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Determinants

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Determinants

- Over the years there have various ways to define the determinant.
- We are going to opt for expedience over elegance and proceed with the classical treatment.
- A **permutation** $p = (p_1, p_2, \cdots, p_n)$ of the numbers $(1, 2, \cdots, n)$ is simply any rearrangement.
- For example, the set

$$\{(1,2,3) \quad (1,3,2) \quad (2,1,3) \quad (2,3,1) \quad (3,1,2) \quad (3,2,1)\}$$

  contains the six distinct permutations of $(1, 2, 3)$.
- In general, the sequence $(1, 2, \cdots, n)$ has $n! = n(n-1)(n-2)\cdots 1$ different permutations.
- Given a permutation, consider the problem of restoring it to natural order by a sequence of pairwise interchanges.

- For example, $(1, 4, 3, 2)$ can be restored to natural order with a single interchange of 2 and 4 or three adjacent interchanges can be used.

- The important thing here is that both 1 and 3 are odd.
- Try to restore $(1, 4, 3, 2)$ to natural order by using an even number of interchanges, and you will discover that it is impossible.
- The parity of a permutation is unique—i.e., if a permutation p can be restored to natural order by an even (odd) number of interchanges, then every other sequence of interchanges that restores p to natural order must also be even (odd).

- Accordingly, the sign of a permutation p is defined to be the number

$$\sigma(p) = \begin{cases} +1 & \text{if } p \text{ can be restored to natural order by an} \\ & \textit{even} \text{ number of interchanges,} \\ -1 & \text{if } p \text{ can be restored to natural order by an} \\ & \textit{odd} \text{ number of interchanges.} \end{cases}$$

Definition of Determinant

For an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, the ***determinant*** of $\mathbf{A}$ is defined to be the scalar

$$\det(\mathbf{A}) = \sum_{p} \sigma(p) a_{1p_1} a_{2p_2} \cdots a_{np_n},$$

where the sum is taken over the $n!$ permutations $p = (p_1, p_2, \ldots, p_n)$ of $(1, 2, \ldots, n)$.

- The determinant of $\mathbf{A}$ can be denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$.
- Note: the determinant of a nonsquare matrix is not defined.

- For example, When $\mathbf{A}$ is 2×2, there are 2 permutations of $(1, 2)$, namely, $\{(1, 2)\ (2, 1)\}$, so det $(\mathbf{A})$ contains the two terms

$$\sigma(1,2)a_{11}a_{22} \quad and \quad \sigma(2,1)a_{12}a_{21}.$$

Since $\sigma(1, 2) = +1$ and $\sigma(2, 1) = -1$, we obtain the familiar formula

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Triangular Determinants

The determinant of a triangular matrix is the product of its diagonal entries. In other words,

$$\begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{vmatrix} = t_{11}t_{22}\cdots t_{nn}.$$

Transposition Doesn't Alter Determinants

- $\det\left(\mathbf{A}^T\right) = \det(\mathbf{A})$ for all $n \times n$ matrices.

Effects of Row Operations

Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}_{n\times n}$ by one of the three elementary row operations:

Type I: Interchange rows i and j.

Type II: Multiply row i by $\alpha \neq 0$.

Type III: Add α times row i to row j.

The value of $\det(\mathbf{B})$ is as follows:

- $\det(\mathbf{B}) = -\det(\mathbf{A})$ for Type I operations.
- $\det(\mathbf{B}) = \alpha \det(\mathbf{A})$ for Type II operations.
- $\det(\mathbf{B}) = \det(\mathbf{A})$ for Type III operations.

- It is now possible to evaluate the determinant of an elementary matrix associated with any of the three types of elementary operations.
- Let $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ be elementary matrices of Types I, II, and III, respectively.
- det$(\mathbf{I}) = 1$, det$(\mathbf{E}) = -1$, det$(\mathbf{F}) = \alpha$ and det$(\mathbf{G}) = 1$.

$$\begin{aligned} \det(\mathbf{EA}) &= -\det(\mathbf{A}) &= \det(\mathbf{E})\det(\mathbf{A}), \\ \det(\mathbf{FA}) &= \alpha\det(\mathbf{A}) &= \det(\mathbf{F})\det(\mathbf{A}), \\ \det(\mathbf{GA}) &= \det(\mathbf{A}) &= \det(\mathbf{G})\det(\mathbf{A}). \end{aligned}$$

- In other words, det $(\mathbf{PA}) =$ det $(\mathbf{P})$det $(\mathbf{A})$ whenever $\mathbf{P}$ is an elementary matrix of Type I, II, or III. It' s easy to extend this observation to any number of these elementary matrices, $\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_k$, by writing

$$\begin{aligned} \det(\mathbf{P}_1\mathbf{P}_2\cdots\mathbf{P}_k\mathbf{A}) &= \det(\mathbf{P}_1)\det(\mathbf{P}_2\cdots\mathbf{P}_k\mathbf{A}) \\ &= \det(\mathbf{P}_1)\det(\mathbf{P}_2)\det(\mathbf{P}_3\cdots\mathbf{P}_k\mathbf{A}) \\ &\vdots \\ &= \det(\mathbf{P}_1)\det(\mathbf{P}_2)\cdots\det(\mathbf{P}_k)\det(\mathbf{A}). \end{aligned}$$

Invertibility and Determinants

- $\mathbf{A}_{n\times n}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$

  or, equivalently,

- $\mathbf{A}_{n\times n}$ is singular if and only if $\det(\mathbf{A}) = 0$.

- It might be easy to get idea that $\det(\mathbf{A})$ is somehow a measure of how close $\mathbf{A}$ is to being singular, but this is not necessarily the case.
- **Small Determinants $\nLeftrightarrow$ Near Singularity.**
- A minor determinant (or simply a minor) of $\mathbf{A}_{m\times n}$ is defined to be the determinant of any $k \times k$ submatrix of $\mathbf{A}$. For example,

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \text{ and } \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -6 \text{ are } 2\times 2 \text{ minors of } \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

An individual entry of $\mathbf{A}$ can be regarded as a 1×1 minor, and $\det(\mathbf{A})$ itself is considered to be a 3×3 minor of $\mathbf{A}$.

- $rank(\mathbf{A})$ = the size of the largest nonzero minor of $\mathbf{A}$.

Problem: Use determinants to compute the rank of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{pmatrix}$.

Solution: Clearly, there are 1×1 and 2×2 minors that are nonzero, so $rank(\mathbf{A}) \geq 2$. In order to decide if the rank is three, we must see if there are any 3×3 nonzero minors. There are exactly four 3×3 minors, and they are

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 7 & 8 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 3 & 1 \\ 4 & 6 & 1 \\ 7 & 9 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 & 1 \\ 5 & 6 & 1 \\ 8 & 9 & 1 \end{vmatrix} = 0.$$

Since all 3×3 minors are 0, we conclude that $rank(\mathbf{A}) = 2$.

Product Rules

- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ for all $n \times n$ matrices.
- $\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \det(\mathbf{A})\det(\mathbf{D})$ if $\mathbf{A}$ and $\mathbf{D}$ are square.

- The product rule provides a practical way to compute determinants.
- The definition of a determinant is purely algebraic, but there is a concrete geometrical interpretation.
- A solid in $\mathfrak{R}^m$ with parallel opposing faces whose adjacent sides are defined by vectors from a linearly independent set $\{x_1, x_2, \cdots, x_n\}$ is called an n-dimensional parallelepiped.
- A two-dimensional parallelepiped is a parallelogram, and a three-dimensional parallelepiped is a skewed rectangular box.
- When $\mathbf{A} \in \mathfrak{R}^{m\times n}$ has linearly independent columns, the volume of the n-dimensional parallelepiped generated by the columns of $\mathbf{A}$ is $V_n = \left[\det(\mathbf{A}^T\mathbf{A})\right]^1/2$. In particular, if $\mathbf{A}$ is square, then $V_n = |\det(\mathbf{A})|$.

- For every nonsingular matrix $\mathbf{A}$, there is a permutation matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{LU}$ in which $\mathbf{L}$ is lower triangular with 1's on its diagonal, and $\mathbf{U}$ is upper triangular with the pivots on its diagonal.
- The product rule guarantees that $\det(\mathbf{P})\det(\mathbf{A}) = \det(\mathbf{L})\det(\mathbf{U})$.

Computing a Determinant

If $\mathbf{PA}_{n\times n} = \mathbf{LU}$ is an LU factorization obtained with row interchanges (use partial pivoting for numerical stability), then

$$\det(\mathbf{A}) = \sigma u_{11} u_{22} \cdots u_{nn}.$$

The u_{ii}'s are the pivots, and σ is the sign of the permutation. That is,

$$\sigma = \begin{cases} +1 & \text{if an } \textit{even} \text{ number of row interchanges are used,} \\ -1 & \text{if an } \textit{odd} \text{ number of row interchanges are used.} \end{cases}$$

If a zero pivot emerges that cannot be removed (because all entries below the pivot are zero), then $\mathbf{A}$ is singular and $\det(\mathbf{A}) = 0$.

- It's sometimes necessary to compute the derivative of a determinant whose entries are differentiable functions.

Derivative of a Determinant

If the entries in $\mathbf{A}_{n\times n} = [a_{ij}(t)]$ are differentiable functions of t, then

$$\frac{d\left(\det\left(\mathbf{A}\right)\right)}{dt} = \det\left(\mathbf{D}_1\right) + \det\left(\mathbf{D}_2\right) + \cdots + \det\left(\mathbf{D}_n\right),$$

where $\mathbf{D}_i$ is identical to $\mathbf{A}$ except that the entries in the i^{th} row are replaced by their derivatives—i.e., $[\mathbf{D}_i]_{k*} = \begin{cases} \mathbf{A}_{k*} & \text{if } i \neq k, \\ d\,\mathbf{A}_{k*}/dt & \text{if } i = k. \end{cases}$

- Evaluate the derivative $d(\det(\mathbf{A}))/dt$ for $\mathbf{A} = \begin{pmatrix} e^t & e^{-t} \\ \cos t & \sin t \end{pmatrix}$.

$$\frac{d\left(\det\left(\mathbf{A}\right)\right)}{dt} = \begin{vmatrix} \mathrm{e}^t & -\mathrm{e}^{-t} \\ \cos t & \sin t \end{vmatrix} + \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ -\sin t & \cos t \end{vmatrix} = \left(\mathrm{e}^t + \mathrm{e}^{-t}\right)\left(\cos t + \sin t\right).$$

Proof. This follows directly from the definition of a determinant by writing

$$\frac{d\big(\det(\mathbf{A})\big)}{dt} = \frac{d}{dt}\sum_p \sigma(p) a_{1p_1} a_{2p_2} \cdots a_{np_n} = \sum_p \sigma(p) \frac{d\big(a_{1p_1} a_{2p_2} \cdots a_{np_n}\big)}{dt}$$

$$= \sum_p \sigma(p) \Big(a'_{1p_1} a_{2p_2} \cdots a_{np_n} + a_{1p_1} a'_{2p_2} \cdots a_{np_n} + \cdots + a_{1p_1} a_{2p_2} \cdots a'_{np_n} \Big)$$

$$= \sum_p \sigma(p) a'_{1p_1} a_{2p_2} \cdots a_{np_n} + \sum_p \sigma(p) a_{1p_1} a'_{2p_2} \cdots a_{np_n} + \cdots + \sum_p \sigma(p) a_{1p_1} a_{2p_2} \cdots a'_{np_n}$$

$$= \det(\mathbf{D}_1) + \det(\mathbf{D}_2) + \cdots + \det(\mathbf{D}_n). \quad \blacksquare$$

Additional Properties of Determinants

- The purpose of this section is to present some additional properties of determinants that will be helpful in later developments.

Block Determinants

If $\mathbf{A}$ and $\mathbf{D}$ are square matrices, then

$$\det\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} = \begin{cases}\det(\mathbf{A})\det\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right) & \text{when } \mathbf{A}^{-1} \text{ exists,}\\ \det(\mathbf{D})\det\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right) & \text{when } \mathbf{D}^{-1} \text{ exists.}\end{cases}$$

The matrices $\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ are called the ***Schur complements*** of $\mathbf{A}$ and $\mathbf{D}$, respectively

Proof. If $\mathbf{A}^{-1}$ exists, then $\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{C}\mathbf{A}^{-1} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{0} & \mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\end{pmatrix}$

- Since the determinant of a product is equal to the product of the determinants, it's only natural to inquire if a similar result holds for sums.
- In other words, is $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$? Almost never!

- Nevertheless, there are still some statements that can be made regarding the determinant of certain types of sums.

Rank-One Updates

If $\mathbf{A}_{n\times n}$ is nonsingular, and if $\mathbf{c}$ and $\mathbf{d}$ are $n \times 1$ columns, then

- $\det\left(\mathbf{I} + \mathbf{c}\mathbf{d}^T\right) = 1 + \mathbf{d}^T\mathbf{c}$,
- $\det\left(\mathbf{A} + \mathbf{c}\mathbf{d}^T\right) = \det\left(\mathbf{A}\right)\left(1 + \mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}\right)$.

- The proof follows by applying the product rules

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{d}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} + \mathbf{c}\mathbf{d}^T & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{d}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{c} \\ \mathbf{0} & 1 + \mathbf{d}^T\mathbf{c} \end{pmatrix}$$

- Write $\mathbf{A} + \mathbf{c}\mathbf{d}^T = \mathbf{A}(\mathbf{I} + \mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T)$.

Problem: For $\mathbf{A} = \begin{pmatrix} 1+\lambda_1 & 1 & \cdots & 1 \\ 1 & 1+\lambda_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+\lambda_n \end{pmatrix}$, $\lambda_i \neq 0$, find $\det(\mathbf{A})$.

Solution: Express $\mathbf{A}$ as a rank-one updated matrix $\mathbf{A} = \mathbf{D} + \mathbf{e}\mathbf{e}^T$, where $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mathbf{e}^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$.

$$\det(\mathbf{D} + \mathbf{e}\mathbf{e}^T) = \det(\mathbf{D})\left(1 + \mathbf{e}^T\mathbf{D}^{-1}\mathbf{e}\right) = \left(\prod_{i=1}^{n} \lambda_i\right)\left(1 + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right).$$

Cramer's Rule

In a nonsingular system $\mathbf{A}_{n\times n}\mathbf{x} = \mathbf{b}$, the i^{th} unknown is

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})},$$

where $\mathbf{A}_i = \left[\mathbf{A}_{*1} \,\middle|\, \cdots \,\middle|\, \mathbf{A}_{*i-1} \,\middle|\, \mathbf{b} \,\middle|\, \mathbf{A}_{*i+1} \,\middle|\, \cdots \,\middle|\, \mathbf{A}_{*n}\right]$. That is, $\mathbf{A}_i$ is identical to $\mathbf{A}$ except that column $\mathbf{A}_{*i}$ has been replaced by $\mathbf{b}$.

Proof. Since $\mathbf{A}_i = \mathbf{A} + (\mathbf{b} - \mathbf{A}_{*i})\,\mathbf{e}_i^T$, where $\mathbf{e}_i$ is the i^{th} unit vector,

$$\det(\mathbf{A}_i) = \det(\mathbf{A})\Big(1 + \mathbf{e}_i^T \mathbf{A}^{-1}(\mathbf{b} - \mathbf{A}_{*i})\Big) = \det(\mathbf{A})\Big(1 + \mathbf{e}_i^T(\mathbf{x} - \mathbf{e}_i)\Big)$$
$$= \det(\mathbf{A})(1 + x_i - 1) = \det(\mathbf{A})\,x_i.$$

Thus $x_i = \det(\mathbf{A}_i)/\det(\mathbf{A})$ because $\mathbf{A}$ being nonsingular insures $\det(\mathbf{A}) \neq 0$

Problem: Determine the value of t for which $x_3(t)$ is minimized in

$$\begin{pmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1/t \\ 1/t^2 \end{pmatrix}.$$

Solution: Only one component of the solution is required, so it's wasted effort to solve the entire system. Use Cramer's rule to obtain

$$x_3(t) = \frac{\begin{vmatrix} t & 0 & 1 \\ 0 & t & 1/t \\ 1 & t^2 & 1/t^2 \end{vmatrix}}{\begin{vmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{vmatrix}} = \frac{1 - t - t^2}{-1} = t^2 + t - 1, \quad \text{and set} \quad \frac{d\,x_3(t)}{dt} = 0$$

- Recall that minor determinants of $\mathbf{A}$ are simply determinants of submatrices of $\mathbf{A}$.
- We are now in a position to see that in an $n \times n$ matrix the $n-1 \times n-1$ minor determinants have a special significance.

Cofactors

The ***cofactor*** of $\mathbf{A}_{n\times n}$ associated with the (i, j)-position is defined as

$$\mathring{A}_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the $n-1 \times n-1$ minor obtained by deleting the i^{th} row and j^{th} column of $\mathbf{A}$. The matrix of cofactors is denoted by $\mathring{\mathbf{A}}$.

Problem: For $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 6 \\ -3 & 9 & 1 \end{pmatrix}$, determine the cofactors $\mathring{A}_{21}$ and $\mathring{A}_{13}$.

Solution:

$\mathring{A}_{21} = (-1)^{2+1} M_{21} = (-1)(-19) = 19$ and $\mathring{A}_{13} = (-1)^{1+3} M_{13} = (+1)(18) = 18$.

The entire matrix of cofactors is $\mathring{\mathbf{A}} = \begin{pmatrix} -54 & -20 & 18 \\ 19 & 7 & -6 \\ -6 & -2 & 2 \end{pmatrix}$.

- The cofactors of a square matrix $\mathbf{A}$ appear naturally in the expansion of det $(\mathbf{A})$. For example,

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
&\qquad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \\
&= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) \\
&\qquad + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \\
&= a_{11}\mathring{A}_{11} + a_{12}\mathring{A}_{12} + a_{13}\mathring{A}_{13}.
\end{aligned}$$

- This expansion is called the cofactor expansion of det$(\mathbf{A})$ in terms of the first row. It can also be written as any other row or column.

Cofactor Expansions

- $\det(\mathbf{A}) = a_{i1}\mathring{A}_{i1} + a_{i2}\mathring{A}_{i2} + \cdots + a_{in}\mathring{A}_{in}$ (about row i).
- $\det(\mathbf{A}) = a_{1j}\mathring{A}_{1j} + a_{2j}\mathring{A}_{2j} + \cdots + a_{nj}\mathring{A}_{nj}$ (about column j).

Problem: Use cofactor expansions to evaluate $\det(\mathbf{A})$ for

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 7 & 1 & 6 & 5 \\ 3 & 7 & 2 & 0 \\ 0 & 3 & -1 & 4 \end{pmatrix}.$$

Solution: To minimize the effort, expand $\det(\mathbf{A})$ in terms of the row or column that contains a maximal number of zeros. For this example, the expansion in terms of the first row is most efficient because

$$\det(\mathbf{A}) = a_{11}\mathring{A}_{11} + a_{12}\mathring{A}_{12} + a_{13}\mathring{A}_{13} + a_{14}\mathring{A}_{14} = a_{14}\mathring{A}_{14} = (2)(-1)\begin{vmatrix} 7 & 1 & 6 \\ 3 & 7 & 2 \\ 0 & 3 & -1 \end{vmatrix}.$$

Now expand this remaining 3×3 determinant either in terms of the first column or the third row. Using the first column produces

$$\begin{vmatrix} 7 & 1 & 6 \\ 3 & 7 & 2 \\ 0 & 3 & -1 \end{vmatrix} = (7)(+1)\begin{vmatrix} 7 & 2 \\ 3 & -1 \end{vmatrix} + (3)(-1)\begin{vmatrix} 1 & 6 \\ 3 & -1 \end{vmatrix} = -91 + 57 = -34,$$

so $\det(\mathbf{A}) = (2)(-1)(-34) = 68$. You may wish to try an expansion using different rows or columns, and verify that the final result is the same.

- In the previous example, we were able to take advantage of the fact that there were zeros in convenient positions.
- However, for a general matrix $\mathbf{A}_{n\times n}$ with no zero entries, it's not difficult to verify that successive application of cofactor expansions requires $n!\left(1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n-1)!}\right)$ multiplications to evaluate det $(\mathbf{A})$.
- Even for moderate values of n, this number is too large for the cofactor expansion to be practical for computational purposes.
- Nevertheless, cofactors can be useful for theoretical developments such as the following determinant formula for $\mathbf{A}^{-1}$.

Determinant Formula for $\mathbf{A}^{-1}$

The ***adjugate*** of $\mathbf{A}_{n\times n}$ is defined to be $\operatorname{adj}(\mathbf{A}) = \mathring{\mathbf{A}}^T$, the transpose of the matrix of cofactors—some older texts call this the ***adjoint*** matrix. If $\mathbf{A}$ is nonsingular, then

$$\mathbf{A}^{-1} = \frac{\mathring{\mathbf{A}}^T}{\det(\mathbf{A})} = \frac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})}.$$

Proof. $[\mathbf{A}^{-1}]_{ij}$ is the i^{th} component in the solution to $\mathbf{Ax} = \mathbf{e}_j$, where $\mathbf{e}_j$ is the j^{th} unit vector. By Cramer's rule, this is

$$[\mathbf{A}^{-1}]_{ij} = x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})},$$

where $\mathbf{A}_i$ is identical to $\mathbf{A}$ except that the i^{th} column has been replaced by $\mathbf{e}_j$, and the cofactor expansion in terms of the i^{th} column implies that

$$\det(\mathbf{A}_i) = \begin{array}{c} \phantom{a_{11}\cdots}\overset{i^{th}}{\downarrow}\phantom{\cdots a_{1n}} \\ \begin{vmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{j1} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} \end{array} = \mathring{A}_{ji}. \quad \blacksquare$$

Problem: For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, determine a general formula for $\mathbf{A}^{-1}$.

Solution: $\mathrm{adj}(\mathbf{A}) = \mathring{\mathbf{A}}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $\det(\mathbf{A}) = ad - bc$, so

$$\mathbf{A}^{-1} = \frac{\mathrm{adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Exercises

1. What is the volume of the parallelepiped generated by the three vectors $x_1 = (3, 0, -4, 0)^T$, $x_2 = (0, 2, 0, -2)^T$ and $x_3 = (0, 1, 0, 1)^T$?
2. If $\mathbf{A}$ is $n \times n$, explain why det$(\alpha\mathbf{A})$=α^ndet$(\mathbf{A})$ for all scalars α.
3. Use a cofactor expansion to evaluate each of the following determinants.

   (a) $\begin{vmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{vmatrix}$, (b) $\begin{vmatrix} 0 & 0 & -2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 0 & 2 & -3 & 0 \end{vmatrix}$.
4. By example, show that det$(\mathbf{A} + \mathbf{B}) \neq$ det$(\mathbf{A}) + det(\mathbf{B})$.
5. Using square matrices, construct an example that shows that

$$\text{det} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \neq \text{det}(\mathbf{A})det(\mathbf{D}) - \text{det}(\mathbf{B})det(\mathbf{C}).$$