacitors C_1 and C_2 (where $C_1 > C_2$) are charged to the same initial potential difference V_i, but with opposite polarity. The charged capacitors are removed from the battery, and their plates are connected as shown in the Figure (a). The switches S_1 and S_2 are then closed, as shown in Figure (b). Find the final potential difference V_f between a and b after the switches are closed.



Solution. We have the initial equations

$$Q_{1i} = C_1 V_i \quad \text{and} \quad Q_{2i} = C_2 V_i.$$

Because the two capacitors in opposite polarity, the final net electric charge is

$$Q_f = Q_{1i} + (-Q_{2i}) = (C_1 - C_2)V_i.$$

Therefore, the final potential difference in figure (b) is

$$V_f = \frac{Q_f}{C_f} = \frac{V_i(C_1 - C_2)}{C_1 + C_2}.$$

98. Prove the invariance of a Minkowski metric under a Lorentz transformation. (i.e., show $\Delta s^2 = \Delta s'^2$)

Solution. Vary the coordinates under a Lorentz transformation:

$$\Delta x = \gamma(\Delta x' + v\Delta t'), \quad \Delta y = \Delta y', \quad \Delta z = \Delta z', \quad \Delta t = \gamma\left(\Delta t' + \frac{v}{c^c}\Delta x'\right)$$

Inserting into a Minkowski metric in K system gives

$$\begin{aligned}
\Delta s^2 &= \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2 \\
&= \gamma^2(\Delta x'^2 + 2v\Delta x'\Delta t' + v^2\Delta t'^2) + \Delta y'^2 + \Delta z'^2 - c^2\gamma^2\left(\Delta t'^2 + \frac{2v}{c^2}\Delta t'\Delta x' + \frac{v^2}{c^4}\Delta x'^2\right) \\
&= \gamma^2\Delta x'^2\left(1 - \frac{v^2}{c^2}\right) + \Delta y'^2 + \Delta z'^2 - \gamma^2 c^2\Delta t'^2\left(-\frac{v^2}{c^2} + 1\right) \\
&= \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2\Delta t'^2 \\
&= \Delta s'^2
\end{aligned}$$

Result exactly a Minkowski metric in a different coordinate system K'.

Note. Almost all the carbon on Earth is ^{12}C, but not quite. The isotope ^{14}C is produced by cosmic rays in the atmosphere. It decays naturally, but is replenished at such a rate that the fraction of ^{14}C in the atmosphere remains constant. Living plants and animals take in both ^{12}C and ^{14}C from the atmosphere and incorporate both into their bodies. Once the living organism dies, it no longer takes in carbon atoms from the atmosphere, and the proportion of ^{14}C gradually falls off as it undergoes radioactive decay. This effect can be used to find the age of dead organisms, or human artifacts made from plants or animals.

99. An archaeologist has found a fossil in which the ratio of ^{14}C to ^{12}C is 1/5 the ratio found in the atmosphere. Given that the half-life of carbon-14 is 5,730 years, approximately how old is the fossil?

Solution. The decay function $N = N_0 e^{-\lambda t}$ becomes

$$\frac{1}{5}N_0 = N_0 e^{-\lambda t} \quad \Rightarrow \quad \frac{1}{5} = e^{-\lambda t}.$$

Since the half-life equation is $\frac{1}{2} = e^{-5730\lambda}$, the equations becomes

$$-\lambda t = \ln\frac{1}{5} \quad \text{and} \quad -5730\lambda = \ln\frac{1}{2}$$

Eliminating λ gives

$$t = \ln\frac{1}{5} \times 5730\left(\ln\frac{1}{2}\right)^{-1} = 13305 \text{ years}$$

The age of the fossil is about 133 hundred years old.

100. (a) Find the rest energy of a proton in electron volts, given that mass of proton is 1.67×10^{-27} kg, charge of electron -1.6×10^{-19} C.

(b) If the total energy of a proton is two and an one-third times of its rest energy, what is the speed of the proton moving?

(c) Find the kinetic energy of the proton in electron volts.

(d) What is the proton's momentum?

Solution. (a)

$$E_{\text{rest}} = m_0c^2 = (1.67 \times 10^{-27})(3 \times 10^8)^2 = 1.50 \times 10^{-10} \text{ J}.$$

One electron volt is the amount of energy of one electron charge under one volt voltage. Thus, in terms of electron volt,

$$E_{\text{rest}} = 1.50 \times 10^{-10} \text{ J} \times \frac{1.00 \text{ eV}}{1.60 \times 10^{-19} \text{ J}} = 0.938 \times 10^9 \text{ eV} = 938 \text{ MeV}.$$

(b) The total energy is

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}}$$

with two and an one-third times its rest energy, i.e., $E = 7/3\, m_0c^2$. Solving for v gives $v = \sqrt{1 - (3/7)^2}c = 2.71 \times 10^8$ m/s.

(c) The kinetic energy

$$\text{K.E.} = E - m_0c^2 = 7/3\, m_0c^2 - m_0c^2 = 4/3\, m_0c^2.$$

By using the value of m_0c^2 in (a), in terms of electron volt the kinetic energy is

$$\frac{4}{3} \times \frac{1.5 \times 10^{-10}}{1.6 \times 10^{-19}} = 1.250 \times 10^9 \text{ eV} = 1250 \text{ MeV}$$

(d) Inserting $E = 7/3\, m_pc^2$ into the relativistic equation,

$$p^2 = \frac{E^2}{c^2} - |\mathbf{p}|^2.$$

Solving for $|\mathbf{p}|$, we have

$$|\mathbf{p}| = \sqrt{\frac{(7/3\, m_0c^2)^2}{c^2} - (m_0c)^2} = \frac{2.108 m_0c^2}{c} = \frac{2.108 \times 938}{c}\text{MeV} = 1977 \text{ MeV/c},$$

or 1.056×10^{-18} kgms^{-1}.