Chapter 1 Exercise Solutions

Ex. 1.1 We prove by contradiction. Suppose that $\sqrt{6}$ is a rational number. Put $\sqrt{6} = m/n$, with m, n being co-prime. This gives $m^2 = 6n^2$, which implies that m is a multiple of 3. Put $m = 3k$. Then we have $2n^2 = 3k^2$. This implies that n is also a multiple of 3. This contradicts to the hypothesis that m and n are co-prime. Thus, the number $\sqrt{6}$ is irrational.

Ex. 1.2 We prove by contradiction. Suppose that the set $\{\sqrt{n} : n \in \mathbb{N}\}$ is bounded. Denote B an upper bound of the set. Thus, we have that $\sqrt{n} \le B$ for all $n \in \mathbb{N}$. It gives that

$$n \le B^2, \qquad \text{for all } n \in \mathbb{N}.$$

This contradicts to the archimedean property.

Ex. 1.3 Suppose E is bounded above, and suppose β_1 and β_2 are two distinct suprema of E.

For any $x \in E$, we have $x \le \beta_1$ and $x \le \beta_2$. If $\beta_1 < \beta_2$, then

$$x \le \beta_1 < \beta_2,$$

so that β_2 is not a supremum of E by definition. This contradicts to the hypothesis. Hence, $\beta_1 \ge \beta_2$.

In the similar manner, we have $\beta_2 \ge \beta_1$.

Therefore, we must have $\beta_2 = \beta_1$.

Ex. 1.4 Denote $A = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \dots\right\}$.

- Clearly, for any $x \in A$, we have $x \ge 1$, so that 1 is a lower bound of A. Furthermore, for any positive ε, since $1 < 1+\varepsilon$ and $1 \in A$, we know that $1+\varepsilon$ is not a lower bound of A. By the definition, we have $\inf A = 1$.
- Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. We prove by induction that $x_{2^n} \ge 1 + \frac{n}{2}$, so that the sequence $\{x_n\}$ is not bounded above. It follows that $\sup A = \infty$.

  In fact, we have $x_2 = 1 + \frac{1}{2}$ so that the desired inequality holds for $n = 1$.

  Assume that $x_{2^k} \ge 1 + \frac{k}{2}$ for an integer $n = k$. Then

$$\begin{aligned} x_{2^{k+1}} &= x_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \ge x_{2^k} + \underbrace{\frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}}_{2^k \text{ terms}} \\ &= x_{2^k} + \frac{2^k}{2^{k+1}} = x_{2^k} + \frac{1}{2} \ge 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}, \end{aligned}$$

  so that the desired inequality holds for $n = k+1$.

Ex. 1.5 (⇒) Assume that $|x-a| < \delta$.

- If $x - a \ge 0$, then the inequality $|x-a| < \delta$ gives $x - a < \delta$, so that $x < a + \delta$. Thus, $a \le x < a + \delta$.
- If $x - a < 0$, then the inequality $|x-a| < \delta$ gives $-(x-a) < \delta$, so that $a - \delta < x$. Thus, $a - \delta < x < a$.

Combining above we conclude that we always have $a - \delta < x < a + \delta$.

(⇐) Assume that $a - \delta < x < a + \delta$.

- If $x - a \ge 0$, then $a \le x < a + \delta$, or $0 \le x - a < \delta$. Thus, we have $|x - a| < \delta$.
- If $x - a < 0$, then $a - \delta < x < a$, or $-\delta < x - a < 0$. Thus, we also have $|x - a| < \delta$.

Combining above we conclude that we always have $|x - a| < \delta$. ◀

Ex. 1.6 Denote $S = \big\{|a+b| : \ a^2 < 2, \, |b+1| < 3\big\}$.

First, we prove that the set is bounded. In fact, by the triangle inequality, we have

$$\begin{aligned} 0 \le |a+b| &= |a+b+1-1| \le |a| + |b+1| + |-1| \\ &= \sqrt{a^2} + |b+1| + 1 < \sqrt{2} + 3 + 1 = 4 + \sqrt{2}. \end{aligned}$$

Next, we find the infimum and the supremum of S.

- Take $a = 0$ and $b = 0$. Then $a^2 < 2$ and $|b+1| < 3$. Clearly, $|a+b| = 0$. Hence, we have $\inf S = 0$.
- For any $0 < \varepsilon < 1$, let
$$a = -\sqrt{2} + \tfrac{1}{4}\varepsilon, \qquad b = -4 + \tfrac{1}{4}\varepsilon.$$
Then $-\sqrt{2} < a < 0 < \sqrt{2}$ and $-3 < b+1 < -2 < 3$. Thus, $a^2 < 2$ and $|b+1| < 3$. Since
$$a + b = -4 - \sqrt{2} + \tfrac{1}{2}\varepsilon < 0,$$
we have
$$|a+b| = 4 + \sqrt{2} - \tfrac{1}{2}\varepsilon > 4 + \sqrt{2} - \varepsilon.$$
This demonstrates that $4 + \sqrt{2} - \varepsilon$ is not an upper bound of S. Hence, by definition, we have $\sup S = 4 + \sqrt{2}$.

Ex. 1.7 By the triangle inequality,

$$|x_1| = |x_1 - x_2 + x_2| \le |x_1 - x_2| + |x_2|,$$

so that $|x_1| - |x_2| \le |x_1 - x_2|$. Similarly, we have $|x_2| - |x_1| \le |x_1 - x_2|$. Combining these two inequalities, we get

$$\big||x_1| - |x_2|\big| \le |x_1 - x_2|.$$

Ex. 1.8 Let $A = \sum a_j^2$, $B = \sum b_j^2$, $C = \sum a_j b_j$.

If $B = 0$, then $b_j = 0$ for $j = 1, \ldots, n$. For $\lambda = 0$ and any $\mu \ne 0$, these values λ and μ are not both zero. Obviously, we have $\lambda a_j = \mu b_j$, $j = 1, 2, \ldots, n$.

If $B \ne 0$, then

$$\begin{aligned} 0 \le \sum_{j=1}^{n} (Ba_j - Cb_j)^2 &= B^2 \sum_{j=1}^{n} a_j^2 - BC \sum_{j=1}^{n} a_j b_j - BC \sum_{j=1}^{n} a_j b_j + C^2 \sum_{j=1}^{n} b_j^2 \\ &= B^2 A - BC^2 - BC^2 + BC^2 \\ &= B(AB - C^2). \end{aligned}$$

Since $AB - C^2 = 0$, we have $Ba_j - Cb_j = 0$, $j = 1, 2, \ldots, n$. If we take $\lambda = B$ and $\mu = C$, then, λ and μ are not both zero such that $\lambda a_j = \mu b_j$, $j = 1, 2, \ldots, n$. ◀

Chapter 2 Quiz Answers

①E: The set of all infinite sequences of 0's and 1's is not countable, by Proposition 2.8.

②B: By Proposition 2.11, we know that $\bigcap_\alpha A_\alpha^c = \left(\bigcup_\alpha A_\alpha\right)^c$. Thus, the given set relation $\bigcup_\alpha A_\alpha = \bigcap_\alpha A_\alpha^c$ is equivalent to $\bigcup_\alpha A_\alpha = \left(\bigcup_\alpha A_\alpha\right)^c$. The latter is equivalent to $\bigcup_\alpha A_\alpha = \varnothing$.

③B: The interior of A is the union of all open sets contained in A. Hence, the interior of A is a subset of A.

④E: Every compact set is closed, by Proposition 2.18.

⑤D: Take $E = (0,1) \subset \mathbb{R}$ and $K = [0,1] \subset \mathbb{R}$. By Theorem 2.21. We know that K is compact in $\mathbb{R}$, but E is not. Obviously, E is a bounded subset of K.

⑥A: Let S be a compact subset of $\mathbb{R}$. By Theorem 2.21, S is bounded and closed. Since $\mathbb{R}$ possesses the least-upper-bound property, $y = \sup S$ is finite. By Proposition 2.15, $y \in \overline{S}$. Because S is closed, by Proposition 2.14, $\overline{S} = S$. Hence, $\sup S \in S$. This means that S has a maximum element.

⑦C: By the corollary of Proposition 2.18, the intersection of a compact set and a closed set is compact.

⑧D: A perfect set is a closed set with no isolated points. Option D is true by definition.

Option A is false: Consider the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. It is compact but not perfect.

Option B is false: $\mathbb{R}$ is perfect, but not compact.

Option C is false: $\mathbb{R}$ is closed, but not compact.

Option E is false: The set of all rational numbers of $\mathbb{R}$ is dense in $\mathbb{R}$, but not closed, so not compact. ◀

Chapter 2 Exercise Solutions

Ex. 2.1 Define a function $f\colon (0,1) \to [0,1]$ by

$$f(x) = \begin{cases} \dfrac{1}{n-2}, & \text{if } x = \dfrac{1}{n},\ n = 3, 4, \ldots, \\ 0, & \text{if } x = \frac{1}{2}, \\ x, & \text{otherwise.} \end{cases}$$

Then f is bijective from $(0,1)$ onto $[0,1]$.

Proof: f is injective. Denote $S = \left\{\frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Let $x_1, x_2 \in (0,1)$, with $x_1 \neq x_2$.

① If $x_1, x_2 \in S$, then, there are distinct $m, n \geq 3$, such that $x_1 = \dfrac{1}{m}$, $x_2 = \dfrac{1}{n}$. Thus, $f(x_1) = \dfrac{1}{m-2} \neq \dfrac{1}{n-2} = f(x_2)$.

② If $x_1 = \dfrac{1}{n} \in S$ $(n \geq 3)$ and $x_2 = \frac{1}{2}$, then $f(x_1) = \dfrac{1}{n-2} \neq 0 = f(x_2)$.

③ If $x_1 = \dfrac{1}{n} \in S$ $(n \geq 3)$ and $x_2 \in (0,1) \backslash \left(S \cup \left\{\frac{1}{2}\right\}\right)$, then $f(x_1) = \dfrac{1}{n-2} \neq x_2 = f(x_2)$.

④ If $x_1 = \frac{1}{2}$ and $x_2 \in (0,1) \backslash \left(S \cup \left\{\frac{1}{2}\right\}\right)$, then $f(x_1) = 0 \neq x_2 = f(x_2)$.

⑤ If $x_1, x_2 \in (0,1) \backslash \left(S \cup \left\{\frac{1}{2}\right\}\right)$, then $f(x_1) = x_1 \neq x_2 = f(x_2)$.

In summary, for all $x_1, x_2 \in (0,1)$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Proof: f is surjective. For any $y \in [0,1]$, we have

$$y = \begin{cases} f(\frac{1}{2}), & \text{if } y = 0, \\ f\left(\dfrac{1}{n+2}\right), & \text{if } y = \dfrac{1}{n},\ n \geq 1, \\ f(y), & \text{if } y \neq 0 \text{ or } y \neq \dfrac{1}{n},\ n \geq 1. \end{cases}$$

Ex. 2.2 ($\Rightarrow$) Assume that E is open.

If $p \in E$, then there exists $r > 0$ such that $N_r(p) \subset E$. This means that p is an interior point, so $p \in E^\circ$. Hence, $E \subset E^\circ$.

On the other hand, interior points in E are necessarily in E, since any neighborhood of a point contains that point. Hence $E^\circ \subset E$.

Therefore, we conclude that $E = E^\circ$.

($\Leftarrow$) Assume that $E = E^\circ$. To show E is open, we only need to prove that E° is open.

Suppose $p \in E^\circ$. Then there exists $r > 0$ such that $N_r(p) \subset E$. Since $N_r(p)$ is open, for any point $x \in N_r(p)$, there exists $\delta > 0$ such that $N_\delta(x) \subset N_r(p) \subset E$. This means that every point in $N_r(p)$ is an interior point of E. Hence $N_r(p) \subset E^\circ$. This implies that E° is open.

Ex. 2.3 We prove that the interior E° of E is the largest open set contained in E by completing the following steps:

1. E° is an open set contained in E.
2. Any open set U contained in E is a subset of E°.

Step 1 Suppose $p \in E^\circ$. Then there exists $r > 0$ such that $N_r(p) \subset E$. Clearly, $p \in N_r(p)$. Thus, $p \in E$. Hence, E° is a subset of E.

Furthermore, since $N_r(p)$ is open, for any point $x \in N_r(p)$, there exists $\delta > 0$ such that $N_\delta(x) \subset N_r(p) \subset E$. This means that every point in $N_r(p)$ is an interior point of E, that is, $N_r(p) \subset E^\circ$. Hence, E° is an open set contained in E. ◀

Step 2 Let U be an open set contained in E. For any point $p \in U$, there exists an open neighborhood $N_r(p) \subset U$. Since U is a subset of E, we have $N_r(p) \subset E$. This means that p is an interior point of E, and hence belongs to E°. Therefore, U is a subset of E°.

Ex. 2.4 We only prove the relation for two subsets A and B, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. By repeating applying the relation for two subsets, one can easily to have the desired relation for n subsets.

Since $A \subset A \cup B$ and $B \subset A \cup B$, we have $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$. Thus, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

On the other hand, since $A \subset \overline{A}$ and $B \subset \overline{B}$, we have $A \cup B \subset \overline{A} \cup \overline{B}$. Thus, $\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}}$. Since the closures $\overline{A}$ and $\overline{B}$ are closed and the union of two closed sets is closed, we know that $\overline{A} \cup \overline{B}$ is closed. By Proposition 2.14, we have $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Hence, $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

Therefore, we conclude that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Ex. 2.5

1. By the archimedean property, we can choose a positive integer N such $\varepsilon > \dfrac{1}{N}$. Then the interval $(-\varepsilon, \varepsilon)$ contains $0, \dfrac{1}{N}, \dfrac{1}{N+1}, \dfrac{1}{N+2}, \dots$. Clearly, the finite collection
$$(-\varepsilon, \varepsilon), (1-\varepsilon, 1+\varepsilon), (\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon), \dots, (\tfrac{1}{N-1}-\varepsilon, \tfrac{1}{N-1}+\varepsilon)$$
is a subcover of S.

2. To show that S is compact, by Theorem 2.21, we only need to prove that S is bounded and closed.

   Proof: S is bounded. For any $x \in S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \right\}$, we have $|x| \le 1$, so that S is bounded.

   Proof: S is closed. By the archimedean property, for any $\varepsilon > 0$, there is a positive integer n, such that $\varepsilon > \dfrac{1}{n}$. Thus, any neighborhood of 0 contains infinitely many points in S, so that 0 is a limit point of S.

   For any $x \in (0, 1]$, it is easy to see that there exists $r > 0$ such that the neighborhood $N_r(x)$ contains at most one point of S. Hence, any point in $(0, 1]$ is not a limit point of S.

   Therefore, 0 is the only limit point of S. Since $0 \in S$, we conclude that S is closed.

Ex. 2.6 For each n, since A_n is a nonempty bounded open subset, there is a bounded closed interval I_n such that $A_n \subset I_n$. Thus, $\overline{A}_n \subset \overline{I}_n = I_n$. Hence, $\overline{A}_n$ is a nonempty bounded closed subset of $\mathbb{R}$, so it is compact by Theorem 2.21. Therefore, $\bigcap_{n=1}^{\infty} \overline{A}_n \neq \varnothing$, by Theorem 2.20.

If we can prove $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A}_n$, then we can conclude that $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$.

Clearly, $A_n \subset \overline{A}_n$ for all $n = 1, 2, \dots$. Thus, $\bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} \overline{A}_n$.

On the other hand, to show that $\bigcap_{n=1}^{\infty} \overline{A}_n \subset \bigcap_{n=1}^{\infty} A_n$, we notice that
$$\bigcap_{n=2}^{\infty} \overline{A}_n \subset \overline{A}_2 \subset A_1 \subset \overline{A}_1$$
and have
$$\bigcap_{n=1}^{\infty} \overline{A}_n = \left(\bigcap_{n=2}^{\infty} \overline{A}_n\right) \cap \overline{A}_1 = \bigcap_{n=2}^{\infty} \overline{A}_n.$$
Because $\overline{A}_n \subset A_{n-1}$ for all $n = 2, 3, \dots$, we have $\bigcap_{n=2}^{\infty} \overline{A}_n \subset \bigcap_{n=1}^{\infty} A_n$. Hence $\bigcap_{n=1}^{\infty} \overline{A}_n \subset \bigcap_{n=1}^{\infty} A_n$.

Therefore, we conclude that $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A}_n$.

Ex. 2.7

1. We only prove that for two perfect sets A and B, the union $A \cup B$ is perfect. By repeating applying the result for two perfect sets, one can easily to have the same result holds for any finite collection of perfect sets. ◀

By definition, a set is perfect if it is closed and if every point of the set is a limit point of the set. We know that, by Proposition 2.13, the union $A \cup B$ is closed since A and B are closed. If $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$, then x is a limit point of A, so it is a limit point of $A \cup B$. Similarly, if $x \in B$, then x is also a limit point of $A \cup B$. Therefore, the union $A \cup B$ is perfect.

2. To see that the union of a countable collection of perfect sets may not be perfect, consider the collection $\{A_n\}$, where

$$A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right], \quad n = 1, 2, \ldots.$$

We claim that

$$\bigcup_{n=1}^{\infty} A_n = (-1, 1).$$

In fact, it is clear that $A_n \subset (-1, 1)$, so that $\bigcup_{n=1}^{\infty} A_n \subset (-1, 1)$.

On the other hand, if $x \in (-1, 1)$, then, by the archimedean property, there is a positive integer N_1 such that $N_1 > \dfrac{1}{1-x}$. For the same reason, there is a positive integer N_2 such that $N_2 > \dfrac{1}{1+x}$. Thus, for any positive integer $n > \max\{N_1, N_2\}$, we have

$$n > \frac{1}{1-x}, \qquad n > \frac{1}{1+x}.$$

These two inequalities give $-1 + \dfrac{1}{n} < x < 1 - \dfrac{1}{n}$. Consequently, we have $x \in A_n \subset \bigcup_{n=1}^{\infty} A_n$. Hence, we have $(-1, 1) \subset \bigcup_{n=1}^{\infty} A_n$.

Therefore, we have $\bigcup_{n=1}^{\infty} A_n = (-1, 1)$.

Since any closed interval is perfect, and any finite open interval is not perfect (because it is not closed), the proved equality shows that the union of a countable collection of perfect sets may not be perfect.

Ex. 2.8 Let G be an open set in $\mathbb{R}$. For each $x \in G$, there are y and z, with $z < x < y$, such that $(z, y) \subset G$. Let $b = \sup\{y \colon (x, y) \subset G\}$ and $a = \inf\{z \colon (z, x) \subset G\}$. Then $-\infty \leq a < x < b \leq \infty$. Put $I_x = (a, b)$. It is clear that I_x is an open interval.

We claim that $b \notin G$. In fact, there is nothing to prove if $b = \infty$. If b is finite, and $b \in G$, then there is some $\delta > 0$ such that $(b - \delta, b + \delta) \subset G$ since G is open. This contradicts to the definition of b. Similarly, $a \notin G$.

We shall prove that $I_x \subset G$. Let $w \in I_x$, say $x < w < b$. By the definition of b, there is $y > x$ such that $(x, y) \subset G$. Hence $w \in G$. We can similarly discuss the case of $a < w < x$.

For each $x \in G$, the above construction yields a collection of open intervals $\{I_x\}$. We claim that $G = \bigcup I_x$. In fact, since each $I_x \subset G$, we have $\bigcup I_x \subset G$. On the other hand, for any $x \in G$, we know there is I_x such that $x \in I_x$. This implies $x \in \bigcup I_x$, so that $G \subset \bigcup I_x$.

It remains to show that the collection of open intervals $\{I_x\}$ is disjoint and at most countable.

To show that $\{I_x\}$ is disjoint, we let (a, b) and (c, d) be any two open intervals in the collection with both containing a common point x. Since $a < x < b$ and $c < x < d$, we have $c < b$ and $a < d$. Since $c \notin G$, it does not belong to (a, b), so that $c \leq a$. The reversed inequality $a \leq c$ holds by the same argument. Hence $a = c$. Similarly, $b = d$. Thus, any two different open intervals in the collection $\{I_x\}$ are disjoint.

To show that the collection $\{I_x\}$ is countable, we choose a rational number in each I_x as its representative. This can be done since $\mathbb{Q}$ is dense in $\mathbb{R}$. Since we have a disjoint collection, each segment contains a different rational number. Hence the collection can be put in one-to-one correspondence with a subset of the rational numbers. Thus it is an at most countable collection. ◀

Chapter 3 Quiz Answers

①D: The sequence $1, 2, 3, \ldots$ contains no convergent subsequence.

②C: By the remark in Definition 3.4, a sequence converges if and only if every subsequence converges to the same limit.

③A: By the definition, every sequence of K has a subsequence that converges to a point in K.

④C: The Cauchy criterion states that a series converges if and only if for every $\varepsilon > 0$, there is an integer N such that

$$\left|\sum_{k=n}^{m} a_k\right| < \varepsilon,$$

if $m \geq n \geq N$. Hence, it is a necessary and sufficient condition for convergence of a series.

⑤E: The Monotone Convergence Theorem states that a monotonic sequence converges if and only if it is bounded. Hence, we know that a bounded monotonic sequence must converge. This gives a sufficient condition for a sequence to converge.

⑥B: If $\underline{\lim}_{n\to\infty} x_n = -\infty$, then there exists a subsequence of $\{x_n\}$ whose limit is $-\infty$. If $\underline{\lim}_{n\to\infty} x_n = -\infty$ and $\overline{\lim}_{n\to\infty} x_n$ is finite, then $\{x_n\}$ must be bounded above but unbounded below.

⑦D: If a series converges as determined by the Comparison Test, the Root Test, or the Ratio Test, then the seres converges absolutely. The Divergence Test cannot be used for convergence.

Dirichlet's Test can be used to determine conditional convergence. A successful example is the convergence of the alternating harmonic series.

⑧E: If the absolute value of a series converges, then the series converges. ◀

Chapter 3 Exercise Solutions

Ex. 3.1 Since $\lim_{n\to\infty} x_n = \alpha$, for $\varepsilon = 1 > 0$, there exists an integer N_1 such that $n \geq N_1$ implies $|x_n - \alpha| < 1$. Thus, by triangle inequality, for $n \geq N_1$, we have

$$|x_n| = |x_n - \alpha + \alpha| \leq |x_n - \alpha| + |\alpha| < 1 + |\alpha|.$$

Hence, for $n \geq N_1$,

$$|x_n - \alpha| \leq |x_n| + |\alpha| < 1 + |\alpha| + |\alpha| = 1 + 2|\alpha|.$$

Again since $\lim_{n\to\infty} x_n = \alpha$, for $\varepsilon > 0$, there is an integer N_2 such that $n \geq N_2$ implies $|x_n - \alpha| < \dfrac{\varepsilon}{1+2|\alpha|}$. Let $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we have

$$\left|x_n^2 - \alpha^2\right| = |x_n - \alpha| \cdot |x_n + \alpha| < |x_n - \alpha| \cdot (1 + 2|\alpha|) < \frac{\varepsilon}{1+2|\alpha|} \cdot (1 + 2|\alpha|) = \varepsilon.$$

Hence, by the definition of limit, we conclude that $\lim_{n\to\infty} x_n^2 = \alpha^2$.

Ex. 3.2 It is clear that $0 < s_1 = \sqrt{2} < 2$. Suppose $0 < s_n < 2$. Then

$$0 < s_{n+1} = \sqrt{2 + \sqrt{s_n}} < \sqrt{2 + \sqrt{2}} < 2,$$

Hence, by induction, we have $0 < s_n < 2$ for all $n \geq 1$.

We shall show that $\{s_n\}$ is an increasing sequence by induction. In fact,

$$s_2 = \sqrt{2 + \sqrt{\sqrt{2}}} > \sqrt{2} = s_1.$$

Suppose $s_n > s_{n-1}$. Then

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

Thus, $\{s_n\}$ is a bounded increasing sequence. By Theorem 3.10, $\{s_n\}$ converges.

Ex. 3.3 We show that

$$(x_{2n-1}, x_{2n}) = \left(\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^{n-1}-1}{2^n}\right), \qquad n \geq 1.$$

In fact, by the definition,

$$(x_1, x_2) = (0, \tfrac{0}{2}) = (0, 0).$$

Suppose the formula holds for $n = k$. Then

$$\begin{aligned}
\left(x_{2(k+1)-1}, x_{2(k+1)}\right) &= \left(\frac{1}{2} + x_{2k}, \frac{x_{2k+1}}{2}\right) = \left(\frac{1}{2} + \frac{2^{k-1}-1}{2^k}, \frac{\frac{1}{2} + x_{2k}}{2}\right) \\
&= \left(\frac{2^{k+1}-2}{2^{k+1}}, \frac{1}{4} + \frac{1}{2} \cdot \frac{2^{k-1}-1}{2^k}\right) \\
&= \left(\frac{2^{(k+1)-1}-1}{2^{(k+1)-1}}, \frac{2^{(k+1)-1}-1}{2^{k+1}}\right).
\end{aligned}$$

The proved expression for $\{x_n\}$ gives:

$$\lim_{n\to\infty} x_{2n-1} = 1, \quad \lim_{n\to\infty} x_{2n} = \frac{1}{2}.$$

Hence, by the definitions of upper and lower limits, we have

$$\limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}}\, x_n = 1, \qquad \liminf_{n\to\infty} x_n = \underline{\lim_{n\to\infty}}\, x_n = \frac{1}{2}.$$

Ex. 3.4 Let $\{x_n\}$ be monotonically increasing with a convergent subsequence $\{x_{n_k}\}$, $\lim_{k\to\infty} x_{n_k} = \alpha$. Since $\{x_{n_k}\}$ is also monotonically increasing, by Theorem 3.10, $\alpha = \sup\{x_{n_k}\}$. We claim that $x_n \leq \alpha$ for all $n \geq 1$. In fact, if there is $x_N > \alpha$, then we have $x_{n_N} \geq x_N > \alpha$. This violates the fact that $\alpha = \sup\{x_{n_k}\}$.

For any $\varepsilon > 0$, there exists an integer K such that $k \geq K$ implies

$$|x_{n_k} - \alpha| < \varepsilon.$$

The last inequality implies that $\alpha - \varepsilon < x_{n_k} < \alpha + \varepsilon$. Since $\{x_n\}$ is monotonically increasing, we have that, for $n \geq n_K$,

$$\alpha - \varepsilon < x_{n_K} \leq x_n \leq \alpha < \alpha + \varepsilon.$$

By definition, this means that $\lim_{n\to\infty} x_n = \alpha$.

Ex. 3.5 By applying Proposition 3.14 and its corollary, we have

$$L = \lim_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n \leq \underline{\lim}_{n\to\infty} y_n \leq \overline{\lim}_{n\to\infty} y_n \leq \overline{\lim}_{n\to\infty} z_n = \lim_{n\to\infty} z_n = L.$$

Thus, $\underline{\lim}_{n\to\infty} y_n = \overline{\lim}_{n\to\infty} y_n = L$, so that $\lim_{n\to\infty} y_n = L$.

1. Let $x_n = \sqrt[n]{n} - 1$. Then $n = (1+x_n)^n$. It is clear that $x_n > 0$ for all $n > 1$. By the binomial formula, we get
$$n = (1+x_n)^n > 1 + nx_n + \frac{n(n-1)}{2}x_n^2 > \frac{n(n-1)}{2}x_n^2, \qquad n > 1.$$
Thus, we have
$$0 < x_n < \sqrt{\frac{2}{n-1}}, \qquad n > 1.$$
It is easy to see that $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$. Hence, the hypotheses of the Squeeze Theorem hold. Consequently, we have $\lim_{n\to\infty} x_n = 0$. Therefore, $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

2. If $n > 2024$, by the binomial formula, for $p > 0$, we have
$$(1+p)^n > 1 + \binom{n}{1}p + \binom{n}{2}p^2 + \cdots + \binom{n}{2024}p^{2024} > \frac{n(n-1)\cdots(n-2023)}{2024!}p^{2024}.$$
Thus, we get
$$0 < \frac{n^{2023}}{(1+p)^n} < \frac{n^{2023}}{\frac{n(n-1)\cdots(n-2023)}{2024!}p^{2024}}, \qquad n > 2024.$$
It is easy to see that $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{n^{2023}}{\frac{n(n-1)\cdots(n-2023)}{2024!}p^{2024}} = 0$. Hence, the hypotheses of the Squeeze Theorem hold. Consequently, we have $\lim_{n\to\infty} \frac{n^{2023}}{(1+p)^n} = 0$.

Ex. 3.6

1. For each $n \geq 1$, the partial sum
$$s_n = \sum_{k=1}^{n} a_k = \sqrt{n+1} - 1.$$
Since $\{s_n\}$ is unbounded, so it diverges. Hence, $\sum a_n$ diverges.

2. Since
$$0 < a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} \leq \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n^{3/2}}.$$
Since $\sum \frac{1}{n^{3/2}}$ is a convergent p-series (with $p = \frac{3}{2}$), by the Comparison Test, we know that $\sum a_n$ converges.

3. Based on the result given in part **1** in Exercise 3.5, we know that $\lim \sqrt[n]{n} = 1$. Hence, there exists N_0 such that for $n \geq N_0$,
$$0 < \sqrt[n]{n} - 1 < \frac{1}{2},$$
which implies that for $n \geq N_0$,
$$0 < a_n = (\sqrt[n]{n} - 1)^n < \left(\frac{1}{2}\right)^n.$$
The series $\sum \left(\frac{1}{2}\right)^n$ is a convergent geometric series, thus, by the Comparison Test, we know that $\sum a_n$ converges.

4. If $|z| \leq 1$, then $|1+z^n| \leq 1 + |z|^n \leq 2$, which implies
$$|a_n| = \left|\frac{1}{1+z^n}\right| \not\to 0.$$
◀

By the Divergence Test, $\sum a_n$ diverges.

If $|z| > 1$, by the estimation

$$|a_n| = \left|\frac{1}{1+z^n}\right| \le \frac{1}{|z|^n - 1},$$

The series $\sum \frac{1}{|z|^n - 1}$ converges by the Ratio Test, since

$$\lim \frac{\frac{1}{|z|^n - 1}}{\frac{1}{|z|^{n-1} - 1}} = \lim \frac{1}{|z|} \cdot \frac{\frac{1}{1 - 1/|z|^n}}{\frac{1}{1 - 1/|z|^{n-1}}} = \frac{1}{|z|} < 1.$$

Hence, we know that $\sum a_n$ converges by the Comparison Test.

Ex. 3.7 It is clear that the following inequality holds

$$0 \le \frac{\sqrt{a_n}}{n} = \sqrt{\frac{a_n}{n^2}} \le \frac{1}{2}a_n + \frac{1}{2} \cdot \frac{1}{n^2}, \qquad n = 1, 2, 3, \ldots.$$

Since $\sum a_n$ is convergent and $\sum \frac{1}{n^2}$ is a convergent p-series (with $p = 2$), by Proposition 3.18, we know that the series $\sum b_n$ converges, where $b_n = \frac{1}{2}a_n + \frac{1}{2} \cdot \frac{1}{n^2}$. Hence, by the Comparison Test, we know that $\sum \frac{\sqrt{a_n}}{n}$ converges.

Ex. 3.8 For convergent series $\sum a_n$, the series $\sum a_n^2$ may not be convergent. This can be illustrated in the following example.

Consider the series $\sum a_n$, with

$$a_{2n-1} = \frac{1}{\sqrt{n}}, \quad a_{2n} = -\frac{1}{\sqrt{n}}, \qquad n \ge 1.$$

It is easy to see that the partial sums of $\sum a_n$ are

$$s_{2n-1} = \frac{1}{\sqrt{n}}, \quad s_{2n} = 0, \qquad n \ge 1.$$

Thus, we have

$$0 \le s_n \le \sqrt{\frac{2}{n+1}}, \qquad n \ge 1.$$

Applying the Squeeze Theorem gives $\lim s_n = 0$ so that $\sum a_n$ converges.

On the other hand, let $t_n = \sum_{k=1}^{n} \frac{1}{k}$. Then $s_{2n} = 2t_n$, where s_n is the n-th partial sum of the series

$$\sum a_n^2 = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \cdots.$$

Since the harmonic series diverges, we know that $\{t_n\}$, which is the sequence of the partial sums of the harmonic series, diverges. Hence, the sequence $\{s_{2n}\}$ diverges. Because the latter is a subsequence of $\{s_n\}$. Hence, $\{s_n\}$ diverges.

Ex. 3.9

1. If $L^* = \overline{\lim_{n\to\infty}} \frac{a_n}{b_n} < \infty$, then there is an integer N_0 such that $n \ge N_0$ implies $\frac{a_n}{b_n} < L^* + 1$. Hence,

$$0 \le a_n < (L^* + 1)b_n, \qquad n \ge N_0.$$

If $\sum b_n$ converges, then $\sum (L^* + 1)b_n$ converges. In this case, by the Comparison Test, we know that $\sum a_n$ converges.

2. If $L_* = \underline{\lim_{n\to\infty}} \frac{a_n}{b_n} > 0$, then there is an integer N_0 such that $n \ge N_0$ implies $\frac{a_n}{b_n} > \frac{1}{2}L_*$. Hence,

$$0 \le \tfrac{1}{2}L_* b_n < a_n, \qquad n \ge N_0.$$

If $\sum b_n$ diverges, then $\sum \frac{1}{2}L_* b_n$ diverges. In this case, by the Comparison Test, we know that $\sum a_n$ diverges. ◀

Ex. 3.10 Let $A_n = \sum_{k=0}^{n} |a_k|$ and $B_n = \sum_{k=0}^{n} |b_k|$. By the hypothesis, we know that $\{A_n\}$ and $\{B_n\}$ are convergent. For their Cauchy product $\sum c_n$, by applying the triangle inequality, we have

$$\sum_{k=0}^{n} |c_k| = \sum_{k=0}^{n} \left| \sum_{j=0}^{k} a_j b_{k-j} \right| \le \sum_{k=0}^{n} \sum_{j=0}^{k} |a_j| \, |b_{k-j}| = A_n B_n.$$

Hence, by the Comparison Test, the Cauchy product $\sum c_n$ is absolutely convergent. ◀

Chapter 4 Quiz Answers

①A: Item A is the continuity of f at x_0. Item C is the uniform continuity of f on E.

②C: Any interval $[a, b]$ is compact. By the Extreme Value Theorem, the continuous function f on the compact set $[a, b]$ attains its maximum value.

③D: The function f is continuous on $\mathbb{R}$. Since

$$\{x \in \mathbb{R} : f(x) \neq 0\} = \{x \in \mathbb{R} : f(x) < 0\} \cup \{x \in \mathbb{R} : f(x) > 0\}$$

Each set on the right-hand side is open. In fact, the complement of $\{x \in \mathbb{R} : f(x) < 0\}$ is $\{x \in \mathbb{R} : f(x) \geq 0\}$. The latter is closed: if $\{x_n\}$ is a consequence in $\{x \in \mathbb{R} : f(x) \geq 0\}$, and $x_n \to x^*$, then by the continuity,

$$f(x^*) = \lim f(x_n) \geq 0,$$

so that $x^* \in \{x \in \mathbb{R} : f(x) \geq 0\}$.

Thus, the set $\{x \in \mathbb{R} : f(x) \neq 0\}$ is a union of two open sets, so it is open.

④E: Since $\overline{E}$ is closed by Proposition 2.14, we know that $(\overline{E})^{\mathsf{c}}$ is open. Thus, by Theorem 4.17, $f^{-1}((\overline{E})^{\mathsf{c}})$ is open.

⑤D: According to Theorem 4.23, a continuous mapping maps a connected subset to a connected subset. Because $E = (a, b)$ is connected, the set $f(E)$ is connected.

⑥A: The interval $[a, b]$ is compact, by the Extreme Value Theorem, the supremum $\sup_{x\in[a,b]} g(x)$ and the infimum $\inf_{x\in[a,b]} g(x)$ can be attained by the function g. In fact, the values of g lie between these two values. So, option A is false.

⑦B: The function $f(x) = 1/x$ is continuous on $(0, 1)$, but not uniformly continuous on $(0, 1)$.

⑧D: Since $\lim_{x\to 0-} f(x) = -1$, $\lim_{x\to 0+} f(x) = 1$, and neither $\lim_{x\to 0+} g(x)$ nor $\lim_{x\to 0-} g(x)$ exists. Hence, at $x = 0$, f has a jump discontinuity and g has an essential discontinuity. ◀

Chapter 4 Exercise Solutions

Ex. 4.1 The original definition of limit is

"for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - q| \leq \varepsilon$ for all points $x \in E$ for which $0 < |x - p| < \delta$."

Thus, if $\lim_{x \to p} f(x) = q$, then for every $\varepsilon^* = \varepsilon^2 > 0$, there exists a $\delta > 0$ such that $|f(x) - q| < \varepsilon^* = \varepsilon^2$ for all points $x \in E$ for which $0 < |x - p| < \delta$. Hence, the modified statement for limit holds.

On the other hand, assume that the modified statement for limit holds. Then, for every $\varepsilon^* > 0$, there exists a $\delta > 0$ such that $|f(x) - q| \leq \varepsilon^{*2}$ for all points $x \in E$ for which $0 < |x - p| < \delta$. In particular, for every $\varepsilon > 0$, if we take $\varepsilon^* = \sqrt{\frac{1}{2}\varepsilon}$, then there exists a $\delta > 0$ such that $|f(x) - q| \leq \frac{1}{2}\varepsilon < \varepsilon$ for all points $x \in E$ for which $0 < |x - p| < \delta$. This means that $\lim_{x \to p} f(x) = q$.

Ex. 4.2 The function $f(x) = \sqrt{x}$ is defined on $[0, \infty)$. We need to prove that $\lim_{x \to p} f(x) = f(p)$ for every $p \in [0, \infty)$.

Let $\varepsilon > 0$ be given.

Case 1: $\lim_{x \to 0} \sqrt{x} = 0$. In this case, we understand that when $x \to 0$, we have $x \geq 0$.

Take $\delta = \varepsilon^2$. If $|x - 0| < \delta$, then $\left|\sqrt{x} - \sqrt{0}\right| = \sqrt{x} < \sqrt{\delta} = \varepsilon$. Thus, by the definition, we have $\lim_{x \to 0} \sqrt{x} = 0$.

Hence, the function f is continuous at $x = 0$.

Case 2: $\lim_{x \to p} \sqrt{x} = \sqrt{p}$ for all $p > 0$.

Take $\delta = \sqrt{p} \cdot \varepsilon$. If $|x - p| < \delta$, then

$$\left|\sqrt{x} - \sqrt{p}\right| = \frac{|x - p|}{\sqrt{x} + \sqrt{p}} \leq \frac{|x - p|}{\sqrt{p}} < \frac{\sqrt{p} \cdot \varepsilon}{\sqrt{p}} = \varepsilon.$$

Thus, by the definition, we have $\lim_{x \to p} \sqrt{x} = \sqrt{p}$ for all $p > 0$.

Hence, the function f is continuous at $x = p$ for all $p > 0$.

Ex. 4.3 Let $\varepsilon > 0$ be given.

For any $p \in (0, \infty)$, since $p > 0$, we take $\delta_1 = \frac{1}{2}p > 0$. If $|x - p| < \delta_1$, then $\frac{1}{2}p = p - \delta_1 < x < p + \delta_1$. It is obvious we have the following

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{p}}\right| = \frac{\left|\sqrt{x} - \sqrt{p}\right|}{\sqrt{x} \cdot \sqrt{p}} = \frac{|x - p|}{\left|\sqrt{x} + \sqrt{p}\right| \cdot \sqrt{x} \cdot \sqrt{p}} < \frac{|x - p|}{\sqrt{p} \cdot \sqrt{x} \cdot \sqrt{p}} = \frac{|x - p|}{\sqrt{x} \cdot p}$$

Take $\delta_2 = \dfrac{p^{3/2} \cdot \varepsilon}{\sqrt{2}} > 0$ and let $\delta = \min\{\delta_1, \delta_2\}$. Then, when $|x - p| < \delta$, we have

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{p}}\right| < \frac{\frac{p^{3/2} \cdot \varepsilon}{\sqrt{2}}}{\sqrt{\frac{1}{2}p} \cdot p} = \varepsilon.$$

Thus, by the definition, we have $\lim_{x \to p} \dfrac{1}{\sqrt{x}} = \dfrac{1}{\sqrt{p}}$ for all $p > 0$.

Hence, the function $\dfrac{1}{\sqrt{x}}$ is continuous on $(0, \infty)$.

Ex. 4.4

1. Assume that f and g are monotonically increasing. Thus, for $a \leq x < y \leq b$, we have

$$f(x) \leq f(y), \qquad g(x) \leq g(y).$$

Hence, for $a \leq x < y \leq b$

$$H(x) = \max\{f(x), g(x)\} \leq \max\{f(y), g(y)\} = H(y),$$

so that H is monotonically increasing.

In the similar manner, we can show that if f and g are monotonically decreasing, so it H.

The analogous results hold for the function h.

2. It is easy to verify that the following identity holds: ◀

$$H(x) = \max\{f(x), g(x)\} = \tfrac{1}{2}\left[f(x) + g(x) + |f(x) - g(x)|\right].$$

According to Item 1 of Proposition 4.18, a linear combination of continuous functions is continuous. Additionally, we know that the absolute value function is continuous. By using Item 3 of Proposition 4.18, we can conclude that the composition of the absolute value function and a continuous function is also continuous. Therefore, we can deduce that the function H is continuous.

In the similar manner, using the identity

$$h(x) = \min\{f(x), g(x)\} = \tfrac{1}{2}\left[f(x) + g(x) - |f(x) - g(x)|\right],$$

we can prove that h is continuous if f and g are continuous.

Ex. 4.5 Since $Z(f) = f^{-1}(\{0\})$, and $\{0\}$ is a closed set in $\mathbb{R}$, by Theorem 4.17, $Z(f)$ is closed if f is continuous.

Another alternative approach is to prove the result directly. Let p be a limit point of $Z(f)$. Then there exists a sequence $\{p_n\}$ in $Z(f)$ such that $|p_n - p| \to 0$. Since f is continuous at p, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - p| < \delta$ implies $|f(x) - f(p)| < \varepsilon$. By $|p_n - p| \to 0$, we know that there exists N such that $n \geq N$ implies $|p_n - p| < \delta$. Hence, if $n \geq N$, then $|f(p_n) - f(p)| < \varepsilon$. Since $f(p_n) = 0$, we know that $|f(p)| < \varepsilon$ for any $\varepsilon > 0$. This implies $f(p) = 0$, or $p \in Z(f)$. Therefore $z(f)$ is closed, since it contains all its limit points.

Ex. 4.6 Put $g(x) = x - f(x)$ for $x \in I$. It is clear that g is continuous on I. If $g(0) = 0$ or $g(1) = 0$, the conclusion of the problem holds either for $x = 0$ or $x = 1$. Otherwise, we have $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$. By the Intermediate Value Theorem, there exists a $x \in (0, 1)$ such that $g(x) = 0$, since $g(0) < 0 < g(1)$. This gives $f(x) = x$ for this x.

Ex. 4.7 Without loss of generality, assume that $x_1 < \cdots < x_n$. By the hypothesis, the function f is continuous on $[x_1, x_n]$. Denote

$$m = \min_{x \in [x_1, x_n]} f(x), \qquad M = \max_{x \in [x_1, x_n]} f(x).$$

Show that

$$m = \frac{m + m + \cdots + m}{n} \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq \frac{M + M + \cdots + M}{n} \leq M.$$

By the Intermediate Value Theorem, there exists $\xi \in [x_1, x_n] \subset (a, b)$ such that

$$f(\xi) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Ex. 4.8 By the hypothesis $\lim\limits_{x\to\infty} f(x) = A$, we know that for $\varepsilon = 1$, there exists a number b, with $b > a$, such that $x \geq b$ implies $|f(x) - A| < 1$. Thus, when $x \geq b$, we have

$$|f(x)| = |f(x) - A + A| \leq |f(x) - A| + |A| < 1 + |A|.$$

On the finite interval $[a, b]$, which is compact, the function f is continuous. By Theorem 4.19, we know that $f([a, b])$ is bounded, so that there is a real number M such that for all $x \in [a, b]$,

$$|f(x)| \leq M.$$

Denote $B = \max\{1 + |A|, M\}$. Then, for all $x \in [a, \infty) = [a, b] \cup [b, \infty)$, we have

$$|f(x)| \leq B,$$

that is, the function f is bounded on $[a, \infty)$.

Ex. 4.9 If f is a uniformly continuous on E, then there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for any $x, y \in E$ satisfying $|x - y| < \delta$. Let $\{x_n\}$ be a Cauchy sequence in E. By definition, there exists an integer N such that $|x_n - x_m| < \delta$ if $n, m \geq N$. Thus, if $n, m \geq N$, we have

$$|f(x_n) - f(x_m)| < \varepsilon.$$

This means that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

Ex. 4.10 Since f is uniformly continuous on E, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ for any $x, y \in E$ satisfying $|x - y| < \delta$. ◀

Because the set E is bounded, it is contained in a bounded closed interval I, that is, $E \subset I$. For each $x \in I$, the collection $\{(x-\delta, x+\delta)\}$ of open intervals is an open cover of I. Since I is compact, there is a finite subcover of I. Obviously, this finite collection is also a cover of E. We keep only the open intervals in the collection which intersect with E, say $(y_1-\delta, y_1+\delta), \ldots, (y_K-\delta, y_K+\delta)$. Let $x_i \in (y_i-\delta, y_i+\delta)$, $i = 1, \ldots, K$, where $x_i \in E$. Denote $M = \max_{1 \le i \le K}\{|f(x_i)|\}$.

For any fixed $x \in E$, there is i_0, $1 \le i_0 \le K$, such that $x \in (x_{i_0}-\delta, x_{i_0}+\delta)$. Thus,

$$|f(x)| \le |f(x) - f(x_{i_0})| + |f(x_{i_0})| < 1 + M,$$

so that f is bounded on E. ◀

Chapter 5 Quiz Answers

①B: If $f'(x)$ exists for $x \in (a,b)$, then both the left-hand derivative and the right-hand derivative exist and equal to $f'(x)$. So, item B is correct.

②A: Since f is differentiable on $[a,b]$, by Preposition 5.2, f is continuous on $[a,b]$. By the Mean Value Theorem, there is a point $x \in (a,b)$ such that

$$f(b) - f(a) = (b-a)f'(x).$$

Since $f(a) = f(b)$, we have $f'(x) = 0$.

③C: By the definition, a critical point of function f is the point x at which either $f'(x) = 0$ or $f'(x)$ is undefined.

④D: The Mean Value Theorem claims that

$$\frac{f(b)-f(a)}{b-a} = f'(x).$$

This can be interpreted that the average rate of change of a function over an interval is equal to its instantaneous rate of change at some point within the interval.

⑤E: By the Mean Value Theorem, for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| = |f'(c)| \cdot |x-y| \le M|x-y|.$$

Thus, for every $\varepsilon > 0$, when $|x-y| < \dfrac{\varepsilon}{M} = \delta$, then $\delta > 0$, and

$$|f(x) - f(y)| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Hence, f is uniformly continuous on $\mathbb{R}$.

⑥D: According to l'Hôpital's rule, if $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

⑦B: Taylor's theorem is a method for approximating a function with a polynomial. It gives the remainder $\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, which measures the accuracy of the approximation.

⑧D: Taylor's theorem gives

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \qquad x_0 \in [a,b].$$

The remainder $R_n(x) = f(x) - \sum\limits_{k=0}^{n} \dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ is the error of the approximation between $f(x)$ and $\sum\limits_{k=0}^{n} \dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k$. For any $x \in [a,b]$, we have

$$|R_n(x)| \le \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \right|.$$

The maximum possible error of the right term is $M(b-a)^{n+1}/(n+1)!$. ◀

Chapter 5 Exercise Solutions

Ex. 5.1 The inequality implies that, for $x \neq y$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|.$$

Taking $x \to y$, we have $|f'(y)| \leq 0$, which implies that $f'(y) = 0$ for all $y \in \mathbb{R}$. We conclude that f is constant, by Item 2 of the Monotone Test.

Ex. 5.2 If there is $x \in (a, \infty)$ such that $f(x) = f(a)$, then, by the Mean Value Theorem, there exists ξ between x and a such that

$$f'(\xi) = \frac{f(x) - f(a)}{x - a} = \frac{f(a) - f(a)}{x - a} = 0.$$

If for all $x \in (a, \infty)$ such that $f(x) \neq f(a)$, then, without loss of generality, we assume that there is a number $c \in (a, \infty)$ such that $f(c) > f(a)$. Denote $\varepsilon = \frac{1}{2}[f(c) - f(a)] > 0$. Since $\lim\limits_{x \to \infty} f(x) = f(a)$, there is $X > c$ such that $|f(x) - f(a)| < \varepsilon$ for $x > X$, so that

$$f(X + 1) < f(a) + \varepsilon = \tfrac{1}{2}[f(a) + f(c)] < f(c).$$

Hence, the function f is continuous on $[a, X + 1]$, differentiable in $(a, X + 1)$, and for $c \in (a, X + 1)$,

$$f(c) > f(a), \qquad f(c) > f(X + 1).$$

By the Extreme Value Theorem, the function f attains its maximum value at some point $\xi \in (a, X + 1)$. By Proposition 5.7, we know that $f'(\xi) = 0$.

Ex. 5.3 Put $M = \sup\limits_{x \in \mathbb{R}} |g'(x)|$. Take any ε satisfying $0 < \varepsilon < \dfrac{1}{2M + 1}$. For $x < y$, by the Mean Value Theorem, we have

$$\begin{aligned} f(y) - f(x) = f'(c)(y - x) &= [1 + \varepsilon g'(c)](y - x) \\ &\geq (1 - \varepsilon M)(y - x) \\ &> \left(1 - \frac{1}{2M + 1} \cdot M\right)(y - x) > \frac{1}{2}(y - x), \end{aligned}$$

which implies that f is strictly increasing. Hence f is one-to-one.

Ex. 5.4 Put

$$P(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n + 1} x^{n+1}.$$

It is clear that the function P is a polynomial, so it is differentiable everywhere on $\mathbb{R}$. We know $P(0) = 0$, and $P(1) = 0$ by the hypothesis. By the Mean Value Theorem, there is $c \in (0, 1)$ such that

$$C_0 + C_1 c + \cdots + C_{n-1} c^{n-1} + C_n c^n = P'(c) = \frac{P(1) - P(0)}{1 - 0} = 0.$$

This completes the proof.

Ex. 5.5 Without loss of generality, we assume that $f'_+(a) > 0$. Then, there is a $\delta_1 > 0$ such that, if $t \in [a, b]$ and $t - a < \delta_1$.

$$\left| \frac{f(t) - f(a)}{t - a} - f'_+(a) \right| < \tfrac{1}{2} f'_+(a).$$

It follows that there is $x_1 \in (a, b)$ such that

$$f(x_1) = f(x_1) - f(a) > \tfrac{1}{2} f'_+(a)(x_1 - a) > 0.$$

Similarly, since $f'_-(b) > 0$, there is a $\delta_2 > 0$ such that, if $t \in [a, b]$ and $b - t < \delta_2$.

$$\left| \frac{f(t) - f(b)}{t - b} - f'_-(b) \right| < \tfrac{1}{2} f'_-(b).$$

It follows that there is $x_2 \in (a, b)$ such that

$$f(x_2) = f(x_2) - f(b) < \tfrac{1}{2} f'_-(b)(x_2 - b) < 0.$$

Clearly, we can choose x_1 and x_2 so that $a < x_1 < x_2 < b$. ◀

Since $f(x_1)\cdot f(x_2)<0$, by the Intermediate Value Theorem or Bolzano's Theorem, there is $\xi\in(x_1,x_2)\subset(a,b)$ such that $f(\xi)=0$.

Ex. 5.6 By the definition of derivative,

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x}=f'(x),\qquad \lim_{t\to x}\frac{g(t)-g(x)}{t-x}=g'(x)\neq 0.$$

Apply Proposition 4.13, we have

$$\lim_{t\to x}\frac{f(t)}{g(t)}=\lim_{t\to x}\frac{\dfrac{f(t)-f(x)}{t-x}}{\dfrac{g(t)-g(x)}{t-x}}=\frac{\lim\limits_{t\to x}\dfrac{f(t)-f(x)}{t-x}}{\lim\limits_{t\to x}\dfrac{g(t)-g(x)}{t-x}}=\frac{f'(x)}{g'(x)}.$$

Ex. 5.7 By Taylor's Theorem, for any $y\in\mathbb{R}$, we have

$$f(y)=f(x)+f'(x)(y-x)+\frac{f''(\xi)}{2}(y-x)^2,$$

where ξ is a number between x and y. Putting $y=x+h,x-h$ into the formula, respectively, we have

$$f(x+h)=f(x)+f'(x)h+\frac{f''(\xi_1)}{2}h^2,$$
$$f(x-h)=f(x)-f'(x)h+\frac{f''(\xi_2)}{2}h^2,$$

where ξ_1 is between x and $x+h$ and ξ_2 is between x and $x-h$. The sum of these two equations yields

$$f(x+h)+f(x-h)=2f(x)+\frac{f''(\xi_1)+f''(\xi_2)}{2}\cdot h^2.$$

Since $f''(x)>0$ for all $x\in\mathbb{R}$, we see that $\dfrac{f''(\xi_1)+f''(\xi_2)}{2}\cdot h^2>0$. Thus,

$$f(x+h)+f(x-h)>2f(x)$$

holds for all $x,h\in\mathbb{R}$. Finally, if we take $x=\frac{1}{2}(x_1+x_2)$ and $h=\frac{1}{2}(x_1-x_2)$, then the desired equality follows.

Ex. 5.8 Let $x_1,x_2\in\mathbb{R}$, with $x_1<x_2$. Denote $\overline{x}=\frac{1}{2}(x_1+x_2)$. By Taylor's Theorem, for any $x\in\mathbb{R}$, we have

$$f(x)=f(\overline{x})+f'(\overline{x})\,(x-\overline{x})+\tfrac{1}{2}f''(\overline{x})\,(x-\overline{x})^2+\tfrac{1}{6}f'''(\xi)(x-\overline{x})^3,$$

where ξ is a number between x and $\overline{x}$. Putting $x=x_1,x_2$ into the formula, respectively, we have

$$f(x_1)=f(\overline{x})+f'(\overline{x})\,(x_1-\overline{x})+\tfrac{1}{2}f''(\overline{x})\,(x_1-\overline{x})^2+\tfrac{1}{6}f'''(\xi_1)(x_1-\overline{x})^3,$$
$$f(x_2)=f(\overline{x})+f'(\overline{x})\,(x_2-\overline{x})+\tfrac{1}{2}f''(\overline{x})\,(x_2-\overline{x})^2+\tfrac{1}{6}f'''(\xi_2)(x_2-\overline{x})^3,$$

where ξ_1 is between x_1 and $\overline{x}$ and ξ_2 is between $\overline{x}$ and x_2. The difference of these two equations yields

$$f(x_2)-f(x_1)=f'(\overline{x})\,(x_2-x_1)+\tfrac{1}{6}[f'''(\xi_2)+f'''(\xi_1)]\cdot 2\cdot\frac{(x_2-x_1)^3}{8}.$$

Since $f'''(x)>0$ for all $x\in\mathbb{R}$, we see that $\frac{1}{6}[f'''(\xi_2)+f'''(\xi_1)]\cdot 2\cdot\dfrac{(x_2-x_1)^3}{8}>0$. Thus,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}>f'\big(\tfrac{1}{2}(x_1+x_2)\big)$$

holds for all $x_1,x_2\in\mathbb{R}$ with $x_2>x_1$.

Ex. 5.9 When $M_0=0$, then $f(x)\equiv 0$, the inequality is trivial.

When $M_2=0$, then $f'(x)$ is constant and $f(x)$ is a linear function, by the Mean Value Theorem. In this case, if $f'(x)\equiv c\neq 0$, then M_0 is infinite, a contradiction to the hypothesis. If $f'(x)\equiv 0$, then $M_1=0$, again we have a trivial inequality.

When $M_0>0$ and $M_2>0$, by Taylor's Theorem, for any $h>0$, there is a $\xi\in(x,x+2h)$ such that

$$f(x+2h)=f(x)+\frac{1}{1!}f'(x)(2h)+\frac{1}{2!}f''(\xi)(2h)^2,$$

◀

which gives

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi).$$

By the triangle inequality, we have, for any $h > 0$,

$$|f'(x)| \le \frac{1}{2h}[|f(x+2h)| + |f(x)|] + h|f''(\xi)| \le \frac{1}{h}M_0 + hM_2.$$

In particular, if we take $h = \sqrt{M_0/M_2}$ in the last inequality, then

$$|f'(x)| \le 2\sqrt{M_0\, M_2}$$

for any $x \in (a, \infty)$. Since x is arbitrary, we have

$$M_1 \le 2\sqrt{M_0\, M_2},$$

which implies

$$M_1^2 \le 4M_0\, M_2.$$

Ex. 5.10

1. If x_1 and x_2 are two fixed points of f, and if $x_1 \ne x_2$, then by the Mean Value Theorem, there is a point ξ between x_1 and x_2 such that

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1.$$

This contradicts to the hypothesis.

2. For the sequence $\{x_n\}$ generated by $x_{n+1} = f(x_n)$, if $x_{n_0+1} = x_{n_0}$ for some n_0, then $x_{n_0+2} = f(x_{n_0+1}) = f(x_{n_0}) = x_{n_0+1}$. In this manner, we see that $x_k = x_{n_0}$ for all $k \ge n$. Hence, in this case, we have $\lim x_n = x_{n_0}$, and $x_{n_0} = x_{n_0+1} = f(x_{n_0})$.

If $x_{n+1} \ne x_n$ for all n, by the Mean Value Theorem, for every n, there is ξ_n between x_n and x_{n+1} such that

$$x_{n+2} - x_{n+1} = f(x_{n+1}) - f(x_n) = f'(\xi_n)(x_{n+1} - x_n).$$

Since $|f'(t)| \le A$ for all t, we have

$$|x_{n+2} - x_{n+1}| \le |f'(\xi_n)| \cdot |x_{n+1} - x_n| \le A \cdot |x_{n+1} - x_n|.$$

Hence,

$$|x_{n+1} - x_n| \le A^{n-1} \cdot |x_2 - x_1|, \qquad n = 1, 2, 3, \cdots.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Indeed, for $0 \le A < 1$, we have $\lim A^n = 0$ (see part 2 in Ex. 3.5). Let $\varepsilon > 0$ be given. Since $\dfrac{1-A}{|x_2 - x_1|} > 0$, there exists N such that $n \ge N$ implies $A^n < \varepsilon \cdot \dfrac{1-A}{|x_2 - x_1|}$. For $n > m \ge N$,

$$\begin{aligned} |x_m - x_n| &\le |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + \cdots + |x_{n-1} - x_n| \\ &\le A^m \cdot |x_2 - x_1| + A^{m+1} \cdot |x_2 - x_1| + \cdots + A^{n-1} \cdot |x_2 - x_1| \\ &= \frac{A^m(1 - A^{n-m})}{1-A} \cdot |x_2 - x_1| \\ &\le A^m \cdot \frac{|x_2 - x_1|}{1-A} < \varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Put $\lim x_n = x$. Then, by the continuity of f,

$$x = \lim x_n = \lim x_{n+1} = \lim f(x_n) = f(x),$$

that is, x is a fixed point of f. ◀

Chapter 6 Quiz Answers

①C: Any Riemann integrable function must be bounded, as stated in the remark of Theorem 6.3.

②A: By item 1 of Theorem 6.4, if f is continuous on $[a,b]$, then $f \in \mathscr{R}[a,b]$.

③B: It is known that $\underline{\int_a^b} f\,\mathrm{d}x \le \overline{\int_a^b} f\,\mathrm{d}x$. Hence, option B implies $\underline{\int_a^b} f\,\mathrm{d}x = \overline{\int_a^b} f\,\mathrm{d}x$, so that $f \in \mathscr{R}[a,b]$.

Option A is not true, since $f \in \mathscr{R}[a,b]$ is equivalent to $\sup_P L(P,f) = \inf_P U(P,f)$.

Option C is not true, since the hypothesis requires only for some positive integer n. This is not sufficient for f being integrable.

Option D is not true for the same reason as option C.

④D: By the hypothesis, we have $g_2 - g_1 \ge 0$, so that $f_1(g_2-g_1) \le f_2(g_2-g_1)$. Applying the monotonicity of the Riemann integral, we have

$$\int_a^b f_1(g_2-g_1)\,\mathrm{d}x \le \int_a^b f_2(g_2-g_1)\,\mathrm{d}x,$$

so that

$$\int_a^b (f_1g_2+f_2g_1)\,\mathrm{d}x \le \int_a^b (f_1g_1+f_2g_2)\,\mathrm{d}x.$$

⑤E: The functions $\frac{1}{1+x^2}, x^2+x^3, |x|, e^x$ are all continuous functions, so Theorem 6.5 applies.

⑥C: Let $F(x) = \int_0^x f(t)\,\mathrm{d}t$. Then $F\big(\alpha(x)\big) = \int_0^{\alpha(x)} f(t)\,\mathrm{d}t$. Thus, by the chain rule,

$$\Big[F\big(\alpha(x)\big)\Big]' = F'\big(\alpha(x)\big)\cdot\alpha'(x).$$

By Part 1 of the Fundamental Theorem of Calculus, we know $F'(x) = f(x)$, sp that

$$F'\big(\alpha(x)\big) = f\big(\alpha(x)\big).$$

Hence, option C is true.

⑦B: Since F_α is an antiderivative of $f\alpha'$, by Part 2 of the Fundamental Theorem of Calculus, we know that option B is true.

The function G and $F\cdot\alpha$ are not antiderivatives of $f\alpha'$. Thus, options A and C are not true.

⑧A: If $f =$ on $[a,b]$, then its antiderivative is a constant, so that $\int_a^b f\,\mathrm{d}x = 0$ by the Fundamental Theorem of Calculus.

The other options are not consequences of the Fundamental Theorem of Calculus. ◀

Chapter 6 Exercise Solutions

Ex. 6.1 ($\Rightarrow$) Suppose $f \in \mathscr{R}[a,b]$. By the integrability criterion, for every $\varepsilon > 0$, there exists a partition of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. Thus, by taking $P_1 = P_2 = P$, we have

$$U(P_2,f) - L(P_1,f) < \varepsilon.$$

($\Rightarrow$) Suppose that for every $\varepsilon > 0$, there are partitions P_1 and P_2 such that $U(P_2,f) - L(P_1,f) < \varepsilon$. Let P be the common refinement of P_1 and P_2. Then, by the corollary of Proposition 6.2, we have

$$L(P_1,f) \leq L(P^*,f) \leq U(P^*,f) \leq U(P_2,f).$$

Thus, we have

$$U(P^*,f) - L(P^*,f) < U(P_2,f) - L(P_1,f) < \varepsilon.$$

This demonstrates that the integrability criterion holds for the function f on $[a,b]$. Hence, $f \in \mathscr{R}[a,b]$.

Ex. 6.2 Suppose $f(x^*) > 0$ for some $x^* \in [a,b]$. Since f is continuous on $[a,b]$, for $\varepsilon = \frac{1}{2}f(x^*) > 0$, there exist $\delta > 0$ such that $|x - x^*| < \delta$ and $x \in [a,b]$ imply

$$|f(x) - f(x^*)| < \tfrac{1}{2}f(x^*).$$

Thus, we know that there is an interval whose length is at least δ, say $[\gamma, \gamma+\delta] \subset [a,b]$, on which

$$f(x) > f(x^*) - \tfrac{1}{2}f(x^*) = \tfrac{1}{2}f(x^*).$$

By the monotonicity of the Riemann integral, we have

$$\int_a^b f\,\mathrm{d}x = \int_a^\gamma f\,\mathrm{d}x + \int_\gamma^{\gamma+\delta} f\,\mathrm{d}x + \int_{\gamma+\delta}^b f\,\mathrm{d}x \geq 0 + \tfrac{1}{2}f(x^*)\delta + 0 > 0,$$

which contradicts to the hypothesis $\displaystyle\int_a^b f\,\mathrm{d}x = 0$. Therefore, for every $x \in [a,b]$, we have $f(x) = 0$.

Ex. 6.3 By the fact that the rational and the irrational numbers are both sense in every $[a,b]$ for any $a < b$, we know that for every partition of $[a,b]$,

$$U(P,f) = \sum_{i=1}^n \sup_{[x_{i-1},x_i]} f(x)\,\Delta x_i = b - a, \qquad L(P,f) = \sum_{i=1}^n \inf_{[x_{i-1},x_i]} f(x)\,\Delta x_i = 0.$$

Hence $\displaystyle\underline{\int_a^b} f\,\mathrm{d}x = 0 < b - a = \overline{\int_a^b} f\,\mathrm{d}x$. By the definition, $f \notin \mathscr{R}[a,b]$.

Ex. 6.4 Claim 1: The condition $f^2 \in \mathscr{R}[a,b]$ does not imply $f \in \mathscr{R}[a,b]$.

Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ -1, & \text{if } x \text{ is irrational.} \end{cases}$$

It is clear that $f^2 \equiv 1$ is a constant function and is integrable on any finite interval $[a,b]$ $(a < b)$. However, the function f is not integrable on $[a,b]$. In fact, since the rational and the irrational numbers are both sense in $[a,b]$, we have

$$U(P,f) = \sum_{i=1}^n \sup_{[x_{i-1},x_i]} f(x)\,\Delta x_i = \sum_{i=1}^n \sup_{[x_{i-1},x_i]} (1)\,\Delta x_i = b - a,$$

$$L(P,f) = \sum_{i=1}^n \inf_{[x_{i-1},x_i]} f(x)\,\Delta x_i = \sum_{i=1}^n \inf_{[x_{i-1},x_i]} (-1)\,\Delta x_i = -(b - a).$$

Hence $\displaystyle\underline{\int_a^b} f\,\mathrm{d}x = -(b-a) < b - a = \overline{\int_a^b} f\,\mathrm{d}x$. By the definition, $f \notin \mathscr{R}[a,b]$.

Claim 2: The condition $f^3 \in \mathscr{R}[a,b]$ implies $f \in \mathscr{R}[a,b]$.

The function $g(x) = x^3 \colon [a,b] \to [a^3, b^3]$ is continuous and bijective, shown as follows.

- Continuous: The function g is an elementary function. All elementary functions are continuous in their domains.

– Injective: If $x_1^3 = x_2^3$, then, from the equality

$$\begin{aligned} 0 &= (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) \\ &= (x_1 - x_2)\left[(x_1 + \tfrac{1}{2}x_2)^2 + \tfrac{3}{4}x_2^2\right], \end{aligned}$$

we have $x_1 = x_2$ (because $(x_1 + \frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 = 0$ also implies $x_1 = x_2$).

– Surjective: For ant $y \in [a^3, b^3]$, let $x = \operatorname{sgn} y \cdot \sqrt[3]{|y|}$. Then $x \in [a, b]$ and $x^3 = y$.

Thus, we apply item 3 in Proposition 4.18 to know that the function $\phi(x) = \sqrt[3]{x}$ is continuous. It is clear that $f = \phi \circ f^3$. Since $f^3 \in \mathscr{R}[a, b]$, we conclude that $f \in \mathscr{R}[a, b]$, by Theorem 6.5.

Ex. 6.5 **1.** Since $f, g \in \mathscr{R}[a, b]$, by the linearity of the Riemann integral, we know $f + g, f - g \in \mathscr{R}[a, b]$. Since the square of integrable function is integrable, we further know that $(f + g)^2, (f - g)^2 \in \mathscr{R}[a, b]$. Applying the linearity of the Riemann integral once more, we know that

$$fg = \tfrac{1}{2}[(f + g)^2 - (f - g)^2] \in \mathscr{R}[a, b].$$

2. By part **1**, it suffices to show that $1/g$ is integrable when g is integrable and $|g| \geq c > 0$. We note the equality

$$\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| = \frac{|g(x) - g(y)|}{|g(x)g(y)|} \leq \frac{1}{c^2}\left|g(x) - g(y)\right|.$$

Thus, for any partition P of $[a, b]$,

$$P: \quad a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

if we denote

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} g(x), \qquad m_i = \inf_{x_{i-1} \leq x \leq x_i} g(x),$$

then we have

$$\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| \leq \frac{1}{c^2}(M_i - m_i), \qquad x, y \in [x_{i-1}, x_i],\ i = 1, 2, \ldots, n.$$

Hence,

$$U(P, 1/g) - L(P, 1/g) \leq \frac{1}{c^2}[U(P, g) - L(P, g)].$$

The integrability of $1/g$ follows from the integrability of g and the integrability criterion.

3. As in part **1**, we know $f + g, f - g \in \mathscr{R}[a, b]$. Since the absolute value function $\phi(x) = |x|$ is continuous, we know $|f - g| \in \mathscr{R}[a, b]$. Applying the linearity of the Riemann integral, we know that

$$\begin{aligned} \max\{f, g\} &= \tfrac{1}{2}\left[f + g + |f - g|\right] \in \mathscr{R}[a, b], \\ \min\{f, g\} &= \tfrac{1}{2}\left[f + g - |f - g|\right] \in \mathscr{R}[a, b]. \end{aligned}$$

Ex. 6.6 Let $h = g - f$. Then h is continuous on $[a, b]$ except possibly at x^*, so that $h \in \mathscr{R}[a, b]$ by item 2 of Theorem 6.4. Thus, the function $g = h + f$ is integrable, by the linearity of the Riemann integral. Furthermore, the desired equality follows from the fact

$$\int_a^b h \,\mathrm{d}x = 0.$$

To show the last equality, we note that $h \equiv 0$ except possibly at x^*. For any partition P of $[a, b]$, it is easy to have the following:

$$\begin{aligned} L(P, h) &= 0, \quad \text{if } h(x^*) \geq 0; \\ U(P, h) &= 0, \quad \text{if } h(x^*) \leq 0, \end{aligned}$$

Thus, either $\underline{\int_a^b} h \,\mathrm{d}x = 0$ if $h(x^*) \geq 0$ or $\overline{\int_a^b} f \,\mathrm{d}x = 0$ if $h(x^*) \leq 0$. Since $h \in \mathscr{R}[a, b]$, we have

◀

$$\int_a^b h\,dx = \underline{\int_a^b} h\,dx = \overline{\int_a^b} h\,dx.$$

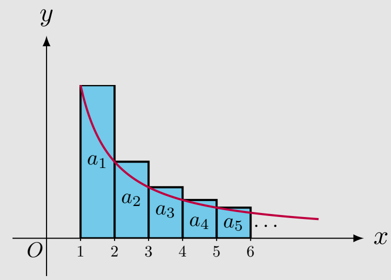
Hence, we conclude that $\int_a^b h\,dx = 0$.

Ex. 6.7 (⇒) Suppose $\sum_{k=1}^{\infty} a_k$ converges. We prove that the limit $\lim_{n\to\infty}\int_1^n f(x)\,dx$ exists and is finite.

For each integer $k \ge 1$, put $g(x) = f(k)$ for $x \in [k, k+1)$. Then g is a function defined on $[1, \infty)$. Since f is monotonically decreasing, it is clear that $g \ge f$. Thus

$$\int_1^n f(x)\,dx \le \int_1^n g(x)\,dx = \sum_{k=1}^{n-1} f(k) \le \sum_{k=1}^{\infty} a_k,$$

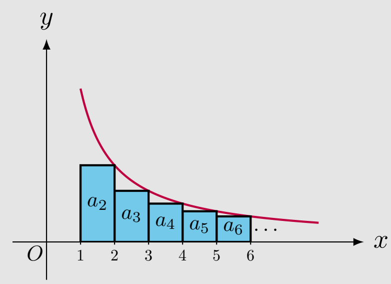
so that the increasing sequence $\left\{\int_1^n f(x)\,dx\right\}$ is bounded above. Hence, the limit $\lim_{n\to\infty}\int_1^n f(x)\,dx$ exists and is finite.



(⇐) Suppose that the limit $\lim_{n\to\infty}\int_1^n f(x)\,dx$ exists and is finite. We prove that $\sum_{k=1}^{\infty} a_k$ converges.

For each integer $k \ge 1$, put $h(x) = f(k+1)$ for $x \in [k, k+1)$. Then h is a function defined on $[1, \infty)$. Since f is monotonically decreasing, we know that $h \le f$. Hence

$$\begin{aligned}\sum_{k=1}^{n} a_k &= f(1) + \sum_{k=2}^{n} f(k)\\ &\le f(1) + \int_1^n h(x)\,dx \le f(1) + \int_1^n f(x)\,dx.\end{aligned}$$



Since

$$\int_1^n f(x)\,dx \le \lim_{n\to\infty}\int_1^n f(x)\,dx,$$

we know that the partial sums of the nonnegative series $\sum a_k$ are bounded above, so that $\sum_{k=1}^{\infty} a_k$ converges.

Ex. 6.8 Since f is continuous on $[a, b]$, by item 1 of Theorem 6.4, we know that $f \in \mathscr{R}[a, b]$. By Ex. 6.5, the product fg is a product of two integrable functions, so it is integrable.

Denote

$$m = \min_{x\in[a,b]} f(x), \qquad M = \max_{x\in[a,b]} f(x).$$

Without loss of generality, we assume that $g(x) \ge 0$ for all $x \in [a, b]$. By the monotonicity of the Riemann integral, we have

$$m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx.$$

If $\int_a^b g(x)\,dx = 0$, then the above inequalities yield $\int_a^b f(x)g(x)\,dx = 0$, so that the desired equality holds for any fixed $\xi \in [a, b]$.

If $\int_a^b g(x)\,dx > 0$, then we have $m \le \lambda \le M$, where

$$\lambda = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$$

By the Intermediate Value Theorem, there is $x \in [a, b]$, such that $f(\xi) = \lambda$. The desired equality follows.

In particular, when $g \equiv 1$, since $\int_a^b 1\,dx = b - a$, we obtain

◀

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = f(\xi).$$

Ex. 6.9 We prove the remainder formula by induction.

For $n=0$, the formula gives

$$\frac{1}{0!}\int_a^x (x-t)^0 f'(t)\,\mathrm{d}t = f(x)-f(a) = f(x)-T_0(x) = R_0(x).$$

So, the formula holds for $n=0$.

Suppose the formula holds for $n=k$:

$$R_k(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,\mathrm{d}t, \qquad x\in I.$$

Then, by integration by parts, we get

$$\begin{aligned}
&\frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\,\mathrm{d}t\\
&= \left[\frac{1}{(k+1)!}(x-t)^{k+1} f^{(k+1)}(t)\right]_{t=a}^{x} + \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,\mathrm{d}t\\
&= R_k(x) - \frac{f^{(k+1)}(a)}{(k+1)}(x-a)^{k+1}\\
&= f(x) - T_k(x) - \frac{f^{(k+1)}(a)}{(k+1)}(x-a)^{k+1}\\
&= f(x) - T_{k+1}(x) = R_{k+1}(x).
\end{aligned}$$

So, the formula holds for $n=k+1$.

Hence, the remainder formula holds for all $n\geq 0$.

Ex. 6.10 By the Newton–Leibniz formula, for any $x\in[a,b]$,

$$\begin{aligned}
f(x)-f(a) &= \int_a^x f'(t)\,\mathrm{d}t,\\
f(x)-f(b) &= -\int_x^b f'(t)\,\mathrm{d}t.
\end{aligned}$$

The sum of these equalities gives

$$2f(x) = 2f(x) - [f(a)+f(b)] = \int_a^x f'(t)\,\mathrm{d}t - \int_x^b f'(t)\,\mathrm{d}t.$$

Integrating both sides of the last equality gives

$$\int_a^b f(x)\,\mathrm{d}x = \frac{1}{2}\int_a^b \left[\int_a^x f'(t)\,\mathrm{d}t - \int_x^b f'(t)\,\mathrm{d}t\right]\mathrm{d}x.$$

Hence,

$$\begin{aligned}
\left|\int_a^b f(x)\,\mathrm{d}x\right| &\leq \frac{1}{2}\int_a^b \left|\int_a^x f'(t)\,\mathrm{d}t - \int_x^b f'(t)\,\mathrm{d}t\right|\mathrm{d}x\\
&\leq \frac{1}{2}\int_a^b \left[\left|\int_a^x f'(t)\,\mathrm{d}t\right| + \left|\int_x^b f'(t)\,\mathrm{d}t\right|\right]\mathrm{d}x\\
&\leq \frac{1}{2}\int_a^b \left[\int_a^x \left|f'(t)\right|\,\mathrm{d}t + \int_x^b \left|f'(t)\right|\,\mathrm{d}t\right]\mathrm{d}x\\
&= \frac{1}{2}\int_a^b \left[\int_a^b \left|f'(t)\right|\,\mathrm{d}t\right]\mathrm{d}x\\
&= \frac{1}{2}\int_a^b \left|f'(t)\right|\,\mathrm{d}t\cdot\int_a^b 1\,\mathrm{d}x = \frac{b-a}{2}\int_a^b \left|f'(t)\right|\,\mathrm{d}t.
\end{aligned}$$

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