## 1. 对于矩阵

$$\mathbf{A}=\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}, \mathbf{B}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{C}=\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix},$$

分别计算 Frobenius-norm, 1-norm, 2-norm, ∞-norm.

### 1.1 Frobenius 范数

Frobenius 范数定义为矩阵中所有元素的平方和的平方根。假设矩阵$M=[m_{ij}]$为$m\times n$的矩阵，则 Frobenius 范数为：

$$\|M\|_F=\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|m_{ij}|^2}$$

对于矩阵$A$:

$\|A\|_F=\sqrt{1^2+(-2)^2+(-1)^2+2^2}=\sqrt{1+4+1+4}=\sqrt{10}\approx 3.162$

对于矩阵$B$：

$\|B\|_F=\sqrt{0^2+1^2+0^2+0^2+0^2+1^2+1^2+0^2+0^2}=\sqrt{0+1+0+0+0+1+1+0+0}=\sqrt{3}\approx 1.732$

对于矩阵$C$：

$\|C\|_F=\sqrt{4^2+(-2)^2+4^2+(-2)^2+1^2+(-2)^2+4^2+(-2)^2+4^2}=\sqrt{81}=9$

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### 1.2 1-范数

1-范数定义为矩阵所有列元素绝对值和的最大值。对于矩阵$M$ ,1-范数为：

$$\|M\|_1=\max_{1\leq j\leq n}\sum_{i=1}^{m}|m_{ij}|$$

对于矩阵$A$：
第一列的和：$|1|+|-1|=1+1=2$
第二列的和：$|-2|+|2|=2+2=4$
$\|A\|_1=4$
对于矩阵$B$：
第一列的和：$|0|+|0|+|1|=1$
第二列的和：$|1|+|0|+|0|=1$
第三列的和：$|0|+|1|+|0|=1$
$\|B\|_1=1$

对于矩阵$C$：
第一列的和：$|4|+|-2|+|4|=4+2+4=10$
第二列的和：$|-2|+|1|+|-2|=2+1+2=5$
第三列的和：$|4|+|-2|+|4|=4+2+4=10$
$\|C\|_1=10$

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### 1.3 2-范数

- The matrix norm induced by the euclidean vector norm is
$\|\mathbf{A}\|_2=\max_{\|\mathbf{x}\|_2=1}\|\mathbf{A}\mathbf{x}\|_2=\sqrt{\lambda_{\max}}$,
where $\lambda_{\max}$ is the largest number $\lambda$ such that $\mathbf{A}^*\mathbf{A}-\lambda I$ is singular.

2-范数是转置共轭矩阵与矩阵A的积的最大特征根的平方根值

$A^T = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} A^T A = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$

$$\begin{aligned}(A^T A - \lambda I) = \begin{pmatrix} 2-\lambda & -4 \\ -4 & 8-\lambda \end{pmatrix} => |f(A)| \\ &= 16 - 10\lambda + \lambda^2 - 16 \\ &= \lambda^2 - 10a \\ &=> \lambda = 0, \lambda = 10 \\ &\sqrt{\lambda_{max}} = \sqrt{10}\end{aligned}$$

**对于矩阵$A$:2-范数约为$\sqrt{10} \approx 3.162.$**

$B^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$B^T B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B^T B - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$

令$|B^T B - \lambda I| = 0 \equiv$ 令 $B^T B - \lambda I$奇异

$1 - \lambda = 0$时$, \lambda = 1$

**对于矩阵$B$:2-范数为 1**

$C^T = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} = C$

$C^T C = \begin{pmatrix} 36 & -34 & 36 \\ -18 & 9 & -18 \\ 36 & -34 & 36 \end{pmatrix}$

$C^T - \lambda I = \begin{pmatrix} 36-\lambda & -34 & 36 \\ -18 & 9-\lambda & -18 \\ 36 & -34 & 36-\lambda \end{pmatrix}$

$$\begin{aligned}|C^T C - \lambda I| &= (-1)^{1+1} \cdot (36-\lambda)[(9-\lambda)(36-\lambda) - 18 \times 34] \\ &+ (-1)^{1+2} \cdot (-34)[(-18)(36-\lambda) + 18 \times 36] \\ &+ (-1)^{1+3} \cdot (36)[(-18)(-34) - (9-\lambda)36] = -(\lambda - 36)((\lambda-36)(\lambda-9) - 612) = 0\end{aligned}$$

**对于矩阵$C$:2-范数为 $\sqrt{36} = 6$**

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## 2.对于向量空间$\mathbf{R}^2 \times 2$,定义$< \mathbf{A}, \mathbf{B} >= trace(\mathbf{A}^T \mathbf{B}).$

(1) 简要说明$< \mathbf{A}, \mathbf{B} >$满足内积定义，为 $\mathbf{R}^{2\times2}$ 空间的一个内积。

(2)证明

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

为向量空间$\mathbf{R}^{2\times2}$的一组标准正交基，并计算矩阵$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$在该组基下的傅里叶展开 (Fouier expansion).

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### (1) 验证$\langle A, B \rangle =$trace$(A^T B)$ 是否为 $\mathbb{R}^{2\times2}$空间的内积。

要验证$\langle A, B \rangle$是否满足内积的定义，我们需要检查它是否满足以下内积的四个性质：

1. 非负性：对于任意矩阵$A$, $\langle A, A \rangle \geq 0$,且当且仅当$A = 0$时$\langle A, A \rangle = 0$。

$$< A,A > = \operatorname{trace}\left(A^T A\right) = \operatorname{trace}\left(\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right)$$
$$= \operatorname{trace}\begin{pmatrix} a_{11}\cdot a_{11} + a_{21}\cdot a_{21}, & a_{11}\cdot a_{12} + a_{21}\cdot a_{22} \\ a_{12}\cdot a_{11} + a_{22}\cdot a_{21}, & a_{12}\cdot a_{12} + a_{22}\cdot a_{22} \end{pmatrix}$$
$$= a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2$$

当$A \neq \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)$时, $< A,A > \neq 0$

2. 共轭对称性: $\langle A,B\rangle = \langle B,A\rangle$。

$$< A,B >= trace(A^T B) = trace\left(\left(\begin{array}{cc} a_{11} & a_{11} \\ a_{12} & a_{22} \end{array}\right)\left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & a_{22} \end{array}\right)\right)$$
$$= a_{11}b_{11} + a_{21}b_{21} + a_{12}\cdot b_{12} + a_{22}\cdot b_{22}$$
$$< B,A > = b_{11}a_{11} + b_{21}a_{21} + b_{12}\cdot a_{12} + b_{22}\cdot a_{22}$$
$$=< A,B >$$

3. 线性性: 对于任意矩阵$A$和$B$,以及实数$c$,有$\langle cA,B\rangle = c\langle A,B\rangle$。

$$< cA,B > = ca_{11}b_{11} + ca_{21}b_{21} + ca_{12}b_{12} + ca_{22}b_{22}$$
$$= c(a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22})$$
$$= c < A,B >$$

4. 可加性: 对于任意矩阵$A$、$B$、$C$,有$\langle A+B,C\rangle = \langle A,C\rangle + \langle B,C\rangle$。

$$< A+B,C > = \operatorname{trace}\left(\left(\begin{array}{cc} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{array}\right)\left(\begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \end{array}\right)\right)$$
$$= (a_{11}+b_{11})c_{11} + (a_{21}+b_{21})c_{21} + (a_{12}+b_{12})c_{12} + (a_{22}+b_{22})c_{22}$$
$$=< A_2C > + < B_2C >$$

验证这四个性质即可说明这是$\mathbb{R}^{2\times 2}$空间的一个内积。

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**(2)证明集合$\mathcal{B}$为$\mathbb{R}^{2\times 2}$的一组标准正交基，并计算矩阵$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$在这组基下的傅里叶展开。**

$$B_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad B_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

1. 正交性: 要验证$\mathcal{B}$中的基向量是否正交，我们需要检查集合中的每对基向量$B_i$和$B_j$是否满足$\langle B_i,B_j\rangle = 0$当$(i \neq j)$时

$$< B_1,B_2 >= 0\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot 0 + \frac{1}{\sqrt{2}}\cdot 0 + 0\cdot\frac{-1}{\sqrt{2}} \quad = 0$$
$$< B_1,B_3 >= 0\cdot\frac{1}{2} + \frac{1}{\sqrt{2}}\cdot\frac{-1}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} + 0\cdot\frac{1}{2} \quad = 0$$
$$< B_1,B_4 >= 0\cdot\frac{1}{2} + \frac{1}{\sqrt{2}}\cdot\frac{-1}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} + 0\cdot\frac{1}{2} \quad = 0$$
$$< B_2,B_3 >= \frac{1}{\sqrt{2}}\cdot\frac{1}{2} + 0\cdot\frac{-1}{2} + 0\cdot\frac{1}{2} + \frac{-1}{\sqrt{2}}\cdot\frac{1}{2} \quad = 0$$
$$< B_3,B_4 >= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{-1}{2} + \frac{-1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} \quad = 0$$

2. 标准化: 如果所有基向量都满足$\langle B_i,B_i\rangle = 1$,那么这些基向量是标准的。

基向量分别为:

$< B_1,B_1 >= 0 + \frac{1}{2} + \frac{1}{2} + 0 = 1$ $< B_3,B_3 >= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$

$< B_2,B_2 >= \frac{1}{2} + 0 + 0 + \frac{1}{2} = 1$ $< B_4,B_4 >= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$

$$A = \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right) = C_1B_1 + C_2B_2 + C_3B_3 + C_4B_4$$
$$= \begin{pmatrix} \frac{\sqrt{2}}{2}C_2 + \frac{-1}{2}C_3 + \frac{1}{2}C_4 & \frac{\sqrt{2}}{2}C_1 + \frac{-1}{2}C_3 + \frac{1}{2}C_4 \\ \frac{\sqrt{2}}{2}C_1 + \frac{1}{2}C_3 + \frac{-1}{2}C_4 & \frac{-\sqrt{2}}{2}C_2 + \frac{1}{2}C_3 + \frac{1}{2}C_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{1}{2} & \frac{-1}{2} \\ 0 & \frac{-\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$A_{[\beta]} = (\sqrt{2}, \sqrt{2}, 2, 2)^T$

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## 3.对于向量组$\left\{ \mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix} \right\}$在三个有效数字情形下，分别使用传统 Gram-Schmidt 和修改后的 Gram-Schmidt 方法，把上述向量组正交化

1. 正交化 $\mathrm{x}_1$
   由于$\mathbf{x_1}$是第一个向量，我们将它归一化得到$\mathbf{q_1}$：$\mathbf{q_1} = \frac{\mathbf{x_1}}{\|\mathbf{x_1}\|}$
   $\|\mathbf{x_1}\| = \sqrt{1^2 + 0^2 + (10^{-3})^2} \approx 1.000$
   因此，$\mathbf{q_1} = \begin{pmatrix} 1 \\ 0 \\ 0.001 \end{pmatrix}$
2. 正交化$\mathrm{x}_2$
   将$\mathbf{x}_2$从$\mathbf{q}_1$中正交化得到$\mathbf{u}_2$：$\mathbf{u_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0.001 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -0.001 \end{pmatrix}$
   计算$\mathbf{q_1} \cdot \mathbf{x_2}$:
   $\mathbf{q}_1 \cdot \mathbf{x}_2 = 1 \times 1 + 0 \times 0 + 0.001 \times 0 = 1.$
   因此，归一化$\mathbf{u}_2$得到$\mathbf{q}_2$：$\mathbf{u_2} = \mathbf{x_2} - \mathrm{proj}_{\mathrm{q}_1}\mathbf{x_2} = \mathbf{x_2} - (\mathbf{q_1} \cdot \mathbf{x_2})\mathbf{q_1}.$
   $\|\mathbf{u_2}\| = \sqrt{0^2 + 0^2 + (-0.001)^2} = 0.001,$
   $\mathbf{q_2} = \frac{\mathbf{u_2}}{0.001} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
3. 正交化$\mathrm{x}_3$
   将$\mathbf{x}_3$从$\mathbf{q}_1$和$\mathbf{q}_2$中正交化得到$\mathbf{u}_3$：
   $$\mathbf{u_3} = \mathbf{x_3} - \mathrm{proj}_{\mathrm{q1}}\mathbf{x_3} - \mathrm{proj}_{\mathrm{q2}}\mathbf{x_3}.$$
   计算$\mathbf{q_1} \cdot \mathbf{x_3}$和$\mathbf{q_2} \cdot \mathbf{x_3}$：
   $$\mathbf{q_1} \cdot \mathbf{x_3} = 1 \times 1 + 0 \times 10^{-3} + 0.001 \times 0 = 1,$$
   $$\mathbf{q_2} \cdot \mathbf{x_3} = 0 \times 1 + 0 \times 10^{-3} + (-1) \times 0 = 0.$$
   因此，
   $$\mathbf{u}_3 = \mathbf{x}_3 - 1 \cdot \mathbf{q}_1 = \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0.001 \end{pmatrix} = \begin{pmatrix} 0 \\ 10^{-3} \\ -0.001 \end{pmatrix}.$$
   归一化$\mathbf{u}_3$得到$\mathbf{q}_3$：
   $$\|\mathbf{u_3}\| = \sqrt{0^2 + (10^{-3})^2 + (-0.001)^2} = \sqrt{2 \times (10^{-3})^2} \approx 0.00141,$$
   $$\mathbf{q_3} = \frac{\mathbf{u_3}}{0.00141} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \end{pmatrix}.$$
4. 最终，正交基向量组为:
   $$\mathbf{q_1} = \begin{pmatrix} 1 \\ 0 \\ 0.001 \end{pmatrix}, \quad \mathbf{q_2} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{q_3} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \end{pmatrix}.$$

**4.试判断矩阵$\begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{-2i}{\sqrt{6}} \end{pmatrix}$是否为酉矩阵。**

$$U = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{-2i}{\sqrt{6}} \end{pmatrix} U^* = \begin{pmatrix} \frac{1-i}{\sqrt{3}} & \frac{-i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{6}} & \frac{2i}{\sqrt{6}} \end{pmatrix}$$

$$\begin{aligned} U^*U &= \begin{pmatrix} \frac{(1-i)(1+i)}{\sqrt{3}\cdot\sqrt{3}} + \frac{(-i+i)}{\sqrt{3}\cdot\sqrt{3}} & \frac{(1-i)(1+i)}{\sqrt{3}\cdot\sqrt{6}} + \frac{(-i)(2i)}{\sqrt{3}\cdot\sqrt{6}} \\ \frac{(1-i)(i+i)}{\sqrt{6}\cdot\sqrt{3}} + \frac{(2i)(i)}{\sqrt{6}\cdot\sqrt{3}} & \frac{(1-i)(1+i)}{\sqrt{6}\cdot\sqrt{6}} + \frac{(2i)(-2i)}{\sqrt{6}\cdot\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+1+1+i-i}{3} & \frac{(1+1+i-i)+2(-1)}{3\sqrt{2}} \\ \frac{1+1+i-i}{3\sqrt{2}} & \frac{1+1+i-i+2-2\cdot(i-1)}{6} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \longrightarrow U\text{是酉矩阵} \end{aligned}$$

**5.从向量$\mathbf{x} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$出发，使用 elementary reflector 构造 $R^3$一组标准正交基。****

给定向量$\mathbf{x} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$

我们希望找到一个 Householder 反射矩阵$H$, 使得 $H\mathbf{x}$变成与$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$平行的向量

1. 计算$\|\mathbf{x}\|$ 向量 $\mathbf{x}$ 的模为:

$$\|\mathbf{x}\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{1} = 1.$$

2. 选择$\alpha$并构造向量 v
   我们令$\alpha = -\|\mathbf{x}\| = -1$,以确保反射唯一。
   接下来构造向量$\mathbf{v} = \mathbf{x} - \alpha\mathbf{e}_1 = \mathbf{x} + \mathbf{e}_1$：

$$\mathbf{v} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}.$$

3. 归一化v 计算 v 的模:
   $\|\mathbf{v}\| = \sqrt{\left(\frac{4}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{16}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{24}{9}} = \frac{\sqrt{24}}{3} = \frac{2\sqrt{6}}{3}$
   因此,单位向量 $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$为:
   $\mathbf{u} = \frac{1}{\frac{2\sqrt{6}}{3}} \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} = \frac{3}{2\sqrt{6}} \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$
4. 构造 Householder 矩阵$H$
   Householder 矩阵的定义为:
   $H = I - 2\mathbf{u}\mathbf{u}^T$
   $\mathbf{u}\mathbf{u}^T = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{4}{6} & -\frac{2}{6} & -\frac{2}{6} \\ -\frac{2}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{2}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$
   计算$\mathbf{H} = I - 2\mathbf{u}\mathbf{u}^T$：
   $H = I - 2\begin{pmatrix} \frac{4}{6} & -\frac{2}{6} & -\frac{2}{6} \\ -\frac{2}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{2}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

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**6.$\mathbf{A}=\begin{pmatrix} 0 & -20 & -14 \\ 3 & 27 & -4 \\ 4 & 11 & -2 \end{pmatrix}$,使用 Given reduction 方法找到一个正交矩阵$\mathbf{P}$,使得$\mathbf{PA}=\mathbf{T}$,这里$\mathbf{T}$为上三角矩阵，且对角元素都为正数。**

$$\mathbf{P}_{ij}=\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \overline{c} & & & \overline{s} & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & -s & & & c & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} \longleftarrow \text{row } irowi$$

$$A=\begin{pmatrix} 0 & -20 & -14 \\ 3 & 27 & -4 \\ 4 & 11 & -2 \end{pmatrix}$$

1. 消去 $a_{31}$
   $$\cos=\frac{0}{\sqrt{0^2+4^2}}=0 \quad , \quad \sin=\frac{4^2}{\sqrt{0^2+4^2}}=1$$
   $$T_{13}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
   $$A'=T_{13}\cdot A=\begin{pmatrix} 4 & 11 & -2 \\ 3 & 27 & -4 \\ 0 & 20 & 14 \end{pmatrix}$$
2. 消去 $a'_{21}$
   $$\cos=0.8\sin=0.6$$
   $$T'_{12}=\begin{pmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0.8 & 1 \end{pmatrix}$$
   $$A''=T'_{12}T_{13}A=\begin{pmatrix} 5 & 8.8+16.2 & -1.6-2.4 \\ -2.4+2.4 & -6.6+21.6 & 1.2-3.2 \\ 0 & 20 & 14 \end{pmatrix}$$
   $$=\begin{pmatrix} 5 & 25 & -4 \\ 0 & 15 & -2 \\ 0 & 20 & 14 \end{pmatrix}$$
3. 消去$a''_{32}, \cos=0.6, \sin=0.8$
   $$T'''_{23}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{pmatrix}$$
   $$A'''=\begin{pmatrix} 5 & 25 & -4 \\ 0 & 9+16 & -2.4-1.6 \\ 0 & -12+12 & 1.6+8.4 \end{pmatrix}=\begin{pmatrix} 5 & 25 & -4 \\ 0 & 25 & -4 \\ 0 & 0 & 10 \end{pmatrix}$$
   $$P=T''_{23}\cdot T'_{12}\cdot T_{13}$$
   $$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{pmatrix}\begin{pmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{12}{25} & -\frac{9}{25} \\ -\frac{3}{5} & -\frac{16}{25} & \frac{12}{25} \end{pmatrix}$$

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**7.对于矩阵 $\mathbf{A}=\begin{pmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{pmatrix}$ 分别使用 Householder reduction和 Givens reduction 实现该矩阵的 QR 分解。**

# $\mathbf{A}=\mathbf{QR}$

1. $\mathbf{H}_1=\mathbf{I}-2\mathbf{u}_1\mathbf{u}_1^T$

$$A=\begin{pmatrix}1&19&-34\\-2&-5&20\\2&8&37\end{pmatrix}$$

$$a_1=\begin{pmatrix}1&-2&2\end{pmatrix}^T\ \alpha_1=\|a\|_2=3$$

$$u_1=\frac{a_1-\alpha_1e_1}{||a_1-\alpha_1e_1||_2}=\frac{\begin{pmatrix}1&-2&2\end{pmatrix}^T}{\sqrt{12}}=\begin{pmatrix}\frac{-\sqrt3}{3}&\frac{-\sqrt3}{3}&\frac{\sqrt3}{3}\end{pmatrix}^T$$

$$H_1=I-2u_1u_1^T=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-2\begin{pmatrix}\frac13&\frac13&-\frac13\\\frac13&\frac13&-\frac13\\-\frac13&-\frac13&\frac13\end{pmatrix}$$

$$=\begin{pmatrix}\frac13&\frac{-2}{3}&\frac23\\\frac{-2}{3}&\frac13&\frac23\\\frac23&\frac23&\frac13\end{pmatrix}$$

$$H_1A=\begin{pmatrix}\frac13&\frac{-2}{3}&\frac23\\\frac{-2}{3}&\frac13&\frac23\\\frac23&\frac23&\frac13\end{pmatrix}\begin{pmatrix}1&19&-34\\-2&-5&20\\2&8&37\end{pmatrix}=\begin{bmatrix}3&15&0\\0&-9&54\\0&12&3\end{bmatrix}$$

$$a_2=\begin{pmatrix}-9&12\end{pmatrix}^T\ \alpha_2=\|a\|_2=15$$

$$u_1=\frac{a_2-\alpha_2e_1'}{||a_2-\alpha_2e_1'||_2}=\frac{\begin{pmatrix}-24&12\end{pmatrix}^T}{12\sqrt5}=\begin{pmatrix}\frac{-2\sqrt5}{5}&\frac{\sqrt5}{5}\end{pmatrix}^T$$

$$\tilde H_2=I_2-2\begin{pmatrix}\frac{-2\sqrt5}{5}\end{pmatrix}\begin{pmatrix}\frac{-2\sqrt5}{5}&\frac{\sqrt5}{5}\end{pmatrix}$$

$$=\begin{pmatrix}1-\frac85&0+\frac45\\0+\frac45&1-\frac25\end{pmatrix}=\begin{pmatrix}\frac{-3}{5}&\frac45\\\frac45&\frac35\end{pmatrix}$$

$$H_2H_1A=\begin{pmatrix}1&0&0\\0&\frac{-3}{5}&\frac45\\0&\frac45&\frac35\end{pmatrix}\begin{pmatrix}3&15&0\\0&-9&54\\0&12&3\end{pmatrix}=\begin{bmatrix}3&15&0\\0&15&-30\\0&0&45\end{bmatrix}=\text{R}$$

$$A=QR$$

$$=(H_2H_1)^{-1}R$$

$$=H_1^T\cdot H_2^TR$$

$$=\begin{pmatrix}\frac13&\frac{-2}{3}&\frac23\\\frac{-2}{3}&\frac13&\frac23\\\frac23&\frac23&\frac13\end{pmatrix}\begin{pmatrix}1&0&0\\0&\frac{-3}{5}&\frac45\\0&\frac45&\frac35\end{pmatrix}=\begin{bmatrix}\frac13&\frac{14}{15}&-\frac{2}{15}\\-\frac23&\frac13&\frac23\\\frac23&-\frac{2}{15}&\frac{11}{15}\end{bmatrix}$$

2. givens reduction

$$A=\begin{pmatrix}1&19&-34\\-2&-5&20\\2&8&37\end{pmatrix}$$

消去 $a_{31}\cos=\frac{\sqrt5}{5}\sin=\frac{2\sqrt5}{5}$

$$T_{13}=\begin{pmatrix}\frac{\sqrt5}{5}&0&\frac{2\sqrt5}{5}\\0&1&0\\\frac{-2\sqrt5}{5}&0&\frac{\sqrt5}{5}\end{pmatrix}$$

$$T_{13}A=\begin{pmatrix}\frac{\sqrt5+4\sqrt5}{5}&\frac{19\sqrt5+16\sqrt5}{5}&\frac{40\sqrt5}{5}\\-2&-5&20\\0&\frac{-38\sqrt5+8\sqrt5}{5}&\frac{68\sqrt5+37\sqrt5}{5}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt5&7\sqrt5&8\sqrt5\\-2&-5&20\\0&6\sqrt5&21\sqrt5\end{pmatrix}=\begin{bmatrix}\sqrt5&7\sqrt5&8\sqrt5\\-2&-5&20\\0&6\sqrt5&21\sqrt5\end{bmatrix}$$

消去 $a_{21}\cos=\frac{\sqrt5}{3}\sin=\frac{-3}{3}$

$$T_{12}=\begin{pmatrix}\frac{\sqrt5}{3}&\frac{-2}{3}&0\\\frac23&\frac{\sqrt5}{3}&0\\0&0&1\end{pmatrix}$$

$$T_{12}T_{13}A=\begin{pmatrix}\frac{5+4}{3}&\frac{35+10}{3}&\frac{40-40}{3}\\0&\frac{14\sqrt5-5\sqrt5}{3}&\frac{16\sqrt5+20\sqrt5}{3}\\0&6\sqrt5&21\sqrt5\end{pmatrix}=\begin{pmatrix}3&15&0\\0&3\sqrt5&12\sqrt5\\0&6\sqrt5&2\sqrt5\end{pmatrix}$$

消去 $a_{32}\cos=\frac{\sqrt5}{5}\sin=\frac{2\sqrt5}{5}$

$$T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ 0 & \frac{-2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{pmatrix}$$

$$T_{23}T_{12}T_B A = \begin{pmatrix} 3 & 15 & 0 \\ 0 & \frac{15+60}{5} & \frac{60+42\cdot 5}{5} \\ 0 & 0 & -24+21 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 15 & 0 \\ 0 & 15 & 54 \\ 0 & 0 & -3 \end{pmatrix} = R$$

$$A = QR$$

$$Q = (T_{23}T_{12}T_{13})^{-1}$$

$$= T_{13}^T T_{12}^T T_{23}^T$$

$$= \begin{pmatrix} \frac{\sqrt{5}}{5} & 0 & \frac{-2\sqrt{5}}{5} \\ 0 & 1 & 0 \\ \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{3} & \frac{2}{3} & 0 \\ \frac{-2}{3} & \frac{\sqrt{5}}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} \\ 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{pmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$