

**Mathematical Analysis**

數學分析

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## Preface 前言

Mathematical analysis is a fundamental branch of mathematics that provides the tools and methods to study the behavior of mathematical objects such as functions, sequences, and series. Analysis lies at the heart of many areas of mathematics, science, and engineering, and its concepts and techniques have numerous applications in fields ranging from finance to physics.

This book aims to provide a thorough introduction to mathematical analysis, covering essential topics such as continuity, differentiation, integration, and infinite series. By presenting these concepts in a clear and concise manner, readers will be equipped with a solid foundation in the principles of analysis and their applications in mathematics and related fields.

Throughout the book, we emphasize the development of a rigorous mathematical framework, with a focus on clear and concise proofs that build upon the basic principles of analysis. We also include appropriate examples and exercises to help readers develop their problem-solving skills and deepen their understanding of the material.

Our hope is that this book will provide students with a solid foundation in mathematical analysis that they can build upon as they progress in their studies of mathematics and related fields. We also hope that this book will serve as a valuable reference for researchers and practitioners who use analysis in their work.

數學分析是數學的基礎學科之一，對研究對象如函數、序列和級數的行為提供工具和方法。分析學在數學、科學和工程的許多領域中都具有重要作用，其概念和技巧在金融、物理等領域也有眾多應用。

本書旨在提供全面的數學分析介紹，涵蓋連續性、微分、積分、無窮級數等基本主題。通過以清晰簡明的方式呈現這些概念，讀者將能夠掌握分析原理及其在數學和相關領域的應用。

在全書中，我們強調發展嚴謹的數學框架，著重於基於分析學基本原理的清晰簡明的證明。我們還包括適當的例子和練習，幫助讀者發展解決問題的技能並加深對材料的理解。

我們希望本書能為學生提供堅實的數學分析基礎，並在他們進一步學習數學和相關領域時發揮作用。我們也希望本書能為在工作中使用分析學的研究人員和實踐者提供有價值的參考。

This book is presented in the form of an e-book, and uses hyperlinks to connect the content together, greatly improving the reader's overall understanding of the mathematical content and enhancing the readability of the e-book.

本書以電子書的形式呈現，並用超鏈接將內容聯繫起來，這大大地改善了讀者對數學內容的整體性的了解，增加了電子書的可讀性。

## Mathematical Arguments 數學論證

## §0.1 Mathematical Proof 數學證明

What is a mathematical proof?

A **mathematical proof** is an inferential argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion. – from Wikipedia

Usually, a mathematical proof uses a few following terminologies to form an argument:

- **definition** (定義)
  - a definition is used to give a precise meaning to a new term, by describing a condition which unambiguously qualifies what a mathematical term is and is not.
- **axiom** (公理)
  - an axiom is a self-evident rule or first principle accepted as true.
- **proposition** (命題)
  - a proposition is a theorem of lesser importance, or one that is considered so elementary or immediately obvious.
- **lemma** (引理)
  - a lemma is a minor proposition which is used as a stepping stone to a larger result.
- **theorem** (定理)
  - a theorem is a statement which has been proved true by a special kind of logical argument.
- **corollary** (推論)
  - a corollary is a theorem connected by a short proof to an existing theorem.

Every proof may use certain assumptions together with certain basic or original axioms and previously established statements such as theorems. It is constructed along with the accepted rules of inference.

Remark

- Presenting many cases in which the statement holds is not constituted for a proof, which must demonstrate that the statement is true in *all* possible cases.

## Remark

- A rigorous proof may exploit one of the following approaches:
  - **Direct Proof** (直接法)
  - **Proof by Induction** (歸納法)
  - **Proof by Contraposition** (逆否命題法)
  - **Proof by Contradiction** (反證法)
  - **Proof by Exhaustion** (窮舉法)
  - **Proof by Counterexample** (反例舉證法)

## Direct Proof 直接法

## Direct Proof

A **direct proof** is a way of showing the truth or falsehood of a given statement by a straightforward combination of established facts, usually axioms, existing lemmas and theorems, without making any further assumptions.

We use a direct proof to prove statements of the form “if  $p$  then  $q$ ” or “ $p$  implies  $q$ ” which we can write as  $p \Rightarrow q$ .

## outline for a direct proof

**Claim:**  $p \Rightarrow q$ .

**Proof:**

- ① Assume the statement  $p$  is true.
- ② Use what we know about  $p$  and other facts as necessary to deduce that another statement  $q$  is true. This accomplishes  $p \Rightarrow q$ .



## Remark

- A direct proof is one of the most familiar forms of proof.

## Proof by Induction 歸納法

## Proof by Induction

A **proof by induction** (also called *mathematical induction*) is to prove statements by showing a logical progression of justifiable steps by first asserting a hypothesis.

There are usually two types of induction, regular and strong, to prove that a statement  $p(n)$  is true for  $n \geq n_0$ ,  $n \in \mathbb{N}_+$ .

Both types of induction consist of two steps: the base step and the inductive step.

- Base step: proves the statement for the initial case without assuming any knowledge of other cases.
- Inductive step: proves the statement for the progressional step under an “inductive hypothesis”.

## Principle of Induction

The base step and the inductive step, together, prove that

$$p(n_0) \Rightarrow p(n_0 + 1) \Rightarrow p(n_0 + 2) \Rightarrow p(n_0 + 3) \Rightarrow \cdots$$

Therefore,  $p(n)$  is true for  $n \geq n_0$ ,  $n \in \mathbb{N}_+$ .

## outline for regular induction

**Claim:**  $p(n)$  for  $n \geq n_0$ ,  $n \in \mathbb{N}$ .**Proof:**

- ① Base Step: Verify that  $p(n_0)$  is true.
- ② Inductive Step:
  - Assume “inductive hypothesis”:  $p(k)$  is true, where  $k \geq n_0$ .
  - Use what we know about the inductive hypothesis and other facts as necessary to deduce that  $p(k+1)$  is true.
- ③ Therefore, by the principle of induction,  $p(n)$  is true for  $n \geq n_0$ ,  $n \in \mathbb{N}$ .

## outline for strong induction

**Claim:**  $p(n)$  for  $n \geq n_0$ ,  $n \in \mathbb{N}$ .**Proof:**

- ① Base Step: Verify that  $p(n_0)$  is true.
- ② Inductive Step:
  - Assume “inductive hypothesis”:  $p(n)$  is true, where  $n_0 \leq n \leq k$ .
  - Use what we know about the inductive hypothesis and other facts as necessary to deduce that  $p(k+1)$  is true.
- ③ Therefore, by the principle of induction,  $p(n)$  is true for  $n \geq n_0$ ,  $n \in \mathbb{N}$ .



## Remark

- The base step of the induction is a factual statement and the inductive step is a conditional statement.
- Usually, the base step begins with  $n_0 = 1$ , but it can be with  $n_0 = 0$  or any fixed integer  $n_0$ .
- Both steps are important. False induction proofs can occur when some minor details are left out.
- In general, strong induction is particularly helpful in cases at the inductive step where your  $p(k+1)$  is defined in terms of more than one previous  $p(n)$  ( $n_0 \leq n \leq k$ ), or in cases where you may not know how many previous statements.

## Proof by Contraposition 逆否命題法

## Proof by Contraposition

A **proof by contraposition** infers the statement “if  $p$  then  $q$ ” by establishing the logically equivalent contrapositive statement: “if  $\neg q$  then  $\neg p$ ”, where  $\neg$  is the standard logical **not** symbol.

## outline for proof by contraposition

**Claim:**  $p \Rightarrow q$ .**Proof:**

- ① We will prove  $\neg q \Rightarrow \neg p$ .
- ② Assume the statement  $\neg q$  is true, that is,  $q$  is false. Use what we know about  $\neg q$  and other facts as necessary to deduce that another statement  $p$  is false. Thus,  $\neg q \Rightarrow \neg p$ .
- ③ Therefore, by contraposition,  $p \Rightarrow q$ .



## Remark

- The contrapositive statement of “ $p \Rightarrow q$ ” is “ $\neg q \Rightarrow \neg p$ ”, while the converse statement is “ $q \Rightarrow p$ ”.

## Remark

- Proof by contraposition is to prove a claim indirectly: one assumes that the conclusion is false then proves that the hypothesis is also false. For readability of the proof, it is helpful to make a simple statement, like “we will prove this result by contraposition”.
- Using contraposition on an “or” statement is useful because it negates and reverses the inference direction, transforming the “or” into an “and” (De Morgan’s law), making the proof process easier.

## Proof by Contradiction 反證法

## Proof by Contradiction

A **proof by contradiction** is to establish the truth of a statement, by showing that assuming the statement to be false leads to a contradiction.

## outline for proof by contradiction

**Claim:**  $p \Rightarrow q$ .

**Proof:**

- ① We will prove that  $\neg q$  leads to a contradiction.
- ② Assume the statement  $\neg q$  is true, that is,  $q$  is false. Use what we know about  $\neg q$ ,  $p$  and other facts as necessary to deduce a contradiction.
- ③ Therefore, by contradiction,  $\neg q$  cannot be true, so  $q$  is true.



## Remark

- Proof by contradiction is to prove a claim indirectly: one assumes that the conclusion is false then proves that the hypothesis leads to a contradiction. For readability of the proof, it is helpful to make a simple statement, like “we will prove this result by contradiction”.
- Proof by contradiction is often used when there is some binary choice between possibilities.
- One of the main advantages of proof by contradiction is its versatility. It can be used to prove a wide variety of mathematical statements, ranging from simple theorems to complex conjectures. Additionally, it often provides a concise and elegant argument that can help to illuminate the underlying structure of a mathematical problem.

It’s important to use proof by contradiction carefully, as it can lead to false results. The assumption of the opposite statement may not be logically contradictory, causing the proof to fail. Moreover, it can be less intuitive and harder to follow than direct proofs, especially for less experienced readers.

## Remark

- Don’t confuse “proof by contraposition” with “proof by contradiction”. While both establish the truth of a statement, they differ in approach. The former shows that negating the conclusion implies negating the hypothesis, while the latter shows that negating the conclusion leads to a logical contradiction.



## Proof by Exhaustion 窮舉法

### Proof by Exhaustion

A **proof by exhaustion** is to validate the conclusion by dividing it into a finite number of cases and proving each one separately.

#### outline for a proof by exhaustion

**Claim:**  $p \Rightarrow q$ .

**Proof:**

- ① Establish the cases that apply to the statement  $q$ .
- ② Use what we know about  $p$  and other facts as necessary to prove that the statement  $q$  is true in each case.
- ③ Since  $q$  is true in every case, we conclude that  $q$  is true.



#### Remark

- A proof by exhaustion is also known as **proof by cases**, or the **brute force method**.
- When the problem is divided into a finite number of cases, one needs to make sure that these cases exhaust the possibilities and to prove the desired result in each case.

## Proof by Counterexample 反例舉證法

### Proof by Counterexample

A **proof by counterexample** is not technically a proof. It is merely a way of showing that a given statement cannot possibly be correct by providing an example where it does not hold.

#### outline for a proof by counterexample

**Claim:** disprove  $p \Rightarrow q$ .

**Proof:**

- ① Find a case where  $p$  is true but  $q$  is false.



#### Remark

- Proof by counterexample is based on the following principle:
 

- Since mathematical statement is true only when it is true 100% of the time, one can prove that it is false by finding a single example where it is not true.
- A counterexample is an example to disprove a statement.
- A “proof by counterexample” is different from a “proof by example”. The latter is usually invalid in mathematical arguments. A single example cannot prove a universal statement (unless the universe comprises only one case!). A single counterexample can disprove a universal statement.

## §0.2 Mathematical Writing 數學寫作

### Guidelines of mathematical writing

The importance of writing in the mathematics classroom cannot be overemphasized. In the process of writing, students clarify their own understanding of mathematics and hone (磨練) their communication skills. They must organize their ideas and thoughts more logically and structure their conclusions in a more coherent way.

– from IDRA

- Although there are no definite rules in mathematical writing, there are some standard guidelines, as compiled below, that will make your writing clearer. Some examples of bad usage (marked with ✗) and good usage (marked with ✓) are used to illustrate these guidelines.

#### ① Use complete sentences.

使用完整的句子。

In order to convey your logical train of thought, you should use complete sentences in your writing.

✗ Since  $x$  is a real number.

✓ Since  $x$  is a real number, we have  $x^2 \geq 0$ .

✗ If differentiable, so continuous.

✓ If a function is differentiable at a point  $a$ , then it is continuous at  $a$ .

#### ② Avoid ambiguity.

避免歧義。

The language of mathematics is precise and unambiguous, and ambiguity can lead to confusion, misunderstandings, and even errors.

✗ Let  $m$  and  $n$  be odd and even.

✓ Let  $m$  and  $n$  be odd and even respectively.

✗ We have  $x^2 + 1 \in S$ ,  $x \in \mathbb{R}$ .

✗ We have  $x^2 + 1 \in S$  for  $x \in \mathbb{R}$ .

✓ We have  $x^2 + 1 \in S$  for all  $x \in \mathbb{R}$ .

✓ We have  $x^2 + 1 \in S$  for some  $x \in \mathbb{R}$ .

#### ③ Never begin a sentence with a mathematical symbol.

不用數學符號開句。

Beginning a sentence with a mathematical symbol can make the writing less clear and more difficult to understand. Mathematical symbols are case sensitive, for instance, since  $x$  and  $X$  can have entirely different meanings, putting such symbols at the beginning of a sentence can lead to ambiguity.

✗  $A$  is a subset of  $B$ .

✓ The set  $A$  is a subset of  $B$ .

✗  $f$  is differentiable in  $(a, b)$ .

✓ The function  $f$  is differentiable in  $(a, b)$ .

- ✗  $x^2 + 2x - 3 = 0$  has two real solutions.
- ✗  $X^2 + 2x - 3 = 0$  has two real solutions.
- ✓ The equation  $x^2 + 2x - 3 = 0$  has two real solutions.

#### ④ Separate mathematical symbols and expressions with words.

將數學符號和表達式用字詞分隔開。

Using words to separate mathematical symbols and expressions can enhance the clarity and comprehensibility. Without such separators, there is a risk of confusion, as the expressions may appear to merge together.

- ✗ Unlike  $A \cup B$ ,  $A \cap B$  equals  $\emptyset$ .
- ✓ Unlike  $A \cup B$ , the set  $A \cap B$  equals  $\emptyset$ .
- ✗ Because  $x^2 + y^2 = r^2$ ,  $x = r \cos \theta$  and  $y = r \sin \theta$ .
- ✓ Because  $x^2 + y^2 = r^2$ , it follows that  $x = r \cos \theta$  and  $y = r \sin \theta$ .

#### ⑤ Avoid using unnecessary symbols.

避免使用無必要的符號。

Avoiding unnecessary symbols in mathematical writing can help to improve clarity, efficiency, consistency, accessibility, and aesthetics (美感).

- ✗ If an integer  $n$  is even, then its square is also even.
- ✓ If an integer is even, then its square is also even.

#### ⑥ Avoid misuse of symbols.

避免錯誤使用符號。

Symbols like  $=$ ,  $\sim$ ,  $\leq$ ,  $\subset$ ,  $\in$ , etc. are not words and are primarily used in mathematical expressions. Using them in other contexts may seem inappropriate.

- ✗ If two sets are  $\sim$ , then they have the same cardinality.
- ✓ If two sets are equivalent, then they have the same cardinality.
- ✗ Since  $m$  is odd and  $n$  odd  $\Rightarrow n^2$  odd,  $m^2$  is odd.
- ✓ Since  $m$  is odd and any odd number squared is odd, we know that  $m^2$  is odd.

#### ⑦ Use first person plural.

使用第一人稱複數。

In mathematical writing, it is common to use the words “we” and “us” rather than “I”, “you” or “me”. This is designed to involve the readers and promote their active participation, creating a sense of collaboration and teamwork.

- ✗ By the Pythagorean theorem, I get the hypotenuse  $c = \sqrt{a^2 + b^2}$ .
- ✓ By the Pythagorean theorem, we get the hypotenuse  $c = \sqrt{a^2 + b^2}$ .

#### ⑧ Use the active voice.

使用主動語態。

The active voice is generally preferred over the passive voice because it can help to make the writing more concise, clear, and direct.

- ✗ The hypotenuse  $c = \sqrt{a^2 + b^2}$  is obtained by taking the square root of the identity  $c^2 = a^2 + b^2$ .
- ✓ Taking the square root of the identity  $c^2 = a^2 + b^2$ , we get the hypotenuse  $c = \sqrt{a^2 + b^2}$ .

⑨ Use conjunction words and conjunctive adverbs properly.

正確使用連接詞與連接副詞。

Conjunction words (like “and”, “or”, “but”) and conjunctive adverbs (like “thus”, “hence”, “however”) are important tools for connecting ideas and creating coherent, well-structured sentences in mathematical writing.

- ✗ The function  $f$  is continuous and not differentiable at  $x = 0$ .
- ✓ The function  $f$  is continuous, but not differentiable at  $x = 0$ .
- ✗ The function is continuous and is not uniformly continuous.
- ✓ The function is continuous; however, it is not uniformly continuous.
- ✗ Although the data is inconsistent, nevertheless, we can still make some observations.
- ✓ Even though the data is inconsistent, we can still make some observations.
- ✓ We can still make some observations, despite the fact that the data is inconsistent.

⑩ Use correct punctuation.

正確使用標點符號。

Using correct punctuation is an important aspect of clear and effective mathematical writing.

- ✗ Hence, we get the equality  $c^2 = a^2 + b^2$
- ✓ Hence, we get the equality  $c^2 = a^2 + b^2$ .
- ✗ Suppose that  $x$  is nonnegative, then we have  $x + 1 > 0$ .
- ✓ Suppose that  $x$  is nonnegative. Then we have  $x + 1 > 0$ .

## Number Systems 數系

## Overview of Chapter 1

In this chapter, we will discuss the appropriate number system, real numbers, on which we will build the foundation for future mathematical analysis.

- Real numbers form an *ordered field* that enables arithmetic operations and the establishment of orders, which are essential for studying limits. Limits involve the idea of approaching a value from above or below, which is only meaningful within an ordered field.
- The *least-upper-bound property* of real numbers makes them ideal for mathematical analysis. Real numbers can measure and compare quantities, and have limits and continuity. In contrast, the set of rational numbers lacks this property, making the concept of limits incomplete and unsuitable for analysis.

## §1.1 Number Sets 數集

## 1.1 Definition: set (集合)

If  $A$  is any **set** (集合) (whose elements may be numbers or any other objects), we write  $x \in A$  to indicate that  $x$  is a member (or an element) of  $A$ . If  $x$  is not a member of  $A$ , we write  $x \notin A$ .

The set which contains no element will be called the *empty* set (空集), denoted as  $\emptyset$ .

If  $A$  and  $B$  are sets, and if every element of  $A$  is an element of  $B$ , we say that  $A$  is a **subset** (子集, 子集合) of  $B$ , and write  $A \subset B$ , or  $B \supset A$ . If in addition, there is an element of  $B$  which is not in  $A$ , then  $A$  is said to be a *proper* subset of  $B$  (真子集), and write  $A \subsetneq B$ , or  $B \supsetneq A$ .

$$\bullet A \subset A.$$

$$\bullet \text{ If } A \subset B \text{ and } B \subset C, \text{ then } A \subset C.$$

$$\bullet \text{ If } A \subset B \text{ and } B \subset A, \text{ then } A = B.$$

## 1.2 Definition: the sets of numbers 數集


- The set of all positive integers (正整數) is the set  $\{1, 2, 3, \dots\}$ , denoted  $\mathbb{N}_+$ .
- The set of all natural numbers (自然數) is the set  $\{0, 1, 2, 3, \dots\}$ , denoted  $\mathbb{N}$ .
- The set of all integers (整數) is the set  $\{0, \pm 1, \pm 2, \dots\}$ , denoted  $\mathbb{Z}$ .
- The set of all rational numbers (有理數) is the set

$$\{m/n : m \in \mathbb{Z}, n \in \mathbb{N}_+, m \text{ and } n \text{ co-prime (互質)}\},$$

denoted  $\mathbb{Q}$ . A number is rational if and only if it is a terminating decimal or a repeating decimal.

- An irrational number (無理數) is a non-terminating, non-repeating decimal. The set of all real numbers (實數) is the set of all rational numbers and irrational numbers, denoted as  $\mathbb{R}$ .

## Remark

- $\mathbb{N}_+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$ . 
- There are infinitely many rational numbers between any two given rational numbers, as the midpoint between them is also a rational number. Nevertheless, the rational number system contains “gaps” that are filled by real numbers.
- We will show that the real number system contains no gaps.

## Remark

- If each point  $x$  of an oriented straight line is put into correspondence with its distance from a given point  $O$  (which is positive if the point is located in a positive direction from  $O$ , and is negative otherwise), the resulting correspondence is a *one-to-one* correspondence (一一對應) between the points on the straight line and the real numbers.

## 1.3 Definition: order and ordered sets (序與有序集)

Let  $S$  be a set. An **order** (序) on  $S$  is a relation, denoted by  $<$ , with the following properties:

- (1) If  $x \in S$  and  $y \in S$ , then one and only one of the following statements is true:

$$x < y, \quad x = y, \quad y < x.$$

- (2) If  $x, y, z \in S$ , if  $x < y$  and  $y < z$ , then  $x < z$ .

## Remark

- The statement “ $x < y$ ” may be read as “ $x$  is less than  $y$ ” or “ $x$  is smaller than  $y$ ”.
- The statement “ $y > x$ ” is equivalent to “ $x < y$ ”.
- The statement “ $x \leq y$ ” indicates that “ $x < y$  or  $x = y$ ”. In other words,  $x \leq y$  is the negation (否定) of  $x > y$ .

An **ordered set** (有序集) is a set  $S$  in which an order is defined.

## Remark

- If we define  $r < s$  to mean that  $s - r > 0$ , then  $\mathbb{Q}$  becomes an ordered set.

## 1.4 Definition: field (域)

A **field** (域) is a set  $F$  with two operations, called **addition** (+) and **multiplication** ( $\times$ ), which satisfy the following one set of axioms for addition, one set of axioms for multiplication, and the **distributive law** (分配律) relating the operations:

## Axioms for addition

- (A1) Closure under + :  $\forall x, y \in F, x + y \in F$   
 (A2) Commutativity of + :  $\forall x, y \in F, x + y = y + x$   
 (A3) Associativity of + :  $\forall x, y, z \in F, (x + y) + z = x + (y + z)$   
 (A4) Identity element for + :  $\exists 0 \in F, \forall x \in F, x + 0 = x$ ,  $0$  is called the zero  
 (A5) Inverse element for + :  $\forall x \in F, \exists x' \in F, x + x' = 0_F$ , write  $x' = -x$  the negative element

## Axioms for multiplication

- (M1) Closure under  $\times$  :  $\forall x, y \in F, x \times y \in F$   
 (M2) Commutativity of  $\times$  :  $\forall x, y \in F, x \times y = y \times x$   
 (M3) Associativity of  $\times$  :  $\forall x, y, z \in F, (x \times y) \times z = x \times (y \times z)$   
 (M4) Identity element for  $\times$  :  $\exists 1 \in F, 1 \neq 0, \forall x \in F, 1 \times x = x$ ,  $1$  is called the unity  
 (M5) Inverse element for  $\times$  :  $\forall x \in F, x \neq 0, \exists x^{-1} \in F, x \times x^{-1} = 1_F$

## The distributive law

- (D) Product is distributive over addition :  $\forall x, y, z \in F, x \times (y + z) = x \times y + x \times z$

## Remark

- The zero  $0$ , the negative element  $-x$  of  $x$ , and the unity  $1$ , and the inverse element  $x^{-1}$  of  $x$  (for  $x \neq 0$ ) are all unique. For instance, for any given  $x$ , there is a unique element  $x'$  such that  $x + x' = 0$ .
- For convenience, one sometimes uses  $x \cdot y$  or  $xy$  to replace  $x \times y$ . One usually writes

$$x - y, \frac{x}{y}, x^3, 3x, x + y + z, xyz$$

in place of  $x + (-y), x \times y^{-1}, xxx, x + x + x, (x + y) + z, (xy)z$ .

## 1.5 Proposition: field properties

## Properties on addition

- If  $x + y = x + z$ , then  $y = z$ .
- If  $x + y = x$ , then  $y = 0$ .
- If  $x + y = 0$ , then  $y = -x$ .
- $-(-x) = x$ .

## Properties on multiplication

- If  $x \neq 0$  and  $xy = xz$ , then  $y = z$ .
- If  $x \neq 0$  and  $xy = x$ , then  $y = 1$ .
- If  $x \neq 0$  and  $xy = 1$ , then  $y = x^{-1}$ .
- If  $x \neq 0$ , then  $(x^{-1})^{-1} = x$ .

## Properties on the zero and the negative elements

- $0x = 0$ .
- If  $x \neq 0$  and  $y \neq 0$ , then  $xy \neq 0$ .
- $(-x)y = -(xy) = x(-y)$ .
- $(-x)(-y) = xy$ .



## 1.6 Definition: ordered field (有序域)

A **ordered field** (有序域) is a field  $F$  on which an *order* is defined, such that

- (1) if  $x, y, z \in F$  and  $y < z$ , then  $x + y < x + z$ ;
- (2) if  $x, y \in F$ ,  $x > 0$ , and  $y > 0$ , then  $xy > 0$ .

If  $x > 0$ , we call  $x$  **positive**; if  $x < 0$ ,  $x$  is **negative**.

## Properties on ordered field

1. If  $x > 0$ , then  $-x < 0$ , and *vice versa*.
2. If  $x > 0$  and  $y < z$ , then  $xy < xz$ .
3. If  $x < 0$  and  $y < z$ , then  $xy > xz$ .
4. If  $x \neq 0$ , then  $x^2 > 0$ . In particular,  $1 > 0$ .
5. If  $0 < x < y$ , then  $0 < y^{-1} < x^{-1}$ . ▶

1.7 Proposition:  $\mathbb{Q}$  and  $\mathbb{R}$  are ordered fields

The set of rational numbers  $\mathbb{Q}$  and the set of real numbers  $\mathbb{R}$  are both ordered fields.

## Remark

- There are other ordered fields. For example, any subfield of an ordered field, such as the real algebraic numbers. (An algebraic number is a number that is a root of a nonzero polynomial in one variable with integer coefficients.)

## §1.2 Least-Upper-Bound Property 確界原理

## 1.8 Definition: bounded set and the least-upper-bound property (有界集與確界原理)

Suppose  $S$  is an ordered set, and  $E \subset S$ .

If there exists a  $\beta \in S$  such that  $x \leq \beta$  for *every*  $x \in E$ , we say that  $E$  is **bounded above** (有上界), and call  $\beta$  an **upper bound** of  $E$ .

If there exists an  $\alpha \in S$  such that  $\alpha \leq x$  for *every*  $x \in E$ , we say that  $E$  is **bounded below** (有下界), and call  $\alpha$  a **lower bound** of  $E$ .

A set is **bounded** (有界) if it has both upper and lower bounds.

Suppose  $E$  is bounded above. If there is a  $\beta \in S$  with the following properties:

- (1)  $\beta$  is an upper bound of  $E$ ;
- (2) if  $\gamma < \beta$ , then  $\gamma$  is not an upper bound of  $E$

then  $\beta$  is called the **least upper bound** of  $E$  or the **supremum** (上確界) of  $E$ , and we write

$$\beta = \sup E.$$

Suppose  $E$  is bounded below. If there is an  $\alpha \in S$  with the following properties:

- (1)  $\alpha$  is a lower bound of  $E$ ;
- (2) if  $\alpha < \gamma$ , then  $\gamma$  is not a lower bound of  $E$

then  $\alpha$  is called the **greatest lower bound** of  $E$  or the **infimum** (下確界) of  $E$ , and we write

$$\alpha = \inf E.$$

## Remark

- The supremum of a set, if it exists, is unique. The same conclusion hold for the infimum. ▶



An ordered set  $S$  is said to have the **least-upper-bound property** (確界原理) if the following is true:  
for any nonempty subset  $E$  of  $S$ , if  $E$  is bounded above, then  $\sup E$  exists in  $S$ .

An ordered set with the least-upper-bound property also has the greatest-lower-bound property. ►

#### Remark

- The least-upper-bound property is a crucial concept in the study of limits and is essential for the development of calculus and real analysis.

### 1.9 Theorem: the least-upper-bound property of $\mathbb{Q}$ and $\mathbb{R}$

1. With the usual addition, multiplication, and the order,  $\mathbb{Q}$  is an ordered field which does *not* have the least-upper-bound property. ►
2. With the usual addition, multiplication, and the order,  $\mathbb{R}$  is an ordered field which *has* the least-upper-bound property. Moreover,  $\mathbb{R}$  contains  $\mathbb{Q}$  as a subfield. ►

#### Remark

- Whether possessing the least-upper-bound property or not is one of the major differences between  $\mathbb{R}$  and  $\mathbb{Q}$ . Without the least-upper-bound property, the set  $\mathbb{Q}$  cannot cover the whole line completely so that the key concept, limit, cannot be *well* defined.

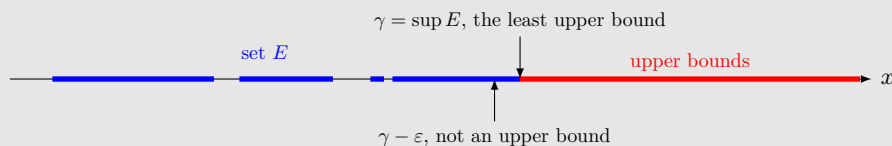
#### Remark

- Suppose  $E \subset \mathbb{R}$  is a bounded above in  $\mathbb{R}$ . Then  $\gamma = \sup E$  is the number that satisfies the following:

(1)' For any  $p \in E$ , the inequality  $p \leq \gamma$  holds.

(2)' For any given  $\varepsilon > 0$ , there exists a number  $q \in E$  such that  $\gamma - \varepsilon < q$ .

Item (1)' means that  $\gamma$  is an upper bound of  $E$ ; Item (2)' means that any number smaller than  $\gamma$  is not an upper bound of  $E$ . These can be illustrated as in the following diagram.



### 1.10 Theorem: the archimedean property of $\mathbb{R}$ and the rational density theorem

1. If  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ , and  $x > 0$ , then there is a positive integer  $n$  such that  $nx > y$ .
2. If  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ , and  $x < y$ , then there exists a  $p \in \mathbb{Q}$  such that  $x < p < y$ . ►

#### Remark

- The first statement is known as **archimedean property** (阿基米德性質).
- The second statement says that  $\mathbb{Q}$  is *dense* in  $\mathbb{R}$  (稠密).

**1.11 Proposition:** existence of  $n$ th root of positive real numbers

For any given positive real number  $x$  and positive integer  $n$ , there is a one and only one real positive  $y$  such that

$$y^n = x.$$

This number  $y$  is called the  **$n$ th root** of  $x$  and denoted as  $\sqrt[n]{x}$  or  $x^{1/n}$ . ▶

**Law of radicals**

For any positive real numbers  $a$  and  $b$  and positive integer  $n$ ,

$$(ab)^{1/n} = a^{1/n} \cdot b^{1/n}. \quad \blacktriangleright$$

**Remark**

- The equation  $y^n = 0$  has only zero solution. Hence, it is meaningful to define  $\sqrt[n]{0} = 0$ .

**1.12 Definition:** the absolute value (絕對值)

If  $x$  is a real number, its **absolute value** (絕對值) is defined as  $|x| = (x^2)^{1/2}$ .

**Remark**

- If  $x$  is real, then  $|x| = x$  for  $x \geq 0$ ;  $|x| = -x$  for  $x < 0$ .
- $|x - a| < \delta \iff a - \delta < x < a + \delta$  for any real  $x, a$ , and  $\delta$  with  $\delta > 0$ . ▶

**Properties of the absolute value**

If  $x$  and  $y$  are real numbers, then

1.  $|x| \geq 0$  and  $|x| = 0$  if and only if  $x = 0$ ;
2.  $|xy| = |x| \cdot |y|$ ;
3.  $|x + y| \leq |x| + |y|$ . ▶

**Remark**

- The inequality  $|x + y| \leq |x| + |y|$  is widely used to estimate quantities in analysis. For instance, it plays a vital role in proving the continuity of many functions.

**Remark**

- If one writes  $x = y_1 - y_2$ ,  $y = y_2 - y_3$ , then  $x + y = y_1 - y_3$ . In this case, the inequality  $|x + y| \leq |x| + |y|$  becomes

$$|y_1 - y_3| \leq |y_1 - y_2| + |y_2 - y_3|.$$

This is analogous to the fact that the length of any side of a triangle cannot exceed the sum of the lengths of the other two sides. For this sake, one often refers the inequality  $|z + w| \leq |z| + |w|$  as the **triangle inequality** (三角不等式).

**1.13 Theorem:** the Cauchy-Schwarz inequality (柯西-許瓦爾茲不等式)

If  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  are real numbers, then the **Cauchy-Schwarz inequality** (柯西-許瓦爾茲不等式) holds:

$$|a_1 b_1 + \dots + a_n b_n|^2 \leq (a_1^2 + \dots + a_n^2) (b_1^2 + \dots + b_n^2). \quad \blacktriangleright$$

**Remark**

- The Cauchy-Schwarz inequality is considered to be one of the most important and widely used inequalities in mathematics.
- The Cauchy-Schwarz inequality has many different versions. Currently, it is on real numbers. Other versions are on complex numbers, euclidean spaces, function spaces, and random variables.

## 1.14 Definition: the extended real number system (擴張實數系)

The **extended real number system** (擴張實數系) comprises the real field  $\mathbb{R}$  and two symbols,  $\infty$  and  $-\infty$ . We preserve the original order in  $\mathbb{R}$ , and define:  $\forall x \in \mathbb{R}, -\infty < x < \infty$ .

Denote  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ . The numbers in  $\mathbb{R}$  are called **finite**, and the symbols  $\infty$  and  $-\infty$  **infinite**.

## Remark

- Clearly,  $\infty$  is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. Thus if, for example,  $E$  is a nonempty set of real numbers which is not bounded above in  $\mathbb{R}$ , then  $\sup E = \infty$ . Similarly, if  $E$  is a nonempty set of real numbers which is not bounded below in  $\mathbb{R}$ , then  $\inf E = -\infty$ .

## Remark

- The extended real number system does not form a field. So, when  $\infty$  and/or  $-\infty$  are involved in operations, one *cannot* treat them the same way as numbers in  $\mathbb{R}$ . However, it is customary to make the following conventions (約定):

$$(1) \text{ If } x \in \mathbb{R}, \text{ then } \quad x + \infty = \infty, \quad x - \infty = -\infty, \quad \frac{x}{\infty} = \frac{x}{-\infty} = 0.$$

$$(2) \text{ If } x > 0 \text{ then } x \cdot (\infty) = \infty, x \cdot (-\infty) = -\infty; \text{ if } x < 0 \text{ then } x \cdot (\infty) = -\infty, x \cdot (-\infty) = \infty.$$

## Addendum 後記

## Addendum of Chapter 1

- In this e-book, we assume that the least-upper-bound property holds in  $\mathbb{R}$  without a rigorous proof. A detailed proof can be found in Rudin's book, *Principles of Mathematical Analysis*.
- Thanks to Proposition 1.11, it is easy to extend the definition of the exponent  $b^r$  to rational  $r$  for  $b > 0$ . In fact, by using the least-upper-bound property in  $\mathbb{R}$ , it can be further extended to real  $r$  for  $b > 0$ . In other words, for all real  $r$  and  $b > 0$ , the exponent  $b^r$  is well-defined as a real number.

– For positive numbers  $a$  and  $b$ , and real numbers  $\alpha$  and  $\beta$ , the following **laws of exponents** (指數定律) hold:

$$1. a^\alpha \cdot a^\beta = a^{\alpha+\beta}.$$

$$2. \frac{a^\alpha}{a^\beta} = a^{\alpha-\beta}.$$

$$3. (a^\alpha)^\beta = a^{\alpha\beta}.$$

$$4. (ab)^\alpha = a^\alpha \cdot b^\alpha.$$

$$5. a^{-\alpha} = \frac{1}{a^\alpha}.$$

$$6. a^1 = a, a^0 = 1.$$

- Given two real numbers  $b > 1$ ,  $y > 0$ , it can be proven that there is a unique real  $x$ , called the **logarithm** of  $y$  to the base  $b$ , denoted as  $\log_b y$ , such that  $b^x = y$ . Hence, by requiring  $\log_b y = -\log_{b^{-1}} y$  for  $0 < b < 1$ , the logarithm  $\log_b y$  is well-defined as a real number for all positive  $b$  and  $y$ , with  $b \neq 1$ .

– For positive numbers  $a$  and  $b$ , with  $a, b \neq 1$ , positive numbers  $A$  and  $B$ , and real number  $\alpha$ , the following **laws of logarithms** (對數定律) holds:

$$1. \log_a (A \cdot B) = \log_a A + \log_a B.$$

$$2. \log_a \left( \frac{A}{B} \right) = \log_a A - \log_a B.$$

$$3. \log_a A^\alpha = \alpha \log_a A.$$

$$4. a^{\log_a A} = A.$$

$$5. \log_a A = \frac{\log_b A}{\log_b a}.$$

$$6. \log_a a = 1, \log_a 1 = 0.$$

## Exercises of Chapter 1 練習題

## Chapter 1: Quiz

30 Minutes

- ① Which of the following is an example of a field?
- the set of integers
  - the set of even integers
  - the set of natural numbers
  - the set of rational numbers
  - the set of irrational numbers
- ② Which of the following sets is an ordered field?
- the set of integers
  - the set of even integers
  - the set of natural numbers
  - the set of real numbers
  - the set of extended real numbers
- ③ Which of the following sets is not closed under addition?
- The set of odd integers
  - The set of even integers
  - The set of natural numbers
  - The set of integers
  - The set of rational numbers
- ④ Which of the following is NOT an example of a bounded set in  $\mathbb{R}$ ?
- $(0, 1]$
  - $\{x : |x| < 1\}$
  - $\{x : x < 3\}$
  - $\{x : -4 \leq x \leq 3\}$
  - $\{x : x^2 \leq 4\}$
- ⑤ Every nonempty bounded-above subset of  $\mathbb{R}$
- has a minimum element
  - has a maximum element
  - has an infimum
  - has a supremum
  - has a unique element
- ⑥ Which of the following sets contains the least upper bound within itself?
- $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$
  - $\{x : x \text{ is a rational number and } x^2 \leq 2\}$
  - $\{x : x \text{ is an irrational number and } x^2 \leq 2\}$
  - $\{x : x \text{ is a rational number and } x^2 < 2\}$
  - $\{x : x \text{ is an irrational number and } x^2 < 2\}$
- ⑦ Which of the following is a violation of the triangle inequality?
- $|x - y| \leq |x| + |y|$
  - $|x + y| \leq |x| + |y|$
  - $|x - y| \geq |x| - |y|$
  - $|x + y| \geq |x| - |y|$
  - $|x + y| \geq |x| + |y|$
- ⑧ Which of the following is NOT a consequence of the Cauchy-Schwarz inequality?
- $|x_1x_2 + x_2x_3 + \dots + x_nx_1| \leq x_1^2 + x_2^2 + \dots + x_n^2$
  - $\left| \frac{x_1 + \dots + x_n}{n} \right| \leq \sqrt{\frac{x_1^2 + \dots + x_n^2}{n}}$
  - $n^2 \leq (x_1^2 + \dots + x_n^2)(x_1^{-2} + \dots + x_n^{-2})$
  - $\left| \frac{x_1}{x_2} + \frac{x_2}{x_3} + \dots + \frac{x_n}{x_1} \right| \leq (x_1^2 + \dots + x_n^2)(x_1^{-2} + \dots + x_n^{-2})$
  - $\frac{(x_1 + \dots + x_n)^2}{y_1^2 + \dots + y_n^2} \leq \frac{x_1^2}{y_1^2} + \dots + \frac{x_n^2}{y_n^2}$

①D, ②D, ③A, ④C, ⑤D, ⑥C, ⑦E, ⑧D

## Chapter 1: Exercises

## Exercise 1.1

(E)

Prove that  $\sqrt{6}$  is irrational.

Hint ①

① Prove by contradiction.

## Exercise 1.2

(E)

Prove that the set  $\{\sqrt{n} : n \in \mathbb{N}\}$  is unbounded.

Hint ①

① Prove by contradiction.

Hint ②

② Apply the archimedean property.

## Exercise 1.3

(I)

Prove the supremum is unique, if it exists.

Hint ①

① Suppose  $E$  is bounded above, and suppose  $\beta_1$  and  $\beta_2$  are two distinct suprema of  $E$ . Prove that  $\beta_1 = \beta_2$ .

## Exercise 1.4

(I)

Prove the set

$$\left\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \dots\right\}$$

is unbounded. Find the infimum and the supremum.

Hint ①

① Let  $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ . Prove by induction that  $x_{2n} \geq 1 + \frac{1}{2}$ .

Hint ②

② Apply the archimedean property.

## Exercise 1.5

(E)

Prove that  $|x - a| < \delta$  if and only if  $a - \delta < x < a + \delta$  for any real  $x$ ,  $a$ , and  $\delta$  with  $\delta > 0$ .

Hint ①

① Consider the cases  $x \geq a$  and  $x < a$ , respectively.

## Exercise 1.6

(D)

Prove that the set  $\{|a + b| : a^2 < 2, |b + 1| < 3\}$  is bounded. Find the infimum and the supremum.

Hint ②

② To show that  $\inf S = 0$ , take some  $a, b$  satisfying  $a^2 < 2, |b + 1| < 3$  such that  $|a + b| = 0$ .

Hint ③

③ To show that  $\sup S = 4 + \sqrt{2}$ , show that for  $0 < \varepsilon < 1$ , the value  $4 + \sqrt{2} - \varepsilon$  is not an upper bound of  $S$ .

Hint ④

④ Take  $a = -\sqrt{2} + \frac{1}{4}\varepsilon$ ,  $b = -4 + \frac{1}{4}\varepsilon$ . Show that  $a^2 < 2, |b + 1| < 3$ , and  $|a + b| > 4 + \sqrt{2} - \varepsilon$ .  
① Denote  $S = \{|a + b| : a^2 < 2, |b + 1| < 3\}$ . Show that  $0 \leq |a + b| < 4 + \sqrt{2}$  for all  $a, b$  satisfying  $a^2 < 2, |b + 1| < 3$ . Apply the triangle inequality.

## Exercise 1.7

(E)

Let  $x_1$  and  $x_2$  be real numbers. Prove that

$$||x_1| - |x_2|| \leq |x_1 - x_2|.$$

Hint ①

① Apply the triangle inequality.

## Exercise 1.8

(I)

Prove that equality holds in the Cauchy-Schwarz inequality if and only if there are two real numbers  $\lambda$  and  $\mu$ , not both zero, such that  $\lambda a_j = \mu b_j$ ,  $j = 1, 2, \dots, n$ .

Hint ①

① Let  $A = \sum a_j^2$ ,  $B = \sum b_j^2$ ,  $C = \sum a_j b_j$ . Show that  $|\sum a_j b_j - C|^2 = B(AB - C^2)$ .

## Basic Topology 拓撲學基礎

## Overview of Chapter 2

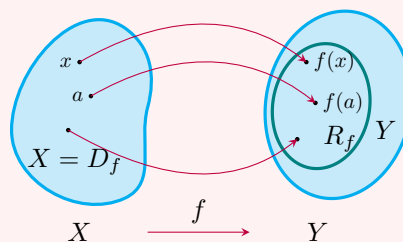
The upcoming chapter will cover the fundamental topology of real numbers, which serves as a crucial basis for mathematical analysis. It establishes a structure for examining the characteristics of mathematical entities, including functions, and enables the exploration of key concepts such as continuity, convergence, compactness, and connectedness, among others.

- We will introduce the concept of cardinality of sets and prove that:
  - The set  $\mathbb{Q}$  is countable, while the set  $\mathbb{R}$  is uncountable.
- We will study in detail about open and closed sets in  $\mathbb{R}$ .
  - A set in  $\mathbb{R}$  is open if and only if its complement is closed, and *vice versa*.
  - The union of any collection of open sets is also open, whereas only a finite intersection of open sets is necessarily open.
- A set in  $\mathbb{R}$  is closed if and only if it is equal to its closure.
- We will characterize the compact sets in  $\mathbb{R}$  by showing the following statements are equivalent:
  - ①  $K$  is closed and bounded.
  - ②  $K$  is compact.
  - ③ Every infinite subset of  $K$  contains a limit point in  $K$ .
- We will prove the Bolzano-Weierstrass theorem: every bounded infinite subset of  $\mathbb{R}$  has at least one limit point in  $\mathbb{R}$ .

## §2.1 Cardinality 基數

## 2.1 Definition: function (函數)

Consider two sets  $X$  and  $Y$ . Suppose that for each element  $x$  of the set  $X$ , there is a unique assigned element  $y$  of the set  $Y$ , denoted by  $y = f(x)$ . Then  $f$  is said to be a **function** (函數) of  $X$  into  $Y$  (or a **mapping** (映射, 映像)), for which we denote  $f: X \rightarrow Y$ .



## Remark

- The terms “*function*” and “*mapping*” are often used interchangeably in math, but their usage depends on context. A function assigns one output to each input, while a mapping describes any relation between two sets that assigns elements from one set to elements from the other set.

- The set  $X$  is called the **domain** (定義域) of  $f$ , denoted as  $D_f$  (that is,  $D_f = X$ ).
- The set  $Y$  is called the **codomain** (陪域) of  $f$ .
- The set of all values of  $f$  is called the **range** (值域) of  $f$ , denoted as  $R_f$ . Generally,  $R_f \neq Y$ .
- The element  $y$  is called the **image** of  $x$  under  $f$ .
- The element  $x$  is called the **inverse image** of  $y$  under  $f$ .
- The element  $f(x)$  is called the **value** (值) of  $f$ .

Let  $E \subset X$ . Denote  $f(E) = \{f(x) : x \in E\}$ , called the **image** of  $E$  under  $f$ .  $R_f = f(A)$ .

Let  $F \subset Y$ . Denote  $f^{-1}(F) = \{x \in X : f(x) \in F\}$ , called the **inverse image** of  $F$  under  $f$ .

If  $R_f = Y$ , that is, every element of  $Y$  is an image of an element of  $X$  under  $f$ , then  $f$  maps  $X$  **onto**  $Y$ , and  $f$  is said to be **surjective** (滿射).

If  $f$  maps distinct elements of its domain to distinct elements, that is,  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ , then  $f$  is said to be **injective** (單射).

If a mapping  $f$  is both surjective and injective, then it is said to be **bijective** (雙射).

#### Remark

- In mathematics, the term “onto” is used as synonymous with “surjective”; while “one-to-one” is used as synonymous with “injective”.
- When  $f$  is injective, it is also said to be a 1-1 (one-to-one) mapping (一對一).
- The term “bijective” is used as synonymous with “**one-to-one correspondence**” (一一對應). One must not be confused with the latter with “one-to-one mapping”.

Suppose  $f: X \rightarrow Y$  is injective so that for every  $y \in R_f$ , there exists a unique  $x \in X$  satisfying  $f(x) = y$ . Then a mapping  $g: R_f \rightarrow X$  can be properly defined as follows: for each  $y \in R_f$ , define  $g(y) = x$ , where  $x$  satisfies  $f(x) = y$ . We call the mapping  $g$  to be the **inverse mapping** (逆映射) of  $f$ , denoted to be  $f^{-1}$ , with domain  $D_{f^{-1}} = R_f$ , and range  $R_{f^{-1}} = X$ .

## 2.2 Definition: cardinal number (基數)

If there exists a bijective mapping from set  $A$  to set  $B$ , then we say that  $A$  and  $B$  have the same **cardinal number** (基數), or briefly, that  $A$  and  $B$  are **equivalent**, and write  $A \sim B$ .

The relation “having the same cardinal number” is an **equivalence relation** (等價關係), that satisfies the following properties

1. **Reflexivity** (自反性):  $A \sim A$ .
2. **Symmetry** (對稱性): if  $A \sim B$ , then  $B \sim A$ .
3. **Transitivity** (傳遞性): if  $A \sim B$  and  $B \sim C$ , then  $A \sim C$ .


Let  $\mathbb{N}_{\leq n} = \{1, 2, \dots, n\}$  for any  $n \in \mathbb{N}_+$ , where  $\mathbb{N}_+ = \{1, 2, 3, \dots\}$ . For any set  $A$ , we say

- $A$  is **finite** (有限集) if  $A \sim \mathbb{N}_{\leq n}$  for some  $n$ . In this case, we write  $|A| = n$ .
- $A$  is **infinite** (無限集) if  $A$  is not finite.
- $A$  is **countable** (可數集) if  $A \sim \mathbb{N}_+$ . In this case, we write  $|A| = \aleph_0$ .
- $A$  is **uncountable** (不可數集) if  $A$  is neither finite nor countable.
- $A$  is **at most countable** (至多可數集) if  $A$  is finite or countable.

## Remark

- When an equivalence relation is applied to a set, it divides the set into multiple separate and non-overlapping subsets, which are referred to as **equivalence classes** (等價類).

## Remark

- For two *infinite* sets  $A$  and  $B$ , “having the same number of elements” becomes quite vague. The existence of bijection from  $A$  to  $B$  makes a rigorous sense.
- For two *finite* sets  $A$  and  $B$ ,  $A \sim B$  if and only if  $|A| = |B|$ .
- Any finite set cannot be equivalent to one of its proper subsets.
- Any infinite set is equivalent to one of its proper subsets. 
- The power set of set  $A$ , denoted as  $2^A$ , is the set that contains all subsets of  $A$ .
  - If  $|A| = n < \infty$ , then  $|2^A| = 2^n$ .

## 2.3 Definition: sequence (序列)

We call a mapping  $f$  **sequence** (序列), if it is defined on the set  $\mathbb{N}_+$  of all positive integers. It is customary to write a sequence as  $\{x_n\}$ , or  $x_1, x_2, x_3, \dots$ , where  $x_n = f(n)$  for  $n \in \mathbb{N}_+$ . The values of  $f$ , that is, the elements  $x_n$ , are called the **terms** of the sequence. If  $x_n \in S$  for all  $n \in \mathbb{N}_+$ , then  $\{x_n\}$  are said to be a **sequence in  $S$** . Usually,  $x_k$  is said to be the  $k$ th term of the sequence  $\{x_n\}$ .

## Remark

- The terms  $x_1, x_2, x_3, \dots$  of a sequence need not be distinct.
- We also regard the sequence  $\{x_n\}$  as a set.
- Every countable set can be regarded as a sequence of distinct terms.
- Sometimes it is convenient to replace  $\mathbb{N}_+$  by  $\mathbb{N}$ , the set of all natural (nonnegative) integers, so that a sequence may start with 0 rather than with 1.

## 2.4 Definition: union and intersection of sets (並集與交集)

Let  $A$  and  $\Omega$  be sets, and suppose that for each element  $\alpha \in A$ , there is a corresponding subset  $E_\alpha$  of  $\Omega$ . We denote the set of all such subsets by  $\{E_\alpha\}$ , which is referred to as a collection or family of sets.

The **union** (並集) of the sets  $E_\alpha$  is the set  $S$  such that  $x \in S$  if and only if  $x \in E_\alpha$  for at least one  $\alpha \in A$ , and denote

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

If  $A = \mathbb{N}_+$ , one usually writes

$$S = \bigcup_{m=1}^{\infty} E_m.$$

If  $A = \mathbb{N}_{\leq n}$ , the usual notation is

$$S = \bigcup_{m=1}^n E_m \quad \text{or} \quad S = E_1 \cup E_2 \cup \dots \cup E_n.$$

The **intersection** (交集) of the sets  $E_\alpha$  is the set  $T$  such that  $x \in T$  if and only if  $x \in E_\alpha$  for every  $\alpha \in A$ , and denote

$$T = \bigcap_{\alpha \in A} E_\alpha.$$

If  $A = \mathbb{N}_+$ , one usually writes

$$T = \bigcap_{m=1}^{\infty} E_m.$$

If  $A = \mathbb{N}_{\leq n}$ , the usual notation is

$$T = \bigcap_{m=1}^n E_m \quad \text{or} \quad T = E_1 \cap E_2 \cap \dots \cap E_n.$$



- We say that  $A$  and  $B$  **intersect** (相交), if  $A \cap B \neq \emptyset$ .
- We say that  $A$  and  $B$  **disjoint** (不相交), if  $A \cap B = \emptyset$ .

**Remark**

- Many properties of unions and intersections are quite similar to those of sums and products. For instance, the commutative, the associative, and the distributive laws hold:

$$A \cup B = B \cup A, (A \cup B) \cup C = A \cup (B \cup C)$$

$$A \cap B = B \cap A, (A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

**2.5 Proposition:** infinite subset of a countable set is countable

Every infinite subset of a countable set is countable. ▶

**2.6 Proposition:** countable union of countable sets is countable

Let  $\{E_n\}, n = 1, 2, 3, \dots$ , be a sequence of countable sets. Then the union  $\bigcup_{n=1}^{\infty} E_n$  is countable. ▶

**Corollary**

Suppose  $A$  is at most countable, and, for every  $\alpha \in A$ ,  $B_\alpha$  is at most countable. Then  $\bigcup_{\alpha \in A} B_\alpha$  is at most countable.

**2.7 Proposition:**  $n$ -tuples of a countable set is countable

Let  $A$  be a countable set, and  $B_n$  be the set of all  $n$ -tuples  $(a_1, \dots, a_n)$ , where  $a_k \in A$  ( $k = 1, \dots, n$ ), and the elements  $a_1, \dots, a_n$  need not be distinct. Then  $B_n$  is countable. ▶

**Corollary**

The  $n$ -dimensional integer lattice  $\mathbb{Z}^n$ , whose points are  $n$ -tuples of integers, is countable.

**Corollary**

The set of all rational numbers  $\mathbb{Q}$  is countable. ▶

**2.8 Proposition:** the set of 0-1 sequences is uncountable

Let  $A$  be the set of all sequences whose elements are the digits 0 and 1. Then  $A$  is uncountable. ▶

**Remark**

- A sequence whose elements are the digits 0 and 1 is like  $1, 0, 0, 1, 0, 1, 1, 0, \dots$
- Real numbers can be expressed in decimal (using 0 to 9) or binary (base 2) representations. The proposition implies that the set of real numbers  $\mathbb{R}$  is uncountable, meaning there are significantly more real numbers than rational numbers.
- We shall give a second proof of this fact in Theorem 2.23.

§2.2 Subsets in  $\mathbb{R}$  實數的子集2.9 Definition: subsets in  $\mathbb{R}$  (實數子集)

All points and sets mentioned below are elements and subsets of  $\mathbb{R}$ .

- The set  $(a, b) = \{x : a < x < b\}$  is called **open interval**.
- The set  $[a, b] = \{x : a \leq x \leq b\}$  is called **closed interval**.
- The **complement** (補集) of a set  $E$ , denoted by  $E^c$ , is the set of all points in  $\mathbb{R}$  but not in  $E$ .  
 $(E^c)^c = E$
- The **difference** (差集) of  $B$  and  $A$ , written  $B \setminus A$  or  $B - A$ , is the set of all points in  $B$  but not in  $A$ .  
 $B \setminus A = B \cap A^c$
- A **neighborhood** (鄰域) of a point  $p$  is the set  $N_r(p) = \{q \in \mathbb{R} : |p - q| < r\} \subset \mathbb{R}$ , where  $r$  is called the *radius* of  $N_r(p)$ .
- A point  $p$  is a **limit point** (極限點) of the set  $E$  if *every* neighborhood of  $p$  contains a point  $q \neq p$  such that  $q \in E$ .
- If  $p \in E$  and  $p$  is not a limit point of  $E$ , then  $p$  is called an **isolated point** (孤立點) of  $E$ .
- A set  $E$  is **closed** (閉集) if every limit point of  $E$  is a point in  $E$ .
- The **closure** (閉包) of  $E$  is the set  $\bar{E} = E \cup E'$ , where  $E'$  the set of all limit points of  $E$  in  $\mathbb{R}$ .
- A point  $p$  is an **interior point** of  $E$  (內點) if there is a neighborhood  $N$  of  $p$  such that  $N \subset E$ . The **interior** (內部) of  $E$ , denoted by  $E^\circ$ , is the set of all its interior points.
- A set  $E$  is **open** (開集) if every point of  $E$  is an interior point of  $E$ .
- A set  $E \subset X \subset \mathbb{R}$  is **relatively open** (相對開集) in  $X$  (or simply “**open in  $X$** ”) if there is an open set  $U$  of  $\mathbb{R}$  such that  $E = U \cap X$ .
- A set  $E$  is **perfect** (完全集) if  $E$  is closed and if every point of  $E$  is a limit point of  $E$ .
- A set  $E$  is **bounded** (有界集) if there is a real number  $M$  such that  $|p| < M$  for all  $p \in E$ . Equivalently,  $E$  is bounded if it is contained in a finite interval.
- A set  $E$  is **dense** (稠密集) in  $X$  if every point of  $X$  is a limit point of  $E$ , or a point of  $E$ , or both.
- A point  $p$  is a **boundary point** of  $E$  (邊界點) if any neighborhood of  $p$  intersects both  $E$  and  $E^c$ . The **boundary** (邊界) of  $E$ , denoted by  $\partial E$ , is the set comprising all boundary points of  $E$ .

## Remark

- A limit point of  $E$ , or a boundary point of  $E$ , is not necessarily a point of  $E$ .
- A *closed* set is a set which is closed for limits. So, any finite set is closed.
- Since a sequence can be regarded as a set, hence, it makes sense to call a sequence to be *bounded*.

## Open and Closed Sets 開集與閉集

## 2.10 Proposition: on neighborhoods and limit points

Every neighborhood is an open set. ▶

A finite point set has no limit points.

If  $p$  is a limit point of  $E$ , then every neighborhood of  $p$  contains infinitely many points of  $E$ . ▶

**2.11 Proposition:** the complement of a union equals the intersection of complements

Let  $\{E_\alpha\}$  be a (finite or infinite) collection of sets in  $\mathbb{R}$ . Then

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^c = \bigcap_{\alpha} E_{\alpha}^c, \quad \left(\bigcap_{\alpha} E_{\alpha}\right)^c = \bigcup_{\alpha} E_{\alpha}^c.$$

**2.12 Proposition:** a set is open if and only if its complement is closed

A set in  $\mathbb{R}$  is open if and only if its complement is closed.

A set in  $\mathbb{R}$  is closed if and only if its complement is open.

**2.13 Proposition:** the union of open sets is open; the intersection of closed sets is closed

Suppose the sets mentioned below are subsets of  $\mathbb{R}$ .

1. For any collection of  $\{G_\alpha\}$  of open sets,  $\bigcup_{\alpha} G_{\alpha}$  is open.
2. For any collection of  $\{F_\alpha\}$  of closed sets,  $\bigcap_{\alpha} F_{\alpha}$  is closed.
3. For any *finite* collection of  $G_1, \dots, G_n$  of open sets,  $\bigcap_{j=1}^n G_j$  is open.
4. For any *finite* collection of  $F_1, \dots, F_n$  of closed sets,  $\bigcup_{j=1}^n F_j$  is closed.

**Remark**

- The hypothesis of finite collection is indispensable in items 3 and 4.

**2.14 Proposition:** characterization of the closure

Suppose  $E \subset \mathbb{R}$ . Then

1.  $\bar{E}$  is closed.
2.  $E = \bar{E}$  if and only if  $E$  is closed.
3.  $\bar{E} \subset F$  for every closed set  $F \subset \mathbb{R}$  such that  $E \subset F$ .

**Remark**

- According to items 1 and 3,  $\bar{E}$  is the *smallest* closed subset of  $\mathbb{R}$  that contains  $E$ .

**Remark**

- Because closed sets are closed under intersection, the closure of a set can be obtained by taking the intersection of all closed sets that contain it.
- The closure  $\bar{E}$  is the union of  $E$  and its boundary  $\partial E$ , that is,  $\bar{E} = E \cup \partial E$ .

**2.15 Proposition:** the supremum of bounded-above set of real numbers is in the closure of the set

Let  $E$  be a nonempty set of *real numbers* which is bounded above. Let  $y = \sup E$ . Then  $y \in \bar{E}$ . Hence  $y \in E$  if  $E$  is closed. An analogue result is also true for the infimum.

**Connected Sets** 連通集**2.16 Definition:** connected sets (連通集)

- Two subsets  $A$  and  $B$  of  $\mathbb{R}$  are **separated** (分離) if both  $A \cap \bar{B}$  and  $\bar{A} \cap B$  are empty.
- A set  $E \subset \mathbb{R}$  is **connected** (連通) if  $E$  is not a union of two nonempty separated sets.

**Remark**

- Separated sets are disjoint, but disjoint sets need not be separated.

**characterization of connected set**

A subset  $E$  of  $\mathbb{R}$  is connected if and only if for any  $x, y \in E$  and  $x < z < y$ , then  $z \in E$ .

## §2.3 Compact Sets 緊致集

### 2.17 Definition: compact set (緊集)

- An **open cover** of a set  $E \subset \mathbb{R}$  is a collection  $\{G_\alpha\}$  of open subsets of  $\mathbb{R}$  such that  $E \subset \bigcup_{\alpha} G_\alpha$ .
- A set  $K \subset \mathbb{R}$  is said to be **compact** (緊集, 緊致集) if every open cover of  $K$  contains a finite subcover. More explicitly, if  $\{G_\alpha\}$  is an open cover of (compact set)  $K$ , then there are finitely many indices  $\alpha_1, \dots, \alpha_n$  such that

$$K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

#### Remark

- Note that the term *open cover* refers to the collection of open sets  $\{G_\alpha\}$  and not to their union  $\bigcup_{\alpha} G_\alpha$ , which is a subset of  $\mathbb{R}$ .
- Every finite set is compact.
- The open interval  $(0, 1)$  is not compact. In general, any open interval  $(a, b)$  is not compact. ▶

#### Remark

- Compactness is a fundamental concept in mathematical analysis. It plays a fundamental role in proving the existence of solutions to equations and systems of equations, particularly in infinite-dimensional spaces. It allows us to establish convergence and to find fixed points of mappings, which are key steps in many existence proofs.

### 2.18 Proposition: the relationship between compact sets and closed sets

Assume all sets mentioned below are subsets of  $\mathbb{R}$ .

1. Compact sets are closed.
2. Closed subsets of compact sets are compact. ▶

#### Corollary

If  $F$  is closed and  $K$  is compact, then  $F \cap K$  is compact.

#### Remark

- Closed sets are not necessarily compact. For instance,  $\mathbb{R}$  is closed but not compact.

### 2.19 Theorem: finite closed intervals are compact

Every finite closed interval is compact. ▶

### 2.20 Theorem: Cantor's intersection theorem 康托爾交集定理

If  $\{K_n\}$  is a sequence of nonempty compact sets such that  $K_n \supset K_{n+1}$ ,  $n = 1, 2, 3, \dots$ , then  $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ . ▶

#### Corollary - the nested intervals theorem 區間套定理

If  $\{I_n\}$  is a sequence of nested closed intervals, that is,  $I_n \supset I_{n+1}$ ,  $n = 1, 2, 3, \dots$ , then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**2.21 Theorem:** characterizations of compact sets in  $\mathbb{R}$  and the Heine-Borel theorem

Let  $K \subset \mathbb{R}$ . The following statements are equivalent:

1.  $K$  is closed and bounded.
2.  $K$  is compact.
3. Every infinite subset of  $K$  contains a limit point in  $K$ .

**Remark**

- The equivalence of items 1 and 2 is also known as the **Heine-Borel theorem** (海涅-博雷爾定理).

**Remark**

- The study of the equivalence of compactness is important because it allows us to apply the same proof techniques to different contexts, connect different areas of mathematics, understand the properties of different types of spaces, and apply mathematics to other fields.

**2.22 Theorem:** the Bolzano-Weierstrass theorem 聚点定理

Every bounded infinite subset of  $\mathbb{R}$  has at least one limit point in  $\mathbb{R}$ .

**2.23 Proposition:** nonempty perfect set is uncountable

Every nonempty perfect set of real numbers is uncountable.

**Corollary**

Every interval  $[a, b]$  ( $a < b$ ) is uncountable.  
The set of all real numbers is uncountable.

**Remark**

- Starting with  $[0, 1]$  and repeatedly removing the middle third open interval of each remaining segment gives rise to the Cantor set. The set is nonempty and perfect, and it contains no nontrivial intervals.

**Addendum 後記****Addendum of Chapter 2**

- Denoting the cardinality of the set of all real numbers as  $\aleph_1$ , we have  $\aleph_0 < \aleph_1$ . The *continuum hypothesis* states that there is no set with a cardinality strictly between that of the integers and the real numbers, and that  $\aleph_1$  is the smallest cardinal number greater than  $\aleph_0$ . However, in the early 1960s, Kurt Gödel and Paul Cohen independently showed that the hypothesis cannot be proven or disproven using the widely accepted Zermelo–Fraenkel set theory with the axiom of choice, which is the most common foundation of mathematics. This discovery had a profound impact on the foundations of mathematics, raising questions about the nature of mathematical truth and the limits of mathematical proof.
- This e-book primarily focuses on continuity, differentiation, and integration on the real line. As a starting point, we examine open sets and compact sets in  $\mathbb{R}$  to establish a foundation. However, it's important to note that these concepts have broad applications in mathematics, including topology, analysis, and geometry. They can be extended to more general spaces like metric spaces. Understanding these concepts is crucial for comprehending the properties of spaces and the behavior of functions defined on them.

## Exercises of Chapter 2 練習題

## Chapter 2: Quiz

30 Minutes

- ① Which of the following sets is not countable?
- the set of prime numbers
  - the set of all positive integers
  - the set of all rational numbers
  - the set of all  $n$ -tuples of integers
  - the set of all infinite sequences of 0's and 1's
- ② Let  $\{A_\alpha\}$  be a collection of sets. Which of the following is equivalent to  $\bigcup_\alpha A_\alpha = \bigcap_\alpha A_\alpha^c$ ?
- |  |  |
|--|--|
| A. $\bigcap_\alpha A_\alpha = \emptyset$   | D. $\bigcap_\alpha A_\alpha^c = \emptyset$ |
| B. $\bigcup_\alpha A_\alpha = \emptyset$   | E. $\bigcup_\alpha A_\alpha = \mathbb{R}$  |
| C. $\bigcup_\alpha A_\alpha^c = \emptyset$ |  |
- ③ Let  $A \subset \mathbb{R}$ . Which of the following is true?
- the closure of  $A$  is equal to  $A$
  - the interior of  $A$  is a subset of  $A$
  - the boundary of  $A$  is an open set
  - if  $A$  is closed, then the boundary of  $A$  is empty
  - if  $A$  is open, then the boundary of  $A$  is empty
- ④ Which of the following statements is true?
- every open set is connected
  - every closed set is compact
  - every bounded set is closed
  - every closed set is bounded
  - every compact set is closed
- ⑤ Which of the following statements is NOT true for a compact set in  $\mathbb{R}$ ?
- every compact set is limit point compact
  - every compact set is closed
  - every compact set is bounded
  - every bounded subset of a compact set is compact
  - every closed subset of a compact set is compact
- ⑥ Which of the following is true in  $\mathbb{R}$ ?
- any nonempty compact subset has a maximum element
  - any nonempty bounded subset has a maximum element
  - any nonempty closed subset has a maximum element
  - any nonempty connected subset has a maximum element
  - none of the above
- ⑦ Let  $A$  and  $B$  be subsets of  $\mathbb{R}$ . Which of the following statements is true?
- if  $A$  is open and  $B$  is closed, then  $A \cap B$  is closed
  - if  $A$  is closed and  $B$  is open, then  $A \cup B$  is open
  - if  $A$  is compact and  $B$  is closed, then  $A \cap B$  is compact
  - if  $A$  is open and  $B$  is compact, then  $A \cap B$  is open
  - if  $A$  is dense and  $B$  is open, then  $A \cap B$  is dense
- ⑧ Which of the following statements is true in  $\mathbb{R}$ ?
- every compact set is a perfect set
  - every perfect set is a compact set
  - every closed set is a perfect set
  - every perfect set is a closed set
  - every dense set is a perfect set

①E, ②B, ③B, ④E, ⑤D, ⑥A, ⑦C, ⑧D

## Chapter 2: Exercises

## Exercise 2.1

(I)

Show that the open interval  $(0, 1)$  and the closed interval  $[0, 1]$  have the same cardinality by constructing a bijective mapping between them.

Hint ①

① Take a countable set  $S$  in  $(0, 1)$  and construct a bijective mapping from  $S$  to  $S \cup \{0, 1\}$ .

## Exercise 2.2

(E)

Prove that  $E$  is open if and only if  $E = E^\circ$ .

Hint ①

①  $\Rightarrow$ : Assume that  $E$  is open. To show  $E = E^\circ$ , prove that  $E \subset E^\circ$  and  $E^\circ \subset E$ .

Hint ②

②  $\Leftarrow$ : Assume that show  $E = E^\circ$ . To show  $E$  is open, prove that  $E^\circ$  is open.

## Exercise 2.3

(I)

Let  $E \subset \mathbb{R}$ . Prove that the interior  $E^\circ$  of  $E$  is the largest open set contained in  $E$ .

Hint ①

① The interior  $E^\circ$  is an open set contained in  $E$ .

Hint ②

② Show that if  $U$  is an open set contained in  $E$ , then

Hint ③

③ Let  $x$  be a point in  $U$ . Show that  $x$  is an interior point of  $E$ .

## Exercise 2.4

(I)

Let  $E_1, \dots, E_n$  be a finite collection of subsets of real numbers. Prove that

$$\overline{\bigcup_{j=1}^n E_j} = \bigcup_{j=1}^n \overline{E_j}.$$

Hint ①

① Only need to prove the relation for two subsets  $A$  and  $B$ .

Hint ②

② Show  $\overline{A \cup B} \subset \overline{A \cup B}$ : Since  $A \subset A \cup B$  and  $B \subset A \cup B$ , we have  $\overline{A} \subset \overline{A \cup B}$  and  $\overline{B} \subset \overline{A \cup B}$ .

Hint ③

③ Show  $\overline{A \cup B} \subset \overline{A \cup B}$ : Since  $A \subset \overline{A}$  and  $B \subset \overline{B}$ , we have  $A \cup B \subset \overline{A \cup B}$ .

Hint ④

④ The closures  $\overline{A}$  and  $\overline{B}$  are closed. The union of two closed sets is closed.

## Exercise 2.5

(I)

Consider the set

$$S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$

1. For the open cover of  $S$ ,

$$(\varepsilon - \varepsilon, \varepsilon), (1 - \varepsilon, 1 + \varepsilon), \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right), \dots,$$

where  $\varepsilon$  satisfies  $0 < \varepsilon < \frac{1}{2}$ , find a finite subcover of  $S$ .

2. Prove that  $S$  is compact.

Hint ①

① For part 1, choose a positive integer  $N$  such that  $\frac{1}{N} < \varepsilon$ . Show that the interval  $(-\varepsilon, \varepsilon)$  contains  $0, \frac{1}{N+1}, \frac{1}{N+2}, \dots$ .

Hint ②

② For part 2, prove that  $S$  is bounded and closed.

Hint ③

③ Prove that  $S$  is closed, by showing that 0 is the only limit point of  $S$  and  $0 \in S$ .

## Exercise 2.6

(I)

Suppose that  $\{A_n\}$  is a sequence of nonempty bounded open subsets of  $\mathbb{R}$  satisfying

$$A_n \subset \overline{A_{n+1}}, \quad n = 1, 2, \dots$$

Prove that  $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ .

Hint ①

① For each  $n$ ,  $\overline{A_n}$  is nonempty and compact.

Hint ②

② Show that  $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A_n}$ .

Exercise 2.7

(I)

1. Prove that the union of a finite collection of perfect sets in  $\mathbb{R}$  is perfect.
2. Construct a countable collection of perfect sets in  $\mathbb{R}$  such that their union is not perfect.

Hint ①

① For part 1, prove the conclusion for two perfect sets  $A$  and  $B$ .

Hint ②

② Suppose  $x$  is a point of  $A \cup B$ . Prove that  $x$  is a limit point of  $A \cup B$ .

Hint ③

③ For part 2, consider the union of  $A_n$ , where  $A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$ ,  $n = 1, 2, \dots$

Exercise 2.8

(D)

Prove that every open set in  $\mathbb{R}$  can be expressed as the union of at most countably many disjoint open intervals.

Hint ③

③ Show that  $G$  equals the union of such open intervals.

Hint ④

④ Show that these open intervals are disjoint.

Hint ⑤

⑤ Use rational numbers as representatives to show that the collection of these open intervals is countable.

Hint ①

① Let  $G$  be an open set in  $\mathbb{R}$ . For each  $x \in G$ , there are  $y$  and  $z$ , with  $z < x < y$ , such that  $(z, y) \subset G$ .

Hint ②

② Show that  $a, b \notin G$ , where  $a = \inf\{z : (z, x) \subset G\}$ ,  $b = \sup\{y : (x, y) \subset G\}$ .



## Sequences and Series 序列與級數

## Overview of Chapter 3

We will study convergence of sequences and series in this chapter.

- For real sequences, we will prove that
  - Every bounded sequence in  $\mathbb{R}$  contains a convergent subsequence.
  - A sequence converges if and only if it is a Cauchy sequence.
  - A monotonic sequence converges if and only if it is bounded.
- We will extend the concept of limit for sequences to the upper and the lower limits, whose values are in  $\overline{\mathbb{R}}$  (the extended real numbers). We will prove that  $\lim_{n \rightarrow \infty} x_n = x$  if and only if  $\overline{\lim}_{n \rightarrow \infty} x_n = \underline{\lim}_{n \rightarrow \infty} x_n = x$ , where  $x \in \overline{\mathbb{R}}$ .
- We will study convergence/divergence of series.
  - We will prove the Comparison Test, the Root Test, and the Ratio Test. These tests can be used to determine whether a series converges absolutely.
  - We will prove Dirichlet's Test, Abel's Test, and the Alternating Series Test. These tests are applicable for determining the conditional convergence.
- We will explain the differences between the absolute convergence and the conditional convergence.

## §3.1 Convergent Sequences 收斂序列

## 3.1 Definition: convergent sequence (收斂序列)

Suppose  $\{x_n\}$  is a sequence in  $\mathbb{R}$ . We say that  $\{x_n\}$  is **convergent** (收斂) if there is a point  $x \in \mathbb{R}$  with the following property:

for every  $\varepsilon > 0$ , there is an integer  $N$  such that  $n \geq N$  implies that  $|x_n - x| < \varepsilon$ .

In this case, we also say that  $\{x_n\}$  converges to  $x$ . The point  $x$  is called a **limit** (極限) of  $\{x_n\}$ , and write  $x_n \rightarrow x$ , or  $\lim_{n \rightarrow \infty} x_n = x$ . ▶

If  $\{x_n\}$  does not converge, it is said to be **divergent** (發散). ▶

## an equivalent statement

for every  $\varepsilon^* > 0$ , there is a number  $N$  such that  $n \geq N$  implies that  $|x_n - x| < f(\varepsilon^*)$ , where  $f$  is some function such that  $f(\varepsilon^*) < \varepsilon$ . ▶

- $\lim_{n \rightarrow \infty} x_n = x$  if and only if  $\lim_{n \rightarrow \infty} (x_n - x) = 0$ .
- $\lim_{n \rightarrow \infty} x_n = 0$  if and only if  $\lim_{n \rightarrow \infty} |x_n| = 0$ .

**3.2 Proposition:** properties of convergent sequence

Let  $\{x_n\}$  be a sequence in  $\mathbb{R}$ .

1. A convergent sequence has a unique limit: if  $x_n \rightarrow x$  and  $x_n \rightarrow x'$ , then  $x = x'$ .
2. Sequence  $\{x_n\}$  converges to  $x \in \mathbb{R}$  if and only if every neighborhood of  $x$  contains  $x_n$  for all but finitely many  $n$ .
3. If  $E \subset \mathbb{R}$  and if  $x$  is a limit point of  $E$ , then there is a sequence  $\{x_n\}$  in  $E$  such that  $x = \lim_{n \rightarrow \infty} x_n$ .
4. If  $\{x_n\}$  converges, then  $\{x_n\}$  is bounded. ▶

**3.3 Proposition:** operations on convergent sequences

Suppose  $\{x_n\}, \{y_n\}$  are real sequences, and  $\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} y_n = y$ . Thus

1.  $\lim_{n \rightarrow \infty} (x_n + y_n) = x + y$ ;
2.  $\lim_{n \rightarrow \infty} cx_n = cx, \lim_{n \rightarrow \infty} (c + x_n) = c + x$ , for any real number  $c$ ;
3.  $\lim_{n \rightarrow \infty} x_n y_n = xy$ ;
4.  $\lim_{n \rightarrow \infty} \frac{1}{x_n} = \frac{1}{x}$ , provided  $x_n \neq 0$  ( $n = 1, 2, 3, \dots$ ), and  $x \neq 0$ . ▶

## Subsequences 子序列

**3.4 Definition:** subsequence (子序列)

For a given sequence  $\{x_n\}$  of real numbers, if some (possibly none) terms are omitted from the sequence, the remaining sequence is called a **subsequence** (子序列) of  $\{x_n\}$ . It is customary to write a subsequence as  $\{x_{n_k}\}$ , where  $n_1 < n_2 < n_3 < \dots$ .

If a subsequence  $\{x_{n_k}\}$  converges, its limit is called a **subsequential limit** of  $\{x_n\}$ .

**Remark**

- Any subsequence of the original sequence retains terms in their original order.
- The indices of any subsequence satisfy:  $n_k < n_{k+1}$  and  $k \leq n_k$ .
- A sequence  $\{x_n\}$  converges to  $x$  if and only if every subsequence of  $\{x_n\}$  converges to  $x$ . ▶

**3.5 Theorem:** compact implies every sequence has a convergent subsequence

If  $K \subset \mathbb{R}$  is compact, then every sequence of  $K$  has a subsequence that converges to a point in  $K$ . ▶

**Corollary**

Every bounded sequence in  $\mathbb{R}$  contains a convergent subsequence. ▶

**3.6 Proposition:** subsequential limits form a closed set

The subsequential limits of any sequence in  $\mathbb{R}$  form a closed subset of  $\mathbb{R}$ . ▶

## Cauchy Sequences 柯西序列

## 3.7 Definition: Cauchy sequence (柯西序列)

A sequence  $\{x_n\}$  in  $\mathbb{R}$  is a **Cauchy sequence** (柯西序列) if

for every  $\varepsilon > 0$  there exists an integer  $N$  such that  $|x_n - x_m| < \varepsilon$  if  $n, m \geq N$ .

## 3.8 Theorem: convergence of Cauchy sequences

A sequence in  $\mathbb{R}$  is convergent if and only if it is a Cauchy sequence. ▶

## 3.9 Definition: monotonic sequence (單調序列)

Let  $\{x_n\}$  be a sequence of real numbers.

If  $x_n \leq x_{n+1}$  for  $n = 1, 2, 3, \dots$ , we call the sequence **monotonically increasing** (單調上升).

If  $x_n \geq x_{n+1}$  for  $n = 1, 2, 3, \dots$ , we call the sequence **monotonically decreasing** (單調下降).

A sequence is **monotonic** (單調) if it is either monotonically increasing or monotonically decreasing.

## 3.10 Theorem: the monotone convergence theorem 單調收斂定理

1. If a sequence is monotonically increasing and bounded above, then its supremum is the limit.
2. If a sequence is monotonically decreasing and bounded below, then its infimum is the limit.
3. A monotonic sequence converges if and only if it is bounded. ▶

## Remark

- A monotonically increasing sequence is always bounded below, and if bounded above, then it is bounded. Similarly, a monotonically decreasing sequence is bounded if bounded below.

## Upper and Lower Limits 上、下極限

## 3.11 Definition: infinite limits (無窮大極限)

Let  $\{x_n\}$  be a sequence of real numbers.

We say that  $\{x_n\}$  is **divergent to  $\infty$**  if for every real  $M$  there is an integer  $N$  such that  $n \geq N$  implies  $x_n \geq M$ . We write  $x_n \rightarrow \infty$ , or  $\lim_{n \rightarrow \infty} x_n = \infty$ .

We say that  $\{x_n\}$  is **divergent to  $-\infty$**  if for every real  $M$  there is an integer  $N$  such that  $n \geq N$  implies  $x_n \leq M$ . We write  $x_n \rightarrow -\infty$ , or  $\lim_{n \rightarrow \infty} x_n = -\infty$ .

## Remark

- A sequence that approaches a finite number has a convergent limit, while a sequence that does not approach a finite number, such as with infinite limits, is *divergent*.

## Remark

- Some operational rules on limits can be easily extended to infinite limits, and some additional rules hold in  $\overline{\mathbb{R}}$ .

– For any constant  $c$ , if  $\lim_{n \rightarrow \infty} x_n = \infty$ , then  $\lim_{n \rightarrow \infty} (c + x_n) = \infty$ ;  
if  $\lim_{n \rightarrow \infty} x_n = -\infty$ , then  $\lim_{n \rightarrow \infty} (c + x_n) = -\infty$ .

– if  $\lim_{n \rightarrow \infty} x_n = \infty$ , then  $\lim_{n \rightarrow \infty} cx_n = \infty$  if  $c > 0$ ;  $\lim_{n \rightarrow \infty} cx_n = -\infty$  if  $c < 0$ .

## 3.12 Definition: upper and lower limits (上極限、下極限)

Let  $\{x_n\}$  be a sequence of real numbers.

In the *extended real number system*, let  $E$  be the set comprising all the numbers  $x$  such that  $x_{n_i} \rightarrow x$  for some subsequence  $\{x_{n_i}\}$ . Define the numbers  $x^*$ ,  $x_*$ :

$$\text{upper limit } x^* = \sup E; \quad \text{lower limit } x_* = \inf E,$$

called the upper limit (上極限) and the lower limit (下極限) of the sequence  $\{x_n\}$ , respectively. We also write

$$x^* = \limsup x_n \text{ or } \overline{\lim}_{n \rightarrow \infty} x_n; \quad x_* = \liminf x_n \text{ or } \underline{\lim}_{n \rightarrow \infty} x_n.$$



## Remark

- One advantage that upper and lower limits have over limits on sequences is their existence.

Every sequence of real numbers must have the upper and the lower limits. ►

## 3.13 Proposition: characterization of the upper limit

Assume that  $\{x_n\}$  is a sequence of real numbers. Let  $E$  be the set comprising all the numbers  $x \in \overline{\mathbb{R}}$  (the extended real numbers) such that  $x_{n_k} \rightarrow x$  for some subsequence  $\{x_{n_k}\}$ , and let  $x^*$  be the upper limit of the sequence  $\{x_n\}$ . Then  $x^*$  has the following two properties:

1.  $x^* \in E$ .
2. If  $y > x^*$ , there is an integer  $N$  such that  $n \geq N$  implies  $x_n < y$ .

Moreover,  $x^*$  is the only number with these two properties. ►

An analogue result is also true for the lower limit  $x_*$ .

## 3.14 Proposition: order rules of upper and lower limits

Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences of real numbers.

If  $x_n \leq y_n$  for  $n \geq N_0$ , where  $N_0$  is fixed, then

$$\underline{\lim}_{n \rightarrow \infty} x_n \leq \underline{\lim}_{n \rightarrow \infty} y_n, \quad \overline{\lim}_{n \rightarrow \infty} x_n \leq \overline{\lim}_{n \rightarrow \infty} y_n. \quad \blacktriangleright$$

## Corollary

The limit  $\lim_{n \rightarrow \infty} x_n = x$  if and only if

$$\overline{\lim}_{n \rightarrow \infty} x_n = \underline{\lim}_{n \rightarrow \infty} x_n = x. \quad \blacktriangleright$$

## Remark

- The statement " $x_n < y_n$ " does not imply " $\lim_{n \rightarrow \infty} x_n < \lim_{n \rightarrow \infty} y_n$ ", even if both limits exist.

## §3.2 Series 級數

## 3.15 Definition: convergent series (收斂級數)

Suppose  $\{a_n\}$  is a sequence of real numbers. Consider the cumulative sum of  $\{a_n\}$ :

$$a_1 + a_2 + a_3 + \cdots,$$

which is called an infinite series. In order to formally make sense of this infinite series, for each positive integer  $n$ , we call the number

$$s_n = \sum_{k=1}^n a_k = a_1 + a_2 + \cdots + a_n$$

the ***n*-th partial sum** (部分和) of the series. If the sequence  $\{s_n\}$  has a limit  $\lim_{n \rightarrow \infty} s_n = s$ , where  $s$  is finite,

then we say the infinite series is **convergent** (收斂), and denote  $\sum_{k=1}^{\infty} a_k = s$ . The value  $s$  is called the

**sum** (級數和) of the series. ▶▶

If the sequence  $\{s_n\}$  is divergent, then we say the series is **divergent** (發散). ▶

## Remark

- $a_1 = s_1$ , and  $a_n = s_n - s_{n-1}$  for  $n > 1$ .
- Sometimes, one may consider  $\sum_{n=0}^{\infty} a_n$ , an infinite series beginning with  $n = 0$ . In many cases, when there is no possible ambiguity, one may simply write  $\sum a_n$ .
- Unless dealing with some comparatively straightforward cases, determining the sum of a given series can be highly challenging.
- The challenge in deciding whether a series converges or diverges lies in the fact that there is no single test that can be used for all series. Rather, there exist numerous tests that are only applicable to specific types of series. The lack of a universal test for all series underscores the intricacy (錯綜複雜) of the subject and emphasizes the importance of scrutinizing and assessing each series on its own merit.

## 3.16 Theorem: the Cauchy Criterion for convergence of series

A series  $\sum a_n$  converges if and only if the **Cauchy Criterion** (柯西準則) holds:

for every  $\varepsilon > 0$ , there exists an integer  $N$  such that  $\left| \sum_{k=n}^m a_k \right| < \varepsilon$  if  $m \geq n \geq N$ . ▶

## Corollary

If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

## Divergence Test

If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum a_n$  diverges. ▶

## Remark

- The converse statement of the corollary is not true. In other words, one cannot conclude that if  $\lim_{n \rightarrow \infty} a_n = 0$ , then  $\sum a_n$  converges. ▶
- The Divergence Test is the contrapositive statement of the corollary.
- When the limit  $\lim_{n \rightarrow \infty} a_n \neq 0$ , it means that either the limit does not exist, or the limit exists but is not equal to 0.

**3.17 Proposition:** convergence of series with nonnegative terms (非負項級數收斂準則)

A series of real numbers with nonnegative terms converges if and only if its partial sums form a bounded sequence. ▶

**3.18 Proposition:** addition and scalar multiplication of series

If  $\sum a_n = A$ , and  $\sum b_n = B$ , then

1.  $\sum (a_n + b_n) = A + B$ ; | 2.  $\sum ca_n = cA$  for any finite number  $c$ .

**Remark**

- This statement demonstrates that treating infinite series as finite series is possible when it comes to addition and scalar multiplication.
- However, the same approach cannot be applied to the product of two convergent series, contrary to what one might assume.

**Euler's Number  $e$**  歐拉數**3.19 Definition:** the number  $e$  (歐拉數)

Define  $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ . This number  $e$  is known as **Euler's number**.

**Remark**

- The series  $\sum_{n=0}^{\infty} \frac{1}{n!}$  converges, and furthermore, the value falls between 2 and 3. ▶

In other words, the number  $e$  is well-defined and  $2 < e < 3$ .

**Remark**

- Euler's number  $e$  is an important universal constant, because it appears in many different areas of mathematics and science. One finds it is quite convenient to use the number  $e$  to describe exponential growth and decay, the solutions of differential equations, and the distribution of random variables.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e. \quad \blacktriangleright$$

**Remark**

- This limit is one of the most important and fundamental limits in mathematics.

**Convergence Tests** 審斂法**3.20 Theorem:** the Comparison Test (比較審斂法)

1. Suppose that  $0 \leq |a_n| \leq b_n$  for  $n \geq N_0$ , where  $N_0$  is some fixed integer. If  $\sum b_n$  converges, then  $\sum a_n$  converges.
2. Suppose that  $0 \leq a_n \leq b_n$  for  $n \geq N_0$ , where  $N_0$  is some fixed integer. If  $\sum a_n$  diverges, then  $\sum b_n$  diverges. ▶

## Remark

- In Item 1, under the same hypothesis, if  $\sum b_n$  converges, then  $\sum |a_n|$  and  $\sum a_n$  both converge.
- The statement in Item 2 is the contrapositive statement in Item 1.

## Remark

- The Comparison Test has a significant limitation in that it necessitates finding another series that is comparable to the series being analyzed and is known to converge or diverge. This can be challenging or unfeasible in certain cases, particularly for intricate series.
- The geometric series and the  $p$ -series are two commonly used series for comparing the convergence of other series.
- The limit version of the Comparison Test may be more convenient in some situations. ▶

## 3.21 Theorem: the Root Test and the Ratio Test

## the Root Test (根值審斂法)

Given  $\sum a_n$ , put  $\alpha = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ . Then

1. if  $\alpha < 1$ ,  $\sum a_n$  converges;
2. if  $\alpha > 1$ ,  $\sum a_n$  diverges;
3. if  $\alpha = 1$ , the test is inconclusive. ▶

## the Ratio Test (比值審斂法)

Given  $\sum a_n$  of nonzero terms,

1. if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then  $\sum a_n$  converges;
2. if  $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$  for all  $n \geq N_0$ , where  $N_0$  is fixed, then  $\sum a_n$  diverges. ▶

## Remark

- When the Root Test or the Ratio Test determine that  $\sum a_n$  converges, the series  $\sum |a_n|$  also converges. This is because the conclusion relies on the Comparison Test.
- The Root Test is beneficial for series that contain terms with  $n$ th powers of the variable, whereas the Ratio Test is advantageous for series that contain terms with ratios of successive terms. ▶

## Remark

- The Root Test and the Ratio Test are closely related as shown in the following result.

## upper-lower limit relations between the Root Test and the Ratio Test

For any sequence  $\{c_n\}$  of positive numbers,

$$\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \leq \lim_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \overline{\lim}_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}. \quad \blacktriangleright$$

$$\text{If } \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = c, \text{ then } \lim_{n \rightarrow \infty} \sqrt[n]{c_n} = c. \quad \blacktriangleright$$

Thus, it follows that if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then  $\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ . Hence, the Root Test also shows convergence whenever the Ratio Test does. However, the converse is not true. ▶

- When the Root Test is inconclusive, it means that the test does not provide enough information to determine whether the series converges or diverges. In such cases, additional tests or techniques may be required to evaluate the series. Analogous to the Root Test, if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , then this gives an inconclusive case for the convergence or divergence of the series. The  $p$ -series with  $p = 1, 2$  provide examples of inconclusive cases for both tests.

## 3.22 Theorem: Dirichlet's Test, Abel's Test, and the Alternating Series Test

Assume  $\{a_n\}$  and  $\{b_n\}$  are two sequence of real numbers and.

## Dirichlet's Test (狄利克雷審斂法)

Suppose

- (1) the partial sums of  $\sum a_n$  form a bounded sequence;
- (2) the sequence  $\{b_n\}$  is monotonically decreasing and  $\lim_{n \rightarrow \infty} b_n = 0$ .

Then  $\sum a_n b_n$  converges. ▶

## the Alternating Series Test (交錯級數審斂法)

Suppose that the series  $\sum c_n$  satisfies the following conditions:

- (1) the terms are alternating:  $c_{2m-1} \geq 0, c_{2m} \leq 0$  for  $m = 1, 2, 3, \dots$ ;
- (2) the sequence  $\{|c_n|\}$  is monotonically decreasing and  $\lim_{n \rightarrow \infty} c_n = 0$ .

Then  $\sum c_n$  converges. ▶

## Abel's Test (阿貝爾審斂法)

Suppose

- (1) the series  $\sum a_n$  converges;
- (2) the sequence  $\{b_n\}$  is monotonic and bounded.

Then  $\sum a_n b_n$  converges. ▶

## Remark

- The Alternating Series Test is a special case of Dirichlet's Test.
- Despite the fact that Abel's Test can be derived from Dirichlet's Test, it is not simply a special case of the latter. There are situations where Abel's Test is applicable but Dirichlet's isn't, and *vice versa*.

## Remark

- The comparison based tests, like the Root Test and the Ratio Test, suggest the convergence of  $\sum a_n$  by showing that  $\sum |a_n|$  converges. However, there are cases where a convergent series has a divergent  $\sum |a_n|$ . On the other hand, Dirichlet's Test and Abel's Test only guarantee the convergence of  $\sum a_n$ , not necessarily  $\sum |a_n|$ . Therefore, if the Root Test or Ratio Test are inconclusive, Dirichlet's Test and Abel's Test can be used instead. ▶ ▶

## Absolute Convergence 絕對收斂

## 3.23 Definition: absolute convergence and conditional convergence (絕對收斂與條件收斂)

Let  $\sum a_n$  be a series of real numbers.

If  $\sum |a_n|$  converges, the series  $\sum a_n$  is said to be **absolutely convergent** (絕對收斂).

If  $\sum a_n$  converges but  $\sum |a_n|$  diverges, the series  $\sum a_n$  is said to be **conditionally convergent** (條件收斂).

## absolute convergence implies convergence

If  $\sum a_n$  converges absolutely, then  $\sum a_n$  converges. ▶

## Remark

- The alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  is an example of conditionally convergent series. The series is convergent, but the absolute value of the series,  $\sum_{n=1}^{\infty} \frac{1}{n}$ , diverges.



**3.24 Definition:** Cauchy product of series (柯西積)

Given  $\sum a_n$  and  $\sum b_n$ , we put

$$c_n = \sum_{k=0}^n a_k b_{n-k}, \quad n = 1, 2, 3, \dots$$

and call  $\sum c_n$  the **Cauchy product** (柯西積) of the two given series.

**Remark**

- This definition may be motivated by a formal multiplication of two power series. If we take two power series  $\sum a_n z^n$  and  $\sum b_n z^n$ , multiply them term by term, and collect terms containing the same power of  $z$ , we get

$$\begin{aligned} \sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) \\ &= a_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \dots \\ &= c_0 + c_1 z + c_2 z^2 + \dots \end{aligned}$$

Setting  $z = 1$ , we arrive the above definition.

**3.25 Proposition:** convergence of the Cauchy product of two infinite series

Let  $\sum a_n$  and  $\sum b_n$  be real series, and let  $\sum c_n$  be their Cauchy product. If  $\sum a_n = A$ ,  $\sum b_n = B$ , and if at least one of them converges absolutely, then  $\sum c_n = AB$ . ▶

**Remark**

- The Cauchy product of two conditionally convergent series may not converge. ▶
- The Cauchy product of two absolutely convergent series must converge absolutely. ▶

**Addendum 後記****Addendum of Chapter 3**

- Infinite series can be used to define important functions, such as exponential functions  $e^x$  and trigonometric functions  $\sin x$  and  $\cos x$ . By a natural extension to complex number field, one can prove the renowned Euler's formula:  $e^{ix} = \cos x + i \sin x$  (歐拉公式), where  $x$  is real and  $i = \sqrt{-1}$  is the imaginary unit satisfying  $i^2 = -1$ . The formula is a powerful tool in dealing with trigonometric functions and is widely used in many areas of mathematics and science, including signal processing, quantum mechanics, and electrical engineering.
- The difference between absolute convergence and conditional convergence is much deeper than the superficial differences given by their definitions.
  - If you are working with series that are absolutely convergent, you can treat them much like finite sums. This means you can rearrange the terms in the series without changing the total sum. When dealing with the product of two absolutely convergent series, you can multiply each term together and rearrange the order of the additions without altering the final sum.

Suppose that real series  $\sum a_n$  converges absolutely. Then every rearrangement of  $\sum a_n$  converges, and they all converge to the same sum.

- However, for conditionally convergent series, one must exercise more caution as these rules no longer hold true. In fact, it is possible to rearrange the terms of a conditionally convergent series to obtain any desired sum, including infinity or negative infinity. This is known as the ***Riemann Rearrangement Theorem***.

the Riemann Rearrangement Theorem (黎曼重排定理)

Suppose that real series  $\sum a_n$  converges conditionally. For  $-\infty \leq \alpha \leq \beta \leq \infty$ , there exists a rearrangement  $\sum a'_n$  with partial sums  $s'_n$  such that

$$\lim_{n \rightarrow \infty} s'_n = \alpha, \quad \overline{\lim}_{n \rightarrow \infty} s'_n = \beta.$$

Therefore, it is important to exercise caution when dealing with conditionally convergent series, and to check for absolute convergence before rearranging the terms of an infinite series.

## Exercises of Chapter 3 練習題

## Chapter 3: Quiz

30 Minutes

- ① Which of the following is NOT true?
- every Cauchy sequence in  $\mathbb{R}$  is convergent
  - every Cauchy sequence in  $\mathbb{R}$  contains a convergent subsequence
  - every Cauchy sequence in  $\mathbb{R}$  is bounded
  - every sequence in  $\mathbb{R}$  contains a convergent subsequence
  - every convergent sequence in  $\mathbb{R}$  is bounded
- ② Which of the following statements is true?
- if a sequence is unbounded, then any of its subsequences diverges
  - if a sequence is bounded, then every subsequence converges
  - if a sequence converges, then every subsequence converges
  - if a sequence diverges, then any subsequence diverges
  - none of the above
- ③ Let  $K$  be a compact subset in  $\mathbb{R}$ . Which of the following statements is true?
- every sequence in  $K$  contains a convergent subsequence
  - every sequence in  $K$  converges to a limit in  $K$
  - every bounded sequence in  $K$  converges
  - every sequence in  $K$  is a Cauchy sequence
  - none of the above
- ④ Which of the following statements is true regarding the Cauchy Criterion for infinite series?
- the Criterion is a sufficient condition for convergence
  - the Criterion is a necessary condition for convergence
  - the Criterion is a necessary and sufficient condition for convergence
  - the Criterion requires that the terms of the series approach zero
  - the Criterion can be applied to series only with nonnegative terms
- ⑤ Which of the following statements best describes the Monotone Convergence Theorem?
- it applies only to finite sequences
  - it applies only to infinite sequences
  - it is for finding the limit of any sequence
  - it gives a necessary condition for a sequence to converge
  - it gives a sufficient condition for a sequence to converge
- ⑥ Let  $\{x_n\}$  be a sequence of real numbers. If  $\lim_{n \rightarrow \infty} x_n = -\infty$  and  $\overline{\lim}_{n \rightarrow \infty} x_n$  is finite, which of the following statements is true?
- $\{x_n\}$  must be bounded
  - $\{x_n\}$  must be bounded above but unbounded below
  - $\{x_n\}$  must be unbounded above but bounded below
  - $\{x_n\}$  must be unbounded above and unbounded below
  - none of the above
- ⑦ Which of the following tests can be used to determine conditional convergence?
- the Ratio Test
  - the Root Test
  - the Comparison Test
  - Dirichlet's Test
  - the Divergence Test
- ⑧ If the series  $\sum |a_n|$  converges, which of the following is true?
- it is not apparent whether the series  $\sum a_n$  converges or diverges
  - the series  $\sum a_n$  converges, provided that all  $a_n$  are positive
  - the series  $\sum a_n$  converges, provided that the sequence  $\{a_n\}$  is monotonic
  - the series  $\sum a_n$  converges, provided that the sequence  $\{a_n\}$  is alternating
  - the series  $\sum a_n$  converges

① D, ② C, ③ A, ④ C, ⑤ E, ⑥ B, ⑦ D, ⑧ E

## Chapter 3: Exercises

## Exercise 3.1

Prove directly by definition that if  $\lim_{n \rightarrow \infty} x_n = \alpha$ , then  $\lim_{n \rightarrow \infty} x_n^2 = \alpha^2$ .

② Use the inequality  $|x_2^n - \alpha_2^n| = |x_2^n + \alpha^n \cdot |x_1^n - \alpha^n| \cdot |x_1^n - \alpha^n| \leq |x_1^n - \alpha^n| \cdot (|x_1^n| + |\alpha^n|)$ .

### Exercise 3.2

Let  $x_1 = \sqrt{2}$ , and

$$x_{n+1} = \sqrt{2 + \sqrt{x_n}}, \quad n = 1, 2, 3, \dots$$

Prove that the sequence  $\{x_n\}$  converges, and find its limit.

② Prove by induction.  
① Prove that the sequence  $\{x_n\}$  is monotonically increasing and is bounded above.

### Exercise 3.3

Find the upper and lower limits of the sequence  $\{x_n\}$  defined by

$$x_1 = 0; \quad x_{2m} = \frac{1}{2}x_{2m-1}; \quad x_{2m+1} = \frac{1}{2} + x_{2m}.$$

② Postulate (假設) expressions for  $x_{2n-1}$  and  $x_{2n}$  and prove them by induction.

① Show that  $\{x_{2n-1}\}$  and  $\{x_{2n}\}$  converge, respectively.

## Exercise 3.4

Prove that if a subsequence of a monotonic sequence converges, then the monotonic sequence itself is also convergent.

③ The sequence  $\{x_n\}$  is monotonically increasing. Prove  $|x_n - \alpha| < \varepsilon$  for  $n > n_{K+1}$ .

② By the hypothesis, for any  $\varepsilon > 0$ , there exists an integer  $K$  such that  $\alpha - \varepsilon < x_{n_k} \leq \alpha$  for  $k > K$ .

① Without loss of generality, assume that  $\{x_n\}$  is monotonically increasing and  $\lim_{n \rightarrow \infty} x_n = \alpha$ . Prove by contradiction that  $x_n \leq \alpha$  for any integer  $n$ .

### Exercise 3.5

Prove the Squeeze Theorem (夾逼定理):

Let  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  be three sequences of real numbers. Suppose the following conditions hold:

- (1)  $x_n \leq y_n \leq z_n$  for  $n \geq N_0$ , where  $N_0$  is some fixed integer;
- (2)  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = L \in \mathbb{R} \cup \{\infty, -\infty\}$ .

Then  $\lim_{n \rightarrow \infty} y_n = L$ .

Apply the Squeeze Theorem to prove the following:

1.  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$

2.  $\lim_{n \rightarrow \infty} \frac{n^{2023}}{(1+p)^n} = 0$   
for  $p > 0.$

④ The binomial formula is 
$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k},$$
 where 
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

③ For part 2, choose a positive integer  $k$  such that  $k > 2023$ . Then, by the binomial formula (二项式公式), for  $n > 2k$ , we have 
$$(1+d)^n = \binom{n}{k} d^k < \frac{n^k d^k}{2^k k!},$$

② For part 1, let  $x_n = \sqrt[n]{n-1}$ . Show that 
$$n = (1+x_n)^n \geq \frac{n(n-1)^{\frac{n}{2}} x_n^{\frac{n}{2}}}{2}$$

① Apply Proposition 3.14 and its corollary.

## Exercise 3.6

(E)

Determine whether the series  $\sum a_n$  is convergent or divergent, if

1.  $a_n = \sqrt{n+1} - \sqrt{n}$ ;

2.  $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$ ;

3.  $a_n = (\sqrt[n]{n} - 1)^n$ ;

4.  $a_n = \frac{1}{1+z^n}$  for real values of  $z$ .

Hint ①

Hint ②

Hint ③

Hint ④

① For part 1, evaluate the partial sums.

② For part 2, show that  $0 < a_n < \frac{n}{3 \cdot 2^n}$ .

③ For part 3, by  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ , show that  $0 < a_n < \left(\frac{2}{3}\right)^n$  for sufficiently large  $n$ .

④ For part 4, show that  $a_n \neq 0$  if  $|z| \leq 1$ ;  $|a_n| \leq \frac{1}{1-|z|}$  if  $|z| > 1$ .

## Exercise 3.7

(E)

Suppose  $\{a_n\}$  is nonnegative. Prove that if  $\sum a_n$  converges, then  $\sum \frac{\sqrt{a_n}}{n}$  converges.

Hint ①

① Use the inequality  $\sqrt{\frac{a_n}{n}} = \sqrt{\frac{a_n}{2} \cdot \frac{2}{n}} \leq \frac{1}{2} \sqrt{a_n} + \frac{1}{n}$ , and the fact that  $\sum \frac{1}{n^2}$  converges.

## Exercise 3.8

(E)

Suppose that  $\sum a_n$  converges. Does the series  $\sum a_n^2$  converge?

Hint ①

① Consider the series  $\sum a_n$ , with  $a_{2n-1} = \frac{\sqrt{n}}{1}$ ,  $a_{2n} = -\frac{\sqrt{n}}{1}$ ,  $n \geq 1$ .

## Exercise 3.9

(I)

Prove the **Limit Comparison Test** (極限比較審斂法):

Suppose that  $\sum a_n$  and  $\sum b_n$  are two series with nonnegative terms. Denote

$$L^* = \overline{\lim}_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad L_* = \underline{\lim}_{n \rightarrow \infty} \frac{a_n}{b_n}.$$

1. If  $L^* < \infty$ , and if  $\sum b_n$  converges, then  $\sum a_n$  converges.

2. If  $L_* > 0$ , and if  $\sum b_n$  diverges, then  $\sum a_n$

Hint ①

Hint ②

diverges.

① For part 1, since  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L^* < \infty$ , there exists  $N$  such that  $\frac{a_n}{b_n} < L^* + 1$  when  $n \geq N$ . Then apply the Comparison Test.

② For part 2, since  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L_* > 0$ , there exists  $N$  such that  $\frac{a_n}{b_n} > \frac{1}{2}L_*$  when  $n \geq N$ . Then apply the Comparison Test.

## Exercise 3.10

(E)

Let  $\sum a_n$  and  $\sum b_n$  be real series, and let  $\sum c_n$  be their Cauchy product. If  $\sum a_n = A$ ,  $\sum b_n = B$ , and both converge absolutely, then  $\sum c_n = AB$  converges absolutely.

Hint ①

① Let  $A_n = \sum_{k=0}^n |a_k|$  and  $B_n = \sum_{k=0}^n |b_k|$ . To show that  $\sum c_n$  converges absolutely, prove that  $\left| \sum_{k=0}^n a_k b_{n-k} \right| \leq A_n B_n$ .

## Continuity 連續性

## Overview of Chapter 4

In this chapter, we will study continuity of real functions.

- Real continuous functions on  $E \subset \mathbb{R}$  can be characterized by the fact that  $f^{-1}(V)$  is open in  $E$  for every open set  $V$  in  $\mathbb{R}$ .
- We will prove that
  - Any real continuous function maps compact set to compact set. This implies that any real continuous function attains its extreme values (the Extreme Value Theorem).
  - Any real continuous function maps connected set to connected set. This implies that any real continuous function on a finite closed  $[a, b]$  assumes all intermediate values between  $f(a)$  and  $f(b)$  (the Intermediate Value Theorem).
- We will prove that continuous functions on a compact set must be uniformly continuous. This result forms the basis for the Riemann integrability of continuous functions.
- Finally, we will discuss discontinuity and monotonic functions.

## §4.1 Functions 函數

## 4.1 Definition: arithmetic operations of functions (函數的算術運算)

Suppose  $f, g: E \subset \mathbb{R} \rightarrow \mathbb{R}$  are two functions. We define  $f + g$ ,  $f - g$ ,  $fg$  and  $f/g$  by

$$\text{addition} : (f + g)(x) = f(x) + g(x);$$

$$\text{subtraction} : (f - g)(x) = f(x) - g(x);$$

$$\text{multiplication} : (fg)(x) = f(x)g(x);$$

$$\text{division} : (f/g)(x) = f(x)/g(x).$$

For the division, it is defined only at those points  $x$  of  $E$  at which  $g(x) \neq 0$ .

## 4.2 Definition: composition (複合函數)

Suppose  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$  and  $g: f(E) \subset \mathbb{R} \rightarrow \mathbb{R}$ . Define  $h = g \circ f: E \subset \mathbb{R} \rightarrow \mathbb{R}$  by

$$h(x) = g(f(x)), \quad x \in E.$$

This function  $h$  is called the **composition function** (複合函數) or *composite* of  $g$  and  $f$ .

## Remark

- We note that the composition of functions is not commutative in general, that is,  $g \circ f \neq f \circ g$ . In fact, even if  $g \circ f$  and  $f \circ g$  are both defined, they may have different values as well as different domains and ranges.
- The function  $g \circ f$  is called the “composite of  $g$  and  $f$ ”. We note that the order of symbol is consistent with the order of the written expression.

## 4.3 Definition: monotonic function (單調函數)

Let  $f$  be a real function on  $(a, b)$ .

We call  $f$  to be **monotonically increasing** (單調上升) on  $(a, b)$  if  $f(x) \leq f(y)$  for any  $x, y$  satisfying  $a < x < y < b$ . If the order  $\leq$  is replaced by the strict order  $<$ , then we call  $f$  to be **strictly increasing** (嚴格上升).

We call  $f$  to be **monotonically decreasing** (單調下降) on  $(a, b)$  if  $f(x) \geq f(y)$  for any  $x, y$  satisfying  $a < x < y < b$ . If the order  $\geq$  is replaced by the strict order  $>$ , then we call  $f$  to be **strictly decreasing** (嚴格下降).

A function is **monotonic** (單調) if it is either increasing or decreasing.

## 4.4 Definition: even, odd and periodic functions (偶函數、奇函數與週期函數)

- Let  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$ , with  $E$  symmetric to the origin.

If  $f(-x) = f(x)$  for every  $x \in E$ , we say that  $f$  is an **even function** (偶函數) on  $E$ .

If  $f(-x) = -f(x)$  for every  $x \in E$ , we say that  $f$  is an **odd function** (奇函數) on  $E$ .

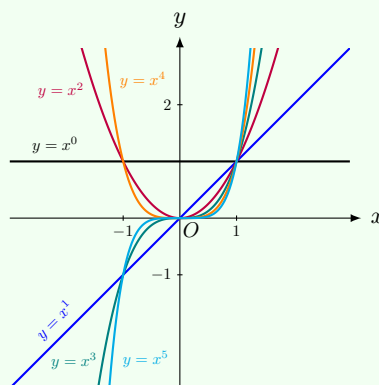
- Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a real function. If there is a positive number  $p$  such that  $f(x+p) = f(x)$  for all  $x \in \mathbb{R}$ , we say that  $f$  is a **periodic function** (週期函數). If there exists a least positive constant  $p$  with this property, it is called the basic period, or simply the **period**.

## Basic Elementary Functions 基本初等函數

## 4.5 Proposition: basic properties of power functions

For fixed  $n \in \mathbb{N}$ , the **power function** (冪函數)  $f(x) = x^n$  has the following properties:

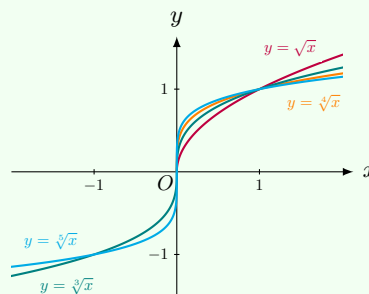
1. The domain is  $(-\infty, \infty)$ . When  $n$  is even, the range is  $[0, \infty)$ ; when  $n$  is odd, the range is  $(-\infty, \infty)$ .
2. When  $n$  is even,  $f$  is monotonically decreasing on  $(-\infty, 0)$  and increasing on  $(0, \infty)$ ; when  $n$  is odd,  $f$  is monotonically increasing on  $(-\infty, \infty)$ .
3. When  $n$  is even,  $f$  is an even function; when  $n$  is odd,  $f$  is an odd function.



## 4.6 Proposition: basic properties of radical functions

For fixed  $n \in \mathbb{N}_+$ , the **radical function** (根函數)  $f(x) = \sqrt[n]{x}$  has the following properties:

1. When  $n$  is even, the domain is  $[0, \infty)$  and the range is  $[0, \infty)$ ; when  $n$  is odd, the domain is  $(-\infty, \infty)$  and the range is  $(-\infty, \infty)$ .
2. The function  $f$  is strictly increasing in its domain;
3. When  $n$  is odd,  $f$  is an odd function.



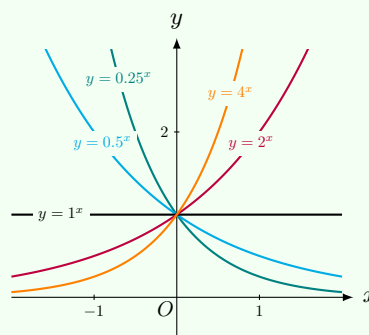
## Remark

- The radical function  $f$  is well-defined, by Proposition 1.11.

## 4.7 Proposition: basic properties of exponential functions

For fixed  $a$  ( $a > 0$ ), the **exponential function** (指數函數)  $f(x) = a^x$  has the following properties:

1. The domain is  $(-\infty, \infty)$ . When  $a \neq 1$ , the range is  $(0, \infty)$ ; when  $a = 1$ , the range is a one-point set  $\{1\}$ .
2. When  $0 < a < 1$ ,  $f$  is strictly decreasing in its domain; when  $a > 1$ ,  $f$  is strictly increasing in its domain; when  $a = 1$ ,  $f$  is constant.
3. Operations on exponential functions follow the laws of exponents.



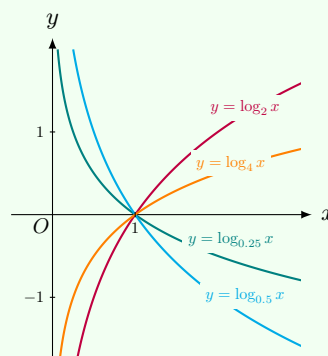
## Remark

- According to the remark in the Addendum of Chapter 1, the exponential function  $f$  is well-defined for  $a > 0$ .

## 4.8 Proposition: basic properties of logarithmic functions

For fixed  $a$  ( $a > 0$  and  $a \neq 1$ ), the **logarithmic function** (對數函數)  $f(x) = \log_a x$  has the following properties:

1. The domain is  $(0, \infty)$  and the range is  $(-\infty, \infty)$ .
2. When  $0 < a < 1$ ,  $f$  is strictly decreasing in its domain; when  $a > 1$ ,  $f$  is strictly increasing in its domain.
3. Operations on logarithmic functions follow the laws of logarithms.



## Remark

- Based on the remark in the Addendum of Chapter 1, the logarithmic function  $f$  is well-defined for  $a > 0$  and  $a \neq 1$ .



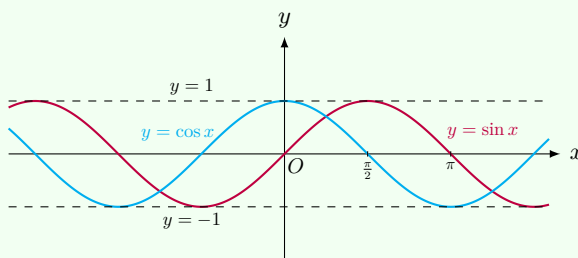
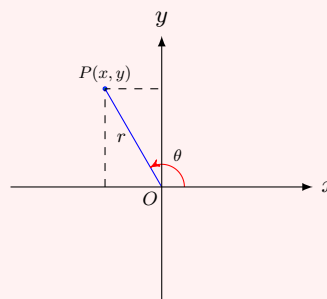
## 4.9 Proposition: basic properties of sine and cosine functions

For an angle  $\theta$ , with  $\theta \in [0, 2\pi)$ , if we let  $P(x, y)$  be the point on the terminal side of  $\theta$  and if let  $r$  be the distance  $|OP|$ , we define the values of the sine and the cosine as

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}.$$

For  $\theta$  outside  $[0, 2\pi)$ , the values of the sine and the cosine are defined in terms of the following periodic property:

$$\sin(\theta + 2\pi) = \sin \theta, \quad \cos(\theta + 2\pi) = \cos \theta.$$



The **sine** function (正弦函數) and the **cosine** function (餘弦函數) have the following properties:

1. The domain of both functions is  $(-\infty, \infty)$ .  
The range of both functions is  $[-1, 1]$ .
2. Both are  $2\pi$ -periodic functions.
3. The values of these two functions for special angles in  $[0, \frac{\pi}{2}]$  are listed as in the table:

degrees (度)	radians (弧度)	$\sin x$	$\cos x$
$0^\circ$	0	0	1
$30^\circ$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$45^\circ$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60^\circ$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$90^\circ$	$\frac{\pi}{2}$	1	0

4. For other special angles, the corresponding values of the functions can be obtained by using the following formulas:

$$\begin{aligned} \sin(x + \frac{\pi}{2}) &= \cos x, & \cos(x + \frac{\pi}{2}) &= -\sin x \\ \sin(x + \pi) &= -\sin x, & \cos(x + \pi) &= -\cos x \end{aligned}$$

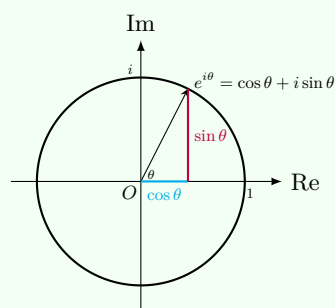
5. Two key equalities hold:

**Pythagorean Identity 畢氏三角恆等式**

$$\sin^2 x + \cos^2 x = 1.$$

**Euler's Formula 歐拉公式**

$$e^{ix} = \cos x + i \sin x, \quad i = \sqrt{-1}.$$



**Remark**

- It is a remarkable fact that these two equalities can be used to derive almost all elementary trigonometric identities.

For other trigonometric functions, they are defined by using the two basic trigonometric functions:

$$\begin{aligned} \text{tangent (正切): } \tan \theta &= \frac{\sin \theta}{\cos \theta}; & \text{cotangent (餘切): } \cot \theta &= \frac{\cos \theta}{\sin \theta}; \\ \text{secant (正割): } \sec \theta &= \frac{1}{\cos \theta}; & \text{cosecant (餘割): } \csc \theta &= \frac{1}{\sin \theta}. \end{aligned}$$

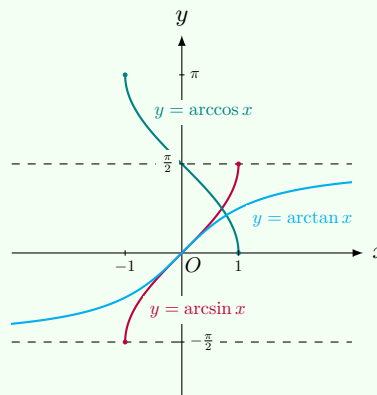
## 4.10 Proposition: basic properties of the inverse sine and the inverse cosine functions

- For each  $x \in [-1, 1]$ , the equation  $x = \sin y$  has a unique solution  $y$  in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . Denote  $y = \arcsin x$ , and call it the **arcsine** function (反正弦函數);
- For each  $x \in [-1, 1]$ , the equation  $x = \cos y$  has a unique solution  $y$  in  $[0, \pi]$ . Denote  $y = \arccos x$ , and call it the **arccosine** function (反餘弦函數).
- For each  $x \in (-\infty, \infty)$ , the equation  $x = \tan y$  has a unique solution  $y$  in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . Denote  $y = \arctan x$ , and call it the **arctangent** function (反正切函數).

They have the following properties:

1. The domain of both the arcsine and arccosine functions is  $[-1, 1]$ . The range of  $\arcsin x$  is  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ , while the range of  $\arccos x$  is  $[0, \pi]$ . On the other hand, the domain of the arctangent function is  $(-\infty, \infty)$ , and its range is  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .
2. In their respective domains, the functions  $\arcsin x$  and  $\arctan x$  are monotonically increasing, while the function  $\arccos x$  is monotonically decreasing.
3. The functions  $\arcsin x$  and  $\arctan x$  are odd, while the function  $\arccos x$  is neither even nor odd.
4. The following equalities hold:

$$\begin{aligned}\sin(\arcsin x) &= x, & \text{for all } x \in [-1, 1]; \\ \cos(\arccos x) &= x, & \text{for all } x \in [-1, 1]; \\ \tan(\arctan x) &= x, & \text{for all } x \in (-\infty, \infty).\end{aligned}$$



## Remark

- When it does not cause ambiguity, one often uses the notations  $\sin^{-1}(x)$ ,  $\cos^{-1}(x)$ , and  $\tan^{-1}(x)$  for these three inverse trigonometric function, respectively.
- There are three additional inverse trigonometric functions,  $\cot^{-1}(x)$ ,  $\sec^{-1}(x)$ ,  $\csc^{-1}(x)$ , that can be defined similarly.

## §4.2 Limits of Functions 函數極限

## 4.11 Definition: limit of function (函數極限)

Suppose  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$  and  $p$  is a limit point of  $E$ . We write  $f(x) \rightarrow q$  as  $x \rightarrow p$ , or  $\lim_{x \rightarrow p} f(x) = q$ , if there is a point  $q \in \mathbb{R}$  with the following property:

for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - q| < \varepsilon$   
for all points  $x \in E$  for which  $0 < |x - p| < \delta$ .

In this case, we say that the **limit** (極限) of function  $f$  at  $p$  equals  $q$ .

## Remark

- Here are some synonyms for “the limit of function  $f$  at  $p$  equals  $q$ ”:
 

– $f(x)$ has a limit of $q$ at $p$	– $f(x)$ approaches $q$ as $x$ approaches $p$
– the limit of $f(x)$ as $x$ approaches $p$ equals $q$	– $f(x)$ converges to $q$ as $x$ approaches $p$
- The limit  $\lim_{x \rightarrow p} f(x)$  can be defined without requiring  $p \in E$ , and even if it does, it is usually not equal to  $f(p)$ .
- $0 < |x - p| < \delta$  for  $x \in E \iff |x - p| < \delta$  for  $x \in E \setminus \{p\}$ .

## 4.12 Proposition: limit of function in term of sequential limits

Suppose  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$ . The limit  $\lim_{x \rightarrow p} f(x) = q$  if and only if  $\lim_{n \rightarrow \infty} f(p_n) = q$  for every sequence  $\{p_n\}$  in  $E \setminus \{p\}$  converging to  $p$ . ▶

## Corollary

If a function has a limit, then the limit is unique. ▶

## 4.13 Proposition: arithmetic operations on limits of functions

Suppose  $E \subset \mathbb{R}$ ,  $p$  is a limit point of  $E$ ,  $f$  and  $g$  are two real functions on  $E$ , and

$$\lim_{x \rightarrow p} f(x) = A, \quad \lim_{x \rightarrow p} g(x) = B.$$

Then

- |  |  |
|--|--|
| 1. $\lim_{x \rightarrow p} (f + g)(x) = A + B$ ;<br>2. $\lim_{x \rightarrow p} (f - g)(x) = A - B$ ; | 3. $\lim_{x \rightarrow p} (fg)(x) = AB$ ;<br>4. $\lim_{x \rightarrow p} (f/g)(x) = A/B$ , if $B \neq 0$ . ▶ |
|--|--|

## 4.14 Definition: limits of function in the extended real number system (在擴張實數系中的函數極限)

For any real  $c$ , the set of real numbers  $x$  such that  $x > c$  is called a **neighborhood** of  $\infty$  and is written  $(c, \infty)$ . Similarly, the set  $(-\infty, c)$  is a neighborhood of  $-\infty$ .

Let  $f$  be a real function defined on  $E \subset \mathbb{R}$ . We write  $f(x) \rightarrow q$  as  $x \rightarrow p$ , or  $\lim_{x \rightarrow p} f(x) = q$ , where  $p$  and  $q$  are in the extended real number system, if

for every neighborhood  $U$  of  $q$ , there is a neighborhood  $V$  of  $p$  satisfying  $V \cap E \neq \emptyset$  such that  $f(x) \in U$  for all  $x \in V \cap E$ ,  $x \neq p$ .

## Remark

- When  $p$  is finite while  $q$  is infinite, the above defines two types of limits:  $\lim_{x \rightarrow p} f(x) = \infty$  and  $\lim_{x \rightarrow p} f(x) = -\infty$ . They are called **infinite limits**.
- An equivalent definition for the infinite limit  $\lim_{x \rightarrow p} f(x) = \infty$  is:

for every  $M$ , there exists a  $\delta > 0$  such that  $f(x) > M$  for all points  $x \in E$  for which  $0 < |x - p| < \delta$ .

## Remark

- When  $p$  is infinite while  $q$  is finite, the above defines two types of limits:  $\lim_{x \rightarrow \infty} f(x) = q$  and  $\lim_{x \rightarrow -\infty} f(x) = q$ . They are called **limits at infinity**.
- An equivalent definition for the limit at infinity  $\lim_{x \rightarrow \infty} f(x) = q$  is:

for every  $\varepsilon > 0$  there exists an  $N$  such that  $|f(x) - q| < \varepsilon$   
for all points  $x \in E$  for which  $x > N$ .

## Remark

- When both  $p$  and  $q$  are infinite, the definition gives another four types of limits:

$$\lim_{x \rightarrow \infty} f(x) = \infty, \quad \lim_{x \rightarrow \infty} f(x) = -\infty, \quad \lim_{x \rightarrow -\infty} f(x) = \infty, \quad \lim_{x \rightarrow -\infty} f(x) = -\infty.$$

They are called **infinite limits at infinity**.

## uniqueness of limit

Suppose  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$ . If  $\lim_{x \rightarrow p} f(x) = q$  and  $\lim_{x \rightarrow p} f(x) = q'$ , where  $p, q, q' \in \mathbb{R} \cup \{\infty, -\infty\}$ , then  $q = q'$ .

## Remark

- When both  $p$  and  $q$  are finite, the above definition coincides with Definition 4.11.

## 4.15 Proposition: arithmetic operations on limits of functions in the extended real number system

Suppose  $E \subset \mathbb{R}$ ,  $f$  and  $g$  are two real functions on  $E$ , and  $\lim_{x \rightarrow p} f(x) = A$ ,  $\lim_{x \rightarrow p} g(x) = B$ , where  $p, A$ , and  $B$  are in  $\mathbb{R} \cup \{\infty, -\infty\}$ . Then

- |   |  |
|---|--|
| 1. $\lim_{x \rightarrow p} (f + g)(x) = A + B;$ | 3. $\lim_{x \rightarrow p} (fg)(x) = AB;$                    |
| 2. $\lim_{x \rightarrow p} (f - g)(x) = A - B;$ | 4. $\lim_{x \rightarrow p} (f/g)(x) = A/B$ , if $B \neq 0$ , |

provided that the right-hand sides of the above equalities are well-defined in  $\mathbb{R} \cup \{\infty, -\infty\}$ .

## Remark

- Note that  $\infty - \infty$ ,  $0 \cdot \infty$ ,  $\infty/\infty$ ,  $A/0$  are not well-defined in  $\mathbb{R} \cup \{\infty, -\infty\}$ .

## §4.3 Continuous Function 連續函數

## 4.16 Definition: continuous function (連續函數)

Suppose  $E \subset \mathbb{R}$ ,  $p \in E$ , and  $f$  maps  $E$  into  $\mathbb{R}$ . Then  $f$  is said to be **continuous** (連續) at  $p$  if

for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - f(p)| < \varepsilon$   
for all points  $x \in E$  for which  $|x - p| < \delta$ .

If  $f$  is continuous at every point of  $E$ , then  $f$  is said to be **continuous on  $E$** .



## continuity at a limit point

Assume that  $p$  is a limit point of  $E \subset \mathbb{R}$ . Then  $f$  is continuous at  $p$  if and only if  $\lim_{x \rightarrow p} f(x) = f(p)$ .

## Remark

- To be continuous at a point, it is necessary for the function to be defined at that point.
- If  $p$  is an isolated point of  $E$ , every functions with domain  $E$  is continuous at  $p$ . ▶

## Remark

- The basic elementary functions (such as power, radical, exponential, logarithmic, trigonometric functions) are continuous in their domains.

## 4.17 Theorem: characterization of continuity

Let  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$ .

1. The function  $f$  is continuous if and only if  $f^{-1}(V)$  is open in  $E$  for every open set  $V$  in  $\mathbb{R}$ .
2. The function  $f$  is continuous if and only if  $f^{-1}(C)$  is closed in  $E$  for every closed set  $C$  in  $\mathbb{R}$ . ▶

## Remark

- A continuous mapping
  - does not necessarily map from closed subsets to closed subsets;
  - does not necessarily map from open subsets to open subsets;
  - does not necessarily map from bounded subsets to bounded subsets. ▶
  - does map from compact subsets to compact subsets;
  - does map from connected subsets to connected subsets.

## 4.18 Proposition: continuity of operations of functions

1. Suppose  $f$  and  $g$  are real continuous functions on  $E \subset \mathbb{R}$ . Then  $f+g$ ,  $f-g$ ,  $fg$ , and  $f/g$  are continuous on  $E$  (for quotient, the denominator is not zero). ▶ ▶
2. Suppose  $f$  and  $g$  are real functions such that the composite of  $g$  and  $f$ ,  $g \circ f$ , is well-defined on  $E \subset \mathbb{R}$ . If  $f$  is continuous at a point  $p \in E$  and if  $g$  is continuous at the point  $f(p)$ , then  $g \circ f$  is continuous at  $p$ . ▶
3. Suppose  $f$  is a continuous bijective mapping of a compact set  $K \subset \mathbb{R}$  onto  $f(K) \subset \mathbb{R}$ . Then the inverse mapping  $f^{-1}$  is a continuous mapping of  $f(K)$  onto  $K$ . ▶

## Remark

- A continuous bijective mapping on *noncompact* set may not have continuous inverse. ▶

## Remark

- Elementary functions are functions that are defined as taking sums, products, roots and compositions of finitely many polynomial, rational, trigonometric, and exponential functions, including possibly their inverse functions. All elementary functions are continuous in their domains.

## §4.4 Continuity and Compactness 連續性與緊致性

## 4.19 Theorem: continuous mapping maps compact set to compact set

Suppose  $f$  is a continuous mapping of a compact set  $K$  into  $\mathbb{R}$ . Then  $f(K)$  is compact, that is, closed and bounded. ▶

A mapping  $f$  of a set  $E \subset \mathbb{R}$  into  $\mathbb{R}$  is said to be **bounded** if there is a real number  $M$  such that  $|f(x)| \leq M$  for all  $x \in E$ .

## Remark

- For any *noncompact* subset of  $\mathbb{R}$ , there exists an unbounded real continuous function. ▶

## 4.20 Theorem: the Extreme Value Theorem (極值定理)

Suppose  $f$  is a real continuous function on a compact set  $K$ . Then there exist points  $p, q \in K$  such that

$$f(p) = \sup_{x \in K} f(x), \quad f(q) = \inf_{x \in K} f(x).$$

That is,  $f$  attains its maximum and its minimum. ▶

## Corollary

Any real continuous function on  $[a, b]$  attains its maximum and minimum.

## Remark

- For any *noncompact* subset of  $\mathbb{R}$ , there exists a continuous function that is bounded but has no maximum value. ▶

## 4.21 Definition: uniformly continuous (一致連續)

Suppose  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$ . We say that  $f$  is **uniformly continuous** (一致連續) on  $E$  if

for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(p) - f(q)| < \varepsilon$  for all points  $p$  and  $q$  in  $E$  for which  $|p - q| < \delta$ .

## Remark

- Continuity and uniform continuity differ significantly. Continuity is defined at a point, while uniform continuity applies to a set. Hence, asking whether a given function is uniformly continuous at a certain point is meaningless.
- For continuity, the value of  $\delta$  depends on the point being considered, while for uniform continuity, a single value of  $\delta$  applies to all points in  $E$ . ▶

## 4.22 Theorem: the relationship between continuity and uniform continuity

Suppose  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$ .

1. If  $f$  is uniformly continuous on  $E$ , then  $f$  is continuous on  $E$ .
2. If  $E$  is compact, and if  $f$  is continuous on  $E$ , then  $f$  is uniformly continuous on  $E$ . ▶

## Remark

- For any *noncompact* subset of  $\mathbb{R}$ , there exists a function that is continuous but not uniformly continuous. ▶

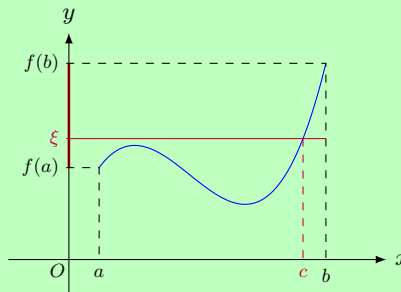
## Continuity and Connectedness 連續性與連通性

## 4.23 Theorem: continuous mapping maps connected set to connected set

Suppose  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$  is continuous. If  $E$  is connected, then  $f(E)$  is connected. ▶

## the Intermediate Value Theorem (介值定理)

Any real continuous function  $f$  on  $[a, b]$  assumes all intermediate values between  $f(a)$  and  $f(b)$ . That is, for any value  $\xi$  between  $f(a)$  and  $f(b)$ , there exists at least one point  $c \in [a, b]$  such that  $f(c) = \xi$ . ▶



## Bolzano's Theorem (零点定理)

Any real continuous function  $f$  on  $[a, b]$  has at least one root if  $f(a) \cdot f(b) < 0$ .

## Remark

- One can modify the Intermediate Value Theorem to have that

Any real continuous function  $f$  on  $[a, b]$  assumes all intermediate values between  $\min_{x \in [a, b]} f(x)$  and  $\max_{x \in [a, b]} f(x)$ .

## Remark

- The converse statement of the Intermediate Value Theorem is not true. In fact, there is a *discontinuous* function  $f$  satisfies that for any two points  $x_1$  and  $x_2$ , with  $x_1 < x_2$ , the function assumes every value between  $f(x_1)$  and  $f(x_2)$ .

## §4.5 Discontinuity and Monotonicity 間斷與單調性

## 4.24 Definition: one-sided limits (單邊極限) and discontinuities (不連續點)

Let  $f$  be defined on open interval  $(a, b)$ . Suppose  $x \in (a, b)$  is fixed.

If  $f(t_n) \rightarrow q$  as  $n \rightarrow \infty$ , for all sequences  $\{t_n\}$  in  $(x, b)$  such that  $t_n \rightarrow x$ , we write  $f(x+) = q$  or  $\lim_{t \rightarrow x+} f(t) = q$  and call the **right-hand limit** (右極限) of  $f$  at  $x$ .

If  $f(t_n) \rightarrow q$  as  $n \rightarrow \infty$ , for all sequences  $\{t_n\}$  in  $(a, x)$  such that  $t_n \rightarrow x$ , we write  $f(x-) = q$  or  $\lim_{t \rightarrow x-} f(t) = q$  and call the **left-hand limit** (左極限) of  $f$  at  $x$ .

The limit  $\lim_{t \rightarrow x} f(t)$  exists if and only if  $f(x+) = f(x-) = \lim_{t \rightarrow x} f(t)$ .

If  $f(x+)$  and  $f(x-)$  exist, the function  $f$  is said to have a **discontinuity of first kind** (第一類不連續點) at  $x$ . It is also called a *simple discontinuity* (簡單不連續點).

If at least one of  $f(x+)$  and  $f(x-)$  does not exist, the function  $f$  is said to have a **discontinuity of second kind** (第二類不連續點) at  $x$ . It is also called an *essential discontinuity* (本性不連續點). ▶

## Remark

- There are two ways in which a function can have a simple discontinuity:
  - ① **removable discontinuity** (可去不連續點):  $f(x+) = f(x-) \neq f(x)$ ;
  - ② **jump discontinuity** (跳躍不連續點):  $f(x+) \neq f(x-)$ .
- A function is called **piecewise continuous** (分段連續) on a given interval  $[a, b]$  if the interval can be broken into a finite number of open intervals on which the function is continuous on each open interval and it has a finite limit at the endpoints of each open interval. The term *piecewise* refers “a piece in a **finite** number of open intervals”.

## 4.25 Proposition: monotonic function always has one-sided limits

Let  $f$  be monotonically increasing on open interval  $(a, b)$ . Then  $f(x+)$  and  $f(x-)$  exist at every point of  $x$  of  $(a, b)$ . More precisely,

$$\sup_{a < t < x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x < t < b} f(t).$$

Furthermore, if  $a < x < y < b$ , then

$$f(x+) \leq f(y-).$$

An analogue result is also true for monotonically decreasing functions. ▶

## Corollary

Monotonic functions have no discontinuities of the second kind.

## Corollary

Any monotonic function on an open interval has at most countably many discontinuities. ▶

## Remark

- Although the discontinuities of a monotonic function on an open interval form an at most countable set, they need not be isolated. In fact, it is possible to construct a function  $f$  on the interval  $(a, b)$  that is monotonic and discontinuous at every point of a given countable subset  $E$  of the interval, even if  $E$  is dense.

## Remark

- In summary, the following properties are true for a monotonic function  $f: \mathbb{R} \rightarrow \mathbb{R}$ :
  - for every point  $x \in \mathbb{R}$ , both  $f(x+)$  and  $f(x-)$  exist;
  - the limits at infinity,  $\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow \infty} f(x)$ , equal either a finite number, or  $\infty$ , or  $-\infty$ ;
  - the function  $f$  has only simple discontinuities;
  - the discontinuities of  $f$  form an at most countable set.

## Addendum 後記

## Addendum of Chapter 4

- Continuity is a fundamental concept in mathematics and has many significant applications in various fields such as calculus, analysis, topology, and geometry.
  - In calculus and analysis, continuity is essential for defining limits, derivatives, and integrals. The concept of continuity allows us to determine the behavior of a function at any point within its



domain, and it helps us to understand the relationship between different functions.

- In topology, continuity is used to define the properties of spaces and the relationships between them. It is a fundamental concept that helps us to understand the topological structure of the universe and the behavior of objects within it.
- In geometry, continuity is used to describe the smoothness and regularity of shapes and surfaces. It is a crucial concept in differential geometry, where it is used to study the curvature and other geometric properties of surfaces and curves.
- Uniform continuity ensures that a function changes gradually over an interval, enabling the Riemann sum (as defined in Chapter 6) to converge to the true value as the subintervals' width approaches zero. In the absence of uniform continuity, the function may oscillate excessively or have sharp spikes, rendering it impossible to accurately approximate the integral using Riemann sums.
- Monotonic functions are a large class of functions, besides continuous functions, that we often encounter in applications. Here are a few examples:
  - In optimization problems, monotonic functions are often used to model constraints or objective functions. This is because they have clear and predictable behavior, making them easier to work with and analyze.
  - In probability theory, monotonic functions are used to transform probability distributions to other distributions with desirable properties. For example, the cumulative distribution function (CDF) of a random variable is a monotonic function, and it can be used to transform any distribution to a uniform distribution.
  - In economics, monotonic functions are used to model utility functions, which describe how individuals make choices based on their preferences. Monotonicity ensures that more of a good is always preferred to less, and that preferences are consistent across different levels of consumption.
  - In computer science and machine learning, monotonic functions are used to model relationships between variables in regression and classification problems. Monotonicity ensures that the relationship between the input and output variables is consistent and predictable, making it easier to interpret and use the model.

## Exercises of Chapter 4 練習題

## Chapter 4: Quiz

30 Minutes

- ① Let  $x_0 \in E$  be a fixed point, and let  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Which of the following statements is true?
- for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $x_1 \in E$  and  $|x_1 - x_0| < \delta$  imply  $|f(x_1) - f(x_0)| < \varepsilon$
  - for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $x_1 \in E$  and  $|x_1 - x_0| < \varepsilon$  imply  $|f(x_1) - f(x_0)| < \delta$
  - for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $x_1, x_2 \in E$  and  $|x_1 - x_2| < \delta$  imply  $|f(x_1) - f(x_2)| < \varepsilon$
  - for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $x_1, x_2 \in E$  and  $|x_1 - x_2| < \varepsilon$  imply  $|f(x_1) - f(x_2)| < \delta$
  - none of the above
- ② Which of the following statements is true for a continuous function  $f$  on an interval  $[a, b]$ ?
- $f$  assumes every value in  $[a, b]$
  - $f$  is increasing on  $(a, b)$
  - $f$  has a maximum value on  $[a, b]$
  - $f$  has a minimum value on  $(a, b)$
  - $f$  is unbounded on  $[a, b]$
- ③ Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Which of the following sets is open?
- |   |   |
|---|---|
| A. $\{x \in \mathbb{R} : f(x) \geq 0\}$ | D. $\{x \in \mathbb{R} : f(x) \neq 0\}$ |
| B. $\{x \in \mathbb{R} : f(x) \leq 0\}$ | E. none of the above                    |
| C. $\{x \in \mathbb{R} : f(x) = 0\}$    |   |
- ④ Suppose  $f$  is continuous on  $E = (a, b)$ . Which of the following is true?
- $f(E)$  is bounded
  - $f(E)$  is open
  - $f(E)$  is compact
  - $f(E)$  is connected
  - none of the above
- ⑤ Let  $g: [a, b] \rightarrow \mathbb{R}$  be a continuous function. Which of the following statements is FALSE?
- $g$  takes every value between  $a$  and  $b$
  - $g$  takes every value between  $g(a)$  and  $g(b)$
  - $g$  takes every value between  $\sup_{x \in [a, b]} g(x)$  and  $\inf_{x \in [a, b]} g(x)$
  - $g([a, b])$  is a closed subset of  $\mathbb{R}$
  - $g([a, b])$  is a bounded subset of  $\mathbb{R}$
- ⑥ Suppose  $f(x) = 1/x$  on  $(0, 1)$ . Which of the following statements is true?
- $f$  is uniformly continuous on  $(0, 1)$
  - $f$  is continuous on  $(0, 1)$  but not uniformly continuous
  - $f$  is not continuous on  $(0, 1)$ .
  - $f$  is uniformly continuous on  $[0, 1]$ .
  - none of the above.
- ⑦ Let  $f(x) = \begin{cases} x^2 - 1, & x \leq 0, \\ x^2 + 1, & x > 0 \end{cases}$  and  $g(x) = \begin{cases} 0, & x = 0, \\ 1/x, & x \neq 0. \end{cases}$  Which of the following statements is true?
- at 0, both  $f$  and  $g$  have jump discontinuities
  - at 0, both  $f$  and  $g$  have removable discontinuities
  - at 0, both  $f$  and  $g$  have essential discontinuities
  - at 0,  $f$  has a jump discontinuity and  $g$  has an essential discontinuity
  - at 0,  $f$  has a removable discontinuity and  $g$  has an essential discontinuity
- ⑧ Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function and  $E$  is a subset of  $\mathbb{R}$ . Which of the following must be true?
- $f(\overline{E})$  is a closed subset in  $\mathbb{R}$
  - $f((\overline{E})^c)$  is an open subset in  $\mathbb{R}$
  - $f(\overline{E^c})$  is a closed subset in  $\mathbb{R}$
  - $f^{-1}((\overline{E})^c)$  is a close subset in  $\mathbb{R}$
  - $f^{-1}((\overline{E})^c)$  is an open subset in  $\mathbb{R}$

①A, ②C, ③D, ④E, ⑤D, ⑥A, ⑦B, ⑧D

## Chapter 4: Exercises

## Exercise 4.1

(E)

Show that the definition of limit can be modified as

“for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - q| \leq \varepsilon^2$  for all points  $x \in E$  for which

$$0 < |x - p| < \delta.”$$

Hint ①

① Show that the modified statement holds if and only if to the original statement holds.

## Exercise 4.2

(I)

Prove directly by definition that the function  $f(x) = \sqrt{x}$  is continuous on  $[0, \infty)$ .

Hint ②

② When  $\alpha > 0$ , use the inequality  $\frac{\sqrt{\alpha} + \sqrt{x}}{\sqrt{\alpha} - \sqrt{x}} \leq \frac{\sqrt{\alpha} + \sqrt{x}}{\sqrt{\alpha} - \sqrt{x}} = \left| \frac{\sqrt{\alpha} + \sqrt{x}}{\sqrt{\alpha} - \sqrt{x}} \right|$

Hint ①

① Prove that the limits  $\lim_{x \rightarrow \alpha^+} \sqrt{x} = \sqrt{\alpha}$  and  $\lim_{x \rightarrow \alpha^-} \sqrt{x} = \sqrt{\alpha}$  for any  $\alpha > 0$ , respectively.

## Exercise 4.3

(I)

Prove directly by definition that the function  $f(x) = \frac{1}{\sqrt{x}}$  is continuous on  $(0, \infty)$ .

Hint ②

② Use the inequality  $\left| \frac{1}{\sqrt{\alpha} + \sqrt{x}} - \frac{1}{\sqrt{\alpha} - \sqrt{x}} \right| = \frac{2\sqrt{\alpha}\sqrt{x}}{|\alpha - x|} \leq \frac{2\sqrt{\alpha}\sqrt{x}}{|\alpha - x|} \cdot \frac{\sqrt{\alpha} + \sqrt{x}}{\sqrt{\alpha} + \sqrt{x}} = \frac{2\sqrt{\alpha}(\sqrt{\alpha} + \sqrt{x})}{|\alpha - x|(\sqrt{\alpha} + \sqrt{x})}$

Hint ①

① Use the definition of limit to prove  $\lim_{x \rightarrow \alpha} \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{\alpha}}$  for any  $\alpha > 0$ .

Hint ③

③ There exists  $\delta_1 > 0$  such that  $x > \frac{1}{2}\alpha$  for all points  $x \in (0, \infty)$  for which  $|x - \alpha| < \delta_1$ .

## Exercise 4.4

(E)

Suppose  $f$  and  $g$  are two real function on  $[a, b]$ . Let

$$H(x) = \max\{f(x), g(x)\},$$

$$h(x) = \min\{f(x), g(x)\}.$$

Prove the following:

1. If  $f$  and  $g$  are monotonic, so are  $H$  and  $h$ .

Hint ①

① For part 1, without loss of generality, suppose  $f$  and  $g$  are monotonically increasing. Use the definition to show that  $H$  is also monotonically increasing.

Hint ②

② For part 2, prove that  $H(x) = \frac{1}{2}[f(x) + g(x) + |f(x) - g(x)|]$ ,  $h(x) = \frac{1}{2}[f(x) + g(x) - |f(x) - g(x)|]$ .

## Exercise 4.5

(E)

Let  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Prove that the zero set of  $f$ ,  $Z(f) = \{x \in E : f(x) = 0\}$ , is

closed.

Hint ①

① Prove that every limit point of  $Z(f)$  is in  $Z(f)$ .

## Exercise 4.6

(E)

Suppose that  $f: [a, b] \rightarrow [a, b]$  is a continuous mapping. Prove that  $f(x) = x$  for at least one  $x \in [a, b]$ .

Hint ①

① Apply Bolzano's Theorem to the function  $F(x) = f(x) - x$ .

## Exercise 4.7

(I)

Suppose that  $f: (a, b) \rightarrow \mathbb{R}$  is continuous. For any  $n$  values  $x_1, \dots, x_n \in (a, b)$ , prove that there exists a value  $\xi \in (a, b)$  such that

$$f(\xi) = \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}.$$

Hint ①

① Apply the Intermediate Value Theorem.

Hint ②

② Denote  $m = \min_{x \in [x_1, x_n]} f(x)$ ,  $M = \max_{x \in [x_1, x_n]} f(x)$ . Show that  $\frac{m}{n} \leq \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n} \leq M$ .

## Exercise 4.8

(I)

Suppose that  $f: [a, \infty)$  is continuous for some  $a \in \mathbb{R}$ , and suppose that  $\lim_{x \rightarrow \infty} f(x) = A \in \mathbb{R}$ . Prove that  $f$  is bounded on  $[a, \infty)$ .

Hint ①

① Use the hypothesis  $\lim_{x \rightarrow \infty} f(x) = A$  to show that there is a number  $b > a$ , the function  $f$  is bounded on  $(b, \infty)$ .

Hint ②

② The function  $f$  is continuous on the compact set  $[a, M]$ . Apply Theorem 4.19 to obtain the boundedness of  $f$  on  $[a, b]$ .

## Exercise 4.9

(I)

Suppose  $f: E \subset \mathbb{R} \rightarrow \mathbb{R}$  is uniformly continuous. Prove that  $\{f(x_n)\}$  is a Cauchy sequence in  $\mathbb{R}$  for every Cauchy sequence  $\{x_n\}$  in  $E$ .

Hint ①

① Directly use the definitions of uniform continuous and Cauchy sequence.

## Exercise 4.10

(I)

Let  $f$  be a real uniformly continuous function on the bounded set  $E$  in  $\mathbb{R}$ . Prove that  $f$  is bounded on  $E$ .

Hint ②

② For any  $\delta > 0$ ,  $E$  is covered by a finite collection of open intervals of length  $2\delta$ . Keep only the open intervals which intersect with  $E$ .

Hint ③

③ Use the uniform continuity and the triangle inequality to show that  $f$  is bounded on  $E$ .

Hint ①

① Since  $E$  is bounded, it is contained in a bounded closed interval.



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 $A \subsetneq B$ ,  $A$  is a proper subset of  $B$  真子集, 9  
 $E'$ , the set of all limit points of  $E$  極限點集, 22  
 $E^\circ$ , the interior of set  $E$  內部, 22  
 $E^c$ , the complement of set  $E$  余集, 補集, 22  
 $N_r(p)$ , a neighborhood of point  $p$  with radius  $r$  鄰域, 22

$|\cdot|$ , absolute value of real number 絕對值, 14  
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## Appendix A: the justifications and the examples


### Example: direct proof 1

Claim: the square of an odd number is also odd.

**Proof** Let  $p$  be the statement that  $n$  is an odd integer and  $q$  be the statement that  $n^2$  is an odd integer. We need to prove that  $p \Rightarrow q$ .

Assume that  $n$  is an odd integer, then by definition  $n = 2k + 1$  for some integer  $k$ . We will now use this to show that  $n^2$  is also an odd integer. In fact, since

$$\begin{aligned}n^2 &= (2k + 1)^2 \\&= 4k^2 + 4k + 1 \\&= 2(2k^2 + 2k) + 1,\end{aligned}$$

we know that  $n^2$  has the form of an odd integer since  $(2k^2 + 2k)$  is an integer. Therefore, we have shown that  $p \Rightarrow q$  and so we have completed our proof. 


**Example:** direct proof 2

Let  $l$ ,  $m$ , and  $n$  be integers. Claim: if  $l$  divides  $m$  and  $l$  divides  $n$  then  $l$  also divides  $m + n$ .

**Proof** Let  $p$  be the statement that  $l$  divides  $m$  and  $l$  divides  $n$ , and  $q$  be the statement that  $l$  divides  $m + n$ . We need to prove that  $p \Rightarrow q$ .

Since  $l$  divides  $m$ , by definition, there is some integer  $k_1$  such that  $m = lk_1$ . Also as  $l$  divides  $n$ , there is some integer  $k_2$  such that  $n = lk_2$ . Thus,

$$m + n = lk_1 + lk_2 = l(k_1 + k_2).$$

Hence,  $l$  divides  $m + n$  since  $(k_1 + k_2)$  is an integer. Therefore, we have shown that  $p \Rightarrow q$  and so we have completed our proof. 

**Example:** a false proof by induction

Claim: for any positive integer  $n$ , the following formula holds:

$$\sum_{j=1}^n j = \frac{1}{2}(n + \frac{1}{2})^2.$$

**Proof** We prove the claim by induction.

**Base Step:**

It is obvious the claim holds when  $n = 1$ .

**Inductive Step:**

Assume that the claim holds for  $n = k$ . Then

$$\begin{aligned} \sum_{j=1}^{k+1} j &= \sum_{j=1}^k j + (k+1) \\ &= \frac{1}{2}(k + \frac{1}{2})^2 + (k+1) \\ &= \frac{1}{2}(k^2 + k + \frac{1}{4} + 2k + 2) \\ &= \frac{1}{2}[(k^2 + 2k + 1) + 2 \cdot (k+1) \cdot \frac{1}{2} + \frac{1}{4}] \\ &= \frac{1}{2}[(k+1) + \frac{1}{2}]^2. \end{aligned}$$

Thus, the claim holds for  $n = k + 1$ , so the induction step is complete.

Consequently, the claim holds for all positive integers  $n$  by the principle of induction.

**Remark**

- The false induction proof illustrated in this example is due to the mistake made in the base step.

**Example:** proof by induction 1

Claim: for any positive integer  $n$ , the following formula holds:

$$\sum_{j=1}^n j = \frac{1}{2}n(n+1).$$

**Proof** We prove the claim by induction.

**Base Step:**

When  $n = 1$ , we have

$$\sum_{j=1}^n j = \sum_{j=1}^1 j = 1 = \frac{1}{2}1(1+1) = \frac{1}{2}n(n+1),$$


so the claim holds .

**Inductive Step:**

Assume that the claim holds for  $n = k$ . Then

$$\begin{aligned} \sum_{j=1}^{k+1} j &= \sum_{j=1}^k j + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}(k+1)(k+2) \\ &= \frac{1}{2}(k+1)[(k+1)+1]. \end{aligned}$$

Thus, the claim holds for  $n = k + 1$ , so the induction step is complete.

Consequently, the claim holds for all positive integers  $n$  by the principle of induction. 

**Example:** proof by induction 2

Claim: for any positive integer  $n$ ,  $11^n - 6$  is divisible by 5.

**Proof** We prove the claim by induction.

**Base Step:**

When  $n = 1$ , we have

$$11^n - 6 = 11^1 - 6 = 5 = 5 \cdot 1,$$

so the claim holds.

**Inductive Step:**

Assume that the claim holds for  $n = k$ . That means  $11^k - 6$  is divisible by 5 and hence  $11^k - 6 = 5m$  for some integer  $m$ . So  $11^k = 5m + 6$ . Then

$$\begin{aligned} 11^{k+1} - 6 &= 11 \cdot 11^k - 6 \\ &= 11 \cdot (5m + 6) - 6 \\ &= 5 \cdot (11m + 12). \end{aligned}$$

As  $(11m + 12)$  is an integer we have that  $11^{k+1} - 6$  is divisible by 5. Thus, the claim holds for  $n = k + 1$ , so the induction step is complete.

Consequently, the claim holds for all positive integers  $n$  by the principle of induction. ◀

**Example:** strong induction

Given  $n \in \mathbb{N}$ , define  $a_n$  recursively as follows:

$$a_0 = 1; \quad a_1 = 3; \quad a_n = 2a_{n-1} - a_{n-2} \quad \text{for } n \geq 2.$$

Claim:  $a_n = 2n + 1$ , for all  $n \geq 1$ .

**Proof** We prove the claim by strong induction.

**Base Step:**

When  $n = 1, 2$ , we have

$$a_0 = 1 = 2(0) + 1;$$

$$a_1 = 3 = 2(1) + 1.$$


So the claim holds for  $n = 1, 2$ .

**Inductive Step:**

Assume that the claim holds for  $1 < 2 \leq n \leq k$ . That means  $a_n = 2n + 1$  for  $1 < 2 \leq n \leq k$ . Then

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} \\ &= 2(2k + 1) - [2(k - 1) + 1] \\ &= 2(k + 1) + 1. \end{aligned}$$

Thus, the claim holds for  $n = k + 1$ , so the induction step is complete.

Consequently, the claim holds for all positive integers  $n$  by the principle of induction. 



**Example:** proof by contraposition 1

Claim: for integers  $m$  and  $n$ , if  $m + n$  is odd, then  $m$  is odd or  $n$  is odd.

**Proof** We prove the claim by contraposition.

Suppose neither  $m$  is odd nor  $n$  is odd. Then  $m$  and  $n$  are both even. So we have  $m = 2k_1$  and  $n = 2k_2$  for some integers  $k_1$  and  $k_2$ . Now  $m + n = 2k_1 + 2k_2 = 2(k_1 + k_2)$ . Since  $(k_1 + k_2)$  is an integer, we see that  $m + n$  must be even.

We have therefore proven that for all integers  $m$  and  $n$ , if  $m + n$  is odd, then  $m$  is odd or  $n$  is odd, by contraposition. ◀

**Example:** proof by contraposition 2

Claim: for three integers  $l$ ,  $m$ , and  $n$ , if their sum is not less than 15, then at least one of these three integers must be greater than or equal to 5.

**Proof** We prove the claim by contraposition.

Suppose none of these three integers,  $l$ ,  $m$ , and  $n$ , is greater than or equal to 5. Then every one of them are less than 5, So we have  $l + m + n < 15$ .

Therefore, at least one of these three integers must be greater than or equal to 5, by contraposition. ◀

**Example:** proof by contraposition 3

Claim: for any integers  $m$ , if  $m^2 + 4m + 1$  is even, then  $m$  is odd.

**Proof** We prove the claim by contraposition.

Suppose  $m$  is even. Then we have  $m = 2k$  for some integer  $k$ . Now

$$m^2 + 4m + 1 = (2k)^2 + 4m + 1 = 2(2k^2 + m) + 1.$$

Since  $(2k^2 + m)$  is an integer, we see that  $m^2 + 4m + 1$  must be odd.

We have therefore proven that for any integers  $m$ , if  $m^2 + 4m + 1$  is even, then  $m$  is odd, by contraposition. ◀

**Example:** proof by contradiction 1

Claim: for any integers  $m$  and  $n$ ,  $m^2 - 4n \neq 2$ .

**Proof** We prove the claim by contradiction.

Assume there exist integers  $m$  and  $n$  such that  $m^2 - 4n = 2$ . Then, we have  $m^2 = 4n + 2 = 2(2n + 1)$ . This means that  $m^2$  is an even number so that  $m$  must be even. Thus there is an integer  $k$  such that  $m = 2k$ . Hence, we have  $(2k)^2 - 4n = 2$ . Dividing by 2, we get  $2k^2 - 2n = 1$ , or  $2(k^2 - n) = 1$ . Since  $k^2 - 2n$  is an integer, the equality implies that 1 is an even number. So we have a contradiction.

Hence, our assumption  $m^2 - 4n = 2$  is false.

Therefore, for any integers  $m$  and  $n$ ,  $m^2 - 4n \neq 2$ . 

**Example:** proof by contradiction 2

Claim: there are infinitely many prime numbers.

**Proof** We prove the claim by contradiction.

Assume there are only finitely many prime numbers. We list them as  $p_1, \dots, p_n$ . Let  $P = p_1 \cdot p_2 \cdots p_n$  be the product of all the listed primes and  $p$  is a prime factor of  $P + 1$ . Since  $p$  must be among the listed primes, we see that  $p$  is also a factor of  $P$ . Then,  $p$  divides both  $P$  and  $P + 1$ , therefore also their difference, which is 1. So we have a contradiction.

Therefore, there are infinitely many prime numbers. ◀

**Example:** proof by exhaustion 1

Claim: for any integer  $n$ , the number  $n^2$  must be of the form  $4k$  or  $4k + 1$  for some  $k \in \mathbb{Z}$ .

**Proof** In order to prove the claim, we consider the following four cases.

Case (i):  $n = 4m$ . It follows that

$$n^2 = (4m)^2 = 16m^2 = 4(4m^2),$$

which is of the form  $4k$ , where  $k = 4m^2 \in \mathbb{Z}$ .

Case (ii):  $n = 4m + 1$ . It follows that

$$n^2 = (4m + 1)^2 = 16m^2 + 8m + 1 = 4(4m^2 + 2m) + 1,$$

which is of the form  $4k + 1$ , where  $k = 4m^2 + 2m \in \mathbb{Z}$ .

Case (iii):  $n = 4m + 2$ . It follows that


$$n^2 = (4m + 2)^2 = 16m^2 + 16m + 4 = 4(4m^2 + 4m + 1),$$

which is of the form  $4k$ , where  $k = 4m^2 + 4m + 1 \in \mathbb{Z}$ .

Case (iv):  $n = 4m + 3$ . It follows that

$$n^2 = (4m + 3)^2 = 16m^2 + 24m + 9 = 4(4m^2 + 6m + 2) + 1,$$

which is of the form  $4k + 1$ , where  $k = 4m^2 + 6m + 2 \in \mathbb{Z}$ .

Since these four cases exhaust the possibilities and since the desired result holds in each case, our proof is complete. 

**Example:** proof by exhaustion 2

Claim: for any integer  $n$ , the number  $2n^2 + n + 1$  is not divisible by 3.

**Proof** In order to prove the claim, we consider the following three cases.

Case (i):  $n = 3m$ . It follows that

$$2n^2 + n + 1 = 2(3m)^2 + 3m + 1 = 3(6m^2 + m) + 1.$$

Since  $(6m^2 + m)$  is an integer, we know that when  $2n^2 + n + 1$  is divided by 3, the remainder is 1. Hence,  $2n^2 + n + 1$  is not divisible by 3.

Case (ii):  $n = 3m + 1$ . It follows that


$$2n^2 + n + 1 = 2(3m + 1)^2 + (3m + 1) + 1 = 3(6m^2 + 5m + 1) + 1.$$

Since  $(6m^2 + 5m + 1)$  is an integer, we know that when  $2n^2 + n + 1$  is divided by 3, the remainder is 1. Hence,  $2n^2 + n + 1$  is not divisible by 3.

Case (iii):  $n = 3m + 2$ . It follows that

$$2n^2 + n + 1 = 2(3m + 2)^2 + (3m + 2) + 1 = 3(6m^2 + 9m + 3) + 2.$$

Since  $(6m^2 + 9m + 3)$  is an integer, we know that when  $2n^2 + n + 1$  is divided by 3, the remainder is 2. Hence,  $2n^2 + n + 1$  is not divisible by 3.

Since these three cases exhaust the possibilities and since the desired result holds in each case, our proof is complete. 

**Example:** proof by counterexample 1

Claim: disprove that all prime numbers are odd numbers.

**Proof** A counterexample to the statement “all prime numbers are odd numbers” is the number 2, as it is a prime number but is not an odd number. ◀



**Example:** proof by counterexample 2

Claim: disprove the equality  $(m + 1)^2 = m^2 + 1$ , where  $m$  is an integer.

**Proof** A counterexample to the equality  $(m + 1)^2 = m^2 + 1$  is  $m = 1$ , as 1 is an integer but  $(1 + 1)^2 \neq 1^2 + 1$ . ◀

**Example:** equation  $p^2 = 2$  has no rational solution

Show that the equation

$$p^2 = 2$$

has no rational solution. In other words, there exists no rational number  $p$  such that  $p^2 = 2$ .

**Proof** We prove by contradiction. Suppose there exists a rational number  $p$  such that  $p^2 = 2$ . Put  $p = m/n$ , with  $m, n$  not both being even. This gives  $m^2 = 2n^2$ , which implies that  $m$  is even. Put  $m = 2k$ . Then we have  $n^2 = 2k^2$ . This implies that  $n$  is also even. This contradicts to the hypothesis that  $m$  and  $n$  are not both even. Thus, the equation  $p^2 = 2$  has no rational solution. ◀

**Justification:** field properties**Properties on addition:**

**Item 1** Assume  $x + y = x + z$ . By the axioms for addition (A),

$$\begin{aligned} y &= 0 + y = (-x + x) + y = -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$

**Item 2** Assume  $x + y = x$ . Then  $x + y = x + 0$ . By Item 1, we get  $y = 0$ .

**Item 3** Assume  $x + y = 0$ . Then  $x + y = x + (-x)$ . By Item 1, we get  $y = -x$ .

**Item 4** Since  $(-x) + x = 0$ , by Item 3, we get  $x = -(-x)$ .

**Properties on multiplication:**

The proof is similar to that for Properties on addition. We omit the details.

**Properties on the zero and the negative elements:**

**Item 1** Assume  $0x = 0$ . By the distributive law and the commutativity, we get

$$0x + 0x = (0 + 0)x = 0x.$$

Thus, by Item 1 of Properties on addition, we have  $0x = 0$ .

**Item 2** Assume  $x \neq 0$ ,  $y \neq 0$ , but  $xy = 0$ . Thus, by Item 1, we get

$$1 = y^{-1}x^{-1}xy = y^{-1}x^{-1}0 = 0,$$

a contradiction. Hence,  $xy \neq 0$ .

**Item 3** Since

$$xy + (-x)y = [x + (-x)]y = 0y = 0,$$

we get  $(-x)y = -(xy)$ , which is the first equality in Item 3. Since

$$xy + x(-y) = x[y + (-y)] = x0 = 0,$$

we get  $x(-y) = -(xy)$ , which is the second equality in Item 3.

**Item 4** Finally, by Item 3 and Item 4 of Properties on addition, we get

$$(-x)(-y) = -[x(-y)] = -[-(xy)] = xy.$$



**Justification: properties on ordered field**

**Item 1** If  $x > 0$ , then  $0 = -x + x > -x + 0$ , so that  $-x < 0$ . Similarly, if  $x < 0$ , then  $0 = -x + x < -x + 0$ , so that  $-x > 0$ . This proves Item 1.

**Item 2** Since  $y < z$ , we have  $z - y > y - y = 0$ . This,  $x(z - y) > 0$ . Hence

$$xz = x(z - y) + xy > 0 + xy = xy.$$

**Item 3** If  $x < 0$ , by Item 1, we know that  $-x > 0$ . By Item 2, we get  $(-x)y < (-x)z$ . Thus,

$$-[x(z - y)] = (-x)(z - y) > 0,$$

so that  $x(z - y) < 0$ . Hence

$$xz = x(z - y) + xy < 0 + xy = xy.$$

**Item 4** If  $x > 0$ , then, by the definition of ordered field, we have  $x^2 > 0$ . If  $x < 0$ , we have  $-x > 0$ , so that  $(-x)^2 > 0$ . By one of the field properties, we have  $x^2 = (-x)^2$ . Thus, for any  $x \neq 0$ , we always have  $x^2 > 0$ .


In particular, since  $1 \neq 0$  and  $1 = 1^2$ , we have  $1 = 1^2 > 0$ .

**Item 5** From Item 2, it is easy to see that if  $y > 0$  and  $w \leq 0$ , then  $yw \leq 0$ . Since  $y \cdot y^{-1} = 1 > 0$ , we know that  $y^{-1} > 0$ .

Similarly, since  $x > 0$ , we know that  $x^{-1} > 0$ .

Furthermore, by multiplying both sides of the inequality  $x < y$  by the positive quantity  $x^{-1}y^{-1}$ , by Item 2, we have

$$y^{-1} < x^{-1}.$$

Hence, we get  $0 < y^{-1} < x^{-1}$ . So, Item 5 holds. 

**Justification:**  $\mathbb{Q}$  does not have the least-upper-bound property

With the usual addition, multiplication, and the order, it is easy to verify the hypotheses in the definition of *ordered field*. We only show that  $\mathbb{Q}$  does not have the least-upper-bound property.

Consider the sets

$$A = \{p : p \text{ is a positive rational number such that } p^2 < 2\},$$

$$B = \{p : p \text{ is a positive rational number such that } p^2 > 2\}.$$

We will show that

1. the set  $A$  contains no largest member and the set  $B$  contains no smallest member;
2. every member in  $B$  is an upper bound of  $A$ , and every member in  $A$  is a lower bound of  $B$ .

If fact, we associate with each positive rational number  $p$  the rational number


$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} > 0.$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}.$$

**Item 1** If  $p \in A$  then  $p^2 < 2$ , and  $q^2 < 2$ . Thus,  $q \in A$ . Since  $q > p$ , we show that  $A$  contains no largest member. Similarly, if  $p \in B$  then  $p^2 > 2$ , and  $q^2 > 2$ . Thus,  $q \in B$ . Since  $q < p$ , we show that  $B$  contains no smallest member.

**Item 2** To see that every member in  $B$  is an upper bound of  $A$ , we take  $p_\alpha \in A$  and  $p_\beta \in B$ . Then  $p_\alpha^2 < 2 < p_\beta^2$ . It gives  $p_\alpha < p_\beta$ , which shows that every member in  $B$  is an upper bound of  $A$ , and every member in  $A$  is a lower bound of  $B$ .


Hence, the set  $A$  is bounded above. Since  $A$  contains no largest member, any member in  $A$  cannot be the least upper bound of  $A$ . Furthermore, since  $B$  contains no smallest member, any member of  $B$  cannot be the least upper bound of  $A$  either. Therefore,  $A$  has no least upper bound in  $\mathbb{Q}$ . Consequently,  $\mathbb{Q}$  does not have the least-upper-bound property. 

**Justification:**  $\mathbb{R}$  has the least-upper-bound property

The statement that  $\mathbb{R}$  contains  $\mathbb{Q}$  as a subfield means that

1.  $\mathbb{Q} \subset \mathbb{R}$ ;
2. the usual addition, multiplication, and the order in  $\mathbb{R}$ , when applied to the members of  $\mathbb{Q}$ , coincide with the usual addition, multiplication, and the order in  $\mathbb{Q}$ .

It is easy to check these two items.

The remaining is to prove that  $\mathbb{R}$  having the least-upper-bound property. Since the proof is rather long and tedious. We omit it here. 

**Example:** supremum and infimum

Find the supremum and the infimum of the set

$$A = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \right\}.$$

**Solution** For any  $n \in \mathbb{N}_+$ ,


$$\frac{1}{2} \leq \frac{n}{n+1} < 1,$$

so that  $\frac{1}{2}$  is a lower bound of  $A$ , and 1 an upper bound.

Since, for any given  $\varepsilon > 0$ ,  $\frac{1}{2} < \frac{1}{2} + \varepsilon$ , we know that  $\frac{1}{2} + \varepsilon$  cannot be a lower bound of  $A$ . Thus,  $\frac{1}{2}$  is the largest lower bound of  $A$ , that is,  $\inf A = \frac{1}{2}$ .

To prove that  $\sup A = 1$ , we only need to show that for any given  $\varepsilon > 0$ ,  $1 - \varepsilon$  cannot be an upper bound of  $A$ . In fact, if we take any positive integer  $n$  such that  $n > \varepsilon^{-1}$ , then

$$\frac{n}{n+1} = 1 - \frac{1}{1+n} > 1 - \frac{1}{n} = 1 - \varepsilon.$$

Hence, 1 is the least upper bound of  $A$ , that is,  $\sup A = 1$ . 

**Justification:** equivalence of the least-upper-bound property and the greatest-lower-bound property

Suppose that  $S$  be an ordered set that possesses the least-upper-bound property. Let  $B$  be a nonempty subset of  $S$  that is bounded below. Denote

$$L(B) = \{x \in S : x \leq b \text{ for all } b \in B\},$$

which is the set of lower bounds for set  $B$ . The set  $L(B)$  is nonempty by assumption.


Then for any fixed  $b_0 \in B$  and any  $l \in L(B)$ , we have  $l \leq b_0$ . This means that  $L(B)$  is bounded above. Since  $S$  possesses the lower-upper-bound property, we know  $l_0 = \sup L(B)$  exists.

**Claim:**  $l_0 = \inf B$ .

For this, we will show that

1.  $l_0$  is a lower bound for  $B$ ;
2.  $l_0$  is greater than any lower bound of  $B$ .

**Item 1** For any  $b \in B$ , as above,  $b$  is an upper bound of  $L(B)$ . Because  $l_0$  is the least upper bound of  $L(B)$ , we have  $l_0 \leq b$ . Thus,  $l_0$  is a lower bound for  $B$ .

**Item 2** Suppose  $l$  is any lower bound of  $B$ . By definition,  $l \in L(B)$ . As  $l_0$  is the least upper bound of  $L(B)$ , we have  $l \leq l_0$ . So,  $l_0$  is greater than any lower bound of  $B$ . 



**Justification:** the archimedean property of  $\mathbb{R}$  and the rational density theorem

**Item 1** We prove the statement by contradiction.

Suppose the statement is false. Denote  $A = \{nx : n \in \mathbb{N}_+\}$ . Then  $y$  would be an upper bound of  $A$ . Thus,  $A$  has a least upper bound, say  $\alpha = \sup A$ . Since  $x > 0$ , we know that  $\alpha - x < \alpha$ . Because  $\alpha - x$  is not an upper bound of  $A$ ,  $\alpha - x < mx$  for some positive integer  $m$ . It follows that  $\alpha < (m+1)x \in A$ , which contradicts that  $\alpha$  is an upper bound of  $A$ .

**Item 2** We will show that there are two integers  $m$  and  $n$  ( $n \neq 0$ ) such that  $x < m/n < y$ .

In fact, since  $x < y$ , we have  $y - x > 0$ . By Item 1, there is a positive integer  $n$  such that  $n(y - x) > 1$ . Again, by Item 1, there are two positive integers  $m_1$  and  $m_2$ , such that

$$m_1 > nx, \quad m_2 > -nx.$$

Thus,  $-m_2 < nx < m_1$ . By progressively decreasing  $m_1$  and increasing  $-m_2$ , we see that there is an integer  $m$ , with  $-m_2 \leq m \leq m_1$ , such that  $m - 1 \leq nx < m$ . Hence, we have two integers  $m$  and  $n$ , such that

$$nx < m \leq 1 + nx < ny.$$

Since  $n > 0$ , it follows that

$$x < \frac{m}{n} < y.$$



**Justification:** existence of  $n$ th root of positive real numbers**Uniqueness:**

This is clear, since, if  $0 < y_1 < y_2$ , then  $y_1^n < y_2^n$ .

**Existence:**

For any given positive real number  $x$  and positive integer  $n$ , let

$$E = \{t : t > 0 \text{ and } t^n < x\}.$$

We prove the existence of the  $n$ th root of  $x$  by showing the following

1.  $E$  is nonempty.
2.  $E$  has an upper bound.
3.  $y = \sup E$  satisfies  $y^n = x$ .

**Item 1:** Take  $t = \frac{x}{1+x}$ . It is clear that  $0 < t < 1$ . Thus,  $t^n < t < x$ . This shows that  $t \in E$ , so that  $E$  is nonempty.

**Item 2:** Take any  $w > 1 + x$ . Then we have  $w^n > w > x$  so that  $w \notin E$ . This indicates that if  $t \in E$ , then  $t < 1 + x < w$ . Thus,  $w$  is an upper bound of  $E$ .

**Item 3:** We show that, for  $y = \sup E$ , each of the inequalities  $y^n < x$  and  $y^n > x$  leads to a contradiction, so that  $y^n = x$ .

Since  $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ , we have

$$b^n - a^n < (b - a)nb^{n-1}$$

when  $0 < a < b$ .

Assume  $y^n < x$ . Choose  $h$  so that  $0 < h < 1$  and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}}.$$

Then, for  $a = y$  and  $b = y + h$ , we have

$$(y + h)^n - y^n < hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n.$$

Thus  $(y + h)^n < x$ , and  $y + h \in E$ . Since  $y + h > y$ , this contradicts to the fact that  $y$  is an upper bound of  $E$ .

Assume  $y^n > x$ . Write

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then  $0 < k < y$ . If  $t \geq y - k$ , then

$$y^n - t^n \leq y^n - (y - k)^n < kny^{n-1} = y^n - x,$$

which implies  $t^n > x$ , and  $t \notin E$ . In other words, if  $t \in E$ , then  $t < y - k$ . Hence  $y - k$  is an upper bound of  $E$ . This contradicts to the hypothesis that  $y$  is the least upper bound of  $E$ .

Hence  $y^n = x$ . ◀

**Justification:** law of radicals

Denote  $\alpha = a^{1/n}$ ,  $\beta = b^{1/n}$ . Then  $ab = \alpha^n \beta^n = (\alpha\beta)^n$ .

The latter equality holds since multiplication is commutative. Hence, by the uniqueness of  $n$ th root of positive real numbers, we get  $\alpha\beta = (ab)^{1/n}$ , so that

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$



**Justification:** properties of the absolute value

**Item 1** When  $x \neq 0$ , we know that  $x^2 > 0$ . Thus, by Proposition 1.11, we have  $|x| = (x^2)^{1/2} > 0$  for nonzero  $x$ .

When  $x = 0$ , the equation  $y^2 = 0$  has unique solution  $y = 0$ , so that  $|x| = 0^{1/2} = 0$ .

**Item 2** We have

$$|xy| = |(xy)^2|^{1/2} = |x^2y^2|^{1/2} = |x^2|^{1/2} \cdot |y^2|^{1/2} = |x| \cdot |y|.$$

**Item 3** We have

$$\begin{aligned} |x + y|^2 &= (x + y)(x + y) \\ &= x^2 + xy + yx + y^2 \\ &= |x|^2 + 2xy + |y|^2 \\ &\leq |x|^2 + 2|x| \cdot |y| + |y|^2 \\ &= (|x| + |y|)^2, \end{aligned}$$

so that  $|x + y| \leq |x| + |y|$ .




**Justification:** the Cauchy-Schwarz Inequality

Denote  $A = \sum_{j=1}^n a_j^2$ ,  $B = \sum_{j=1}^n b_j^2$ ,  $C = \sum_{j=1}^n a_j b_j$ .

If  $B = 0$ , then  $b_1 = \dots = b_n = 0$ . In this case, the conclusion holds trivially.

If  $B \neq 0$ , then  $B > 0$ . By the properties of real numbers, we have

$$\begin{aligned} 0 \leq \sum_{j=1}^n (Ba_j - Cb_j)^2 &= B^2 \sum_{j=1}^n a_j^2 - BC \sum_{j=1}^n a_j b_j - BC \sum_{j=1}^n a_j b_j + C^2 \sum_{j=1}^n b_j^2 \\ &= B^2 A - BC^2 - BC^2 + BC^2 \\ &= B(AB - C^2). \end{aligned}$$

Thus, we have  $AB - C^2 \geq 0$ , which gives the desired inequality. 

**Example:** infinite set is equivalent to one of its proper subsets

Show that

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\} \sim \mathbb{N}_+ = \{1, 2, 3, \dots\}.$$

That is,  $\mathbb{Z}$  is equivalent to one of its proper subsets.

**Proof** Define the mapping  $f: \mathbb{N}_+ \rightarrow \mathbb{Z}$  as following:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

It suffices to show that  $f$  is bijective, i.e., both surjective and injective.

**Surjective:**


Suppose  $k \in \mathbb{Z}$ . We need to show that there exists a number  $l \in \mathbb{N}_+$  such that  $f(l) = k$ .

In fact, if  $k = 0 \in \mathbb{Z}$ , then  $l = 1 \in \mathbb{N}_+$ , and  $f(l) = f(1) = -\frac{1-1}{2} = 0 = k$ . If  $k \in \mathbb{Z}$  is a positive integer, then  $l = 2k \in \mathbb{N}_+$ , and  $f(l) = f(2k) = \frac{2k}{2} = k$ . If  $k \in \mathbb{Z}$  is a negative integer, then then  $l = -2k + 1 \in \mathbb{N}_+$ , and  $f(l) = f(-2k + 1) = -\frac{(-2k + 1) - 1}{2} = k$ . Hence,  $f$  is surjective.

**Injective:**

Suppose  $f(n_1) = f(n_2)$ , with  $n_1, n_2 \in \mathbb{N}_+$ . We need to show that  $n_1 = n_2$ .

If  $n_1$  is even, then  $n_2$  must be also even, otherwise  $f(n_1) = \frac{n_1}{2} > 0 > -\frac{n_2-1}{2} = f(n_2)$  which is impossible. It follows that  $\frac{n_1}{2} = f(n_1) = f(n_2) = \frac{n_2}{2}$ , so that  $n_1 = n_2$ .

If  $n_1$  is odd, then  $n_2$  must be also odd, otherwise  $f(n_1) = -\frac{n_1-1}{2} < 0 < \frac{n_2}{2} = f(n_2)$  which is impossible. It follows that  $-\frac{n_1-1}{2} = f(n_1) = f(n_2) = -\frac{n_2-1}{2}$ , so that  $n_1 = n_2$ . Hence,  $f$  is injective. 

**Justification:** infinite subset of a countable set is countable

Let  $A$  be a countable set, and  $E \subset A$  an infinite subset.

First arrange the elements  $x$  of  $A$  in a sequence  $\{x_n\}$  of distinct elements.

Next, construct a subsequence  $\{x_{n_k}\}$  as follows:

1. Let  $n_1$  be the smallest positive integer such that  $x_{n_1} \in E$ . Having chosen  $n_1, \dots, n_{k-1}$ , let  $n_k$  be the smallest integer greater than  $n_{k-1}$  such that  $x_{n_k} \in E$ .
2. Put  $f(k) = x_{n_k}$ ,  $k = 1, 2, 3, \dots$

It is obvious that  $f$  is a bijective mapping from  $\mathbb{N}_+$  to  $E$ . Consequently,  $E$  is countable. ◀

**Justification:** the distributive law for union and intersection of sets

The distributive law:

Denote  $S = A \cap (B \cup C)$ ,  $T = (A \cap B) \cup (A \cap C)$ . To show that  $S = T$ , it suffices to show that both  $S \subset T$  and  $T \subset S$  hold.

Suppose  $x \in S$ . Then  $x \in A$  and  $x \in B \cup C$ . The latter implies  $x \in B$  or  $x \in C$  (possibly both). Hence,  $x \in A \cap B$  or  $x \in A \cap C$ , so that  $x \in (A \cap B) \cup (A \cap C)$ . That is,  $x \in T$ . Thus,  $S \subset T$ .

Suppose  $x \in T$ . Then  $x \in A \cap B$  or  $x \in A \cap C$ . That is,  $x \in A$ , and  $x \in B \cup C$ . Hence  $x \in A \cap (B \cup C) = S$ , so that  $T \subset S$ . 



**Justification:** countable union of countable sets is countable


Denote  $S = \bigcup_{n=1}^{\infty} E_n$ .

Let  $E_n$  be arranged in a sequence  $\{x_{n,k}\}$ ,  $k = 1, 2, 3, \dots$ , and consider the infinite array

$$\begin{array}{rcll}
 E_1 & \rightarrow & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & \dots \\
 E_2 & \rightarrow & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & \dots \\
 E_3 & \rightarrow & x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & \dots \\
 E_4 & \rightarrow & x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & \dots \\
 & & \vdots & & & & \\
 E_n & \rightarrow & x_{n,1} & x_{n,2} & x_{n,3} & x_{n,4} & \dots \\
 & & \vdots & & & & 
 \end{array}$$

These elements can be arranged in one single sequence:

$$x_{1,1}; x_{2,1}, x_{1,2}; x_{3,1}, x_{2,2}, x_{1,3}; x_{4,1}, x_{3,2}, x_{2,3}, x_{1,4}; \dots$$

By this arrangement as a sequence, we know that if any two sets  $E_n$  have no elements in common, then  $S \sim \mathbb{N}_+$ . Otherwise, there is a subset  $T \subset \mathbb{N}_+$  such that  $S \sim T$ . Thus  $S$  is at most countable. Since  $E_1 \subset S$ , and  $E_1$  is infinite, so  $S$  is infinite. Hence  $S$  is countable. 

**Justification:**  $n$ -tuples of a countable set

We prove the proposition by mathematical induction.

Since  $B_1 = A$ ,  $B_1$  is countable.

Suppose  $B_{n-1}$  is countable ( $n = 2, 3, 4, \dots$ ). To show that  $B_n$  is countable, we notice that the elements of  $B_n$  are of the form  $(b, a)$  with  $b \in B_{n-1}$  and  $a \in A$ . For every fixed  $b$ , the set of pairs  $(b, a)$  is equivalent to  $A$ , and hence countable. Thus  $B_n$  is the union of a countable set of countable sets, which is countable. ◀

**Justification:** the set of all rational numbers is countable

Consider the set  $S$  comprising all ordered pairs  $(m, n)$ , with  $m, n \in \mathbb{N}$ . We know that  $S$  is countable. Since every rational number is of the form  $m/n$ , the set of all rational numbers is equivalent to an infinite subset of  $S$ . Hence, the set of all rational numbers is countable. ◀

**Justification:** the set of 0-1 sequences

We prove the proposition by a contradiction. Suppose  $A$  is countable. We can arrange the elements of  $A$  as a sequence  $s_1, s_2, s_3, \dots$ . Here each  $s_k$  is a 0-1 sequence, like  $1, 0, 1, 1, 1, 0, \dots$ .

We construct a 0-1 sequence  $s$  as follows. If the first digit of  $s_1$  is 1, we let the first digit of  $s$  be 0, and vice versa. In general, if the  $n$ th digit in  $s_n$  is 1, we let the  $n$  digit of  $s$  be 0, and vice versa. Then the sequence  $s$  differs from every member of  $A$ , so that  $s \notin A$ . This contradicts to the definition of  $A$ , the set of all 0-1 sequences.

Consequently, the elements of  $A$  cannot be arranged as a sequence. Therefore,  $A$  is uncountable. ◀

**Example:** a limit point of a set is not necessarily a point in the set

Consider  $E = (0, 1) \subset \mathbb{R}$ . It is obvious that 1 is a limit point of  $E$ , but  $1 \notin E$ . The point 1 is also a boundary point of  $E$ .

**Example - various subsets in  $\mathbb{R}$** 

Consider the following subsets of  $\mathbb{R}$ :

- (a) The set of all real  $x$  such that  $|x| < 1$ .
- (b) The set of all real  $x$  such that  $|x| \leq 1$ .
- (c) A finite set.
- (d) The set of all integers.
- (e) The set consists of the numbers  $1/n$  ( $n = 1, 2, 3, \dots$ ).
- (f) The set of all real numbers,  $\mathbb{R}$ .
- (g) The interval  $(a, b)$ .

Determine whether they are closed, open, perfect, and/or bounded subsets in  $\mathbb{R}$ .

**Solution**

	closed	open	perfect	bounded
(a)	no	yes	no	yes
(b)	yes	no	yes	yes
(c)	yes	no	no	yes
(d)	yes	no	no	no
(e)	no	no	no	yes
(f)	yes	yes	yes	no
(g)	no	yes	no	yes




**Justification:** neighborhoods are open sets

Suppose  $N_r(p)$  is a neighborhood. Let  $q$  be any point in  $N_r(p)$ . Then there is a positive number  $h$  such that

$$|p - q| = r - h.$$

To show that  $q$  is an interior point of  $N_r(p)$ , we just need to show that the neighborhood  $N_h(q)$  of  $q$  is in  $N_r(p)$ . In fact, for  $z \in N_h(q)$ , since

$$|p - z| \leq |p - q| + |q - z| < r - h + h = r,$$

we see that  $z \in N_r(p)$ . Thus  $N_h(q) \subset N_r(p)$ . 

**Justification:** neighborhood of a limit point contains infinitely many points

We prove the result by contradiction.

Suppose there is a neighborhood  $N$  of  $p$  which contains only a finite number of points of  $E$ . Denote them as  $q_1, \dots, q_n \in N \cap E$ , which are distinct from  $p$ , and put

$$r = \min_{1 \leq m \leq n} |q_m - p|.$$

It is obvious that  $r > 0$ .

Now the neighborhood  $N_r(p)$  contains no point in  $E$  except  $p$ . This contradicts to the hypothesis that  $p$  is a limit of  $E$ . ◀



**Justification:** the complement of a union equals the intersection of complements

The first equality:

If  $x \in \left(\bigcup_{\alpha} E_{\alpha}\right)^c$ , then  $x \notin \bigcup_{\alpha} E_{\alpha}$ . Thus,  $x \notin E_{\alpha}$  for any  $\alpha$  so that  $x \in E_{\alpha}^c$  for every  $\alpha$ . Hence,  $x \in \bigcap_{\alpha} E_{\alpha}^c$ . So, we know that  $\left(\bigcup_{\alpha} E_{\alpha}\right)^c \subset \bigcap_{\alpha} E_{\alpha}^c$ .

Conversely, if  $x \in \bigcap_{\alpha} E_{\alpha}^c$ , then  $x \in E_{\alpha}^c$  for every  $\alpha$ . Thus  $x \notin E_{\alpha}$  for any  $\alpha$  so that  $x \notin \bigcup_{\alpha} E_{\alpha}$ . Hence  $x \in \left(\bigcup_{\alpha} E_{\alpha}\right)^c$ . So, we know that  $\bigcap_{\alpha} E_{\alpha}^c \subset \left(\bigcup_{\alpha} E_{\alpha}\right)^c$ .


Therefore  $\left(\bigcup_{\alpha} E_{\alpha}\right)^c = \bigcap_{\alpha} E_{\alpha}^c$ .

The second equality:

Denote  $F_{\alpha} = E_{\alpha}^c$ . Then  $F_{\alpha}^c = E_{\alpha}$ . Thus, by the first equality, we have  $\left(\bigcup_{\alpha} F_{\alpha}^c\right)^c = \bigcap_{\alpha} F_{\alpha}$ , so that  $\bigcup_{\alpha} F_{\alpha}^c = \left(\bigcap_{\alpha} F_{\alpha}\right)^c$ . Hence, if writing  $E_{\alpha}$  in place of  $F_{\alpha}$ , we get the second equality  $\left(\bigcap_{\alpha} E_{\alpha}\right)^c = \bigcup_{\alpha} E_{\alpha}^c$ . ◀

**Justification:** a set is open if and only if its complement is closed

First, suppose that  $E^c$  is closed. We need to show that every point in  $E$  is an interior point of  $E$ . In fact, let  $x \in E$ . Since  $x \notin E^c$ , we know that  $x$  is not a limit point of  $E^c$ . Hence there is a neighborhood  $N$  of  $x$  such that  $N \cap E^c = \emptyset$ , or  $N \subset E$ . Thus,  $x$  is an interior point of  $E$ . So  $E$  is open.

Next, suppose  $E$  is open. To show that  $E^c$  is closed, we need to show that every limit point of  $E^c$  must be still in  $E^c$ . In fact, if  $x$  is a limit point of  $E^c$ . Then every neighborhood of  $x$  contains a point  $p \neq x$ ,  $p \in E^c$ . Hence  $x$  cannot be an interior point of  $E$ . Since  $E$  is open, this means that  $x \notin E$ , or  $x \in E^c$ . Thus  $E^c$  is closed. 

**Justification:** the union of open sets is an open set

**Item 1** Suppose  $x \in \bigcup_{\alpha} G_{\alpha}$ . Then  $x \in G_{\alpha}$  for some  $\alpha$ . Since  $G_{\alpha}$  is open, there is a neighborhood  $N$  of  $x$  such that  $N \subset G_{\alpha}$ . It follows that  $N \subset \bigcup_{\alpha} G_{\alpha}$ . Thus  $x$  is an interior point of  $\bigcup_{\alpha} G_{\alpha}$ . Therefore  $\bigcup_{\alpha} G_{\alpha}$  is open.

**Item 2** By Proposition 2.12, the complement of a closed set is open, so that  $F_{\alpha}^c$  is open for any  $\alpha$ . By Item 1, we know that  $\bigcup_{\alpha} F_{\alpha}^c$  is open. By Proposition 2.11, we have

$$\left(\bigcap_{\alpha} F_{\alpha}\right)^c = \bigcup_{\alpha} F_{\alpha}^c.$$

Hence  $\left(\bigcap_{\alpha} F_{\alpha}\right)^c$  is open. Again by Proposition 2.12, the complement of an open set is closed, so that  $\bigcap_{\alpha} F_{\alpha}$  is closed.

**Item 3** Suppose  $x \in \bigcap_{j=1}^n G_j$ . We know that  $x \in G_j$  for every  $j$ ,  $1 \leq j \leq n$ . For each  $j$ , since  $G_j$  is open, there is a neighborhood  $N_j$  of  $x$ , with radius  $r_j$ , such that  $N_j \subset G_j$ . Put

$$r = \min\{r_1, \dots, r_n\}.$$

Then  $r > 0$ . For the neighborhood  $N_r(x)$ , we have  $N_r(x) \subset N_j$  for  $1 \leq j \leq n$ . Thus  $N_r(x) \subset \bigcap_{j=1}^n G_j$ .

This means that  $x$  is an interior point of  $\bigcap_{j=1}^n G_j$ . Hence,  $\bigcap_{j=1}^n G_j$  is open.

**Item 4** The proof is similar to that of Item 2. It follows from Item 3, the equality  $\left(\bigcap_{j=1}^n F_j\right)^c = \bigcup_{j=1}^n F_j^c$  (Proposition 2.11), and the fact that the complement of a closed (open) set is open (closed) (Proposition 2.12).



**Example - a limit point of a set is not necessarily a point in the set**

- (1) Consider a collection of open sets  $G_j = \left(-\frac{1}{j}, \frac{1}{j}\right)$ ,  $j = 1, 2, 3, \dots$ , in  $\mathbb{R}$ . It is easy to see that  $\bigcap_{j=1}^{\infty} G_j = \{0\}$ , that is not open in  $\mathbb{R}^1$ .
- (2) Consider a collection of closed sets  $F_j = \left[\frac{1}{j+1}, 1 - \frac{1}{j+2}\right]$ ,  $j = 1, 2, 3, \dots$ , in  $\mathbb{R}$ . It is easy to see that  $\bigcup_{j=1}^{\infty} F_j = (0, 1)$ , that is not closed in  $\mathbb{R}^1$ .


**Justification:** properties of closure of a set

**Item 1** To show  $\overline{E}$  is closed, we only need to show that  $\overline{E}^c$  is open. Let  $p \in \overline{E}^c \subset \mathbb{R}$ . By Proposition 2.11,

$$\overline{E}^c = (E \cup E')^c = E^c \cap (E')^c,$$

so we have  $p \notin E$  and  $p$  is not a limit point of  $E$ . Hence, there is a neighborhood  $N$  of  $p$  such that  $N \cap (E \cup E') = \emptyset$ . This means that  $N \subset \overline{E}^c$ .

**Item 2** If  $E = \overline{E}$ , by Item 1, we know that  $E$  is closed. Conversely, if  $E$  is closed, then  $E' \subset E$ . So  $\overline{E} = E \cup E' = E$ .

**Item 3** If  $F$  is closed and  $E \subset F$ , then  $E' \subset F' \subset F$ . Thus  $\overline{E} = E \cup E' \subset F$ . 

**Justification:** the supremum of bounded-above set of real numbers is in the closure

If  $y \in E$ , then  $y \in \overline{E}$ . If  $y \notin E$ , then for every  $h > 0$ , there exists a point  $x \in E$  such that  $y - h < x < y$ , for otherwise  $y - h$  would be an upper bound of  $E$ . By the definition,  $y$  is a limit point of  $E$ , or  $y \in E' \subset \overline{E}$ . ◀

**Justification:** nonempty perfect set is uncountable

Since  $P$  has limit points,  $P$  must be infinite.


Suppose  $P$  is countable, and denote the points of  $P$  as  $x_1, x_2, x_3, \dots$ . We now construct a sequence  $\{V_n\}$  of neighborhoods, as follows, to have a contradiction eventually.

Let  $V_1$  be a neighborhood  $N_r(x_1)$  for some  $r > 0$ .

Suppose  $V_n$  has been constructed, so that  $V_n \cap P \neq \emptyset$ . Since every point of  $P$  is a limit point of  $P$ , we take  $V_{n+1}$  to be a neighborhood such that

$$(i) \quad \overline{V}_{n+1} \subset V_n, \quad (ii) \quad x_n \notin \overline{V}_{n+1}, \quad (iii) \quad V_{n+1} \cap P \neq \emptyset.$$

It is obvious that (iii) is the induction hypothesis to proceed the construction.

Put  $K_n = \overline{V}_n \cap P$ . Since  $\overline{V}_n$  is closed and bounded, and  $P$  is closed, we know that  $K_n$  is compact. Since  $K_n \subset P$ , we have  $\bigcap_{n=1}^{\infty} K_n \subset P$ . The fact  $x_n \notin K_{n+1}$  implies that  $\bigcap_{n=1}^{\infty} K_n$  contains no point of  $P$ . Hence  $\bigcap_{n=1}^{\infty} K_n = \emptyset$ . But (iii) implies that each  $K_n$  is nonempty, and (i) implies that  $K_n \supset K_{n+1}$ . These imply, by Cantor's intersection theorem, that  $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ , a contradiction. 

**Example - two sets disjoint but not separated**

Suppose  $A$  and  $B$  are separated:

$$A \cap \overline{B} = \overline{A} \cap B = \emptyset.$$

Then,

$$A \cap B \supset A \cap \overline{B} = \emptyset,$$

so that  $A \cap B = \emptyset$ , i.e.  $A$  and  $B$  are disjoint.

However, two disjoint sets are not necessarily separated. In fact, the interval  $[0, 1]$  and the open interval  $(1, 2)$  are not separated, since

$$[0, 1] \cap \overline{(1, 2)} = [0, 1] \cap [1, 2] = \{1\} \neq \emptyset,$$

but they are disjoint:  $[0, 1] \cap (1, 2) = \emptyset$ .





**Justification:** characterization of connected set

( $\Rightarrow$ ) Suppose  $E$  is connected, and suppose  $x \in E$ ,  $y \in E$ , and  $x < z < y$ , but  $z \notin E$ . We will deduce a contradiction. In fact, if we put

$$A_z = E \cap (-\infty, z), \quad B_z = E \cap (z, \infty),$$

then  $E = A_z \cup B_z$ . Since  $x \in A_z$  and  $y \in B_z$ ,  $A$  and  $B$  both are nonempty. Since

$$A_z \cap \overline{B_z} \subset (-\infty, z) \cap [z, \infty) = \emptyset,$$

$$\overline{A_z} \cap B_z \subset (-\infty, z] \cap (z, \infty) = \emptyset,$$

we get  $A_z \cap \overline{B_z} = \overline{A_z} \cap B_z = \emptyset$ . Thus  $E$  is not connected, a contradiction to the hypothesis.

( $\Leftarrow$ ) Suppose that for any  $x \in E$ ,  $y \in E$ , and  $x < z < y$ , we always have  $z \in E$ . In this case, if  $E$  is not connected, we will deduce a contradiction. In fact, for  $E$  being not connected, there are nonempty separated sets  $A$  and  $B$  such that  $A \cup B = E$ . Pick  $x \in A$ ,  $y \in B$ . Without loss of generality, we assume that  $x < y$ . Define

$$z = \sup(A \cap [x, y]).$$

By Proposition 2.15,  $z \in \overline{A \cap [x, y]} \subset \overline{A}$ . Since  $A$  and  $B$  are separated,  $z \notin B$ . Hence  $x \leq z < y$ .


If  $z \notin A$ , then  $z \notin A \cup B = E$ . Since in this case we have  $x < z < y$ , by the hypothesis,  $z \in E$ , a contradiction.

If  $z \in A$ , then  $z \notin \overline{B} = B \cup B'$ . By the fact that  $z \notin B'$ , there exists  $z_1$  such that  $z < z_1 < y$  and  $z_1 \notin B$ . Hence  $x < z_1 < y$  and  $z_1 \notin A \cup B = E$ , again a contradiction to the hypothesis. ◀

**Example - open interval is not compact**

For any  $n \in \mathbb{N}_+$ , the subset  $U_n = (1/n, 1) \subset (0, 1)$  is open. Since

$$(0, 1) = \bigcup_{n=1}^{\infty} U_n,$$

the collection  $\{U_n\}$  is an open cover of  $(0, 1)$ . It is obvious that there are no finite subcovers, so that  $(0, 1)$  is not compact. 

**Justification:** the relationship between compact sets and closed sets

**Item 1** Let  $K$  be a compact subset of  $\mathbb{R}$ . To show that  $K$  is closed, we only need to show that  $K^c$  is open.

Suppose  $p \in K^c \subset \mathbb{R}$ . For each  $q \in K$ , we define a neighborhood  $W_q$  of  $q$ , with the radius less than  $\frac{1}{2}|p - q|$ . Then  $\{W_q\}$  is an open cover of  $K$ . Since  $K$  is compact, there are finitely many points  $q_1, \dots, q_n$  in  $K$  such that

$$K \subset W_{q_1} \cup \dots \cup W_{q_n}.$$

For each of these  $q_i$ ,  $1 \leq i \leq n$ , we define a neighborhood  $V_{q_i}$  of  $p$ , with the radius also less than  $\frac{1}{2}|p - q_i|$ . Then  $V_{q_i} \cap W_{q_i} = \emptyset$ . Set  $V = V_{q_1} \cap \dots \cap V_{q_n}$ . Then  $V$  is a neighborhood of  $p$ . The construction of  $\{W_{q_i}\}$  and  $\{V_{q_i}\}$  imply  $p \in V$ . We also have  $V \cap K = \emptyset$ , since

$$V \cap K \subset V \cap (W_{q_1} \cup \dots \cup W_{q_n}) = (V \cap W_{q_1}) \cup \dots \cup (V \cap W_{q_n}) = \emptyset \cup \dots \cup \emptyset = \emptyset.$$

We conclude that  $p$  is an interior point of  $K^c$ . Therefore  $K^c$  is open.

**Item 2** Suppose  $F \subset K \subset \mathbb{R}$ , with  $F$  being closed, and  $K$  compact. Let  $\{V_\alpha\}$  be an open cover of  $F$ . Notice  $F^c$  is open. So, if we add  $F^c$  into the open cover to obtain a collection  $\{V_\alpha\} \cup \{F^c\}$ , it is still an open cover of  $F$ . Since  $K$  is compact, we have a finite subcollection of  $\{V_\alpha\} \cup \{F^c\}$  to cover  $K$ . This implies that


$$K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \cup F^c,$$

for finitely many indices  $\alpha_1, \dots, \alpha_n$ . Since  $F$  is a subset of  $K$ , we have

$$F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \cup F^c,$$

From  $F \cap F^c = \emptyset$ , we have

$$F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}.$$

Thus,  $F$  is compact. 

**Justification:** finite closed intervals are compact

Let  $I = [a, b]$  be a finite closed interval. Put  $\delta = |b - a|$ . Then  $|x - y| \leq \delta$  for  $x, y \in I$ .

If  $I$  is not compact, then there is an open cover  $\{G_\alpha\}$  of  $I$  which contains no finite subcover of  $I$ . Put  $c = (a + b)/2$ . Then at least one of the finite closed intervals  $[a, c]$  and  $[c, b]$ , call it  $I_1$ , cannot be covered by any finite subcollection of  $\{G_\alpha\}$ . Next we subdivide  $I_1$  and continue the process. Then we obtain a sequence  $\{I_n\}$  with the following properties:

- (a)  $I_1 \supset I_2 \supset I_3 \supset \cdots$ ;
- (b)  $I_n$  is not covered by any finite subcollection of  $\{G_\alpha\}$ ;
- (c)  $|x - y| \leq 2^{-n}\delta$  for any  $x, y \in I_n$ .

Write  $I_n = [a_n, b_n]$  and  $E = \{a_n\}$ . Since  $I_n \supset I_{n+1}$ , we know that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b_1.$$


Hence  $E$  is nonempty and bounded above. Put  $x^* = \sup E$ . The inequalities imply  $x^* \leq b_m$  for each  $m$ . It is obvious that  $a_m \leq x^*$  for each  $m$ . Hence  $x^* \in I_m$  for  $m = 1, 2, 3, \dots$ . This implies that  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

Since  $\{G_\alpha\}$  covers  $I$ , there is some  $\alpha$  such that  $x^* \in G_\alpha$ . Since  $G_\alpha$  is open, there exists  $r > 0$  such that  $|y - x^*| < r$  implies that  $y \in G_\alpha$ . Let  $n$  be a positive integer so that  $2^{-n}\delta < r$ . Then item (c) implies that  $I_n \subset G_\alpha$ , which contradicts to item (b). ◀

**Justification:** Cantor's intersection theorem

Fix a member  $K_1$  of  $\{K_\alpha\}$  and put  $G_\alpha = K_\alpha^c$ . Assume that  $\bigcap_{\alpha} K_\alpha = \emptyset$ . Then there is no point of  $K_1$  that belongs to every  $K_\alpha$ . This implies that  $\{G_\alpha\}$  forms an open cover of  $K_1$ . Since  $K_1$  is compact, there are finitely many indices  $\alpha_1, \dots, \alpha_n$  such that

$$K_1 \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

This relation means  $K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$ . This contradicts to the hypothesis. 

**Justification:** characterizations of compact sets in  $\mathbb{R}$  and the Heine-Borel theorem**Item 1  $\Rightarrow$  Item 2:**

Since  $K$  is bounded, there is a finite closed interval  $I$  such that  $K \subset I$ . By Theorem 2.19,  $I$  is compact. By item 2 of Proposition 2.18, we know that  $K = K \cap I$  is compact.

**Item 2  $\Rightarrow$  Item 3:**

We prove this by contradiction.

Assume that an infinite subset  $E$  of a compact set  $K$  has no limit point in  $K$ . Then for each  $q \in K$ , there is a neighborhood  $V_q$  of  $q$  which contains at most one point of  $E$  (this point is actually  $q$  if  $q \in E$ ). Since  $E$  is an infinite set, any finite subcollection of the collection  $\{V_q\}$  cannot cover  $E$ . The same is true for  $K$  since  $E \subset K$ . This contradicts to the facts that  $\{V_q\}$  is an open cover of  $K$  and  $K$  is compact.

**Item 3  $\Rightarrow$  Item 1:**


If  $K$  is not bounded, then for each integer  $n > 0$ , there exists a point  $x_n \in K$  such that

$$|x_n| > n.$$

The set  $S$  consisting of these points  $\{x_n\}$  is infinite and obviously has no limit point in  $\mathbb{R}$ . Hence  $S$  has no limit point in  $K$ . This contradicts to Item 3.

If  $K$  is not closed, then there is a point  $x_0 \in \mathbb{R}$  which is a limit point of  $K$  but not a point in  $K$ . Hence, for each integer  $n > 0$ , there is a point  $x_n \in K$  such that  $|x_n - x_0| < 1/n$ . By Proposition 2.10, we can further assume that any two points in  $\{x_n\}$  are distinct. Hence, if let  $S$  be the set of  $\{x_n\}$ , then  $S$  is infinite, with  $x_0$  being one of its limit points. However,  $S$  has no other limit point in  $\mathbb{R}$ , since for any  $y \in \mathbb{R}$  with  $y \neq x_0$ ,

$$\begin{aligned} |x_n - y| &\geq |x_0 - y| - |x_n - x_0| \\ &\geq |x_0 - y| - 1/n \geq \frac{1}{2}|x_0 - y| \end{aligned}$$

for all but finitely many  $n$ . This implies that  $x_0$  is the only limit of  $S$ . Thus,  $S$  has no limit point in  $K$ , a contradiction to Item 3. 

**Justification:** the Bolzano-Weierstrass theorem

Let  $E$  be a bounded subset in  $\mathbb{R}$ . Then there is a finite closed interval  $I$  such that  $E \subset I$ . By Theorem 2.19,  $I$  is compact, and so  $E$  has a limit point in  $I$ , by Theorem 2.21. ◀

**Example:** a simple limit

For any positive rational number  $p$ , the limit  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ .

**Solution** Let  $p = l/m$ , where  $l, m$  are positive integers. For any  $\varepsilon > 0$ , take an integer  $N$  such that  $N > \frac{1}{\varepsilon^{m/l}}$ , then  $n \geq N$  implies that

$$\left| \frac{1}{n^p} - 0 \right| = \frac{1}{n^{l/m}} \leq \frac{1}{N^{l/m}} < \varepsilon.$$

Hence, by the definition of limit, we conclude that  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ .





**Example:** divergent sequences

- (1) The sequence  $\{(-1)^n\}$  is divergent.

This can be proved by contradiction. If  $\lim_{n \rightarrow \infty} (-1)^n = \alpha$ . Then, for  $\varepsilon = 1$ , there is an integer  $N$  such that  $n \geq N$  implies that  $|(-1)^n - \alpha| < 1$ . It follows that, for any even integer  $n$  satisfying  $n \geq N$ ,

$$|\alpha - 1| < 1 \implies 0 < \alpha < 2.$$

Similarly, for any odd integer  $n$  satisfying  $n \geq N$ ,


$$|\alpha + 1| < 1 \implies -2 < \alpha < 0,$$

There is a contradiction because the two statements “ $0 < \alpha < 2$ ” and “ $-2 < \alpha < 0$ ” cannot both be true at the same time.

- (2) The sequence  $\{n\}$  is divergent.

This can also be proved by contradiction. If  $\lim_{n \rightarrow \infty} n = \alpha$ . Then, for  $\varepsilon = 1$ , there is an integer  $N$  such that  $n \geq N$  implies that  $|n - \alpha| < 1$ . Thus, for any even integer  $n$  satisfying  $n \geq N$ , we have

$$|n - \alpha| < 1 \implies -1 + \alpha < n < \alpha + 1.$$

There is a contradiction because the statement “ $-1 + \alpha < n < \alpha + 1$ ” implies that there are infinitely many integers located between two numbers  $-1 + \alpha$  and  $\alpha + 1$ , which violates the archimedean property. 

**Example:** apply an equivalent statement of the definition

Assume  $\lim_{n \rightarrow \infty} x_n = \alpha$ , where  $x_n \geq 0, n = 1, 2, \dots$ . Prove that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{\alpha}$ .

**Solution** Let  $\varepsilon > 0$  be given.

**Case 1:** when  $\lim_{n \rightarrow \infty} x_n = 0$ .

Use the original definition:

There exists an integer  $N$  such that  $n \geq N$  implies  $|x_n - 0| < \varepsilon^2$ . Thus, whenever  $n \geq N$ , we have

$$|\sqrt{x_n} - \sqrt{0}| = \sqrt{x_n} < \sqrt{\varepsilon^2} = \varepsilon,$$

so that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{0}$  by definition.

Use the equivalent statement:

For every  $\varepsilon^* > 0$ , there exists an integer  $N$  such that  $n \geq N$  implies  $|x_n - 0| < \varepsilon^*$ . Thus, whenever  $n \geq N$ , we have

$$|\sqrt{x_n} - \sqrt{0}| = \sqrt{x_n} < \sqrt{\varepsilon^*}.$$

Hence, if we choose  $\varepsilon^*$  to satisfy  $\varepsilon^* < \varepsilon^2$ , then, whenever  $n \geq N$ , we have

$$|\sqrt{x_n} - \sqrt{0}| < \sqrt{\varepsilon^2} = \varepsilon,$$

so that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{0}$  by definition.

**Case 2:** when  $\lim_{n \rightarrow \infty} x_n = \alpha > 0$ .

Use the original definition:

There exists an integer  $N$  such that  $n \geq N$  implies  $|x_n - \alpha| < \sqrt{\alpha} \cdot \varepsilon$ . Thus, whenever  $n \geq N$ , we have

$$\begin{aligned} |\sqrt{x_n} - \sqrt{\alpha}| &= \frac{|x_n - \alpha|}{\sqrt{x_n} + \sqrt{\alpha}} \\ &\leq \frac{|x_n - \alpha|}{\sqrt{\alpha}} < \frac{\sqrt{\alpha} \cdot \varepsilon}{\sqrt{\alpha}} = \varepsilon. \end{aligned}$$

Hence, by the definition of limit, we conclude that

$$\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{\alpha}.$$

Use the equivalent statement:

For every  $\varepsilon^* > 0$ , there exists an integer  $N$  such that  $n \geq N$  implies  $|x_n - \alpha| < \varepsilon^*$ . Thus, whenever  $n \geq N$ , we have

$$|\sqrt{x_n} - \sqrt{\alpha}| = \frac{|x_n - \alpha|}{\sqrt{x_n} + \sqrt{\alpha}} \leq \frac{|x_n - \alpha|}{\sqrt{\alpha}} < \frac{\varepsilon^*}{\sqrt{\alpha}}.$$

Hence, if we choose  $\varepsilon^*$  to satisfy  $\frac{\varepsilon^*}{\sqrt{\alpha}} < \varepsilon$ , then, whenever  $n \geq N$ , we have

$$|\sqrt{x_n} - \sqrt{\alpha}| < \varepsilon,$$

so that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{\alpha}$  by the definition of limit. ◀

**Justification:** properties of convergent sequence

**Item 1** For any given  $\varepsilon > 0$ , there exist integers  $N, N'$  such that

$$\begin{aligned} n \geq N & \text{ implies } |x_n - x| < \varepsilon/2, \\ n \geq N' & \text{ implies } |x_n - x'| < \varepsilon/2. \end{aligned}$$

Hence, if  $n \geq \max\{N, N'\}$ , then

$$|x - x'| \leq |x_n - x| + |x_n - x'| \leq \varepsilon.$$

Since  $\varepsilon$  is arbitrary, we have  $x = x'$ .


**Item 2** ( $\Rightarrow$ ) Let  $V = N_\varepsilon(x)$  be any neighborhood of  $x$  for some  $\varepsilon > 0$ . Since  $x = \lim_{n \rightarrow \infty} x_n$ , there exists  $N$  such that  $n \geq N$  implies  $|x_n - x| < \varepsilon$ . Thus  $n \geq N$  implies  $x_n \in V$ .

( $\Leftarrow$ ) Suppose every neighborhood of  $x$  contains all but finitely many of  $x_n$ . For any  $\varepsilon > 0$ , for this neighborhood  $V = N_\varepsilon(x)$ , there exists  $N$  such that  $x_n \in V$  if  $n \geq N$ . Thus  $|x_n - x| < \varepsilon$  if  $n \geq N$ . Hence  $x_n \rightarrow x$  by the definition.

**Item 3** For each positive integer  $n$ , there is a point  $x_n \in E$  such that  $|x_n - x| < 1/n$ . Now we show that the sequence  $\{x_n\}$  converges to  $x$ . In fact, for any  $\varepsilon > 0$ , we choose an integer  $N$  such that  $N \cdot \varepsilon > 1$ . Then for  $n \geq N$ ,  $|x_n - x| < 1/n \leq 1/N < \varepsilon$ . Thus, by the definition, we conclude that  $x_n \rightarrow x$ .

**Item 4** For  $\varepsilon = 1$ , there exists an integer  $N$  such that  $n \geq N$  implies  $|x_n - x| < 1$ . Put

$$r = \max\{1, |x_1 - x|, \dots, |x_{N-1} - x|\}.$$

Then  $|x_n - x| \leq r$  for  $n = 1, 2, 3, \dots$ . Hence,  $\{x_n\}$  is bounded. 

**Justification:** operations on convergent sequences

**Item 1** For any given  $\varepsilon > 0$ , there exist integers  $N_1, N_2$  such that

$$\begin{aligned} n \geq N_1 & \text{ implies } |x_n - x| < \varepsilon/2, \\ n \geq N_2 & \text{ implies } |y_n - y| < \varepsilon/2. \end{aligned}$$

Denote  $N = \max\{N_1, N_2\}$ . If  $n \geq N$ , then

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves Item 1.

**Item 2** The proof is trivial.

**Item 3** For any given  $\varepsilon > 0$ , there exist integers  $N_1, N_2$  such that

$$\begin{aligned} n \geq N_1 & \text{ implies } |x_n - x| < \sqrt{\varepsilon}, \\ n \geq N_2 & \text{ implies } |y_n - y| < \sqrt{\varepsilon}. \end{aligned}$$

Denote  $N = \max\{N_1, N_2\}$ . If  $n \geq N$ , then

$$|(x_n - x)(y_n - y)| = |x_n - x| \cdot |y_n - y| < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon.$$

Thus,  $\lim_{n \rightarrow \infty} (x_n - x)(y_n - y) = 0$ . Applying Items 1 and 2 to the identity

$$x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)$$

gives

$$\begin{aligned} \lim_{n \rightarrow \infty} (x_n y_n - xy) &= \lim_{n \rightarrow \infty} (x_n - x)(y_n - y) + \lim_{n \rightarrow \infty} x(y_n - y) + \lim_{n \rightarrow \infty} y(x_n - x) \\ &= 0 + x \cdot 0 + y \cdot 0 = 0. \end{aligned}$$

This proves Item 3.

**Item 4** Since  $x \neq 0$  and  $\lim_{n \rightarrow \infty} x_n = x$ , we can choose an integer  $N_1$  such that  $|x_n - x| < \frac{1}{2}|x|$  when  $n \geq N_1$ . Thus, if  $n \geq N_1$ , then, by the triangle inequality

$$|x_n| \geq |x| - |x_n - x| > |x| - \frac{1}{2}|x| = \frac{1}{2}|x|.$$

For any given  $\varepsilon > 0$ , there is an integer  $N_2$  such that


$$n \geq N_2 \text{ implies } |x_n - x| < \frac{1}{2}|x|^2 \varepsilon.$$

Denote  $N = \max\{N_1, N_2\}$ . If  $n \geq N$ , then


$$\left| \frac{1}{x_n} - \frac{1}{x} \right| = \left| \frac{x_n - x}{x_n x} \right| < \frac{\frac{1}{2}|x|^2 \varepsilon}{\frac{1}{2}|x| \cdot |x|} = \varepsilon.$$

This proves Item 4. 

**Justification:** a sequence converges iff every subsequence converges to the same limit

- ( $\Rightarrow$ ) Suppose that the sequence  $\{x_n\}$  converges to  $x$  and  $\{x_{n_k}\}$  is any subsequence. Then, for every  $\varepsilon > 0$ , there exists  $N$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ . Thus, if  $k \geq N$ , then  $n_k \geq n_N \geq N$  so that  $|x_{n_k} - x| < \varepsilon$ . This shows that every subsequence converges to  $x$ .
- ( $\Leftarrow$ ) Suppose that every subsequence converges to  $x$ . Then the sequence  $\{x_k\}$  converges  $x$  since  $\{x_k\}$  is itself a subsequence of itself. 

**Justification:** compact implies every sequence has a convergent subsequence

Assume that  $\{x_n\}$  is a sequence in  $K$ . If  $\{x_n\}$  contains a subsequence  $\{x_{n_i}\}$  such that  $x_{n_1} = x_{n_2} = \cdots = x$ , then it is clear that this subsequence converges to  $x$ , a point in  $K$ . Otherwise, the set  $E$  comprising all the terms in the sequence  $\{x_n\}$  is infinite. By Theorem 2.21, the set  $E$  has a limit point  $x \in K$ . To construct a subsequence which converges to  $x$ , we first choose  $x_{n_1} \in E$  such that  $|x_{n_1} - x| < 1$ . Having chosen  $n_1, \dots, n_{i-1}$ , by Proposition 2.10, any neighborhood of  $x$  contains infinitely many points of  $\{x_n\}$ , we can choose an integer  $n_i > n_{i-1}$  such that  $|x_{n_i} - x| < 1/i$ . Then  $\{x_{n_i}\}$  converges to  $x$ . 

**Justification:** bounded sequence in  $\mathbb{R}$  contains a convergent subsequence


Any bounded sequence in  $\mathbb{R}$  lies in some finite closed interval, and every finite closed interval is compact by Theorem 2.19. The conclusion follows immediately from the theorem that every sequence of a compact set has a subsequence that converges to a point in the compact set. ◀

**Justification:** subsequential limits form a closed set

Assume that  $\{x_n\}$  is a sequence in  $\mathbb{R}$ . Let  $E^*$  be the set of all subsequential limits of  $\{x_n\}$  and let  $y$  be a limit point of  $E^*$ . We will show that  $y \in E^*$ , so that  $E^*$  is a closed subset in  $\mathbb{R}$ .

If  $E^*$  contains only one point, then  $E^*$  is obviously closed. Otherwise, choose  $n_1$  so that  $x_{n_1} \neq y$ , and put  $\delta = |x_{n_1} - y|$ . Suppose  $n_1, \dots, n_{i-1}$  are chosen. Since  $y$  is a limit point of  $E^*$ , there is an  $z \in E^*$  such that  $|y - z| < 2^{-i}\delta$ . Because  $z \in E^*$ , there is an  $n_i > n_{i-1}$  such that  $|x_{n_i} - z| < 2^{-i}\delta$ . Thus,

$$|x_{n_i} - y| \leq |y - z| + |x_{n_i} - z| < 2^{1-i}\delta.$$

Hence we obtain a subsequence  $\{x_{n_i}\}$  that converges to  $y$ . Thus,  $y \in E^*$ . 



**Justification:** convergence of Cauchy sequences

( $\Rightarrow$ ) Suppose that  $x_n \rightarrow x$ . Then for every  $\varepsilon > 0$ , there is an integer  $N$  such that  $|x_n - x| < \varepsilon/2$  if  $n \geq N$ .

Hence, if  $n, m \geq N$ , then

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \varepsilon.$$

Thus  $\{x_n\}$  is a Cauchy sequence.

( $\Leftarrow$ ) Suppose  $\{x_n\}$  is a Cauchy sequence in  $\mathbb{R}$ . We prove that  $\{x_n\}$  converges to some  $x \in \mathbb{R}$  by completing the following steps:

1. The sequence  $\{x_n\}$  is bounded.
2. The sequence  $\{x_n\}$  contains a subsequence  $\{x_{n_k}\}$  that converges to some  $x \in \mathbb{R}$ .
3. The sequence  $\{x_n\}$  converges to  $x$ .

**Step 1** Since  $\{x_n\}$  is a Cauchy sequence in  $\mathbb{R}$ , for  $\varepsilon = 1$ , there is an integer  $N$  such that  $|x_n - x_m| < 1$  if  $n, m \geq N$ . Thus, for  $n \geq N$ ,

$$|x_n| \leq |x_n - x_N| + |x_N| < 1 + |x_N|.$$

Let  $B = \max\{|x_1|, \dots, |x_{N-1}|, 1 + |x_N|\}$ . Then, for  $n = 1, 2, \dots$ ,

$$|x_n| \leq B,$$

that is,  $\{x_n\}$  is bounded.

**Step 2** We know that  $\{x_n\}$  is bounded from Step 1. By the corollary of Theorem 3.5, we know that there is a subsequence  $\{x_{n_k}\}$  that converges to some point  $x$  in  $\mathbb{R}$ .

**Step 3** From Step 2, for every  $\varepsilon > 0$ , there is an integer  $K$  such that  $|x_{n_k} - x| < \varepsilon/2$  if  $k \geq K$ . For the same  $\varepsilon$ , there is an integer  $M$  such that  $|x_n - x_m| < \varepsilon/2$  if  $n, m \geq M$ . Put  $N = \max\{M, K\}$ . If we take an index  $n_k$  with  $k \geq N$ , then we have  $n_k \geq n_N \geq N$ . Hence, for  $n \geq N$ , we have

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \varepsilon.$$

that is,  $x_n \rightarrow x$ .




**Justification:** the monotone convergence theorem

**Item 1** Let  $\{x_n\}$  be a monotonically increasing sequence. Assume that  $\{x_n\}$  is nonempty and bounded above. By the least-upper-bound property of real numbers,  $x = \sup\{x_n\}$  exists and is finite. For any given  $\varepsilon > 0$ , there exists an integer  $N$  such that  $x - \varepsilon < x_N$ , since otherwise  $x - \varepsilon$  is an upper bound of  $\{x_n\}$ , which contradicts the definition of  $x$ . Because  $\{x_n\}$  is monotonically increasing, and  $x$  is its upper bound, for every  $n > N$ , we have  $x - \varepsilon < x_n \leq x < x + \varepsilon$ . Thus, for every  $n > N$ ,  $|x_n - x| < \varepsilon$ . Hence, by definition, the limit of  $\{x_n\}$  is  $x = \sup\{x_n\}$ .

**Item 2** The proof of Item 2 is similar to that of Item 1.

**Item 3** ( $\Rightarrow$ ) Suppose that a monotonic sequence converges. Then, by Item 1 of Proposition 3.2, the sequence is bounded.

( $\Leftarrow$ ) Suppose that a sequence  $\{x_n\}$  is bounded. If the sequence is monotonically increasing, then, by Item 1,  $\{x_n\}$  converges to the supremum of the sequence. If the sequence is monotonically decreasing, then, by Item 2,  $\{x_n\}$  converges to the infimum of the sequence. 

**Example:** upper and lower limits

Let  $x_n = \frac{(-1)^n n}{n+1}$ . Find the set  $E$  comprising all the subsequential limits (in  $\overline{\mathbb{R}}$ ) of  $\{x_n\}$ ,  $\overline{\lim}_{n \rightarrow \infty} x_n$  and  $\underline{\lim}_{n \rightarrow \infty} x_n$ .

**Solution** Since

$$\begin{aligned}\lim_{k \rightarrow \infty} x_{2k} &= \lim_{k \rightarrow \infty} \frac{2k}{2k+1} = 1, \\ \lim_{k \rightarrow \infty} x_{2k+1} &= \lim_{k \rightarrow \infty} \frac{-(2k+1)}{2k+2} = -1,\end{aligned}$$

we know that  $E = \{-1, 1\}$ . So,

$$\overline{\lim}_{n \rightarrow \infty} x_n = 1, \quad \underline{\lim}_{n \rightarrow \infty} x_n = -1.$$



**Example:** upper and lower limits

Let  $\{x_n\}$  be a sequence containing all rational numbers. Find the set  $E$  comprising all the subsequential limits in  $\overline{\mathbb{R}}$  (the extended real numbers),  $\overline{\lim}_{n \rightarrow \infty} x_n$  and  $\underline{\lim}_{n \rightarrow \infty} x_n$ .

**Solution** By Theorem 1.10, we know that every real number is a subsequential limit of  $\{x_n\}$ . The infinities,  $\infty$  and  $-\infty$ , are also in  $E$ . Hence,  $E = \mathbb{R} \cup \{\infty, -\infty\}$ . It follows that

$$\overline{\lim}_{n \rightarrow \infty} x_n = \infty, \quad \underline{\lim}_{n \rightarrow \infty} x_n = -\infty.$$



**Justification:** existence of the upper and the lower limits

Let  $\{x_n\}$  be a sequence in  $\mathbb{R}$ . Denote  $E$  to be the set comprising all the subsequential limits in  $\overline{\mathbb{R}}$  (the extended real numbers) of the sequence  $\{x_n\}$ . To show the existence of the upper and lower limits of the sequence, we consider the following three cases:

1. The sequence  $\{x_n\}$  is bounded above, but unbounded below.
2. The sequence  $\{x_n\}$  is bounded below, but unbounded above.
3. The sequence  $\{x_n\}$  is bounded above and below.

**Case 1** If the sequence is unbounded below, then it has a subsequence converging to  $-\infty$ , so that

$\varliminf_{n \rightarrow \infty} x_n = -\infty$ . Since the sequence is bounded above, by Proposition 2.15, we know that  $\sup E$  is a finite number, and  $\varlimsup_{n \rightarrow \infty} x_n = \sup E$ .

**Case 2** The proof is similar to that of Case 1.

**Case 3** If  $\{x_n\}$  is bounded, by Proposition 2.15, we know that both  $\sup E$  and  $\inf E$  are finite, and

$$\varlimsup_{n \rightarrow \infty} x_n = \sup E, \quad \varliminf_{n \rightarrow \infty} x_n = \inf E.$$



**Justification:** characterization of the upper limit

**Item 1** If  $x^* = \infty$ , then  $E$  is not bounded above. Hence  $\{x_n\}$  is not bounded above. This implies that there is a subsequence  $\{x_{n_i}\}$  such that  $x_{n_i} \rightarrow \infty$ . Thus,  $x^* = \infty \in E$ .

If  $x^*$  is a finite number, then  $E$  is bounded above. By Proposition 2.15,  $x^* \in \overline{E}$ . Proposition 3.6 says that the subsequential limits form a closed subset of  $\mathbb{R}$ , so that  $\overline{E} = E$ . Thus,  $x^* \in E$ .

If  $x^* = -\infty$ , then  $x_* = -\infty$ . Thus,  $E$  contains only one element  $-\infty$ . Hence, for any real number  $M$ ,  $x_n > M$  for at most finitely many  $n$ . This means that  $\lim_{n \rightarrow \infty} x_n = -\infty$ , so that  $-\infty = x^* \in E$ .

**Item 2** If  $y > x^*$ , and if there are infinitely many  $n$  such that  $x_n \geq y$ , then there exists  $z \in E$  such that  $z \geq y > x^*$ . This contradicts to the definition of  $x^*$ .

**Uniqueness:**

Suppose that there are two numbers,  $p$  and  $q$ , which satisfy both Items 1 and 2, and suppose  $p < q$ . We choose a number  $x$  such that  $p < x < q$ . Since  $p$  satisfies Item 2, we have an integer  $N$  such that  $n \geq N$  implies  $x_n < p$ . Hence  $x_n < x$  for  $n \geq N$ . This implies that  $q$  cannot satisfy Item 1. ◀

**Justification:** order rules of upper and lower limits - Corollary

Let  $E$  be the set comprising all the subsequential limits of  $\{x_n\}$  in  $\overline{\mathbb{R}}$  (the extended real numbers), and let  $x^* = \overline{\lim}_{n \rightarrow \infty} x_n$ ,  $x_* = \underline{\lim}_{n \rightarrow \infty} x_n$ .

( $\Rightarrow$ ) Assume that  $\lim_{n \rightarrow \infty} x_n = x$ . Then  $E = \{x\}$ . Thus,  $\overline{\lim}_{n \rightarrow \infty} x_n = \sup E = x = \inf E = \underline{\lim}_{n \rightarrow \infty} x_n$ .

( $\Leftarrow$ ) Assume that  $\overline{\lim}_{n \rightarrow \infty} x_n = \underline{\lim}_{n \rightarrow \infty} x_n = x$ . To show  $\lim_{n \rightarrow \infty} x_n = x$ , we consider the following three cases:

1.  $x = \infty$ .
2.  $x = -\infty$ .
3.  $x$  is finite.


**Case 1** In this case, if  $\lim_{n \rightarrow \infty} x_n \neq \infty$ , then for some  $M > 0$ , no matter how large  $N$  is, there is an integer  $K \geq N$  such that  $x_K < M$ . Thus, there is a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  satisfying  $x_{n_i} < M$ . It follows that  $\underline{\lim}_{n \rightarrow \infty} x_n \leq M$ . This contradicts to the hypothesis  $\underline{\lim}_{n \rightarrow \infty} x_n = \infty$ . Therefore,  $\lim_{n \rightarrow \infty} x_n = \infty = x$ .

**Case 2** The proof is similar to that of Case 1.

**Case 3** In this case, for any given  $\varepsilon > 0$ , by Proposition 3.13, for  $x + \varepsilon > x = x^*$ , there is an integer  $N_1$  such that  $n \geq N_1$  implies  $x_n < x + \varepsilon$ . Similarly, for  $x - \varepsilon < x = x_*$ , there is an integer  $N_2$  such that  $n \geq N_2$  implies  $x_n > x - \varepsilon$ . Denote  $N = \max\{N_1, N_2\}$ . Then, we have that  $n \geq N$  implies

$$x - \varepsilon < x_n < x + \varepsilon,$$

or equivalently,  $|x_n - x| < \varepsilon$ . Therefore,  $\lim_{n \rightarrow \infty} x_n = x$ .

Consequently, under the assumption that  $\overline{\lim}_{n \rightarrow \infty} x_n = \underline{\lim}_{n \rightarrow \infty} x_n = x$ , we always have  $\lim_{n \rightarrow \infty} x_n = x$ . 

**Justification:** order rules of upper and lower limits

We only show that  $\overline{\lim}_{n \rightarrow \infty} x_n \leq \overline{\lim}_{n \rightarrow \infty} y_n$ . The other inequality can be proved similarly.

Let  $\overline{\lim}_{n \rightarrow \infty} x_n = x$ . Then there is a subsequence  $\{x_{n_k}\}$  such that  $x_{n_k} \rightarrow x$ . To prove the desired inequality, we consider the following three cases:

1.  $x = \infty$ .
2.  $x = -\infty$ .
3.  $x$  is finite.


**Case 1** In this case, for every  $M$ , there exists an integer  $K$  such that  $k \geq K$  implies  $x_{n_k} \geq M$ . Put  $N = \max\{N_0, K\}$ . Then, for  $k \geq N$ , we have  $y_{n_k} \geq x_{n_k} \geq M$ . This implies that  $y_{n_k} \rightarrow \infty$ . Thus, in this case,  $\overline{\lim}_{n \rightarrow \infty} x_n = \overline{\lim}_{n \rightarrow \infty} y_n$ .

**Case 2** In this case,  $\overline{\lim}_{n \rightarrow \infty} x_n = -\infty$ . Apparently, we have

$$\overline{\lim}_{n \rightarrow \infty} x_n = -\infty \leq \overline{\lim}_{n \rightarrow \infty} y_n.$$

**Case 3** In this case, since  $x$  is a finite real number, then for every  $\varepsilon > 0$ , there is an integer  $K$  such that  $k \geq K$  implies  $|x_{n_k} - x| < \varepsilon$ . Put  $N = \max\{N_0, K\}$ . Then, for  $k \geq N$ , we have  $x < x_{n_k} + \varepsilon < y_{n_k} + \varepsilon$ . Thus,  $\overline{\lim}_{k \rightarrow \infty} y_{n_k} \geq x - \varepsilon$ . Since  $\varepsilon$  is arbitrary, we have  $\overline{\lim}_{k \rightarrow \infty} y_{n_k} \geq x$ . Hence,

$$\overline{\lim}_{n \rightarrow \infty} x_n = x \leq \overline{\lim}_{k \rightarrow \infty} y_{n_k} \leq \overline{\lim}_{n \rightarrow \infty} y_n.$$

Hence, the desired inequality holds in all three cases. 



**Example:** geometric series

Let  $a$  and  $r$  be complex, with  $a \neq 0$ . Show that the **geometric series** (幾何級數)  $\sum_{k=1}^{\infty} ar^{k-1}$  is convergent if and only if  $|r| < 1$ .

**Proof** When  $r \neq 1$ , a partial sum of the series  $\sum ar^{n-1}$  is

$$s_n = \sum_{k=1}^n ar^{k-1} = a \cdot \frac{1 - r^n}{1 - r}.$$

It is easy to see that the sequence  $\{r^n\}$  converges if  $|r| < 1$ ; diverges otherwise. Hence, when  $r \neq 1$ , the sequence  $\{s_n\}$  is convergent if and only if  $|r| < 1$ .

When  $r = 1$ , a partial sum of the series  $\sum ar^{n-1}$  is

$$s_n = \sum_{k=1}^n ar^{k-1} = a \cdot n.$$

Obviously, when  $r = 1$ , the sequence  $\{s_n\}$  is divergent.

Therefore, the geometric infinite series  $\sum_{k=1}^{\infty} ar^{k-1}$  is convergent if and only if  $|r| < 1$ .



**Example:** telescoping series

- (1) Show that the **telescoping series** (裂項級數)  $\sum_{k=1}^{\infty} (u_k - u_{k+1})$  converges if and only if  $\{u_k\}$  converges.
- (2) Find the sum of the series  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ .

**Solution**

- (1) A partial sum of the series is

$$\begin{aligned} s_n &= \sum_{k=1}^n (u_k - u_{k+1}) \\ &= (u_1 - u_2) + (u_2 - u_3) + \cdots + (u_n - u_{n+1}) = u_1 - u_{n+1}. \end{aligned}$$

Obviously, the sequence  $\{s_n\}$  is converges if and only if  $\{u_n\}$  converges.

- (2) The given series  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$  is a telescoping series, since

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

It is easy to obtain a partial sum of the series

$$\begin{aligned} s_n &= \sum_{k=1}^n \frac{1}{k(k+1)} \\ &= \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}. \end{aligned}$$

Hence, the sum of the series is  $s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n+1} \right) = 1$ .

**Example:**  $p$  series

Show that the  **$p$ -series**  $\sum \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Remark**

- The most common way to study the convergence of the  $p$ -series is to use the so-called *integral test*. However, since the justification of the test cannot be done before we formally introduce the Riemann integral, here we use the Cauchy's condensation test.

**Cauchy's Condensation Test**

Suppose that  $\{a_n\}$  is a monotonically decreasing nonnegative sequence, i.e.  $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ . Then the series  $\sum a_n$  converges if and only if the condensed series  $\sum 2^n a_{2^n}$  converges. ▶

**Proof** If  $p \leq 0$ , by the Divergence Test, the  $p$ -series is divergent. If  $p > 0$ , one can apply the Cauchy's Condensation Test to the  $p$ -series. In fact, we have

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

It is obvious that  $2^{1-p} < 1$  if and only if  $p > 1$ . The desired result on the  $p$ -series follows by that of the geometric series. ◀

**Justification:** Cauchy's condensation test

Let  $s_n$  and  $\sigma_n$  denote the partial sums of  $\sum a_n$  and  $\sum 2^n a_{2^n}$ , respectively. Since  $\{a_n\}$  is a nonnegative sequence, we know that both  $\{s_n\}$  and  $\{\sigma_n\}$  are monotonically increasing.


( $\Rightarrow$ ) Suppose  $\sum a_n$  converges. By Proposition 3.17,  $\{s_n\}$  is bounded above, that is, there is some number  $M$  such that  $s_n \leq M$  for all integer  $n$ . Since  $\{a_n\}$  is monotonically decreasing, we have, for all integer  $n$ ,

$$\begin{aligned}\sigma_n &= a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \\ &= 2 \cdot \left(\frac{1}{2}a_1\right) + 2 \cdot (a_2) + 2 \cdot (2a_4) + \cdots + 2 \cdot (2^{n-1}a_{2^n}) \\ &\leq 2 \cdot (a_1) + 2 \cdot (a_2) + 2 \cdot (a_3 + a_4) + \cdots + 2 \cdot (a_{2^{n-1}+1} + \cdots + a_{2^n}) \\ &\leq 2 \cdot (a_1 + a_2 + a_3 + \cdots + a_{2^n}) = 2s_{2^n} \leq 2M.\end{aligned}$$

Thus, the monotonically increasing sequence  $\{\sigma_n\}$  is bounded above. By Theorem 3.10 (the monotone convergence theorem), the sequence  $\{\sigma_n\}$  converges. Hence, the series  $\sum 2^n a_{2^n}$  converges.

( $\Leftarrow$ ) Suppose  $\sum 2^n a_{2^n}$  converges. By Proposition 3.17,  $\{\sigma_n\}$  is bounded above, that is, there is some number  $M$  such that  $\sigma_n \leq M$  for all integer  $n$ . For every integer  $n$ , take  $k$  such that  $n < 2^k$ . Since  $\{a_n\}$  is monotonically decreasing, we have


$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = \sigma_k \leq M.\end{aligned}$$

Thus, the monotonically increasing sequence  $\{s_n\}$  is bounded above. By Theorem 3.10, the sequence  $\{s_n\}$  converges. Hence, the series  $\sum a_n$  converges. 

**Justification:** Cauchy Criterion for convergence of series

( $\Rightarrow$ ) Suppose that  $\sum a_n$  is convergent. By Item 1 of Theorem 3.8,  $\{s_n\}$  is a Cauchy sequence. Thus, for every  $\varepsilon > 0$  there is an integer  $K$  such that  $|s_m - s_n| < \varepsilon$  if  $n, m \geq K$ . Notice that  $\sum_{k=n}^m a_k = s_m - s_{n-1}$ .

If we take  $N = K + 1$ , then we get that  $\left| \sum_{k=n}^m a_k \right| < \varepsilon$  if  $m \geq n \geq N$ .


( $\Leftarrow$ ) Suppose that for every  $\varepsilon > 0$ , there is an integer  $N$  such that  $\left| \sum_{k=n}^m a_k \right| < \varepsilon$  if  $m \geq n \geq N$ . Since  $\sum_{k=n}^m a_k = s_m - s_{n-1}$ , the sequence  $\{s_n\}$  is a Cauchy sequence. By Item 3 of Theorem 3.8, we know that  $\{s_n\}$  is convergent, so that the series  $\sum a_n$  is convergent. 

**Example:** divergence test

Show that the series  $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+1}$  diverges.

**Solution** For the given series  $\sum a_n$ , we have  $a_n = \frac{(-1)^n n}{n+1}$ . Since, as  $m \rightarrow \infty$ ,

$$a_{2m} = \frac{(-1)^{2m} 2m}{2m+1} = \frac{1}{1+1/(2m)} \rightarrow 1,$$

we know that the sequence  $\{a_n\}$  has a subsequence whose limit is not 0. Thus,  $\lim_{n \rightarrow \infty} a_n \neq 0$ . Hence, by the Divergence Test, the given series is divergent. 

**Example:** harmonic series is divergent

Show that the harmonic series  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges.

**Proof** We can show that the partial sums  $s_2, s_4, s_8, s_{16}, s_{32}, \dots$  become unbounded, so the sequence  $\{s_n\}$  does not have a finite limit. For the first few terms, we see that

$$\begin{aligned} s_2 &= 1 + \frac{1}{2}, \\ s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}, \\ s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}, \\ s_{16} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16}\right) = 1 + \frac{4}{2}. \end{aligned}$$


In general, we can prove by induction that for all positive integer  $n$ ,

$$s_{2^n} \geq 1 + \frac{n}{2}.$$

In fact, we have already seen that the inequality holds for  $n = 1$ . Suppose that the inequality holds for some positive integer  $n = k$ . Then,

$$\begin{aligned} s_{2^{k+1}} &= s_{2^k} + \left( \frac{1}{2^k + 1} + \frac{1}{2^k + 2} + \dots + \frac{1}{2^k + 2^k} \right) \\ &\geq 1 + \frac{k}{2} + \left( \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \dots + \frac{1}{2^{k+1}} \right) \\ &= 1 + \frac{k}{2} + 2^k \cdot \frac{1}{2^{k+1}} = 1 + \frac{k+1}{2}. \end{aligned}$$


This shows that the desired inequality also holds for  $n = k + 1$ . Hence, the desired inequality holds for all positive integer  $n$ .

Consequently,  $\{s_n\}$  does not have a finite limit. That is, the harmonic series diverges. 

**Justification:** convergence of series with nonnegative terms

Suppose that  $\sum a_n$  is a series of real numbers with nonnegative terms, i.e.  $a_n \geq 0$  for all  $n$ . Let  $s_n = \sum_{k=1}^n a_k$  be the  $n$ -th partial sum of the series. Since  $a_n \geq 0$  for all  $n$ , the sequence  $\{s_n\}$  is monotonically increasing.

( $\Rightarrow$ ) Suppose that  $\sum a_n$  converges. If  $\{s_n\}$  is not bounded, then it is not bounded above since it is monotonically increasing. Thus, the sequence  $\{s_n\}$  must tend to  $\infty$ , so the series does not have a finite limit. Hence,  $\{s_n\}$  must be bounded.

( $\Leftarrow$ ) Suppose that  $\{s_n\}$  is bounded. Then it is bounded above. Since it is monotonically increasing, by Theorem 3.10 (the Monotone Convergence Theorem), the sequence  $\{s_n\}$  converges, so the series  $\sum a_n$  converges. 



**Justification:**  $e$  is well-defined

The series  $\sum_{n=0}^{\infty} \frac{1}{n!}$  has all positive terms. Thus, by Theorem 3.10 (the Monotone Convergence Theorem), it suffices to show that the sequence  $\{s_n\}$  of the partial sums,  $s_n = \sum_{k=0}^n \frac{1}{k!}$ , is bounded above. Actually, when  $n \geq 2$ , we have

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 + \frac{1}{1} - \frac{1}{n} < 3. \end{aligned}$$

To show that the sum of the series falls between 2 and 3, in the same manner, for  $n \geq 5$ , we have


$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5 \cdot 4} + \frac{1}{6 \cdot 5} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{4} - \frac{1}{n} = \frac{71}{24} - \frac{1}{n}. \end{aligned}$$

Obviously,  $s_n \geq 1 + 1 + \frac{1}{2!}$ . Thus, for  $n \geq 5$ ,

$$\frac{5}{2} \leq s_n \leq \frac{71}{24}.$$

By Propositions 3.14 and its corollary, we get

$$\frac{5}{2} \leq s \leq \frac{71}{24}.$$

Therefore,  $2 < s < 3$ . 

**Justification:** Euler's number as the limit of a sequence

Let

$$s_n = \sum_{k=0}^n \frac{1}{k!}, \quad t_n = \left(1 + \frac{1}{n}\right)^n.$$

By the binomial formula, for every positive integer  $n$ ,

$$\begin{aligned} t_n &= 1 + \binom{n}{1} \cdot \left(\frac{1}{n}\right) + \binom{n}{2} \cdot \left(\frac{1}{n}\right)^2 + \binom{n}{3} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \binom{n}{n} \cdot \left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{n(n-1)}{2!} \cdot \left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \frac{n(n-1)(n-2) \cdots 2 \cdot 1}{n!} \cdot \left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right). \end{aligned}$$

Hence,  $t_n \leq s_n$  for every positive integer  $n$ , so that

$$\overline{\lim}_{n \rightarrow \infty} t_n \leq \overline{\lim}_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} s_n = e,$$

by Propositions 3.14 and its corollary.

Next, if  $n \geq m$ , we see that

$$t_n \geq 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right).$$

If letting  $n \rightarrow \infty$  while keeping  $m$  fixed, by Propositions 3.14 and its corollary, then we get

$$\underline{\lim}_{n \rightarrow \infty} t_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!},$$

that is,

$$s_m \leq \underline{\lim}_{n \rightarrow \infty} t_n.$$

Letting  $m \rightarrow \infty$ , we finally get

$$e \leq \underline{\lim}_{n \rightarrow \infty} t_n.$$

Hence, we get

$$e \leq \underline{\lim}_{n \rightarrow \infty} t_n \leq \overline{\lim}_{n \rightarrow \infty} t_n \leq e.$$

It follows that  $\lim_{n \rightarrow \infty} t_n = e$ .



**Justification: Comparison Test**


**Item 1** Since  $\sum b_n$  converges, by the Cauchy criterion, for any given  $\varepsilon > 0$ , there exists an integer  $N_1$  such that

$$\left| \sum_{k=n}^m b_k \right| < \varepsilon,$$

if  $m \geq n \geq N_1$ . Denote  $N = \max\{N_0, N_1\}$ . Thus, if  $m \geq n \geq N$ , then

$$\left| \sum_{k=n}^m a_k \right| \leq \sum_{k=n}^m |a_k| \leq \sum_{k=n}^m b_k < \varepsilon.$$

Hence, by the Cauchy criterion again,  $\sum a_n$  converges.

**Item 2** Item 2 is the contrapositive statement of Item 1 when  $\{a_n\}$  is a sequence of real numbers. 

**Justification: Root Test**

**Item 1** If  $\alpha < 1$ , we can choose  $\beta$  such that  $\alpha < \beta < 1$ . By Item 2 of Theorem 3.13, there is an integer  $N$  such that  $n \geq N$  implies  $\sqrt[n]{|a_n|} < \beta$ , or  $|a_n| < \beta^n$ . Since  $0 < \beta < 1$ ,  $\sum \beta^n$  converges. By Theorem 3.20 (the Comparison Test),  $\sum a_n$  converges.

**Item 2** If  $\alpha > 1$ , then, by Theorem 3.13, there is a sequence  $\{n_k\}$  such that  $\sqrt[n_k]{|a_{n_k}|} \rightarrow \alpha$ . Hence  $|a_n| > 1$  for infinitely many values of  $n$ , so that  $\lim a_n \neq 0$ . By the Divergence Test,  $\sum a_n$  diverges.

**Item 3** Consider two series  $\sum \frac{1}{n}$  and  $\sum \frac{1}{n^2}$ . It is easy to see that

$$\begin{aligned}\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\left|\frac{1}{n}\right|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = 1, \\ \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\left|\frac{1}{n^2}\right|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{(\sqrt[n]{n})^2} = 1.\end{aligned}$$

We know that  $\sum \frac{1}{n}$  is a divergent  $p$ -series ( $p = 1$ ), while  $\sum \frac{1}{n^2}$  is a convergent  $p$ -series ( $p = 2$ ). These two series demonstrate that the test is inconclusive about the convergence of  $\sum a_n$  if  $\alpha = 1$ . ◀

**Justification: Ratio Test**

**Item 1** If  $\overline{\lim}_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , we choose  $\beta$  such that  $\overline{\lim}_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < \beta < 1$ . By Proposition 3.13, there is an integer  $N$ , such that  $n \geq N$  implies

$$\left| \frac{a_{n+1}}{a_n} \right| < \beta.$$


Hence, for  $n \geq N$  and any positive integer  $p$ , we have

$$|a_{n+p}| < \beta |a_{n+p-1}| < \beta^2 |a_{n+p-2}| < \cdots < \beta^{n+p-N} |a_N|.$$

That is,

$$|a_n| < |a_N| \beta^{-N} \cdot \beta^n, \quad n \geq N.$$

This implies that  $\sum a_n$  converges by Theorem 3.20 (the Comparison Test).

**Item 2** If  $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$  for all  $n \geq N_0$ , then  $|a_{n+1}| \geq |a_n|$  for  $n \geq N_0$ , so that  $\lim_{n \rightarrow \infty} a_n \neq 0$ . By the Divergence Test,  $\sum a_n$  diverges. 

**Example:** the Root Test and the Ratio Test

Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots.$$

Direct calculations give

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left( \frac{2}{3} \right)^n = 0,$$

$$\overline{\lim}_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{3}{2} \right)^n = \infty,$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \left( \frac{1}{3^n} \right)^{1/(2n)} = \frac{1}{\sqrt{3}},$$

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \left( \frac{1}{2^n} \right)^{1/(2n)} = \frac{1}{\sqrt{2}}.$$

We see that the root test indicates convergence, but the ratio test does not apply. 

**Example - the Root Test and the Ratio Test**

For each of the following series, determine whether converges or diverges.

(1)  $\sum_{n=1}^{\infty} \left( \frac{3n-1}{4n+1} \right)^n.$

(2)  $\sum_{n=1}^{\infty} \frac{b^n}{n!},$  where  $b > 0.$

**Solution**

(1) The given series  $\sum_{n=1}^{\infty} \left( \frac{3n-1}{4n+1} \right)^n$  has all positive terms. It is easy to have

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{3n-1}{4n+1} \right)^n} = \lim_{n \rightarrow \infty} \frac{3n-1}{4n+1} = \frac{3}{4} < 1.$$

By the Root Test, the series  $\sum_{n=1}^{\infty} \left( \frac{3n-1}{4n+1} \right)^n$  is convergent.

(2) For any  $b > 0$ , the given series  $\sum_{n=1}^{\infty} \frac{b^n}{n!}$  has all positive terms. Since,

$$\lim_{n \rightarrow \infty} \frac{\frac{b^{n+1}}{(n+1)!}}{\frac{b^n}{n!}} = \lim_{n \rightarrow \infty} \frac{b}{n+1} = 0,$$

by the Ratio Test, the series  $\sum_{n=1}^{\infty} \frac{b^n}{n!}$  converges for any positive  $b.$

**Justification:** upper-lower limit relations between the Root Test and the Ratio Test

In the sequence of the inequalities, the second is trivial. We only show the third, since the first can be shown similarly.

$$\text{Put } \alpha = \overline{\lim}_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}.$$

If  $\alpha = \infty$ , there is nothing to prove.

If  $\alpha$  is finite or  $-\infty$ , we fix a number  $\beta > \alpha$ . By Proposition 3.13, there is an integer  $N$  such that  $n \geq N$  implies

$$\frac{c_{n+1}}{c_n} \leq \beta.$$

The inequality implies that for  $n \geq N$  and any positive integer  $p$

$$c_{n+p} \leq \beta c_{n+p-1} \leq \beta^2 c_{n+p-2} \leq \cdots \leq \beta^{n+p-N} c_N.$$

Hence, for  $n \geq N$ ,

$$c_n \leq c_N \beta^{-N} \cdot \beta^n,$$

or

$$\sqrt[n]{c_n} \leq \sqrt[n]{c_N \beta^{-N}} \cdot \beta.$$

Thus,

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \beta.$$

Since the last inequality holds for every  $\beta > \alpha$ , thus, we have

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \alpha = \overline{\lim}_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}.$$





**Example:** compute the root by using the ratio

Compute the limit  $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}}$ .

**Solution** Let  $c_n = \frac{n^n}{n!}$ . Then

$$\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Thus, we get

$$\lim_{n \rightarrow \infty} \sqrt[n]{c_n} = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = e,$$

that is,  $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e$ .



**Example:** the Root Test vs the Ratio Test

For each of the following series, determine whether converges or diverges.

$$(1) \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n^2}.$$

$$(2) \sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

**Solution**

(1) The given series  $\sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n^2}$  has all positive terms. Since

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{n}{n+1} \right)^{n^2}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left( 1 + \frac{1}{n} \right)^n} = \frac{1}{e} < 1.$$

By the **Root Test**, the series  $\sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n^2}$  converges.

(2) All terms in the series  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$  are positive as well. Since

$$\lim_{n \rightarrow \infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left( 1 + \frac{1}{n} \right)^n} = \frac{1}{e} < 1.$$

By the **Ratio Test**, the series  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$  converges.

**Remark**

- For each of the above series, you might as well try a different test!

**Justification:** Dirichlet's Test, Abel's Test, and Alternating Series Test**Dirichlet's Test:****Remark**

- A key step to prove Dirichlet's Test is to convert a finite sum by a so-called “summation-by-parts formula”.

**summation-by-parts formula**

Given two sequence  $\{a_n\}$ ,  $\{b_n\}$ , put

$$A_n = \begin{cases} \sum_{k=0}^n a_k, & \text{if } n \geq 0, \\ 0, & \text{if } n = -1. \end{cases}$$

Then, for  $0 \leq p \leq q$ , we have

$$\sum_{n=p}^q a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p. \quad \blacktriangleright$$

Since the partial sums  $A_n$  of  $\sum a_n$  form a bounded sequence, there is a positive number  $M$  such that  $|A_n| \leq M$  for all  $n$ . Because the sequence  $\{b_n\}$  is monotonically decreasing and  $\lim_{n \rightarrow \infty} b_n = 0$ , we know that  $\{b_n\}$  is a nonnegative sequence, and for any given  $\varepsilon > 0$ , there is a positive integer  $N$  such that  $b_N < (\varepsilon/2M)$ . Thus, for  $p, q \geq N$ , we have

$$\begin{aligned} \left| \sum_{n=p}^q a_n b_n \right| &= \left| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right| \\ &\leq \sum_{n=p}^{q-1} |A_n| |b_n - b_{n+1}| + |A_q| |b_q| + |A_{p-1}| |b_p| \\ &\leq M \left( \sum_{n=p}^{q-1} |b_n - b_{n+1}| + |b_q| + |b_p| \right) \\ &= M \left( \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p \right) \\ &= 2M b_p \leq 2M b_N < \varepsilon. \end{aligned}$$

Hence, the series  $\sum a_n b_n$  satisfies the Cauchy Criterion, so that the series  $\sum a_n b_n$  converges.  $\blacktriangleleft$

**Abel's Test:**

Suppose  $\{b_n\}$  is monotonically decreasing, otherwise we use  $-b_n$  to replace  $b_n$  in the proof. Since  $\{b_n\}$  is also bounded, by Theorem 3.10, it converges. Denote the limit to be  $b$ . Then the series  $\sum a_n (b_n - b)$  satisfies the conditions in Dirichlet's Test. Hence  $\sum a_n (b_n - b)$  converges. The series  $\sum a_n b_n$  converges follows from  $\sum a_n b_n = \sum a_n (b_n - b) + b \sum a_n$ .  $\blacktriangleleft$

**Alternating Series Test:**

Put  $a_n = (-1)^{n+1}$ ,  $b_n = |c_n|$ . Then the series  $\sum c_n$  is  $\sum a_n b_n$ , and Dirichlet's Test applies.  $\blacktriangleleft$

**Justification:** summation-by-parts formula

The elementary calculations give

$$\begin{aligned}
 \sum_{n=p}^q a_n b_n &= \sum_{n=p}^q (A_n - A_{n-1}) b_n \\
 &= \sum_{n=p}^q A_n b_n - \sum_{n=p}^q A_{n-1} b_n \\
 &= \left( \sum_{n=p}^{q-1} A_n b_n + A_q b_q \right) - \left( \sum_{n=p}^{q-1} A_n b_{n+1} - A_{p-1} b_p \right) \\
 &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.
 \end{aligned}$$

**Example:** the alternating harmonic series

Determine the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  whether converges or diverges.

**Solution** For the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ , denote

$$a_n = (-1)^{n-1}, \quad b_n = \frac{1}{n}, \quad n = 1, 2, 3, \dots$$

Then we have the following:

- (1) Since the terms  $a_n$  are  $\pm 1$ , the partial sums  $A_n$  of  $\sum a_n$  satisfy  $|A_n| \leq 1$ , so bounded.
- (2) The sequence  $\{b_n\}$  is monotonically decreasing and  $\lim_{n \rightarrow \infty} b_n = 0$ .

Thus, we can apply **Dirichlet's Test** to conclude that the series converges.

**Remark**

- Although one can use the Alternating Series Test to prove the convergence of the alternating harmonic series, the application of Abel's Test to the series may not be straightforward.

**Example:** Abel's Test

Determine the convergence of the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1 + \frac{1}{n}\right)^n$ .

**Solution** In the given series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(1 + \frac{1}{n}\right)^n}{n}$ , denote

$$a_n = \frac{(-1)^{n-1}}{n}, \quad b_n = \left(1 + \frac{1}{n}\right)^n, \quad n = 1, 2, 3, \dots$$

Then we have the following:

- (1) The series  $\sum a_n$  is the alternating harmonic series that is convergent.
- (2) The sequence  $\{b_n\}$  is bounded, since  $\lim_{n \rightarrow \infty} b_n = e$  exists. To show that the sequence  $\{b_n\}$  is monotonic, we apply the binomial formula to have, for every positive integer  $n$ ,

$$\begin{aligned} b_n &= 1 + \binom{n}{1} \cdot \left(\frac{1}{n}\right) + \binom{n}{2} \cdot \left(\frac{1}{n}\right)^2 + \binom{n}{3} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \binom{n}{n} \cdot \left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{n(n-1)}{2!} \cdot \left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \frac{n(n-1)(n-2) \cdots 2 \cdot 1}{n!} \cdot \left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right). \end{aligned}$$

The last expression is a summation of  $n+1$  positive terms. Obviously, each of these terms gets larger if replacing  $n$  by  $n+1$ , while  $b_{n+1}$  yields a summation of  $n+2$  positive terms. Thus,  $b_n \leq b_{n+1}$ , so the sequence  $\{b_n\}$  is monotonically increasing.

Hence, we can apply Abel's Test to conclude that the series converges.

**Remark**

- It is not a straightforward task to apply Dirichlet's Test to this series.

**Justification:** absolute convergence implies convergence

The result follows from

$$\left| \sum_{k=n}^m a_k \right| \leq \sum_{k=n}^m |a_k|$$

and the Cauchy Criterion.



**Justification:** convergence of Cauchy product of two infinite series

Without loss of generality, we assume that  $\sum a_n$  converges absolutely. Put

$$A_n = \sum_{k=0}^n a_k, \quad B_n = \sum_{k=0}^n b_k, \quad C_n = \sum_{k=0}^n c_k,$$

and denote  $\beta_n = B_n - B$ . We re-write the partial sum of  $\sum c_n$  as follows:

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots + a_n (B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} A_n = A$ , if we can prove that, as  $n \rightarrow \infty$ ,

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \rightarrow 0,$$

then we have  $\lim_{n \rightarrow \infty} C_n = AB$ .

Put  $\alpha = \sum_{n=0}^{\infty} |a_n|$ . Let  $\varepsilon > 0$  be given. Since  $\lim_{n \rightarrow \infty} \beta_n = 0$ , there is an integer  $N$  such that  $n \geq N$  implies  $|\beta_n| < \varepsilon$ . Thus, for  $n \geq N$ , we have

$$\begin{aligned} |\gamma_n| &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1}| |a_{n-N-1}| + \cdots + |\beta_n| |a_0| \\ &< |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon \alpha. \end{aligned}$$

Keeping  $N$  fixed and letting  $n \rightarrow \infty$ , since  $\lim_{k \rightarrow \infty} a_k = 0$ , we get

$$\overline{\lim}_{n \rightarrow \infty} |\gamma_n| \leq \varepsilon \alpha.$$

Because  $\varepsilon$  is arbitrary, it follows that  $\overline{\lim}_{n \rightarrow \infty} |\gamma_n| = 0$ , so that  $\lim_{n \rightarrow \infty} \gamma_n = 0$ .





**Example:** divergent Cauchy product

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges by Dirichlet's Test. Its absolute value series

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

diverges since it is a  $p$ -series with  $p = \frac{1}{2}$ . Thus, the series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$  converges conditionally.

Consider the Cauchy product of this series with itself:

$$\sum_{n=0}^{\infty} c_n = 1 - \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + \left( \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}} \right) - \left( \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{4}} + \frac{1}{\sqrt{4}} \right) + \cdots.$$

We have that, for  $n = 0, 1, 2, \dots$ ,

$$c_n = \sum_{k=0}^n \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^n \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

Since


$$(n-k+1)(k+1) = \left( \frac{n}{2} + 1 \right)^2 - \left( \frac{n}{2} - k \right)^2 \leq \left( \frac{n}{2} + 1 \right)^2,$$

we have

$$|c_n| \geq \sum_{k=0}^n \frac{2}{n+2} = \frac{2(n+1)}{n+2},$$

so that  $\lim_{n \rightarrow \infty} c_n \neq 0$ . Hence, by the Divergence Test, the series  $\sum c_n$  diverges.

**Justification:** limit of function in term of sequential limits

- ( $\Rightarrow$ ) Suppose  $\lim_{x \rightarrow p} f(x) = q$ , and suppose a sequence  $\{p_n\}$  is in  $E \setminus \{p\}$  such that  $\lim_{n \rightarrow \infty} p_n = p$ . Then for any given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - q| < \varepsilon$  if  $x \in E$  and  $0 < |x - p| < \delta$ . Also, there exists an integer  $N$  such that  $n \geq N$  implies  $0 < |p_n - p| < \delta$ . Thus, for  $n \geq N$ , we have  $|f(p_n) - q| < \varepsilon$ . This means that  $\lim_{n \rightarrow \infty} f(p_n) = q$ .
- ( $\Leftarrow$ ) Suppose  $\lim_{n \rightarrow \infty} f(p_n) = q$  for every sequence  $\{p_n\}$  in  $E \setminus \{p\}$  such that  $\lim_{n \rightarrow \infty} p_n = p$ . If  $\lim_{x \rightarrow p} f(x) \neq q$ , then there exists some  $\varepsilon > 0$  such that for every  $\delta > 0$ , there exists a point  $x \in E$  for which  $|f(x) - q| \geq \varepsilon$  but  $0 < |x - p| < \delta$ . Take  $\delta_n = 1/n$ ,  $n = 1, 2, 3, \dots$ , we obtain a sequence  $\{x_n\}$  in  $E \setminus \{p\}$ , with  $\lim_{n \rightarrow \infty} x_n = p$ , such that  $|f(x_n) - q| \geq \varepsilon$ . Thus  $\lim_{n \rightarrow \infty} f(x_n) \neq q$ . This contradicts to the hypothesis. 

**Justification:** uniqueness of limit of function

The uniqueness follows from the uniqueness of limit for sequences (item 1 of Proposition 3.2) and Proposition 4.12. ◀

**Justification:** arithmetic operations on limits of functions

In view of Proposition 4.12, these assertions follow immediately from the analogous properties of sequences (Proposition 3.3). ◀

**Justification:** every function is continuous at isolated points

If  $p$  is an isolated point of  $E$ , then there exists a number  $\delta > 0$  such that the only point  $x \in E$  for which  $|x - p| < \delta$  is  $x = p$ . Thus, for every  $\varepsilon > 0$ , if we take this  $\delta$ , then

$$|f(x) - f(p)| = 0 < \varepsilon,$$

for all points  $x \in E$  for which  $|x - p| < \delta$ .



**Example:** continuity of the absolute value function on  $\mathbb{R}$ 

Let  $f(x) = |x|$ . The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is the so-called **absolute value function** on  $\mathbb{R}$ . Show that  $f$  is continuous at every point.

**Solution** The continuity of the norm function  $f$  follows from the triangle inequality

$$|f(x) - f(y)| = ||x| - |y|| \leq |x - y|.$$



**Justification:** continuity of arithmetic operations

At isolated points of  $E$  there is nothing to prove. At limit points, the result follows from Proposition 4.13 and the equalities  $\lim_{x \rightarrow p} f(x) = f(p)$  and  $\lim_{y \rightarrow q} g(y) = g(q)$ .



**Example:** continuity of polynomials and rational functions on  $\mathbb{R}$ 

- (1) Show that every polynomial

$$P(x) = \sum_{k=1}^n c_k x^k, \quad x \in \mathbb{R},$$

is continuous on  $\mathbb{R}$ .

- (2) Show that every rational function, that is, every quotient of two polynomials, is continuous on
- $\mathbb{R}$
- wherever the denominator is different from zero.

**Proof**

- (1) If
- $x \in \mathbb{R}$
- , the
- identity*
- functions
- $\phi$
- defined by

$$\phi(x) = x$$


is continuous on  $\mathbb{R}$ , since

$$|\phi(x) - \phi(y)| \leq |x - y|.$$

By applying Proposition 4.18 repeatedly, we know that the power function  $x^k$  is continuous on  $\mathbb{R}$ . The same is true of constant multiples to the power function, since constants are evidently continuous. Hence, for real coefficients  $c_k$  with  $k$  being nonnegative integers, every polynomial  $P$ , given by

$$P(x) = \sum_{k=0}^n c_k x^k$$

is continuous on  $\mathbb{R}$ .

- (2) The result follows from part (1) and Proposition 4.18.
- 



**Justification:** continuity of composition

Denote  $h = g \circ f$ . Since  $g$  is continuous at  $f(p)$ , for every  $\varepsilon > 0$ , there exists  $\eta > 0$  such that  $|y - f(p)| < \eta$  and  $y \in f(E)$  imply  $|g(y) - g(f(p))| < \varepsilon$ . Since  $f$  is continuous at  $p$ , there exists  $\delta > 0$  such that  $|x - p| < \delta$  and  $x \in E$  implies  $|f(x) - f(p)| < \eta$ . Hence, if  $|x - p| < \delta$  and  $x \in E$ , then we have

$$|h(x) - h(p)| = |g(f(x)) - g(f(p))| < \varepsilon.$$

Therefore,  $h$  is continuous at  $p$ . 

**Justification:** characterization of continuity

**Item 1** ( $\Rightarrow$ ) Suppose  $f$  is continuous on  $E$  and  $V$  is an open set in  $\mathbb{R}$ . We have to show that every point of  $f^{-1}(V)$  is an interior point of  $f^{-1}(V)$ . Let  $p \in f^{-1}(V)$ . Then  $f(p) \in V$ . Since  $V$  is open in  $\mathbb{R}$ , there exists  $\varepsilon > 0$  such that  $y \in V$  if  $|f(p) - y| < \varepsilon$ . Since  $f$  is continuous at  $p$ , there exists  $\delta > 0$  such that  $|f(x) - f(p)| < \varepsilon$  if  $|x - p| < \delta$ . Thus, if  $|x - p| < \delta$ , then  $f(x) \in V$ , or  $x \in f^{-1}(V)$ . This means that  $p$  is an interior point of  $f^{-1}(V)$ .

( $\Leftarrow$ ) Suppose  $f^{-1}(V)$  is open in  $\mathbb{R}$  for every open set  $V$  in  $\mathbb{R}$ . Fix  $p \in \mathbb{R}$ . For  $\varepsilon > 0$ , let  $V$  be the set of all  $y \in \mathbb{R}$  such that  $|y - f(p)| < \varepsilon$ . For this open set  $V$ ,  $f^{-1}(V)$  is open in  $\mathbb{R}$ . Since  $p \in f^{-1}(V)$ , there exists  $\delta > 0$  such that  $x \in f^{-1}(V)$  if  $|x - p| < \delta$ . But  $x \in f^{-1}(V)$  implies  $f(x) \in V$ , or  $|f(x) - f(p)| < \varepsilon$ . Thus, if  $|x - p| < \delta$ , we have  $|f(x) - f(p)| < \varepsilon$ .


**Item 2** Let us show that for every  $E \subset \mathbb{R}$ ,

$$f^{-1}(E^c) = [f^{-1}(E)]^c.$$

In fact, if  $x \in f^{-1}(E^c)$ , then  $f(x) \in E^c$ , or  $f(x) \notin E$ . This implies  $x \notin f^{-1}(E)$ , or  $x \in [f^{-1}(E)]^c$ . Thus,  $f^{-1}(E^c) \subset [f^{-1}(E)]^c$ . Conversely, if  $x \in [f^{-1}(E)]^c$ , then  $x \notin f^{-1}(E)$ . Hence  $f(x) \notin E$ , or  $f(x) \in E^c$ . This indicates that  $x \in f^{-1}(E^c)$ . Thus,  $[f^{-1}(E)]^c \subset f^{-1}(E^c)$ .

Now the result follows from Item 1 and the fact that  $E$  is closed if and only if  $E^c$  is open. ◀

**Example:** images of continuous function

- (1) Consider the function  $f: (0, 1) \rightarrow \mathbb{R}$  defined by  $f(x) = x$ . It is easy to see that  $E = (0, 1)$  is a closed subset in  $(0, 1)$ , but  $f(E) = (0, 1)$  is not a closed subset in  $\mathbb{R}$ .
- (2) Consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$ . It is easy to see that  $E = (-1, \infty)$  is an open subset in  $\mathbb{R}$ , but  $f(E) = [0, \infty)$  is not an open subset in  $\mathbb{R}$ .
- (3) Consider the function  $f: (0, 1) \rightarrow \mathbb{R}$  defined by  $f(x) = 1/x$ . It is easy to see that  $E = (0, 1)$  is a bounded subset in  $(0, 1)$ , but  $f(E) = (1, \infty)$  is not a bounded subset in  $\mathbb{R}$ . 

**Justification:** continuous mapping maps compact space to compact space

Let  $\{V_\alpha\}$  be an open cover of  $f(K)$ . Since  $f$  is continuous, by Theorem 4.17, each  $f^{-1}(V_\alpha)$  is open. By

$$f(K) \subset \bigcup_{\alpha} V_{\alpha},$$

we know that

$$K \subset f^{-1}\left(\bigcup_{\alpha} V_{\alpha}\right) = \bigcup_{\alpha} f^{-1}(V_{\alpha}).$$

In other words,  $\{f^{-1}(V_{\alpha})\}$  is an open cover of  $K$ . Since  $K$  is compact, there are finitely many indices, say  $\alpha_1, \dots, \alpha_n$ , such that

$$K \subset f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n}).$$

It is known that  $f(f^{-1}(E)) \subset E$ . Thus, we have

$$f(K) \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}.$$

Hence,  $f(K)$  is compact.

**Remark**

- For any  $E \subset \mathbb{R}$ , the set relation holds:  $f(f^{-1}(E)) \subset E$ .

Let  $y \in f(f^{-1}(E))$ . By the definition of  $f(f^{-1}(E))$ , there exists an  $x \in f^{-1}(E)$  such that  $y = f(x)$ .

Since  $x \in f^{-1}(E)$  implies  $f(x) \in E$ , it follows that  $y = f(x) \in E$ .


- For any  $E \subset \mathbb{R}$ , the set relation holds:  $f^{-1}(f(E)) \supset E$ .

Let  $x \in E$ . Then  $f(x) \in f(E)$ . By the definition of  $f^{-1}(f(E))$ ,  $x \in f^{-1}(f(E))$ .

**Example:** continuous function not bounded on noncompact set

Suppose that  $E$  is a noncompact subset in  $\mathbb{R}$ .

Case 1: The subset  $E$  is unbounded. In this case, consider the function  $f$  defined by  $f(x) = x$ . It is easy to see that  $f$  is continuous on  $E$ , but  $f$  is unbounded.

Case 2. The subset  $E$  is bounded. In this case, since  $E$  is noncompact, it is not closed. Thus, there is a limit point  $x_0$  of  $E$  which is not a point of  $E$ . Consider the function  $f$  defined by  $f(x) = \frac{1}{x - x_0}$ . Then  $f$  is continuous on  $E$ , but it is unbounded. 

**Justification:** Extreme Value Theorem

By Theorem 4.19,  $f(K)$  is compact, so it is closed and bounded. By Proposition 2.15, we know that

$$\sup_{x \in K} f(x) \in \overline{f(K)} = f(K).$$


The proof is analogous to the infimum.



**Example:** bounded continuous function has no maximum on noncompact set

Suppose that  $E$  is a noncompact subset in  $\mathbb{R}$ .

Case 1: The subset  $E$  is unbounded. In this case, consider the function  $g$  defined by  $g(x) = \frac{x^2}{1+x^2}$ . It is easy to see that  $g$  is bounded continuous on  $E$ , and  $\sup_{x \in E} g(x) = 1$ . Since  $g(x) < 1$  for all  $x \in E$ , the function  $g$  has no maximum on  $E$ .

Case 2. The subset  $E$  is bounded. In this case, since  $E$  is noncompact, it is not closed. Thus, there is a limit point  $x_0$  of  $E$  which is not a point of  $E$ . Consider the function  $g$  defined by  $g(x) = \frac{1}{1+(x-x_0)^2}$ . Then  $g$  is bounded continuous on  $E$ , and  $\sup_{x \in E} g(x) = 1$ . Since  $g(x) < 1$  for all  $x \in E$ , the function  $g$  has no maximum on  $E$ . 


**Justification:** continuity of bijection

To show that  $f^{-1}$  is continuous, by Theorem 4.17, we show that  $f(V)$  is open in  $Y$  if  $V$  is open in  $X$ .

In fact, for  $V^c$  being closed in  $X$ , we know  $V^c$  is compact by item 2 of Proposition 2.18, since  $X$  is compact. Hence  $f(V^c)$  is compact by Theorem 4.19, since  $f$  is continuous. By item 1 of Proposition 2.18,  $f(V^c)$  is closed. Since  $f$  is one-to-one and onto, the complement of  $f(V)$  is  $f(V^c)$ . Thus  $f(V)$  is open. ◀



**Example:** continuous bijective mapping does not have continuous inverse

Let  $E = (-1, 0] \cup [1, 2]$ . Consider the function  $f: E \rightarrow [0, 4]$ , where  $f(x) = x^2$  for  $x \in E$ . It is easy to check that  $f$  is continuous and bijective. However, the inverse is not continuous, since  $[0, 4]$  is connected but  $E$  is not. 

**Example:** delta dependence on epsilon

The function  $f: (0, 1) \rightarrow \mathbb{R}$ , defined by  $f(x) = \frac{1}{x}$ , is continuous at every point in  $(0, 1)$ , so it is continuous on  $(0, 1)$ . For any  $x, y \in (0, 1)$ , since

$$|f(x) - f(y)| = \frac{|x - y|}{xy}.$$

It is easy to see that there is not single value  $\delta$  such that  $|f(x) - f(y)| < \varepsilon$  for all  $x, y \in (0, 1)$  with  $|x - y| < \delta$ . This is because one can choose  $x, y$  sufficiently close to 0 such that  $|f(x) - f(y)|$  is larger than any finite number.

In fact, for any  $\varepsilon > 0$ , with  $\varepsilon < 1$ , no matter how small  $\delta > 0$  is, we can choose  $n$  sufficient large such that  $n > \max\{1, 1/(2\delta)\}$ . Then, for  $x = 1/(2n)$ ,  $y = 1/n$ , we have  $|x - y| = 1/(2n) < \delta$ , but

$$|f(x) - f(y)| = |2n - n| = n > \varepsilon,$$

so that  $f$  is not uniformly continuous on  $(0, 1)$ .



**Justification:** the relationship between continuity and uniform continuity

**Item 1** This is evident from the definitions of continuity and uniform continuity.

**Item 2** Let  $\varepsilon > 0$  be given. Since  $f$  is continuous on  $E$ , for each  $p \in E$ , there is an associated  $\phi(p) > 0$ , such that  $q \in E$  and  $|p - q| < \phi(p)$  imply  $|f(p) - f(q)| < \frac{1}{2}\varepsilon$ . Let  $J(p)$  be the set of all  $q \in E$  for which  $|p - q| < \frac{1}{2}\phi(p)$ . Thus, the collection of all sets  $J(p)$  forms an open cover of  $E$ . Since  $E$  is compact, there is a finite sets of points  $p_1, \dots, p_n$  in  $E$ , such that

$$E \subset J(p_1) \cup \dots \cup J(p_n).$$


Put  $\delta = \frac{1}{2} \min\{\phi(p_1), \dots, \phi(p_n)\}$ . Then  $\delta > 0$ .

For any two points  $p, q \in E$ , with  $|p - q| < \delta$ , we claim that  $|f(p) - f(q)| < \varepsilon$ . In fact, for the above finite cover, there is a point  $p_m$ ,  $1 \leq m \leq n$ , such that  $p \in J(p_m)$ . Thus,  $|p - p_m| < \frac{1}{2}\phi(p_m)$ . This gives

$$|q - p_m| \leq |p - q| + |p - p_m| \leq \delta + \frac{1}{2}\phi(p_m) < \phi(p_m),$$

which implies

$$|f(p) - f(q)| \leq |f(p) - f(p_m)| + |f(p_m) - f(q)| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

By the definition,  $f$  is uniformly continuous on  $E$ . 

**Example:** continuous function is not uniformly continuous on noncompact set

Suppose that  $E$  is a noncompact subset in  $\mathbb{R}$ .

Case 1: The subset  $E$  is unbounded. In this case, consider the function  $h$  defined by  $h(x) = x^2$ . It is easy to see that  $h$  is continuous on  $E$ .

To see that the function  $h$  is not uniformly continuous on  $E$ . Let  $\varepsilon > 0$  and  $\delta > 0$  be arbitrary. Taking  $t$  close enough to  $x$ , we can then make the difference  $|h(t) - h(x)| = |t + x| \cdot |t - x|$  greater than  $\varepsilon$ , although  $|t - x| < \delta$ . Since this is true for every  $\delta > 0$ , consequently, the function  $h$  is not uniformly continuous on  $E$ .

Case 2. The subset  $E$  is bounded.

In this case, since  $E$  is noncompact, it is not closed. Thus, there is a limit point  $x_0$  of  $E$  which is not a point of  $E$ . Consider the function  $h$  defined by  $h(x) = \frac{1}{x - x_0}$ . Then  $h$  is continuous on  $E$ .

To see that the function  $h$  is not uniformly continuous on  $E$ . Let  $\varepsilon > 0$  and  $\delta > 0$  be arbitrary, and choose a point  $x \in E$  such that  $|x - x_0| < \delta$ . Taking  $t$  close enough to  $x_0$ , we can then make the difference  $|h(t) - h(x)| = \frac{|t - x|}{|t - x_0| \cdot |x - x_0|}$  greater than  $\varepsilon$ , although  $|t - x| < \delta$ . Since this is true for every  $\delta > 0$ , consequently, the function  $h$  is not uniformly continuous on  $E$ . ◀

**Justification:** continuous mapping maps connected subset to connected subset


Assume, on the contrary, that  $f(E) = A \cup B$ , where  $A$  and  $B$  are nonempty separated subsets of  $Y$ . Put  $G = E \cap f^{-1}(A)$ ,  $H = E \cap f^{-1}(B)$ .

It is easy to see that

$$\begin{aligned} G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\ &= E \cap (f^{-1}(A) \cup f^{-1}(B)) \\ &= E \cap f^{-1}(A \cup B) \\ &\supset E \cap E = E. \end{aligned}$$


On the other hands, both  $G$  and  $H$  are subsets of  $E$ . Hence  $E = G \cup H$ .

Since  $A \neq \emptyset$ , by the definition of  $A$ , we know that there are points  $p$  of  $E$  such that  $f(p) \in A$ . This implies  $G \neq \emptyset$ . Similarly,  $H \neq \emptyset$ .

Since  $A \subset \overline{A}$ , we have  $G \subset f^{-1}(\overline{A})$ . Since  $f$  is continuous, we know that  $f^{-1}(\overline{A})$  is closed. Hence,  $\overline{G} \subset f^{-1}(\overline{A})$ . It follows that  $f(\overline{G}) \subset \overline{A}$ . Since  $f(H) \subset f(f^{-1}(B)) \subset B$  and  $\overline{A} \cap B = \emptyset$ , we have  $f(\overline{G} \cap H) = f(\overline{G}) \cap f(H) \subset \overline{A} \cap B = \emptyset$ . Thus,  $\overline{G} \cap H = \emptyset$ . Similarly, we also have  $G \cap \overline{H} = \emptyset$ . Thus,  $G$  and  $H$  are separated. This contradicts to the hypothesis that  $E$  is connected. 

**Justification:** Intermediate Value Theorem

Without loss of generality, we assume  $f(a) < f(b)$ , otherwise we only need to consider the function  $-f$ .

We know that the interval  $[a, b]$  is connected. Hence, we know that  $f([a, b])$  is a connected subset of  $\mathbb{R}$ , by Theorem 4.23, and the assertion follows. 

**Example:** discontinuous functions

- (1) The **Dirichlet function**  $D: \mathbb{R} \rightarrow \mathbb{R}$  defined by


$$D(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

It has a discontinuity of the second kind at every point, since neither  $D(x+)$  nor  $D(x-)$  exists.

- (2) Consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = xD(x)$ , where  $D$  is the Dirichlet function. The function  $f$  is continuous at  $x = 0$  and has a discontinuity of the second kind at every other point.

- (3) The **sign function**  $\text{sgn}: \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

It has a simple discontinuity at  $x = 0$  and is continuous at every other point. 

**Justification:** monotonic function always has one-sided limits

Let  $x$  be a given point in  $(a, b)$ . Since  $f$  is monotonically increasing, the set of numbers  $f(t)$ , where  $a < t < x$ , is bounded above by  $f(x)$ . Let  $A = \sup_{a < t < x} f(t)$  be the least upper bounded. Then  $A \leq f(x)$ . We prove that  $A = f(x-)$ .

For any fixed  $\varepsilon > 0$ , by the definition of  $A$ , there is a number  $\tilde{x} \in (a, x)$  such that

$$A - \varepsilon < f(\tilde{x}) \leq A.$$

(Note:  $\tilde{x} \neq x$  since the least upper bound of  $f$  is taken in the open interval  $(a, x)$ .) Denote  $\delta = x - \tilde{x}$ . Then  $\delta > 0$ . Since  $f$  is monotonically increasing, for  $x - \delta < t < x$ , we have

$$f(x - \delta) \leq f(t) \leq f(x).$$

Thus, for  $x - \delta < t < x$ ,

$$A - \varepsilon < f(t) < A.$$

This implies that  $f(x-) = A$ . Therefore, we have

$$\sup_{a < t < x} f(t) = f(x-) \leq f(x).$$

It is similar to prove

$$f(x) \leq f(x+) = \inf_{x < t < b} f(t).$$

Next, if  $a < x < y < b$ , by the first part proved, we have

$$f(x+) = \inf_{x < t < b} f(t) = \inf_{x < t < y} f(t).$$

Similarly,

$$f(y-) = \sup_{a < t < y} f(t) = \sup_{x < t < y} f(t).$$

Thus

$$f(x+) \leq f(y-).$$





**Justification:** monotonic function has at most countably many discontinuities


We assume that  $f$  is increasing. It will be similar to prove if  $f$  is decreasing. Let  $E$  be the set of points at which  $f$  is discontinuous. With every point  $x$  of  $E$  we associate a rational number  $r(x)$  such that

$$f(x-) < r(x) < f(x+).$$

Notice that  $x_1 < x_2$  implies  $f(x_1+) < f(x_2-)$ . Hence, if  $x_1$  and  $x_2$  are two distinct discontinuities with  $x_1 < x_2$ , then

$$f(x_1-) < r(x_1) < f(x_1+) < f(x_2-) < r(x_2) < f(x_2+),$$

which implies  $r(x_1) \neq r(x_2)$ .

Therefore, we have shown that there is a one-to-one correspondence between the elements in set  $E$  and a subset of the rational numbers, which is a known countable set. 

## Appendix B: solutions to the exercises

### Chapter 1 Quiz Answers

- ①D: The set of rational numbers is an example of an ordered field, by Proposition 1.7. An ordered field is a field.
- ②D: The set of real numbers is an ordered field, by Proposition 1.7.
- ③A: The set of odd numbers is not closed under addition, since the sum of two odd number is not an odd number.
- ④C: The set  $\{x : x < 3\}$  is not a bounded set in  $\mathbb{R}$  because it is not bounded below.
- ⑤D: Every nonempty bounded-above subset of  $\mathbb{R}$  has a least upper bound (supremum). This is not necessarily true for minimum or maximum elements. For example, the set  $(0, 1)$  has no minimum or maximum element, but it does have a supremum (which is 1). The set  $[0, 1)$  has a minimum element (which is 0) but no maximum element, and its supremum is 1.
- ⑥C: The least upper bound of the set,  $\{x : x \text{ is an irrational number and } x^2 \leq 2\}$ , is  $\sqrt{2}$ , which is irrational and hence contained within the set.
- ⑦E: The inequality  $|x + y| \geq |x| + |y|$  violates the triangle inequality because the sum of the absolute values of two real numbers should always be greater than or equal to the absolute value of the sum of these two real numbers.
- ⑧D: The Cauchy-Schwarz inequality is

$$|a_1b_1 + \cdots + a_nb_n|^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

By taking  $a_k = x_k$ ,  $b_k = x_{k+1}^{-1}$ ,  $1 \leq k \leq n$  (denote  $x_{n+1} = x_1$ ), we get

$$a_1b_1 + \cdots + a_nb_n = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1},$$

$$a_1^2 + \cdots + a_n^2 = x_1^2 + \cdots + x_n^2,$$

$$b_1^2 + \cdots + b_n^2 = x_2^{-2} + \cdots + x_{n+1}^{-2} = x_1^{-2} + \cdots + x_n^{-2}.$$

Thus, it follows from the Cauchy-Schwarz inequality that

$$\left| \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \right|^2 \leq (x_1^2 + \cdots + x_n^2)(x_1^{-2} + \cdots + x_n^{-2}).$$

The left-hand side of option D should be squared.

## Chapter 1 Exercise Solutions

**Ex. 1.1** We prove by contradiction. Suppose that  $\sqrt{6}$  is a rational number. Put  $\sqrt{6} = m/n$ , with  $m, n$  being co-prime. This gives  $m^2 = 6n^2$ , which implies that  $m$  is a multiple of 3. Put  $m = 3k$ . Then we have  $2n^2 = 3k^2$ . This implies that  $n$  is also a multiple of 3. This contradicts to the hypothesis that  $m$  and  $n$  are co-prime. Thus, the number  $\sqrt{6}$  is irrational.

**Ex. 1.2** We prove by contradiction. Suppose that the set  $\{\sqrt{n} : n \in \mathbb{N}\}$  is bounded. Denote  $B$  an upper bound of the set. Thus, we have that  $\sqrt{n} \leq B$  for all  $n \in \mathbb{N}$ . It gives that

$$n \leq B^2, \quad \text{for all } n \in \mathbb{N}.$$

This contradicts to the archimedean property.

**Ex. 1.3** Suppose  $E$  is bounded above, and suppose  $\beta_1$  and  $\beta_2$  are two distinct suprema of  $E$ .

For any  $x \in E$ , we have  $x \leq \beta_1$  and  $x \leq \beta_2$ . If  $\beta_1 < \beta_2$ , then

$$x \leq \beta_1 < \beta_2,$$

so that  $\beta_2$  is not a supremum of  $E$  by definition. This contradicts to the hypothesis. Hence,  $\beta_1 \geq \beta_2$ .

In the similar manner, we have  $\beta_2 \geq \beta_1$ .

Therefore, we must have  $\beta_2 = \beta_1$ .

**Ex. 1.4** Denote  $A = \left\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \dots\right\}$ .

– Clearly, for any  $x \in A$ , we have  $x \geq 1$ , so that 1 is a lower bound of  $A$ . Furthermore, for any positive  $\varepsilon$ , since  $1 < 1 + \varepsilon$  and  $1 \in A$ , we know that  $1 + \varepsilon$  is not a lower bound of  $A$ . By the definition, we have  $\inf A = 1$ .

– Let  $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ . We prove by induction that  $x_{2^n} \geq 1 + \frac{n}{2}$ , so that the sequence  $\{x_n\}$  is not bounded above. It follows that  $\sup A = \infty$ .

In fact, we have  $x_2 = 1 + \frac{1}{2}$  so that the desired inequality holds for  $n = 1$ .

Assume that  $x_{2^k} \geq 1 + \frac{k}{2}$  for an integer  $n = k$ . Then

$$\begin{aligned} x_{2^{k+1}} &= x_{2^k} + \frac{1}{2^k + 1} + \dots + \frac{1}{2^{k+1}} \geq x_{2^k} + \underbrace{\frac{1}{2^{k+1}} + \dots + \frac{1}{2^{k+1}}}_{2^k \text{ terms}} \\ &= x_{2^k} + \frac{2^k}{2^{k+1}} = x_{2^k} + \frac{1}{2} \geq 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}, \end{aligned}$$

so that the desired inequality holds for  $n = k + 1$ .

**Ex. 1.5** ( $\Rightarrow$ ) Assume that  $|x - a| < \delta$ .

\* If  $x - a \geq 0$ , then the inequality  $|x - a| < \delta$  gives  $x - a < \delta$ , so that  $x < a + \delta$ . Thus,  $a \leq x < a + \delta$ .

\* If  $x - a < 0$ , then the inequality  $|x - a| < \delta$  gives  $-(x - a) < \delta$ , so that  $a - \delta < x$ . Thus,  $a - \delta < x < a$ .

Combining above we conclude that we always have  $a - \delta < x < a + \delta$ .

( $\Leftarrow$ ) Assume that  $a - \delta < x < a + \delta$ .

\* If  $x - a \geq 0$ , then  $a \leq x < a + \delta$ , or  $0 \leq x - a < \delta$ . Thus, we have  $|x - a| < \delta$ .

\* If  $x - a < 0$ , then  $a - \delta < x < a$ , or  $-\delta < x - a < 0$ . Thus, we also have  $|x - a| < \delta$ .

Combining above we conclude that we always have  $|x - a| < \delta$ . ▶

**Ex. 1.6** Denote  $S = \{|a + b| : a^2 < 2, |b + 1| < 3\}$ .

First, we prove that the set is bounded. In fact, by the triangle inequality, we have

$$\begin{aligned} 0 \leq |a + b| &= |a + b + 1 - 1| \leq |a| + |b + 1| + |-1| \\ &= \sqrt{a^2} + |b + 1| + 1 < \sqrt{2} + 3 + 1 = 4 + \sqrt{2}. \end{aligned}$$

Next, we find the infimum and the supremum of  $S$ .

– Take  $a = 0$  and  $b = 0$ . Then  $a^2 < 2$  and  $|b + 1| < 3$ . Clearly,  $|a + b| = 0$ . Hence, we have  $\inf S = 0$ .

– For any  $0 < \varepsilon < 1$ , let

$$a = -\sqrt{2} + \frac{1}{4}\varepsilon, \quad b = -4 + \frac{1}{4}\varepsilon.$$

Then  $-\sqrt{2} < a < 0 < \sqrt{2}$  and  $-3 < b + 1 < -2 < 3$ . Thus,  $a^2 < 2$  and  $|b + 1| < 3$ . Since

$$a + b = -4 - \sqrt{2} + \frac{1}{2}\varepsilon < 0,$$

we have

$$|a + b| = 4 + \sqrt{2} - \frac{1}{2}\varepsilon > 4 + \sqrt{2} - \varepsilon.$$

This demonstrates that  $4 + \sqrt{2} - \varepsilon$  is not an upper bound of  $S$ . Hence, by definition, we have  $\sup S = 4 + \sqrt{2}$ .

**Ex. 1.7** By the triangle inequality,

$$|x_1| = |x_1 - x_2 + x_2| \leq |x_1 - x_2| + |x_2|,$$

so that  $|x_1| - |x_2| \leq |x_1 - x_2|$ . Similarly, we have  $|x_2| - |x_1| \leq |x_1 - x_2|$ . Combining these two inequalities, we get


$$||x_1| - |x_2|| \leq |x_1 - x_2|.$$

**Ex. 1.8** Let  $A = \sum a_j^2$ ,  $B = \sum b_j^2$ ,  $C = \sum a_j b_j$ .

If  $B = 0$ , then  $b_j = 0$  for  $j = 1, \dots, n$ . For  $\lambda = 0$  and any  $\mu \neq 0$ , these values  $\lambda$  and  $\mu$  are not both zero. Obviously, we have  $\lambda a_j = \mu b_j$ ,  $j = 1, 2, \dots, n$ .

If  $B \neq 0$ , then

$$\begin{aligned} 0 \leq \sum_{j=1}^n (Ba_j - Cb_j)^2 &= B^2 \sum_{j=1}^n a_j^2 - BC \sum_{j=1}^n a_j b_j - BC \sum_{j=1}^n a_j b_j + C^2 \sum_{j=1}^n b_j^2 \\ &= B^2 A - BC^2 - BC^2 + BC^2 \\ &= B(AB - C^2). \end{aligned}$$

Since  $AB - C^2 = 0$ , we have  $Ba_j - Cb_j = 0$ ,  $j = 1, 2, \dots, n$ . If we take  $\lambda = B$  and  $\mu = C$ , then,  $\lambda$  and  $\mu$  are not both zero such that  $\lambda a_j = \mu b_j$ ,  $j = 1, 2, \dots, n$ . 

## Chapter 2 Quiz Answers

- ①E: The set of all infinite sequences of 0's and 1's is not countable, by Proposition 2.8.
- ②B: By Proposition 2.11, we know that  $\bigcap_{\alpha} A_{\alpha}^c = \left(\bigcup_{\alpha} A_{\alpha}\right)^c$ . Thus, the given set relation  $\bigcup_{\alpha} A_{\alpha} = \bigcap_{\alpha} A_{\alpha}^c$  is equivalent to  $\bigcup_{\alpha} A_{\alpha} = \left(\bigcup_{\alpha} A_{\alpha}\right)^c$ . The latter is equivalent to  $\bigcup_{\alpha} A_{\alpha} = \emptyset$ .
- ③B: The interior of  $A$  is the union of all open sets contained in  $A$ . Hence, the interior of  $A$  is a subset of  $A$ .
- ④E: Every compact set is closed, by Proposition 2.18.
- ⑤D: Take  $E = (0, 1) \subset \mathbb{R}$  and  $K = [0, 1] \subset \mathbb{R}$ . By Theorem 2.21. We know that  $K$  is compact in  $\mathbb{R}$ , but  $E$  is not. Obviously,  $E$  is a bounded subset of  $K$ .
- ⑥A: Let  $S$  be a compact subset of  $\mathbb{R}$ . By Theorem 2.21,  $S$  is bounded and closed. Since  $\mathbb{R}$  possesses the least-upper-bound property,  $y = \sup S$  is finite. By Proposition 2.15,  $y \in \overline{S}$ . Because  $S$  is closed, by Proposition 2.14,  $\overline{S} = S$ . Hence,  $\sup S \in S$ . This means that  $S$  has a maximum element.
- ⑦C: By the corollary of Proposition 2.18, the intersection of a compact set and a closed set is compact.
- ⑧D: A perfect set is a closed set with no isolated points. Option D is true by definition.
- Option A is false: Consider the set  $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ . It is compact but not perfect.
- Option B is false:  $\mathbb{R}$  is perfect, but not compact.
- Option C is false:  $\mathbb{R}$  is closed, but not compact.
- Option E is false: The set of all rational numbers of  $\mathbb{R}$  is dense in  $\mathbb{R}$ , but not closed, so not compact. ◀

## Chapter 2 Exercise Solutions

**Ex. 2.1** Define a function  $f: (0, 1) \rightarrow [0, 1]$  by

$$f(x) = \begin{cases} \frac{1}{n-2}, & \text{if } x = \frac{1}{n}, n = 3, 4, \dots, \\ 0, & \text{if } x = \frac{1}{2}, \\ x, & \text{otherwise.} \end{cases}$$

Then  $f$  is bijective from  $(0, 1)$  onto  $[0, 1]$ .

**Proof:**  $f$  is injective. Denote  $S = \{\frac{1}{3}, \frac{1}{4}, \dots\}$ . Let  $x_1, x_2 \in (0, 1)$ , with  $x_1 \neq x_2$ .

- ① If  $x_1, x_2 \in S$ , then, there are distinct  $m, n \geq 3$ , such that  $x_1 = \frac{1}{m}$ ,  $x_2 = \frac{1}{n}$ . Thus,  $f(x_1) = \frac{1}{m-2} \neq \frac{1}{n-2} = f(x_2)$ .
- ② If  $x_1 = \frac{1}{n} \in S$  ( $n \geq 3$ ) and  $x_2 = \frac{1}{2}$ , then  $f(x_1) = \frac{1}{n-2} \neq 0 = f(x_2)$ .
- ③ If  $x_1 = \frac{1}{n} \in S$  ( $n \geq 3$ ) and  $x_2 \in (0, 1) \setminus (S \cup \{\frac{1}{2}\})$ , then  $f(x_1) = \frac{1}{n-2} \neq x_2 = f(x_2)$ .
- ④ If  $x_1 = \frac{1}{2}$  and  $x_2 \in (0, 1) \setminus (S \cup \{\frac{1}{2}\})$ , then  $f(x_1) = 0 \neq x_2 = f(x_2)$ .
- ⑤ If  $x_1, x_2 \in (0, 1) \setminus (S \cup \{\frac{1}{2}\})$ , then  $f(x_1) = x_1 \neq x_2 = f(x_2)$ .

In summary, for all  $x_1, x_2 \in (0, 1)$ , if  $x_1 \neq x_2$ , then  $f(x_1) \neq f(x_2)$ .

**Proof:**  $f$  is surjective. For any  $y \in [0, 1]$ , we have

$$y = \begin{cases} f(\frac{1}{2}), & \text{if } y = 0, \\ f\left(\frac{1}{n+2}\right), & \text{if } y = \frac{1}{n}, n \geq 1, \\ f(y), & \text{if } y \neq 0 \text{ or } y \neq \frac{1}{n}, n \geq 1. \end{cases}$$

**Ex. 2.2** ( $\Rightarrow$ ) Assume that  $E$  is open.

If  $p \in E$ , then there exists  $r > 0$  such that  $N_r(p) \subset E$ . This means that  $p$  is an interior point, so  $p \in E^\circ$ . Hence,  $E \subset E^\circ$ .

On the other hand, interior points in  $E$  are necessarily in  $E$ , since any neighborhood of a point contains that point. Hence  $E^\circ \subset E$ .

Therefore, we conclude that  $E = E^\circ$ .

( $\Leftarrow$ ) Assume that  $E = E^\circ$ . To show  $E$  is open, we only need to prove that  $E^\circ$  is open.

Suppose  $p \in E^\circ$ . Then there exists  $r > 0$  such that  $N_r(p) \subset E$ . Since  $N_r(p)$  is open, for any point  $x \in N_r(p)$ , there exists  $\delta > 0$  such that  $N_\delta(x) \subset N_r(p) \subset E$ . This means that every point in  $N_r(p)$  is an interior point of  $E$ . Hence  $N_r(p) \subset E^\circ$ . This implies that  $E^\circ$  is open.

**Ex. 2.3** We prove that the interior  $E^\circ$  of  $E$  is the largest open set contained in  $E$  by completing the following steps:

1.  $E^\circ$  is an open set contained in  $E$ .
2. Any open set  $U$  contained in  $E$  is a subset of  $E^\circ$ .

**Step 1** Suppose  $p \in E^\circ$ . Then there exists  $r > 0$  such that  $N_r(p) \subset E$ . Clearly,  $p \in N_r(p)$ . Thus,  $p \in E$ . Hence,  $E^\circ$  is a subset of  $E$ .

Furthermore, since  $N_r(p)$  is open, for any point  $x \in N_r(p)$ , there exists  $\delta > 0$  such that  $N_\delta(x) \subset N_r(p) \subset E$ . This means that every point in  $N_r(p)$  is an interior point of  $E$ , that is,  $N_r(p) \subset E^\circ$ . Hence,  $E^\circ$  is an open set contained in  $E$ . ▶

**Step 2** Let  $U$  be an open set contained in  $E$ . For any point  $p \in U$ , there exists an open neighborhood  $N_r(p) \subset U$ . Since  $U$  is a subset of  $E$ , we have  $N_r(p) \subset E$ . This means that  $p$  is an interior point of  $E$ , and hence belongs to  $E^\circ$ . Therefore,  $U$  is a subset of  $E^\circ$ .

**Ex. 2.4** We only prove the relation for two subsets  $A$  and  $B$ ,  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ . By repeating applying the relation for two subsets, one can easily to have the desired relation for  $n$  subsets.

Since  $A \subset A \cup B$  and  $B \subset A \cup B$ , we have  $\overline{A} \subset \overline{A \cup B}$  and  $\overline{B} \subset \overline{A \cup B}$ . Thus,  $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ .

On the other hand, since  $A \subset \overline{A}$  and  $B \subset \overline{B}$ , we have  $A \cup B \subset \overline{A} \cup \overline{B}$ . Thus,  $\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}}$ . Since the closures  $\overline{A}$  and  $\overline{B}$  are closed and the union of two closed sets is closed, we know that  $\overline{A} \cup \overline{B}$  is closed. By Proposition 2.14, we have  $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ . Hence,  $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ .

Therefore, we conclude that  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ .

**Ex. 2.5** 1. By the archimedean property, we can choose a positive integer  $N$  such  $\varepsilon > \frac{1}{N}$ . Then the interval  $(-\varepsilon, \varepsilon)$  contains  $0, \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \dots$ . Clearly, the finite collection

$$(-\varepsilon, \varepsilon), (1 - \varepsilon, 1 + \varepsilon), (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon), \dots, (\frac{1}{N-1} - \varepsilon, \frac{1}{N-1} + \varepsilon)$$

is a subcover of  $S$ .

2. To show that  $S$  is compact, by Theorem 2.21, we only need to prove that  $S$  is bounded and closed.

**Proof:  $S$  is bounded.** For any  $x \in S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ , we have  $|x| \leq 1$ , so that  $S$  is bounded.

**Proof:  $S$  is closed.** By the archimedean property, for any  $\varepsilon > 0$ , there is a positive integer  $n$ , such that  $\varepsilon > \frac{1}{n}$ . Thus, any neighborhood of 0 contains infinitely many points in  $S$ , so that 0 is a limit point of  $S$ .

For any  $x \in (0, 1]$ , it is easy to see that there exists  $r > 0$  such that the neighborhood  $N_r(x)$  contains at most one point of  $S$ . Hence, any point in  $(0, 1]$  is not a limit point of  $S$ .

Therefore, 0 is the only limit point of  $S$ . Since  $0 \in S$ , we conclude that  $S$  is closed.

**Ex. 2.6** For each  $n$ , since  $A_n$  is a nonempty bounded open subset, there is a bounded closed interval  $I_n$  such that  $A_n \subset I_n$ . Thus,  $\overline{A_n} \subset \overline{I_n} = I_n$ . Hence,  $\overline{A_n}$  is a nonempty bounded closed subset of  $\mathbb{R}$ , so it is compact by Theorem 2.21. Therefore,  $\bigcap_{n=1}^{\infty} \overline{A_n} \neq \emptyset$ , by Theorem 2.20.

If we can prove  $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A_n}$ , then we can conclude that  $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ .

Clearly,  $A_n \subset \overline{A_n}$  for all  $n = 1, 2, \dots$ . Thus,  $\bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} \overline{A_n}$ .

On the other hand, to show that  $\bigcap_{n=1}^{\infty} \overline{A_n} \subset \bigcap_{n=1}^{\infty} A_n$ , we notice that

$$\bigcap_{n=2}^{\infty} \overline{A_n} \subset \overline{A_2} \subset A_1 \subset \overline{A_1}$$

and have

$$\bigcap_{n=1}^{\infty} \overline{A_n} = \left( \bigcap_{n=2}^{\infty} \overline{A_n} \right) \cap \overline{A_1} = \bigcap_{n=2}^{\infty} \overline{A_n}.$$

Because  $\overline{A_n} \subset A_{n-1}$  for all  $n = 2, 3, \dots$ , we have  $\bigcap_{n=2}^{\infty} \overline{A_n} \subset \bigcap_{n=1}^{\infty} A_n$ . Hence  $\bigcap_{n=1}^{\infty} \overline{A_n} \subset \bigcap_{n=1}^{\infty} A_n$ .

Therefore, we conclude that  $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A_n}$ .

**Ex. 2.7** 1. We only prove that for two perfect sets  $A$  and  $B$ , the union  $A \cup B$  is perfect. By repeating applying the result for two perfect sets, one can easily to have the same result holds for any finite collection of perfect sets.

By definition, a set is perfect if it is closed and if every point of the set is a limit point of the set. We know that, by Proposition 2.13, the union  $A \cup B$  is closed since  $A$  and  $B$  are closed. If  $x \in A \cup B$ , then either  $x \in A$  or  $x \in B$ . If  $x \in A$ , then  $x$  is a limit point of  $A$ , so it is a limit point of  $A \cup B$ . Similarly, if  $x \in B$ , then  $x$  is also a limit point of  $A \cup B$ . Therefore, the union  $A \cup B$  is perfect.

2. To see that the union of a countable collection of perfect sets may not be perfect, consider the collection  $\{A_n\}$ , where

$$A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right], \quad n = 1, 2, \dots$$

We claim that

$$\bigcup_{n=1}^{\infty} A_n = (-1, 1).$$

In fact, it is clear that  $A_n \subset (-1, 1)$ , so that  $\bigcup_{n=1}^{\infty} A_n \subset (-1, 1)$ .

On the other hand, if  $x \in (-1, 1)$ , then, by the archimedean property, there is a positive integer  $N_1$  such that  $N_1 > \frac{1}{1-x}$ . For the same reason, there is a positive integer  $N_2$  such that  $N_2 > \frac{1}{1+x}$ . Thus, for any positive integer  $n > \max\{N_1, N_2\}$ , we have

$$n > \frac{1}{1-x}, \quad n > \frac{1}{1+x}.$$

These two inequalities give  $-1 + \frac{1}{n} < x < 1 - \frac{1}{n}$ . Consequently, we have  $x \in A_n \subset \bigcup_{n=1}^{\infty} A_n$ . Hence,

we have  $(-1, 1) \subset \bigcup_{n=1}^{\infty} A_n$ .

Therefore, we have  $\bigcup_{n=1}^{\infty} A_n = (-1, 1)$ .

Since any closed interval is perfect, and any finite open interval is not perfect (because it is not closed), the proved equality shows that the union of a countable collection of perfect sets may not be perfect.

**Ex. 2.8** Let  $G$  be an open set in  $\mathbb{R}$ . For each  $x \in G$ , there are  $y$  and  $z$ , with  $z < x < y$ , such that  $(z, y) \subset G$ . Let  $b = \sup\{y: (x, y) \subset G\}$  and  $a = \inf\{z: (z, x) \subset G\}$ . Then  $-\infty \leq a < x < b \leq \infty$ . Put  $I_x = (a, b)$ . It is clear that  $I_x$  is an open interval.

We claim that  $b \notin G$ . In fact, there is nothing to prove if  $b = \infty$ . If  $b$  is finite, and  $b \in G$ , then there is some  $\delta > 0$  such that  $(b - \delta, b + \delta) \subset G$  since  $G$  is open. This contradicts to the definition of  $b$ . Similarly,  $a \notin G$ .

We shall prove that  $I_x \subset G$ . Let  $w \in I_x$ , say  $x < w < b$ . By the definition of  $b$ , there is  $y > x$  such that  $(x, y) \subset G$ . Hence  $w \in G$ . We can similarly discuss the case of  $a < w < x$ .

For each  $x \in G$ , the above construction yields a collection of open intervals  $\{I_x\}$ . We claim that  $G = \bigcup I_x$ . In fact, since each  $I_x \subset G$ , we have  $\bigcup I_x \subset G$ . On the other hand, for any  $x \in G$ , we know there is  $I_x$  such that  $x \in I_x$ . This implies  $x \in \bigcup I_x$ , so that  $G \subset \bigcup I_x$ .

It remains to show that the collection of open intervals  $\{I_x\}$  is disjoint and at most countable.

To show that  $\{I_x\}$  is disjoint, we let  $(a, b)$  and  $(c, d)$  be any two open intervals in the collection with both containing a common point  $x$ . Since  $a < x < b$  and  $c < x < d$ , we have  $c < b$  and  $a < d$ . Since  $c \notin G$ , it does not belong to  $(a, b)$ , so that  $c \leq a$ . The reversed inequality  $a \leq c$  holds by the same argument. Hence  $a = c$ . Similarly,  $b = d$ . Thus, any two different open intervals in the collection  $\{I_x\}$  are disjoint.

To show that the collection  $\{I_x\}$  is countable, we choose a rational number in each  $I_x$  as its representative. This can be done since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Since we have a disjoint collection, each segment contains a different rational number. Hence the collection can be put in one-to-one correspondence with a subset of the rational numbers. Thus it is an at most countable collection. ◀



## Chapter 3 Quiz Answers

①D: The sequence  $1, 2, 3, \dots$  contains no convergent subsequence.

②C: By the remark in Definition 3.4, a sequence converges if and only if every subsequence converges to the same limit.

③A: By the definition, every sequence of  $K$  has a subsequence that converges to a point in  $K$ .

④C: The Cauchy Criterion states that a series converges if and only if for every  $\varepsilon > 0$ , there is an integer  $N$  such that

$$\left| \sum_{k=n}^m a_k \right| < \varepsilon,$$

if  $m \geq n \geq N$ . Hence, it is a necessary and sufficient condition for convergence of a series.

⑤E: The Monotone Convergence Theorem states that a monotonic sequence converges if and only if it is bounded. Hence, we know that a bounded monotonic sequence must converge. This gives a sufficient condition for a sequence to converge.

⑥B: If  $\lim_{n \rightarrow \infty} x_n = -\infty$ , then there exists a subsequence of  $\{x_n\}$  whose limit is  $-\infty$ . If  $\lim_{n \rightarrow \infty} x_n = -\infty$  and  $\lim_{n \rightarrow \infty} x_n$  is finite, then  $\{x_n\}$  must be bounded above but unbounded below.

⑦D: If a series converges as determined by the Comparison Test, the Root Test, or the Ratio Test, then the series converges absolutely. The Divergence Test cannot be used for convergence.

Dirichlet's Test can be used to determine conditional convergence. A successful example is the convergence of the alternating harmonic series.

⑧E: If the absolute value of a series converges, then the series converges. 

## Chapter 3 Exercise Solutions

**Ex. 3.1** Since  $\lim_{n \rightarrow \infty} x_n = \alpha$ , for  $\varepsilon = 1 > 0$ , there exists an integer  $N_1$  such that  $n \geq N_1$  implies  $|x_n - \alpha| < 1$ . Thus, by triangle inequality, for  $n \geq N_1$ , we have

$$|x_n| = |x_n - \alpha + \alpha| \leq |x_n - \alpha| + |\alpha| < 1 + |\alpha|.$$

Hence, for  $n \geq N_1$ ,

$$|x_n - \alpha| \leq |x_n| + |\alpha| < 1 + |\alpha| + |\alpha| = 1 + 2|\alpha|.$$

Again since  $\lim_{n \rightarrow \infty} x_n = \alpha$ , for  $\varepsilon > 0$ , there is an integer  $N_2$  such that  $n \geq N_2$  implies  $|x_n - \alpha| < \frac{\varepsilon}{1 + 2|\alpha|}$ .

Let  $N = \max\{N_1, N_2\}$ . Then, for  $n \geq N$ , we have

$$|x_n^2 - \alpha^2| = |x_n - \alpha| \cdot |x_n + \alpha| < |x_n - \alpha| \cdot (1 + 2|\alpha|) < \frac{\varepsilon}{1 + 2|\alpha|} \cdot (1 + 2|\alpha|) = \varepsilon.$$

Hence, by the definition of limit, we conclude that  $\lim_{n \rightarrow \infty} x_n^2 = \alpha^2$ .

**Ex. 3.2** It is clear that  $0 < s_1 = \sqrt{2} < 2$ . Suppose  $0 < s_n < 2$ . Then

$$0 < s_{n+1} = \sqrt{2 + \sqrt{s_n}} < \sqrt{2 + \sqrt{2}} < 2,$$

Hence, by induction, we have  $0 < s_n < 2$  for all  $n \geq 1$ .

We shall show that  $\{s_n\}$  is an increasing sequence by induction. In fact,

$$s_2 = \sqrt{2 + \sqrt{\sqrt{2}}} > \sqrt{2} = s_1.$$

Suppose  $s_n > s_{n-1}$ . Then

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

Thus,  $\{s_n\}$  is a bounded increasing sequence. By Theorem 3.10,  $\{s_n\}$  converges.

**Ex. 3.3** We show that

$$(x_{2n-1}, x_{2n}) = \left( \frac{2^{n-1} - 1}{2^{n-1}}, \frac{2^{n-1} - 1}{2^n} \right), \quad n \geq 1.$$

In fact, by the definition,

$$(x_1, x_2) = \left(0, \frac{0}{2}\right) = (0, 0).$$

Suppose the formula holds for  $n = k$ . Then

$$\begin{aligned} (x_{2(k+1)-1}, x_{2(k+1)}) &= \left( \frac{1}{2} + x_{2k}, \frac{x_{2k+1}}{2} \right) = \left( \frac{1}{2} + \frac{2^{k-1} - 1}{2^k}, \frac{\frac{1}{2} + x_{2k}}{2} \right) \\ &= \left( \frac{2^{k+1} - 2}{2^{k+1}}, \frac{1}{4} + \frac{1}{2} \cdot \frac{2^{k-1} - 1}{2^k} \right) \\ &= \left( \frac{2^{(k+1)-1} - 1}{2^{(k+1)-1}}, \frac{2^{(k+1)-1} - 1}{2^{k+1}} \right). \end{aligned}$$

The proved expression for  $\{x_n\}$  gives:

$$\lim_{n \rightarrow \infty} x_{2n-1} = 1, \quad \lim_{n \rightarrow \infty} x_{2n} = \frac{1}{2}.$$

Hence, by the definitions of upper and lower limits, we have

$$\limsup_{n \rightarrow \infty} x_n = \overline{\lim}_{n \rightarrow \infty} x_n = 1, \quad \liminf_{n \rightarrow \infty} x_n = \underline{\lim}_{n \rightarrow \infty} x_n = \frac{1}{2}.$$

**Ex. 3.4** Let  $\{x_n\}$  be monotonically increasing with a convergent subsequence  $\{x_{n_k}\}$ ,  $\lim_{k \rightarrow \infty} x_{n_k} = \alpha$ . Since  $\{x_{n_k}\}$  is also monotonically increasing, by Theorem 3.10,  $\alpha = \sup\{x_{n_k}\}$ . We claim that  $x_n \leq \alpha$  for all  $n \geq 1$ . In fact, if there is  $x_N > \alpha$ , then we have  $x_{n_N} \geq x_N > \alpha$ . This violates the fact that  $\alpha = \sup\{x_{n_k}\}$ .

For any  $\varepsilon > 0$ , there exists an integer  $K$  such that  $k \geq K$  implies

$$|x_{n_k} - \alpha| < \varepsilon.$$

The last inequality implies that  $\alpha - \varepsilon < x_{n_k} < \alpha + \varepsilon$ . Since  $\{x_n\}$  is monotonically increasing, we have

that, for  $n \geq n_K$ ,

$$\alpha - \varepsilon < x_{n_K} \leq x_n \leq \alpha < \alpha + \varepsilon.$$

By definition, this means that  $\lim_{n \rightarrow \infty} x_n = \alpha$ .

**Ex. 3.5** By applying Proposition 3.14 and its corollary, we have

$$L = \lim_{n \rightarrow \infty} x_n = \varliminf_{n \rightarrow \infty} x_n \leq \varliminf_{n \rightarrow \infty} y_n \leq \overline{\lim}_{n \rightarrow \infty} y_n \leq \overline{\lim}_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} z_n = L.$$

Thus,  $\varliminf_{n \rightarrow \infty} y_n = \overline{\lim}_{n \rightarrow \infty} y_n = L$ , so that  $\lim_{n \rightarrow \infty} y_n = L$ .

1. Let  $x_n = \sqrt[n]{n} - 1$ . Then  $n = (1 + x_n)^n$ . It is clear that  $x_n > 0$  for all  $n > 1$ . By the binomial formula, we get

$$n = (1 + x_n)^n > 1 + nx_n + \frac{n(n-1)}{2}x_n^2 > \frac{n(n-1)}{2}x_n^2, \quad n > 1.$$

Thus, we have

$$0 < x_n < \sqrt{\frac{2}{n-1}}, \quad n > 1.$$

It is easy to see that  $\lim_{n \rightarrow \infty} 0 = \lim_{n \rightarrow \infty} \sqrt{\frac{2}{n-1}} = 0$ . Hence, the hypotheses of the Squeeze Theorem hold. Consequently, we have  $\lim_{n \rightarrow \infty} x_n = 0$ . Therefore,  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ .

2. If  $n > 2024$ , by the binomial formula, for  $p > 0$ , we have

$$(1+p)^n > 1 + \binom{n}{1}p + \binom{n}{2}p^2 + \cdots + \binom{n}{2024}p^{2024} > \frac{n(n-1) \cdots (n-2023)}{2024!}p^{2024}.$$

Thus, we get

$$0 < \frac{n^{2023}}{(1+p)^n} < \frac{n^{2023}}{\frac{n(n-1) \cdots (n-2023)}{2024!}p^{2024}}, \quad n > 2024.$$

It is easy to see that  $\lim_{n \rightarrow \infty} 0 = \lim_{n \rightarrow \infty} \frac{n^{2023}}{\frac{n(n-1) \cdots (n-2023)}{2024!}p^{2024}} = 0$ . Hence, the hypotheses of the

Squeeze Theorem hold. Consequently, we have  $\lim_{n \rightarrow \infty} \frac{n^{2023}}{(1+p)^n} = 0$ .

**Ex. 3.6** 1. For each  $n \geq 1$ , the partial sum

$$s_n = \sum_{k=1}^n a_k = \sqrt{n+1} - 1.$$

Since  $\{s_n\}$  is unbounded, so it diverges. Hence,  $\sum a_n$  diverges.

2. Since

$$0 < a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \leq \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{n^{3/2}}.$$

Since  $\sum \frac{1}{n^{3/2}}$  is a convergent  $p$ -series (with  $p = \frac{3}{2}$ ), by the Comparison Test, we know that  $\sum a_n$  converges.

3. Based on the result given in part 1 in Exercise 3.5, we know that  $\lim \sqrt[n]{n} = 1$ . Hence, there exists  $N_0$  such that for  $n \geq N_0$ ,

$$0 < \sqrt[n]{n} - 1 < \frac{1}{2},$$

which implies that for  $n \geq N_0$ ,

$$0 < a_n = (\sqrt[n]{n} - 1)^n < \left(\frac{1}{2}\right)^n.$$

The series  $\sum \left(\frac{1}{2}\right)^n$  is a convergent geometric series, thus, by the Comparison Test, we know that  $\sum a_n$  converges.

4. If  $|z| \leq 1$ , then  $|1 + z^n| \leq 1 + |z|^n \leq 2$ , which implies

$$|a_n| = \left| \frac{1}{1 + z^n} \right| \rightarrow 0.$$

By the Divergence Test,  $\sum a_n$  diverges.

If  $|z| > 1$ , by the estimation

$$|a_n| = \left| \frac{1}{1+z^n} \right| \leq \frac{1}{|z|^n - 1},$$

The series  $\sum \frac{1}{|z|^n - 1}$  converges by the Ratio Test, since

$$\lim \frac{\frac{1}{|z|^n - 1}}{\frac{1}{|z|^{n-1} - 1}} = \lim \frac{1}{|z|} \cdot \frac{1 - 1/|z|^n}{1 - 1/|z|^{n-1}} = \frac{1}{|z|} < 1.$$

Hence, we know that  $\sum a_n$  converges by the Comparison Test.

**Ex. 3.7** It is clear that the following inequality holds

$$0 \leq \frac{\sqrt{a_n}}{n} = \sqrt{\frac{a_n}{n^2}} \leq \frac{1}{2}a_n + \frac{1}{2} \cdot \frac{1}{n^2}, \quad n = 1, 2, 3, \dots$$

Since  $\sum a_n$  is convergent and  $\sum \frac{1}{n^2}$  is a convergent  $p$ -series (with  $p = 2$ ), by Proposition 3.18, we know that the series  $\sum b_n$  converges, where  $b_n = \frac{1}{2}a_n + \frac{1}{2} \cdot \frac{1}{n^2}$ . Hence, by the Comparison Test, we know that  $\sum \frac{\sqrt{a_n}}{n}$  converges.

**Ex. 3.8** For convergent series  $\sum a_n$ , the series  $\sum a_n^2$  may not be convergent. This can be illustrated in the following example.

Consider the series  $\sum a_n$ , with

$$a_{2n-1} = \frac{1}{\sqrt{n}}, \quad a_{2n} = -\frac{1}{\sqrt{n}}, \quad n \geq 1.$$

It is easy to see that the partial sums of  $\sum a_n$  are

$$s_{2n-1} = \frac{1}{\sqrt{n}}, \quad s_{2n} = 0, \quad n \geq 1.$$

Thus, we have

$$0 \leq s_n \leq \sqrt{\frac{2}{n+1}}, \quad n \geq 1.$$

Applying the Squeeze Theorem gives  $\lim s_n = 0$  so that  $\sum a_n$  converges.

On the other hand, let  $t_n = \sum_{k=1}^n \frac{1}{k}$ . Then  $s_{2n} = 2t_n$ , where  $s_n$  is the  $n$ -th partial sum of the series

$$\sum a_n^2 = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \dots$$

Since the harmonic series diverges, we know that  $\{t_n\}$ , which is the sequence of the partial sums of the harmonic series, diverges. Hence, the sequence  $\{s_{2n}\}$  diverges. Because the latter is a subsequence of  $\{s_n\}$ . Hence,  $\{s_n\}$  diverges.


**Ex. 3.9** 1. If  $L^* = \overline{\lim}_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty$ , then there is an integer  $N_0$  such that  $n \geq N_0$  implies  $\frac{a_n}{b_n} < L^* + 1$ . Hence,

$$0 \leq a_n < (L^* + 1)b_n, \quad n \geq N_0.$$

If  $\sum b_n$  converges, then  $\sum (L^* + 1)b_n$  converges. In this case, by the Comparison Test, we know that  $\sum a_n$  converges.

2. If  $L_* = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} > 0$ , then there is an integer  $N_0$  such that  $n \geq N_0$  implies  $\frac{a_n}{b_n} > \frac{1}{2}L_*$ . Hence,

$$0 \leq \frac{1}{2}L_*b_n < a_n, \quad n \geq N_0.$$

If  $\sum b_n$  diverges, then  $\sum \frac{1}{2}L_*b_n$  diverges. In this case, by the Comparison Test, we know that  $\sum a_n$  diverges. 

**Ex. 3.10** Let  $A_n = \sum_{k=0}^n |a_k|$  and  $B_n = \sum_{k=0}^n |b_k|$ . By the hypothesis, we know that  $\{A_n\}$  and  $\{B_n\}$  are convergent. For their Cauchy product  $\sum c_n$ , by applying the triangle inequality, we have

$$\sum_{k=0}^n |c_k| = \sum_{k=0}^n \left| \sum_{j=0}^k a_j b_{k-j} \right| \leq \sum_{k=0}^n \sum_{j=0}^k |a_j| |b_{k-j}| = A_n B_n.$$

Hence, by the Comparison Test, the Cauchy product  $\sum c_n$  is absolutely convergent. 