Mathematical Analysis 數學分析

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Preface 前言

Mathematical analysis is a fundamental branch of mathematics that provides the tools and methods to study the behavior of mathematical objects such as functions, sequences, and series. Analysis lies at the heart of many areas of mathematics, science, and engineering, and its concepts and techniques have numerous applications in fields ranging from finance to physics.

數學分析是數學的基礎學科之一，對研究對象如函數、序列和級數的行為提供工具和方法。分析學在數學、科學和工程的許多領域中都具有重要作用，其概念和技巧在金融、物理等領域也有眾多應用。

This book aims to provide a thorough introduction to mathematical analysis, covering essential topics such as continuity, differentiation, integration, and infinite series. By presenting these concepts in a clear and concise manner, readers will be equipped with a solid foundation in the principles of analysis and their applications in mathematics and related fields.

本書旨在提供全面的數學分析介紹，涵蓋連續性、微分、積分、無窮級數等基本主題。通過以清晰簡明的方式呈現這些概念，讀者將能夠掌握分析原理及其在數學和相關領域的應用。

Throughout the book, we emphasize the development of a rigorous mathematical framework, focusing on clear and concise proofs based on the fundamental principles of analysis. Almost all discussions in this book are built on the foundation of the real number field, and most of the topics on the continuity, differentiation, and integration of functions can be extended to higher-dimensional spaces, and even more general spaces. We also include appropriate examples and exercises to help readers develop problem-solving skills and deepen their understanding of the material.

在全書中，我們強調發展嚴謹的數學框架，著重於基於分析學基本原理的清晰簡明的證明。本書的幾乎所有討論都建立的實數域的基礎上，大部分關於函數的連續性、微分及積分可以推廣到高維空間，甚至更一般的空間。我們還包括適當的例子和練習，幫助讀者發展解決問題的技能並加深對內容的理解。

Our hope is that this book will provide students with a solid foundation in mathematical analysis that they can build upon as they progress in their studies of mathematics and related fields. We also hope that this book will serve as a valuable reference for researchers and practitioners who use analysis in their work, especially in the fields of engineering, economics, and finance.

我們希望本書能為學生提供堅實的數學分析基礎，並在他們進一步學習數學和相關領域時發揮作用。我們也希望本書能為在工作中使用分析學的研究人員和實踐者，特別是在工程、經濟與金融等領域，提供有價值的參考。

This book is presented in the form of an e-book, and uses hyperlinks to connect the content together, greatly improving the reader's overall understanding of the mathematical content and enhancing the readability of the e-book.

本書以電子書的形式呈現，並用超鏈接將內容聯繫起來，這大大地改善了讀者對數學內容的整體性的了解，增加了電子書的可讀性。

CHAPTER

第 0 章

Mathematical Arguments 數學論證

§0.1 Mathematical Proof 數學證明

What is a mathematical proof?

A ***mathematical proof*** is an inferential argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion. – from Wikipedia

Usually, a mathematical proof uses a few following terminologies to form an argument:

- ***definition*** (定義)
  – a definition is used to give a precise meaning to a new term, by describing a condition which unambiguously qualifies what a mathematical term is and is not.
- ***axiom*** (公理)
  – an axiom is a self-evident rule or first principle accepted as true.
- ***proposition*** (命題)
  – a proposition is a theorem of lesser importance, or one that is considered so elementary or immediately obvious.
- ***lemma*** (引理)
  – a lemma is a minor proposition which is used as a stepping stone to a larger result.
- ***theorem*** (定理)
  – a theorem is a statement which has been proved true by a special kind of logical argument.
- ***corollary*** (推論)
  – a corollary is a theorem connected by a short proof to an existing theorem.

Every proof may use certain assumptions together with certain basic or original axioms and previously established statements such as theorems. It is constructed along with the accepted rules of inference.

Remark

- Presenting many cases in which the statement holds is not constituted for a proof, which must demonstrate that the statement is true in *all* possible cases.

Remark

- A rigorous proof may exploit one of the following approaches:
  - ***Direct Proof*** (直接法)
  - ***Proof by Induction*** (歸納法)
  - ***Proof by Contraposition*** (逆否命題法)
  - ***Proof by Contradiction*** (反證法)
  - ***Proof by Exhaustion*** (窮舉法)
  - ***Proof by Counterexample*** (反例舉證法)

Direct Proof 直接法

Direct Proof

A ***direct proof*** is a way of showing the truth or falsehood of a given statement by a straightforward combination of established facts, usually axioms, existing lemmas and theorems, without making any further assumptions.

We use a direct proof to prove statements of the form "if p then q" or "p implies q" which we can write as $p \Rightarrow q$.

outline for a direct proof

Claim: $p \Rightarrow q$.

Proof:

① Assume the statement p is true.

② Use what we know about p and other facts as necessary to deduce that another statement q is true. This accomplishes $p \Rightarrow q$.

► ►

Remark

- A direct proof is one of the most familiar forms of proof.

Proof by Induction 歸納法

Proof by Induction

A ***proof by induction*** (also called *mathematical induction*) is to prove statements by showing a logical progression of justifiable steps by first asserting a hypothesis.

There are usually two types of induction, regular and strong, to prove that a statement $p(n)$ is true for $n \geq n_0$, $n \in \mathbb{N}_+$.

Both types of induction consist of two steps: the base step and the inductive step.

- Base step: proves the statement for the initial case without assuming any knowledge of other cases.
- Inductive step: proves the statement for the progressional step under an "inductive hypothesis".

Principle of Induction

The base step and the inductive step, together, prove that

$$p(n_0) \Rightarrow p(n_0+1) \Rightarrow p(n_0+2) \Rightarrow p(n_0+3) \Rightarrow \cdots .$$

Therefore, $p(n)$ is true for $n \geq n_0$, $n \in \mathbb{N}_+$.

outline for regular induction

Claim: $p(n)$ for $n \geq n_0$, $n \in \mathbb{N}$.

Proof:

① Base Step: Verify that $p(n_0)$ is true.

② Inductive Step:

- Assume "inductive hypothesis": $p(k)$ is true, where $k \geq n_0$.
- Use what we know about the inductive hypothesis and other facts as necessary to deduce that $p(k+1)$ is true.

③ Therefore, by the principle of induction, $p(n)$ is true for $n \geq n_0$, $n \in \mathbb{N}$.

outline for strong induction

Claim: $p(n)$ for $n \geq n_0$, $n \in \mathbb{N}$.

Proof:

① Base Step: Verify that $p(n_0)$ is true.

② Inductive Step:

- Assume "inductive hypothesis": $p(n)$ is true, where $n_0 \leq n \leq k$.
- Use what we know about the inductive hypothesis and other facts as necessary to deduce that $p(k+1)$ is true.

③ Therefore, by the principle of induction, $p(n)$ is true for $n \geq n_0$, $n \in \mathbb{N}$.

► ► ►

Remark

- The base step of the induction is a factual statement and the inductive step is a conditional statement.
- Usually, the base step begins with $n_0 = 1$, but it can be with $n_0 = 0$ or any fixed integer n_0.
- Both steps are important. False induction proofs can occur when some minor details are left out. ►
- In general, strong induction is particularly helpful in cases at the inductive step where your $p(k+1)$ is defined in terms of more than one previous $p(n)$ $(n_0 \leq n \leq k)$, or in cases where you may not know how many previous statements.

Proof by Contraposition 逆否命題法

Proof by Contraposition

A ***proof by contraposition*** infers the statement "if p then q" by establishing the logically equivalent contrapositive statement: "if $\neg q$ then $\neg p$", where $\neg$ is the standard logical **not** symbol.

outline for proof by contraposition

Claim: $p \Rightarrow q$.

Proof:

① We will prove $\neg q \Rightarrow \neg p$.

② Assume the statement $\neg q$ is true, that is, q is false. Use what we know about $\neg q$ and other facts as necessary to deduce that another statement p is false. Thus, $\neg q \Rightarrow \neg p$.

③ Therefore, by contraposition, $p \Rightarrow q$.

► ► ►

Remark

- The contrapositive statement of "$p \Rightarrow q$" is "$\neg q \Rightarrow \neg p$", while the converse statement is "$q \Rightarrow p$".

Remark

- Proof by contraposition is to prove a claim indirectly: one assumes that the conclusion is false then proves that the hypothesis is also false. For readability of the proof, it is helpful to make a simple statement, like "we will prove this result by contrapositon".
- Using contraposition on an "or" statement is useful because it negates and reverses the inference direction, transforming the "or" into an "and" (De Morgan's law), making the proof process easier.

Proof by Contradiction 反證法

Proof by Contradiction

A ***proof by contradiction*** is to establish the truth of a statement, by showing that assuming the statement to be false leads to a contradiction.

outline for proof by contradiction

Claim: $p \Rightarrow q$.

Proof:

① We will prove that $\neg q$ leads to a contradiction.

② Assume the statement $\neg q$ is true, that is, q is false. Use what we know about $\neg q$, p and other facts as necessary to deduce a contradiction.

③ Therefore, by contradiction, $\neg q$ cannot be true, so q is true.

Remark

- Proof by contradiction is to prove a claim indirectly: one assumes that the conclusion is false then proves that the hypothesis leads to a contradiction. For readability of the proof, it is helpful to make a simple statement, like "we will prove this result by contradiction".
- Proof by contradiction is often used when there is some binary choice between possibilities.
- One of the main advantages of proof by contradiction is its versatility. It can be used to prove a wide variety of mathematical statements, ranging from simple theorems to complex conjectures. Additionally, it often provides a concise and elegant argument that can help to illuminate the underlying structure of a mathematical problem.

  It's important to use proof by contradiction carefully, as it can lead to false results. The assumption of the opposite statement may not be logically contradictory, causing the proof to fail. Moreover, it can be less intuitive and harder to follow than direct proofs, especially for less experienced readers.

Remark

- Don't confuse "proof by contraposition" with "proof by contradiction". While both establish the truth of a statement, they differ in approach. The former shows that negating the conclusion implies negating the hypothesis, while the latter shows that negating the conclusion leads to a logical contradiction.

Proof by Exhaustion 窮舉法

Proof by Exhaustion

A ***proof by exhaustion*** is to validate the conclusion by dividing it into a finite number of cases and proving each one separately.

outline for a proof by exhaustion

Claim: $p \Rightarrow q$.

Proof:

① Establish the cases that apply to the statement q.

② Use what we know about p and other facts as necessary to prove that the statement q is true in each case.

③ Since q is true in every case, we conclude that q is true.

Remark

- A proof by exhaustion is also known as **proof by cases**, or the **brute force method**.
- When the problem is divided into a finite number of cases, one needs to make sure that these cases exhaust the possibilities and to prove the desired result in each case.

Proof by Counterexample 反例舉證法

Proof by Counterexample

A ***proof by counterexample*** is not technically a proof. It is merely a way of showing that a given statement cannot possibly be correct by providing an example where it does not hold.

outline for a proof by counterexample

Claim: disprove $p \Rightarrow q$.

Proof:

① Find a case where p is true but q is false.

Remark

- Proof by counterexample is based on the following principle:
  - Since mathematical statement is true only when it is true 100% of the time, one can prove that it is false by finding a single example where it is not true.
- A counterexample is an example to disprove a statement.
- A "proof by counterexample" is different from a "proof by example". The latter is usually invalid in mathematical arguments. A single example cannot prove a universal statement (unless the universe comprises only one case!). A single counterexample can disprove a universal statement.

§0.2 Mathematical Writing 數學寫作

Guidelines of mathematical writing

> The importance of writing in the mathematics classroom cannot be overemphasized. In the process of writing, students clarify their own understanding of mathematics and hone (磨練) their communication skills. They must organize their ideas and thoughts more logically and structure their conclusions in a more coherent way. – from IDRA

- Although there are no definite rules in mathematical writing, there are some standard guidelines, as compiled below, that will make your writing clearer. Some examples of bad usage (marked with ✗) and good usage (marked with ✓) are used to illustrate these guidelines.

① **Use complete sentences.**

使用完整的句子.

In order to convey your logical train of thought, you should use complete sentences in your writing.

✗ Since x is a real number.
✓ Since x is a real number, we have $x^2 \geq 0$.

✗ If differentiable, so continuous.
✓ If a function is differentiable at a point a, then it is continuous at a.

② **Avoid ambiguity.**

避免歧義.

The language of mathematics is precise and unambiguous, and ambiguity can lead to confusion, misunderstandings, and even errors.

✗ Let m and n be odd and even.
✓ Let m and n be odd and even respectively.

✗ We have $x^2 + 1 \in S$, $x \in \mathbb{R}$.
✗ We have $x^2 + 1 \in S$ for $x \in \mathbb{R}$.
✓ We have $x^2 + 1 \in S$ for all $x \in \mathbb{R}$.
✓ We have $x^2 + 1 \in S$ for some $x \in \mathbb{R}$.

③ **Never begin a sentence with a mathematical symbol.**

不用數學符號開句.

Beginning a sentence with a mathematical symbol can make the writing less clear and more difficult to understand. Mathematical symbols are case sensitive, for instance, since x and X can have entirely different meanings, putting such symbols at the beginning of a sentence can lead to ambiguity.

✗ A is a subset of B.
✓ The set A is a subset of B.

✗ f is differentiable in (a, b).
✓ The function f is differentiable in (a, b).

✗ $x^2 + 2x - 3 = 0$ has two real solutions.
✗ $X^2 + 2x - 3 = 0$ has two real solutions.
✓ The equation $x^2 + 2x - 3 = 0$ has two real solutions.

④ **Separate mathematical symbols and expressions with words.**

將數學符號和表達式用字詞分隔開.

Using words to separate mathematical symbols and expressions can enhance the clarity and comprehensibility. Without such separators, there is a risk of confusion, as the expressions may appear to merge together.

✗ Unlike $A \cup B$, $A \cap B$ equals $\varnothing$.
✓ Unlike $A \cup B$, the set $A \cap B$ equals $\varnothing$.

✗ Because $x^2 + y^2 = r^2$, $x = r\cos\theta$ and $y = r\sin\theta$.
✓ Because $x^2 + y^2 = r^2$, it follows that $x = r\cos\theta$ and $y = r\sin\theta$.

⑤ **Avoid using unnecessary symbols.**

避免使用無必要的符號.

Avoiding unnecessary symbols in mathematical writing can help to improve clarity, efficiency, consistency, accessibility, and aesthetics (美感).

✗ If an integer n is even, then its square is also even.
✓ If an integer is even, then its square is also even.

⑥ **Avoid misuse of symbols.**

避免錯誤使用符號.

Symbols like $=$, $\sim$, $\leq$, $\subset$, $\in$, etc. are not words and are primarily used in mathematical expressions. Using them in other contexts may seem inappropriate.

✗ If two sets are $\sim$, then they have the same cardinality.
✓ If two sets are equivalent, then they have the same cardinality.

✗ Since m is odd and n odd $\Rightarrow n^2$ odd, m^2 is odd.
✓ Since m is odd and any odd number squared is odd, we know that m^2 is odd.

⑦ **Use first person plural.**

使用第一人稱複數.

In mathematical writing, it is common to use the words “we” and “us” rather than “I”, “you” or “me”. This is designed to involve the readers and promote their active participation, creating a sense of collaboration and teamwork.

✗ By the Pythagorean theorem, I get the hypotenuse $c = \sqrt{a^2 + b^2}$.
✓ By the Pythagorean theorem, we get the hypotenuse $c = \sqrt{a^2 + b^2}$.

⑧ **Use the active voice.**

使用主動語態.

The active voice is generally preferred over the passive voice because it can help to make the writing more concise, clear, and direct.

✗ The hypotenuse $c = \sqrt{a^2 + b^2}$ is obtained by taking the square root of the identity $c^2 = a^2 + b^2$.
✓ Taking the square root of the identity $c^2 = a^2 + b^2$, we get the hypotenuse $c = \sqrt{a^2 + b^2}$.

⑨ **Use conjunction words and conjunctive adverbs properly.**

正確使用連接詞與連接副詞.

Conjunction words (like "and", "or", "but") and conjunctive adverbs (like "thus", "hence", "however") are important tools for connecting ideas and creating coherent, well-structured sentences in mathematical writing.

✗ The function f is continuous and not differentiable at $x = 0$.
✓ The function f is continuous, but not differentiable at $x = 0$.

✗ The function is continuous and is not uniformly continuous.
✓ The function is continuous; however, it is not uniformly continuous.

✗ Although the data is inconsistent, nevertheless, we can still make some observations.
✓ Even though the data is inconsistent, we can still make some observations.
✓ We can still make some observations, despite the fact that the data is inconsistent.

⑩ **Use correct punctuation.**

正確使用標點符號.

Using correct punctuation is an important aspect of clear and effective mathematical writing.

✗ Hence, we get the equality $c^2 = a^2 + b^2$
✓ Hence, we get the equality $c^2 = a^2 + b^2$.

✗ Suppose that x is nonnegative, then we have $x + 1 > 0$.
✓ Suppose that x is nonnegative. Then we have $x + 1 > 0$.

CHAPTER

第 1 章

Number Systems 數系

Overview of Chapter 1

In this chapter, we will discuss the appropriate number system, real numbers, on which we will build the foundation for future mathematical analysis.

- Real numbers form an *ordered field* that enables arithmetic operations and the establishment of orders, which are essential for studying limits. Limits involve the idea of approaching a value from above or below, which is only meaningful within an ordered field.
- The *least-upper-bound property* of real numbers makes them ideal for mathematical analysis. Real numbers can measure and compare quantities, and have limits and continuity. In contrast, the set of rational numbers lacks this property, making the concept of limits incomplete and unsuitable for analysis.

§1.1 Number Sets 數集

1.1 Definition: set (集合)

If A is any **set** (集合) (whose elements may be numbers or any other objects), we write $x \in A$ to indicate that x is a member (or an element) of A. If x is not a member of A, we write $x \notin A$.

The set which contains no element will be called the ***empty*** set (空集), denoted as $\varnothing$.

If A and B are sets, and if every element of A is an element of B, we say that A is a **subset** (子集, 子集合) of A, and write $A \subset B$, or $B \supset A$. If in addition, there is an element of B which is not in A, then A is said to be a ***proper*** subset of B (真子集), and write $A \subsetneq B$, or $B \supsetneq A$.

- $A \subset A$.
- If $A \subset B$ and $B \subset C$, then $A \subset C$.
- If $A \subset B$ and $B \subset A$, then $A = B$.

1.2 Definition: the sets of numbers (數集)

- The set of all positive integers (正整數) is the set $\{1, 2, 3, \ldots\}$, denoted $\mathbb{N}_+$.
- The set of all natural numbers (自然數) is the set $\{0, 1, 2, 3, \ldots\}$, denoted $\mathbb{N}$.
- The set of all integers (整數) is the set $\{0, \pm 1, \pm 2, \ldots\}$, denoted $\mathbb{Z}$.
- The set of all rational numbers (有理數) is the set

$$\{m/n : \ m \in \mathbb{Z}, \, n \in \mathbb{N}_+, \, m \text{ and } n \text{ co-prime (互質)}\},$$

denoted $\mathbb{Q}$. A number is rational *if and only if* it is a terminating decimal or a repeating decimal.
- An irrational number (無理數) is a non-terminating, non-repeating decimal. The set of all real numbers (實數) is the set of all rational numbers and irrational numbers, denoted as $\mathbb{R}$.

Remark

- $\mathbb{N}_+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$. ▶
- There are infinitely many rational numbers between any two given rational numbers, as the midpoint between them is also a rational number. Nevertheless, the rational number system contains "gaps" that are filled by real numbers.
- We will show that the real number system contains no gaps.

Remark

- If each point x of an oriented straight line is put into correspondence with its distance from a given point O (which is positive if the point is located in a positive direction from O, and is negative otherwise), the resulting correspondence is a *one-to-one* correspondence (一一對應) between the points on the straight line and the real numbers.

1.3 Definition: order and ordered sets (序、有序集)

Let S be a set. An **order** (序) on S is a relation, denoted by $<$, with the following properties:

(1) If $x \in S$ and $y \in S$, then one and only one of the following statements is true:

$$x < y, \qquad x = y, \qquad y < x.$$

(2) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Remark

- The statement "$x < y$" may be read as "x is less than y or "x is smaller than y".
- The statement "$y > x$" is equivalent to "$x < y$".
- The statement "$x \leq y$" indicates that "$x < y$ or $x = y$". In other words, $x \leq y$ is the negation (否定) of $x > y$.

An **ordered set** (有序集) is a set S in which an order is defined.

Remark

- If we define $r < s$ to mean that $s - r > 0$, then $\mathbb{Q}$ becomes an ordered set.

1.4 Definition: field (域)

A $\boxed{\textbf{field}}$ (域) is a set F with two operations, called ***addition*** $(+)$ and ***multiplication*** $(\times)$, which satisfy the following one set of axioms for addition, one set of axioms for multiplication, and the ***distributive law*** (分配律) relating the operations:

Axioms for addition

(A1)	Closure under $+$:	$\forall x, y \in F,\ x + y \in F$	
(A2)	Commutativity of $+$:	$\forall x, y \in F,\ x + y = y + x$	
(A3)	Associativity of $+$:	$\forall x, y, z \in F,\ (x + y) + z = x + (y + z)$	
(A4)	Identity element for $+$:	$\exists\, 0 \in F,\ \forall x \in F,\ x + 0 = x,$	0 is called the zero
(A5)	Inverse element for $+$:	$\forall x \in F,\ \exists\, x' \in F,\ x + x' = 0_F,$	write $x' = -x$ the negative element

Axioms for multiplication

(M1)	Closure under $\times$:	$\forall x, y \in F,\ x \times y \in F$	
(M2)	Commutativity of $\times$:	$\forall x, y \in F,\ x \times y = y \times x$	
(M3)	Associativity of $\times$:	$\forall x, y, z \in F,\ (x \times y) \times z = x \times (y \times z)$	
(M4)	Identity element for $\times$:	$\exists\, 1 \in F,\ 1 \neq 0,\ \forall x \in F,\ 1 \times x = x,$	1 is called the unity
(M5)	Inverse element for $\times$:	$\forall x \in F,\ x \neq 0,\ \exists\, x^{-1} \in F,\ x \times x^{-1} = 1_F$	

The distributive law

(D) Product is distributive over addition : $\forall x, y, z \in F,\ x \times (y + z) = x \times y + x \times z$

Remark

- The zero 0, the negative element $-x$ of x, and the unity 1, and the inverse element x^{-1} of x (for $x \neq 0$) are all unique. For instance, for any given x, there is a unique element x' such that $x + x' = 0$.
- For convenience, one sometimes uses $x \cdot y$ or xy to replace $x \times y$. One usually writes
$$x - y, \frac{x}{y}, x^3, 3x, x + y + z, xyz$$
in place of $x + (-y), x \times y^{-1}, xxx, x + x + x, (x + y) + z, (xy)z$.

1.5 Proposition: field properties

Properties on addition

1. If $x + y = x + z$, then $y = z$.
2. If $x + y = x$, then $y = 0$.
3. If $x + y = 0$, then $y = -x$.
4. $-(-x) = x$.

Properties on multiplication

1. If $x \neq 0$ and $xy = xz$, then $y = z$.
2. If $x \neq 0$ and $xy = x$, then $y = 1$.
3. If $x \neq 0$ and $xy = 1$, then $y = x^{-1}$.
4. If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Properties on the zero and the negative elements

1. $0x = 0$.
2. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.
3. $(-x)y = -(xy) = x(-y)$.
4. $(-x)(-y) = xy$.

1.6 Definition: ordered field (有序域)

A **ordered field** (有序域) is a field F on which an *order* is defined, such that

(1) if $x, y, z \in F$ and $y < z$, then $x + y < x + z$;
(2) if $x, y \in F$, $x > 0$, and $y > 0$, then $xy > 0$.

If $x > 0$, we call x ***positive***; if $x < 0$, x is ***negative***.

Properties on ordered field

1. If $x > 0$, then $-x < 0$, and *vice versa*.
2. If $x > 0$ and $y < z$, then $xy < xz$.
3. If $x < 0$ and $y < z$, then $xy > xz$.
4. If $x \neq 0$, then $x^2 > 0$. In particular, $1 > 0$.
5. If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

1.7 Proposition: $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields

The set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$ are both ordered fields.

Remark

- There are other ordered fields. For example, any subfield of an ordered field, such as the real algebraic numbers. (An algebraic number is a number that is a root of a nonzero polynomial in one variable with integer coefficients.)

§1.2 Least-Upper-Bound Property 確界原理

1.8 Definition: bounded set and the least-upper-bound property (有界集、確界原理)

Suppose S is an ordered set, and $E \subset S$.

If there exists a $\beta \in S$ such that $x \leq \beta$ for *every* $x \in E$, we say that E is **bounded above** (有上界), and call β an ***upper bound*** of E.

If there exists an $\alpha \in S$ such that $\alpha \leq x$ for *every* $x \in E$, we say that E is **bounded below** (有下界), and call α a ***lower bound*** of E.

A set is **bounded** (有界) if it has both upper and lower bounds.

Suppose E is bounded above. If there is a $\beta \in S$ with the following properties:

(1) β is an upper bound of E;
(2) if $\gamma < \beta$, then γ is not an upper bound of E

then β is called the ***least upper bound*** of E or the **supremum** (上確界) of E, and we write

$$\beta = \sup E.$$

Suppose E is bounded below. If there is an $\alpha \in S$ with the following properties:

(1) α is a lower bound of E;
(2) if $\alpha < \gamma$, then γ is not a lower bound of E

then α is called the ***greatest lower bound*** of E or the **infimum** (下確界) of E, and we write

$$\alpha = \inf E.$$

Remark

- The supremum of a set, if it exists, is unique. The same conclusion hold for the infimum.

An ordered set S is said to have the **least-upper-bound property** (確界原理) if the following is true:

for any nonempty subset E of S, if E is bounded above, then $\sup E$ exists in S.

An ordered set with the least-upper-bound property also has the greatest-lower-bound property. ▶

Remark

- The least-upper-bound property is a crucial concept in the study of limits and is essential for the development of calculus and real analysis.

1.9 Theorem: the least-upper-bound property of $\mathbb{Q}$ and $\mathbb{R}$ ▶

1. With the usual addition, multiplication, and the order, $\mathbb{Q}$ is an ordered field which does *not* have the least-upper-bound property. ▶
2. With the usual addition, multiplication, and the order, $\mathbb{R}$ is an ordered field which *has* the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield. ▶

Remark

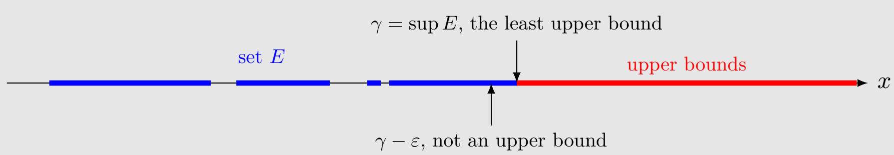
- Whether possessing the least-upper-bound property or not is one of the major differences between $\mathbb{R}$ and $\mathbb{Q}$. Without the least-upper-bound property, the set $\mathbb{Q}$ cannot cover the whole line completely so that the key concept, limit, cannot be *well* defined.

Remark

- Suppose $E \subset \mathbb{R}$ is a bounded above in $\mathbb{R}$. Then $\gamma = \sup E$ is the number that satisfes the following:

  (1)' For any $p \in E$, the inequality $p \leq \gamma$ holds.
  (2)' For any given $\varepsilon > 0$, there exits a number $q \in E$ such that $\gamma - \varepsilon < q$.

  Item (1)' means that γ is an upper bound of E; Item (2)' means that any number smaller than γ is not an upper bound of E. These can be illustrated as in the following diagram.



1.10 Theorem: the archimedean property of $\mathbb{R}$ and the rational density theorem

1. If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$.
2. If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. ▶

Remark

- The first statement is known as **archimedean property** (阿基米德性質).
- The second statement says that $\mathbb{Q}$ in *dense* in $\mathbb{R}$ (稠密). ▶

1.11 Proposition: existence of nth root of positive real numbers

For any given positive real number x and positive integer n, there is a one and only one real positive y such that

$$y^n = x.$$

This number y is called the **nth root** of x and denoted as $\sqrt[n]{x}$ or $x^{1/n}$.

Law of radicals

For any positive real numbers a and b and positive integer n,

$$(ab)^{1/n} = a^{1/n} \cdot b^{1/n}.$$

Remark

- The equation $y^n = 0$ has only zero solution. Hence, it is meaningful to define $\sqrt[n]{0} = 0$.

1.12 Definition: the extended real number system (擴張實數系)

The ***extended real number system*** (擴張實數系) comprises the real field $\mathbb{R}$ and two symbols, ∞ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define: $\forall x \in \mathbb{R}$, $-\infty < x < \infty$.

Denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. The numbers in $\mathbb{R}$ are called ***finite***, and the symbols ∞ and $-\infty$ ***infinite***.

Remark

- Clearly, ∞ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. Thus if, for example, E is a nonempty set of real numbers which is not bounded above in $\mathbb{R}$, then $\sup E = \infty$. Similarly, if E is a nonempty set of real numbers which is not bounded below in $\mathbb{R}$, then $\inf E = -\infty$.

Remark

- The extended real number system does not form a field. So, when ∞ and/or $-\infty$ are involved in operations, one *cannot* treat them the same way as numbers in $\mathbb{R}$. However, it is customary to make the following conventions (約定):
  (1) If $x \in \mathbb{R}$, then
  $$x + \infty = \infty, \qquad x - \infty = -\infty, \qquad \frac{x}{\infty} = \frac{x}{-\infty} = 0.$$
  (2) If $x > 0$ then $x \cdot (\infty) = \infty$, $x \cdot (-\infty) = -\infty$; if $x < 0$ then $x \cdot (\infty) = -\infty$, $x \cdot (-\infty) = \infty$.

§1.3 Extensions of Real Numbers 實數的擴展

Complex Number Field 複數域

1.13 Definition: complex numbers (複數)

A **complex number** (複數) is an ordered pair (a, b) of real numbers. Denote

$$\mathbb{C} = \{(a, b) : \ a \text{ and } b \text{ are real numbers}\},$$

and define two operations in $\mathbb{C}$, $+$ and $\times$, as following:

$$(a, b) + (c, d) = (a + c, b + d),$$

$$(a, b) \times (c, d) = (ac - bd, ad + bc).$$

In $\mathbb{C}$, if two complex numbers, (a, b) and (c, d), equal if and only if $a = c$ and $b = d$.

Remark

- "Ordered" means that (a,b) and (b,a) are regarded as distinct if $a \neq b$.
- Requiring that $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$ is not superfluous (多餘的). One may think of equality of rational numbers, represented as quotients of integers (整數商), as a counterexample (反例).

1.14 Proposition: the complex field $\mathbb{C}$

1. Under the given addition and the multiplication operations, the set of all complex numbers, $\mathbb{C}$, is a field, called the **complex field** (複數域). The zero in $\mathbb{C}$ is $(0,0)$, while the unity in $\mathbb{C}$ is $(1,0)$.
2. For any real numbers a and b,

$$(a,0)+(b,0) = (a+b,0), \qquad (a,0)\times(b,0) = (ab,0).$$

Remark

- This shows that the complex numbers in the form $(a,0)$ have the same arithmetic properties (算術性質) as the corresponding real numbers a. Hence, the operations $+$ and $\times$ in $\mathbb{C}$ can be considered as an extension of the operations $+$ and $\times$ in $\mathbb{R}$. Moreover, one can identify $(a,0)$ with a. Under this identification, the real field $\mathbb{R}$ is a subfield of the complex field $\mathbb{C}$.

1.15 Definition: i - the unit imaginary number (單位虛數)

Denote $i = (0,1)$, the **unit imaginary number** (單位虛數).

Properties of the unit imaginary number

1. $i^2 = i \times i = -1$. For this reason, we write $i = \sqrt{-1}$.
2. $(a,b) = a + bi$ for real numbers a and b.

Remark

- We will use the more customary form $a+bi$ to represent the notation (a,b) for complex numbers.

1.16 Definition: complex conjugate and the absolute value (共軛復數、絕對值)

Let $z = a+bi$ be a complex number, where a and b are real. The complex number $a - bi$ is called the **conjugate** of z (共軛復數), denoted as $\overline{z}$. The real numbers a and b are often referred to as the ***real part*** and the ***imaginary part*** of z (實部, 虛部), respectively, and denoted as

$$a = \operatorname{Re}(z), \qquad b = \operatorname{Im}(z).$$

Properties of complex conjugates

If z and w are complex numbers, then

1. $\overline{z+w} = \overline{z} + \overline{w}$,
2. $\overline{zw} = \overline{z} \cdot \overline{w}$,
3. $z + \overline{z} = 2\operatorname{Re}(z)$, $z - \overline{z} = 2i\operatorname{Im}(z)$,
4. If $z \neq 0$, then $z\overline{z}$ is real and positive.

Remark

- For $z \neq 0$, it is easy to show that $z^{-1} = \dfrac{\overline{z}}{z \cdot \overline{z}}$. Hence,

$$\operatorname{Re}(z^{-1}) = \frac{\operatorname{Re}(z)}{z\cdot\overline{z}}, \quad \operatorname{Im}(z^{-1}) = -\frac{\operatorname{Im}(z)}{z\cdot\overline{z}}.$$

Remark

- By the existence of nth root of positive real numbers and the fact that $z\overline{z}$ is real and positive for nonzero complex number z, so it is meaningful to introduce the absolute value of a complex number.

If z is a complex number, its **absolute value** (絕對值) is defined as $|z| = (z\overline{z})^{1/2}$.

Properties of the absolute value

If z and w are complex numbers, then

1. $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$,
2. $|\overline{z}| = |z|$,
3. $|zw| = |z| \cdot |w|$,
4. $|\operatorname{Re}(z)| \leq |z|$, $|\operatorname{Im}(z)| \leq |z|$,
5. $|z + w| \leq |z| + |w|$.

Remark

- It is easy to see that if $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.
- When x is real, then $\overline{x} = x$, hence $|x| = \sqrt{x^2}$. Thus, $|x| = x$ if $x \geq 0$; $|x| = -x$ if $x < 0$.

If x, a are real, and δ is positive, then

$$|x - a| < \delta \iff a - \delta < x < a + \delta.$$

Remark

- In any ordered field, it is known that $x^2 > 0$ for $x \neq 0$. Hence, the complex field $\mathbb{C}$ is *not* an ordered field due to the fact $i^2 = -1$.

Remark

- If one writes $z = y_1 - y_2$, $w = y_2 - y_3$, then $z + w = y_1 - y_3$. In this case, the inequality $|z + w| \leq |z| + |w|$ becomes

$$|y_1 - y_3| \leq |y_1 - y_2| + |y_2 - y_3|.$$

One can interpret this by identifying a complex number (a, b) as a point in a coordinate plane, then geometrically, the inequality says that the length of any side of a triangle cannot be larger than the sum of the lengths of the other two sides. For this sake, one often refers the inequality $|z + w| \leq |z| + |w|$ as the ***triangle inequality*** (三角不等式).

1.17 Theorem: the Cauchy-Schwarz inequality (柯西-許瓦爾茲不等式)

If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then the **Cauchy-Schwarz inequality** (柯西-許瓦爾茲不等式) holds:

$$\left|\sum_{j=1}^{n} a_j \overline{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \cdot \sum_{j=1}^{n} |b_j|^2.$$

In particular, if all numbers are real, then

$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 \leq \sum_{j=1}^{n} a_j^2 \cdot \sum_{j=1}^{n} b_j^2.$$

Remark

- The Cauchy – Schwarz inequality has many different versions. Currently, it is on complex numbers. Other versions are on euclidean spaces, function spaces, and random variables.

Euclidean Spaces 歐氏空間

1.18 Definition: euclidean spaces (歐氏空間)

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples, $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, where $x_1, \ldots, x_k$ are real numbers, called the ***coordinates*** of $\mathbf{x}$ (座標). The elements of $\mathbb{R}^k$ are called points, or vectors, especially when $k > 1$. We shall use boldfaced letters to denote vectors.

If $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, and α is a real number, put

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_k + y_k), \qquad \alpha\mathbf{x} = (\alpha x_1, \ldots, \alpha x_k),$$

so that $\mathbf{x} + \mathbf{y} \in \mathbb{R}^k$ and $\alpha\mathbf{x} \in \mathbb{R}^k$. These define two operations, called ***addition*** and ***scaler multiplication*** (數乘), respectively, in $\mathbb{R}^k$. It is easy to check that these operations satisfy the commutative, associative, and distributive laws and make $\mathbb{R}^k$ into a **vector space** (向量空間) *over the real field*. The zero element of $\mathbb{R}^k$ (sometimes called the ***origin*** or the ***null vector***) is $\mathbf{0} = (0, \ldots, 0)$.

In $\mathbb{R}^k$, a so-called **inner product** or ***dot product*** (內積, 點積) is defined as

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_i y_i,$$

and furthermore the **norm** (範數, 模) is defined as

$$\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \sqrt{\sum_{j=1}^{n} x_i^2}.$$

The vector space $\mathbb{R}^k$ with the above inner product and norm is called **euclidean space** (歐氏空間, 歐幾里德空間), or euclideann k-space.

Properties of euclidean space $\mathbb{R}^k$

Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then the following hold:

1. $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

2. $\|\alpha\mathbf{x}\| = |\alpha|\ \|\mathbf{x}\|$;

3. the triangle inequality:

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|;$$

4. the Cauchy-Schwarz inequality:

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\ \|\mathbf{y}\|.$$

Remark

- Because of Items 1, 2, and 3, the euclidean space $\mathbb{R}^k$ is regarded as a ***normed space*** (赋範空間), which is a special type of ***metric space*** (度量空間).
- Values of inner product are real numbers, not necessarily non-negative.
- The set $\mathbb{R}^1 = \mathbb{R}$ consists of all real numbers, and is usually called the *line*. The norm in $\mathbb{R}^1$ is just the absolute value of real number.
- The set $\mathbb{R}^2$ consists of all ordered pairs of real numbers and is commonly referred to as the "plane." At times, it is useful to view $\mathbb{C}$ as $\mathbb{R}^2$. The norm in $\mathbb{R}^2$ is equivalent to the absolute value of a complex number. In fact, as vector spaces, they are identical or isomorphic (同構). However, it is important to note that $\mathbb{C}$ is a field, while $\mathbb{R}^2$ is not. Division of two complex numbers is possible, but dividing two points is not.

Addendum 後記

Addendum of Chapter 1

- In this e-book, we assume that the least-upper-bound property holds in $\mathbb{R}$ without a rigorous proof. A detailed proof can be found in Rudin's book, *Principles of Mathematical Analysis*.
- To keep things simple, Chapters 1–6 will focus solely on real numbers. Complex numbers will appear

more often from Chapter 7. Chapter 9 will cover a broader range of analytical results in the euclidean spaces.

- Thanks to Proposition 1.11, one can extend the definition of the exponent b^r to rational r for $b > 0$. In fact, by using the least-upper-bound property in $\mathbb{R}$, it can be further extended to real r. In other words, for all real r and $b > 0$, the exponent b^r is well-defined as a real number.
  - For positive numbers a and b, and real numbers α and β, the following laws of exponents (指數定律) hold: ▶

    1. $a^\alpha \cdot a^\beta = a^{\alpha+\beta}$.
    2. $\dfrac{a^\alpha}{a^\beta} = a^{\alpha-\beta}$.
    3. $(a^\alpha)^\beta = a^{\alpha\cdot\beta}$.
    4. $(ab)^\alpha = a^\alpha \cdot b^\alpha$.
    5. $a^{-\alpha} = \dfrac{1}{a^\alpha}$.
    6. $a^1 = a$, $a^0 = 1$.

- Given two real numbers $b > 1$, $y > 0$, it can be proven that there is a unique real x, called the **logarithm** of y to the base b, denoted as $\log_b y$, such that $b^x = y$. Hence, by requiring $\log_b y = -\log_{b^{-1}} y$ for $0 < b < 1$, the logarithm $\log_b y$ is well-defined as a real number for all positive b and y, with $b \neq 1$.
  - For positive numbers a and b, with $a, b \neq 1$, positive numbers A and B, and real number α, the following laws of logarithms (對數定律) holds: ▶

    1. $\log_a(A \cdot B) = \log_a A + \log_a B$.
    2. $\log_a \left(\dfrac{A}{B}\right) = \log_a A - \log_a B$.
    3. $\log_a A^\alpha = \alpha \log_a A$.
    4. $a^{\log_a A} = A$.
    5. $\log_a A = \dfrac{\log_b A}{\log_b a}$.
    6. $\log_a a = 1$, $\log_a 1 = 0$.

Exercises of Chapter 1 練習題

Chapter 1: Quiz **30 Minutes**

① Which of the following is an example of a field?

A. the set of integers
B. the set of even integers
C. the set of natural numbers
D. the set of rational numbers
E. the set of irrational numbers

② Which of the following sets is an ordered field?

A. the set of integers
B. the set of even integers
C. the set of natural numbers
D. the set of real numbers
E. the set of extended real numbers

③ Which of the following sets is not closed under addition?

A. The set of odd integers
B. The set of even integers
C. The set of natural numbers
D. The set of integers
E. The set of rational numbers

④ Which of the following is NOT an example of a bounded set in $\mathbb{R}$?

A. $(0,1]$
B. $\{x:\ |x|<1\}$
C. $\{x:\ x<3\}$
D. $\{x:\ -4\le x\le 3\}$
E. $\{x:\ x^2\le 4\}$

⑤ Every nonempty bounded-above subset of $\mathbb{R}$

A. has a minimum element
B. has a maximum element
C. has an infimum
D. has a supremum
E. has a unique element

⑥ Which of the following sets contains the least upper bound within itself?

A. $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$
B. $\{x:\ x \text{ is a rational number and } x^2\le 2\}$
C. $\{x:\ x \text{ is an irrational number and } x^2\le 2\}$
D. $\{x:\ x \text{ is a rational number and } x^2< 2\}$
E. $\{x:\ x \text{ is an irrational number and } x^2< 2\}$

⑦ Which of the following is a violation of the triangle inequality?

A. $|x-y|\le|x|+|y|$
B. $|x+y|\le|x|+|y|$
C. $|x-y|\ge|x|-|y|$
D. $|x+y|\ge|x|-|y|$
E. $|x+y|\ge|x|+|y|$

⑧ Which of the following is NOT a consequence of the Cauchy-Schwarz inequality?

A. $\left|x_1x_2+x_2x_3+\cdots+x_nx_1\right|\le x_1^2+x_2^2+\cdots+x_n^2$
B. $\left|\dfrac{x_1+\cdots+x_n}{n}\right|\le\sqrt{\dfrac{x_1^2+\cdots+x_n^2}{n}}$
C. $n^2\le\left(x_1^2+\cdots+x_n^2\right)\left(x_1^{-2}+\cdots+x_n^{-2}\right)$
D. $\left|\dfrac{x_1}{x_2}+\dfrac{x_2}{x_3}+\cdots+\dfrac{x_n}{x_1}\right|\le\left(x_1^2+\cdots+x_n^2\right)\left(x_1^{-2}+\cdots+x_n^{-2}\right)$
E. $\dfrac{(x_1+\cdots+x_n)^2}{y_1^2+\cdots+y_n^2}\le\dfrac{x_1^2}{y_1^2}+\cdots+\dfrac{x_n^2}{y_n^2}$

①D, ②D, ③A, ④C, ⑤D, ⑥C, ⑦E, ⑧D

Chapter 1: Exercises

Exercise 1.1 (E)

Prove that $\sqrt{6}$ is irrational.

Hint ① Prove by contradiction.

Exercise 1.2 (E)

Prove that the set $\{\sqrt{n} : n \in \mathbb{N}\}$ is unbounded.

Hint ① Prove by contradiction.

Hint ② Apply the archimedean property.

Exercise 1.3 (I)

Prove the supremum is unique, if it exists.

Hint ① Suppose E is bounded above, and suppose β_1 and β_2 are two distinct suprema of E. Prove that $\beta_1 = \beta_2$.

Exercise 1.4 (I)

Prove the set

$$\left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \ldots\right\}$$

is unbounded. Find the infimum and the supremum.

Hint ① Let $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Prove by induction that $x_{2^n} \geq 1 + \frac{n}{2}$.

Hint ② Apply the archimedean property.

Exercise 1.5 (E)

Prove that $|x-a| < \delta$ if and only if $a - \delta < x < a + \delta$ for any real x, a, and δ with $\delta > 0$.

Hint ① Consider the cases $x \geq a$ and $x < a$, respectively.

Exercise 1.6 (D)

Prove that the set

$$\left\{|a+b| : a^2 < 2, |b+1| < 3, a, b \text{ real}\right\}$$

is bounded. Find the infimum and the supremum.

Hint ① Denote $S = \left\{|a+b| : a^2 < 2, |b+1| < 3\right\}$. Show that $0 \leq |a+b| < 4 + \sqrt{2}$ for all a, b satisfying $a^2 < 2$, $|b+1| < 3$. Apply the triangle inequality.

Hint ② To show that $\inf S = 0$, take some a, b satisfying $a^2 < 2$, $|b+1| < 3$ such that $|a+b| = 0$.

Hint ③ To show that $\sup S = 4 + \sqrt{2}$, show that for $0 < \varepsilon < 1$, the value $4 + \sqrt{2} - \varepsilon$ is not an upper bound of S.

Hint ④ Take $a = -\sqrt{2} + \frac{1}{4}\varepsilon$, $b = -4 + \frac{1}{4}\varepsilon$. Show that $a^2 < 2$, $|b+1| < 3$, and $|a+b| > 4 + \sqrt{2} - \varepsilon$.

Exercise 1.7 (E)

Let x_1 and x_2 be real numbers. Prove that

$$\big||x_1| - |x_2|\big| \leq |x_1 - x_2|.$$

Hint ① Apply the triangle inequality.

Exercise 1.8 (I)

Prove that equality holds in the Cauchy-Schwarz inequality for real numbers if and only if there are two real numbers λ and μ, not both zero, such that $\lambda a_j = \mu b_j$, $j = 1, 2, \ldots, n$.

Hint ① Let $A = \sum a_j^2$, $B = \sum b_j^2$, $C = \sum a_j b_j$. Show that $\sum |Ba_j - Cb_j|^2 = B(AB - C^2)$.

CHAPTER

第 2 章

Basic Topology 拓撲學基礎

Overview of Chapter 2

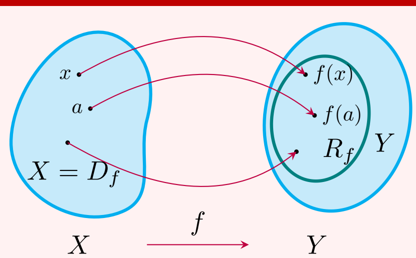
The upcoming chapter will cover the fundamental topology of real numbers, which serves as a crucial basis for mathematical analysis. It establishes a structure for examining the characteristics of mathematical entities, including functions, and enables the exploration of key concepts such as continuity, convergence, compactness, and connectedness, among others.

- We will introduce the concept of cardinality of sets and prove that:
  - The set $\mathbb{Q}$ is countable, while the set $\mathbb{R}$ is uncountable.
- We will study in detail about open and closed sets in $\mathbb{R}$.
  - A set in $\mathbb{R}$ is open if and only if its complement is closed, and *vice versa*.
  - The union of any collection of open sets is also open, whereas only a finite intersection of open sets is necessarily open.
  - A set in $\mathbb{R}$ is closed if and only if it is equal to its closure.
- We will characterize the compact sets in $\mathbb{R}$ by showing the following statements are equivalent:
  ① K is closed and bounded.
  ② K is compact.
  ③ Every infinite subset of K contains a limit point in K.
- We will prove the Bolzano-Weierstrass theorem: every bounded infinite subset of $\mathbb{R}$ has at least one limit point in $\mathbb{R}$.

§2.1 Cardinality 基數

2.1 Definition: function (函數)

Consider two sets X and Y. Suppose that for each element x of the set X, there is a unique assigned element y of the set Y, denoted by $y = f(x)$. Then f is said to be a **function** (函數) of X into Y (or a ***mapping*** (映射, 映像)), for which we denote $f\colon X \to Y$.



Remark

- The terms "*function*" and "*mapping*" are often used interchangeably in math, but their usage depends on context. A function assigns one output to each input, while a mapping describes any relation between two sets that assigns elements from one set to elements from the other set.

- The set X is called the **domain** (定义域) of f, denoted as D_f (that is, $D_f = X$).
- The set Y is called the **codomain** (陪域) of f.
- The set of all values of f is called the **range** (值域) of f, denoted as R_f. Generally, $R_f \neq Y$.
- The element y is called the ***image*** of x under f.
- The element x is called the ***inverse image*** of y under f.
- The element $f(x)$ is called the ***value*** (值) of f.

Let $E \subset X$. Denote $f(E) = \{f(x) : x \in E\}$, called the *image* of E under f. $R_f = f(A)$.
Let $F \subset Y$. Denote $f^{-1}(F) = \{x \in X : f(x) \in F\}$, called the *inverse image* of F under f.

If $R_f = Y$, that is, every element of Y is an image of an element of X under f, then f maps X ***onto*** Y, and f is said to be **surjective** (滿射).

If f maps distinct elements of its domain to distinct elements, that is, $f(x_1) = f(x_2)$ implies $x_1 = x_2$, then f is said to be **injective** (單射).

If a mapping f is both surjective and injective, then it is said to be **bijective** (雙射).

Remark

- In mathematics, the term "onto" is used as synonymous with "surjective"; while "one-to-one" is used as synonymous with "injective".
- When f is injective, it is also said to be a 1-1 (one-to-one) mapping (一對一).
- The term "bijective" is used as synonymous with "***one-to-one correspondence***" (一一對應). One must not be confused with the latter with "one-to-one mapping".

Suppose $f : X \to Y$ is injective so that for every $y \in R_f$, there exists a unique $x \in X$ satisfying $f(x) = y$. Then a mapping $g : R_f \to X$ can be properly defined as follows: for each $y \in R_f$, define $g(y) = x$, where x satisfies $f(x) = y$. We call the mapping g to be the **inverse mapping** (逆映射) of f, denoted to be f^{-1}, with domain $D_{f^{-1}} = R_f$, and range $R_{f^{-1}} = X$.

2.2 Definition: cardinality and cardinal number (基數)

If there exists a bijective mapping from set A to set B, then we say that A and B have the same **cardinality** (基數), or briefly, that A and B are ***equivalent***, and write $A \sim B$. We use the term ***cardinal number*** to refer the size or quantity of a set A, and write it as $|A|$.

Two sets "having the same cardinal number" is an **equivalence relation** (等價關係), that satisfies the following properties

1. ***Reflexivity*** (自反性): $A \sim A$.
2. ***Symmetry*** (對稱性): if $A \sim B$, then $B \sim A$.
3. ***Transitivity*** (傳遞性): if $A \sim B$ and $B \sim C$, then $A \sim C$.

Let $\mathbb{N}_{\le n} = \{1, 2, \ldots, n\}$ for any $n \in \mathbb{N}_+$, where $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. For any set A, we say

- A is ***finite*** (有限集) if $A \sim \mathbb{N}_{\le n}$ for some n. In this case, we have $|A| = n$.
- A is ***infinite*** (無限集) if A is not finite.
- A is ***countable*** (可數集) if $A \sim \mathbb{N}_+$. In this case, we write $|A| = \aleph_0$.
- A is ***uncountable*** (不可數集) if A is neither finite nor countable.
- A is ***at most countable*** (至多可數集) if A is finite or countable.

Remark

- When an equivalence relation is applied to a set, it divides the set into multiple separate and non-overlapping subsets, which are referred to as **equivalence classes** (等價類).

Remark

- For two *infinite* sets A and B, "having the same number of elements" becomes quite vague. The existence of bijection from A to B makes a rigorous sense.
- For two *finite* sets A and B, $A \sim B$ if and only if $|A| = |B|$.
- Any finite set cannot be equivalent to one of its proper subsets.
- Any infinite set is equivalent to one of its proper subsets.
- The power set of set A, denoted as 2^A, is the set that contains all subsets of A.
  - If $|A| = n < \infty$, then $\left|2^A\right| = 2^n$.

2.3 Definition: sequence (序列)

We call a mapping f **sequence** (序列), if it is defined on the set $\mathbb{N}_+$ of all positive integers. It is customary to write a sequence as $\{x_n\}$, or $x_1, x_2, x_3, \dots$, where $x_n = f(n)$ for $n \in \mathbb{N}_+$. The values of f, that is, the elements x_n, are called the ***terms*** of the sequence. If $x_n \in S$ for all $n \in \mathbb{N}_+$, then $\{x_n\}$ are said to be a ***sequence in*** S. Usually, x_k is said to be the kth term of the sequence $\{x_n\}$.

Remark

- The terms $x_1, x_2, x_3, \dots$ of a sequence need not be distinct.
- We also regard the sequence $\{x_n\}$ as a set.
- Every countable set can be regarded as a sequence of distinct terms.
- Sometimes it is convenient to replace $\mathbb{N}_+$ by $\mathbb{N}$, the set of all natural (nonnegative) integers, so that a sequence may start with 0 rather than with 1.

2.4 Definition: union and intersection of sets (並集和交集)

Let A and Ω be sets, and suppose that for each element $\alpha \in A$, there is a corresponding subset E_α of Ω. We denote the set of all such subsets by $\{E_\alpha\}$, which is referred to as a collection or family of sets.

The **union** (並集) of the sets E_α is the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$, and denote

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

If $A = \mathbb{N}_+$, one usually writes

$$S = \bigcup_{m=1}^{\infty} E_m.$$

If $A = \mathbb{N}_{\le n}$, the usual notation is

$$S = \bigcup_{m=1}^{n} E_m \quad \text{or} \quad S = E_1 \cup E_2 \cup \cdots \cup E_n.$$

The **intersection** (交集) of the sets E_α is the set T such that $x \in T$ if and only if $x \in E_\alpha$ for every $\alpha \in A$, and denote

$$T = \bigcap_{\alpha \in A} E_\alpha.$$

If $A = \mathbb{N}_+$, one usually writes

$$T = \bigcap_{m=1}^{\infty} E_m.$$

If $A = \mathbb{N}_{\le n}$, the usual notation is

$$T = \bigcap_{m=1}^{n} E_m \quad \text{or} \quad T = E_1 \cap E_2 \cap \cdots \cap E_n.$$

- We say that A and B ***intersect*** (相交), if $A \cap B \neq \varnothing$.
- We say that A and B ***disjoint*** (不相交), if $A \cap B = \varnothing$.

Remark

- Many properties of unions and intersections are quite similar to those of sums and products. For instance, the commutative, the associative, and the distributive laws hold:

$$A \cup B = B \cup A, \; (A \cup B) \cup C = A \cup (B \cup C)$$

$$A \cap B = B \cap A, \; (A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

▶

2.5 Proposition: infinite subset of a countable set is countable

Every infinite subset of a countable set is countable. ▶

2.6 Proposition: countable union of countable sets is countable

Let $\{E_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of countable sets. Then the union $\bigcup_{n=1}^{\infty} E_n$ is countable. ▶

Corollary

Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Then $\bigcup_{\alpha \in A} B_\alpha$ is at most countable.

2.7 Proposition: n-tuples of a countable set is countable

Let A be a countable set, and B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable. ▶

Corollary

The n-dimensional integer lattice $\mathbb{Z}^n$, whose points are n-tuples of integers, is countable.

Corollary

The set of all rational numbers $\mathbb{Q}$ is countable. ▶

2.8 Proposition: the set of 0-1 sequences is uncountable

Let A be the set of all sequences whose elements are the digits 0 and 1. Then A is uncountable. ▶

Remark

- A sequence whose elements are the digits 0 and 1 is like $1, 0, 0, 1, 0, 1, 1, 0 \ldots$.
- Real numbers can be expressed in decimal (using 0 to 9) or binary (base 2) representations. The proposition implies that the set of real numbers $\mathbb{R}$ is uncountable, meaning there are significantly more real numbers than rational numbers.
- We shall give a second proof of this fact in Theorem 2.23.

§2.2 Subsets in ℝ 實數的子集

2.9 Definition: subsets in ℝ (實數的子集)

All points and sets mentioned below are elements and subsets of ℝ.

- The set $(a, b) = \{x : a < x < b\}$ is called ***open interval***.
- The set $[a, b] = \{x : a \leq x \leq b\}$ is called ***closed interval***.
- The sets
$$(a, b] = \{x : a < x \leq b\},$$
$$[a, b) = \{x : a \leq x < b\}$$
are both called ***half-open intervals***.

- The **complement** (補集) of a set E, denoted by E^c, is the set of all points in ℝ but not in E. $\boxed{(E^c)^c = E}$
- The **difference** (差集) of B and A, written $B\backslash A$ or $B - A$, is the set of all points in B but not in A. $\boxed{B\backslash A = B \cap A^c}$

- A **neighborhood** (鄰域) of a point p is the set $N_r(p) = \{q \in \mathbb{R} : |p - q| < r\} \subset \mathbb{R}$, where r is called the *radius* of $N_r(p)$.
- A point p is a **limit point** (極限點) of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.
- If $p \in E$ and p is not a limit point of E, then p is called an **isolated point** (孤立點) of E.
- A set E is **closed** (閉集) if every limit point of E is a point in E.
- The **closure** (閉包) of E is the set $\overline{E} = E \cup E'$, where E' the set of all limit points of E in ℝ.
- A point p is an ***interior point*** of E (內點) if there is a neighborhood N of p such that $N \subset E$. The **interior** (內部) of E, denoted by E°, is the set of all its interior points.
- A set E is **open** (開集) if every point of E is an interior point of E.
- A set $E \subset X \subset \mathbb{R}$ is **relatively open** (相對開集) in X (or simply "***open in*** X") if there is an open set U of ℝ such that $E = U \cap X$.
- A set E is **perfect** (完全集) if E is closed and if every point of E is a limit point of E.
- A set E is **bounded** (有界集) if there is a real number M such that $|p| < M$ for all $p \in E$. Equivalently, E is bounded if it is contained in a bounded interval.
- A set E is **dense** (稠密集) in X if every point of X is a limit point of E, or a point of E, or both.
- A point p is a ***boundary point*** of E (邊界點) if any neighborhood of p intersects both E and E^c. The **boundary** (邊界) of E, denoted by ∂E, is the set comprising all boundary points of E.

Remark

- A limit point of E, or a boundary point of E, is not necessarily a point of E.
- A *closed* set is a set which is closed for limits. So, any finite set is closed.
- The only subsets in ℝ that are both open and closed are the empty set $\varnothing$ and the entire real line ℝ.
- Since $E = \mathbb{R} \cap E$, so every set is relatively open to itself, even for closed sets in ℝ.
- Since a sequence can be regarded as a set, hence, it makes sense to call a sequence to be *bounded*.

Remark

- The concepts of neighborhood, open set, closed set, dense set, etc., can be extended from the real numbers to metric spaces. We will start applying these general concepts in Chapter 7.

Open and Closed Sets 開集與閉集

2.10 Proposition: on neighborhoods and limit points

Every neighborhood is an open set. ▶

A finite point set has no limit points.

If p is a limit point of E, then every neighborhood of p contains infinitely many points of E. ▶

2.11 Proposition: the complement of a union equals the intersection of complements

Let $\{E_\alpha\}$ be a (finite ot infinite) collection of sets in $\mathbb{R}$. Then

$$\Big(\bigcup_\alpha E_\alpha\Big)^{\mathsf{c}} = \bigcap_\alpha E_\alpha^{\mathsf{c}}, \qquad \Big(\bigcap_\alpha E_\alpha\Big)^{\mathsf{c}} = \bigcup_\alpha E_\alpha^{\mathsf{c}}.$$ ▶

2.12 Proposition: a set is open if and only if its complement is closed

A set in $\mathbb{R}$ is open if and only if its complement is closed. ▶

A set in $\mathbb{R}$ is closed if and only if its complement is open. ▶

2.13 Proposition: the union of open sets is open; the intersection of closed sets is closed

Suppose the sets mentioned below are subsets of $\mathbb{R}$.

1. For any collection of $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

2. For any collection of $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

3. For any *finite* collection of $G_1, \dots, G_n$ of open sets, $\bigcap_{j=1}^{n} G_j$ is open.

4. For any *finite* collection of $F_1, \dots, F_n$ of closed sets, $\bigcup_{j=1}^{n} F_j$ is closed. ▶

Remark

- The hypothesis of finite collection is indispensable in items 3 and 4. ▶

2.14 Proposition: characterization of the closure

Suppose $E \subset \mathbb{R}$. Then

1. $\overline{E}$ is closed. **2.** $E = \overline{E}$ if and only if E is closed.

3. $\overline{E} \subset F$ for every closed set $F \subset \mathbb{R}$ such that $E \subset F$. ▶

Remark

- According to items **1** and **3**, $\overline{E}$ is the *smallest* closed subset of $\mathbb{R}$ that contains E.

Remark

- Because closed sets are closed under intersection, the closure of a set can be obtained by taking the intersection of all closed sets that contain it.
- The closure $\overline{E}$ is the union of E and its boundary ∂E, that is, $\overline{E} = E \cup \partial E$.

2.15 Proposition: the supremum of bounded-above set of real numbers is in the closure of the set

Let E be a nonempty set of *real numbers* which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed. An analogue result is also true for the infimum. ▶

Connected Sets 連通集

2.16 Definition: connected sets (連通集)

- Two subsets A and B of $\mathbb{R}$ are **separated** (分離) if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.
- A set $E \subset \mathbb{R}$ is **connected** (連通) if E is not a union of two nonempty separated sets.

Remark

- Separated sets are disjoint, but disjoint sets need not be separated. ▶

characterization of connected set

A subset E of $\mathbb{R}$ is connected if and only if for any $x, y \in E$ and $x < z < y$, then $z \in E$. ▶

§2.3 Compact Sets 緊致集

2.17 Definition: compact set (緊集)

- An ***open cover*** of a set $E \subset \mathbb{R}$ is a collection $\{G_\alpha\}$ of open subsets of $\mathbb{R}$ such that $E \subset \bigcup_\alpha G_\alpha$.
- A set $K \subset \mathbb{R}$ is said to be **compact** (緊集, 緊致集) if every open cover of K contains a finite subcover. More explicitly, if $\{G_\alpha\}$ is an open cover of (compact set) K, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that
$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

Remark

- Note that the term *open cover* refers to the collection of open sets $\{G_\alpha\}$ and not to their union $\bigcup_\alpha G_\alpha$, which is a subset of $\mathbb{R}$.
- Every finite set is compact.
- The open interval $(0, 1)$ is not compact. In general, any open interval (a, b) is not compact. ▶

Remark

- Compactness is a fundamental concept in mathematical analysis. It plays a fundamental role in proving the existence of solutions to equations and systems of equations, particularly in infinite-dimensional spaces. It allows us to establish convergence and to find fixed points of mappings, which are key steps in many existence proofs.

2.18 Proposition: the relationship between compact sets and closed sets

Assume all sets mentioned below are subsets of $\mathbb{R}$.

1. Compact sets are closed.

2. Closed subsets of compact sets are compact. ▶

Corollary

If F is closed and K is compact, then $F \cap K$ is compact.

Remark

- Closed sets are not necessarily compact. For instance, $\mathbb{R}$ is closed but not compact.

2.19 Theorem: finite closed intervals are compact

Every bounded closed interval is compact. ▶

2.20 Theorem: Cantor's intersection theorem 康托爾交集定理

If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, $n = 1, 2, 3, \ldots$, then $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$. ▶

Corollary - the nested intervals theorem 區間套定理

If $\{I_n\}$ is a sequence of nested closed intervals, that is, $I_n \supset I_{n+1}$, $n = 1, 2, 3, \ldots$, then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

2.21 Theorem: charaterizations of compact sets in $\mathbb{R}$ and the Heine-Borel theorem ▶

Let $K \subset \mathbb{R}$. The following statements are equivalent:

1. K is closed and bounded.

2. K is compact.

3. Every infinite subset of K contains a limit point in K. ▶

Remark

- The equivalence of items **1** and **2** is also known as the Heine-Borel theorem (海涅-博雷爾定理).

Remark

- The study of the equivalence of compactness is important because it allows us to apply the same proof techniques to different contexts, connect different areas of mathematics, understand the properties of different types of spaces, and apply mathematics to other fields.

2.22 Theorem: the Bolzano-Weierstrass theorem (聚点定理) ▶

Every bounded infinite subset of $\mathbb{R}$ has at least one limit point in $\mathbb{R}$. ▶

2.23 Proposition: nonempty perfect set is uncountable

Every nonempty perfect set of real numbers is uncountable. ▶

Corollary

Every interval $[a, b]$ $(a < b)$ is uncountable. The set of all real numbers is uncountable.

Remark

- Starting with $[0, 1]$ and repeatedly removing the middle third open interval of each remaining segment gives rise to the Cantor set. The set is nonempty and perfect, and it contains no nontrivial intervals.

Addendum 後記

Addendum of Chapter 2

- Denoting the cardinality of the set of all real numbers as $\aleph_1$, we have $\aleph_0 < \aleph_1$. The *continuum hypothesis* states that there is no set with a cardinality strictly between that of the integers and the real numbers, and that $\aleph_1$ is the smallest cardinal number greater than $\aleph_0$. However, in the early 1960s, Kurt Gödel and Paul Cohen independently showed that the hypothesis cannot be proven or disproven using the widely accepted Zermelo–Fraenkel set theory with the axiom of choice, which is the most common foundation of mathematics. This discovery had a profound impact on the foundations of mathematics, raising questions about the nature of mathematical truth and the limits of mathematical proof.

- This e-book primarily focuses on continuity, differentiation, and integration on the real line. As a starting point, we examine open sets and compact sets in $\mathbb{R}$ to establish a foundation. However, it's important to note that these concepts have broad applications in mathematics, including topology, analysis, and geometry. They can be extended to more general spaces like metric spaces. Understanding these concepts is crucial for comprehending the properties of spaces and the behavior of functions defined on them.

Exercises of Chapter 2 練習題

Chapter 2: Quiz **30 Minutes**

① Which of the following sets is not countable?

A. the set of prime numbers
B. the set of all positive integers
C. the set of all rational numbers
D. the set of all n-tuples of integers
E. the set of all infinite sequences of 0's and 1's

② Let $\{A_\alpha\}$ be a collection of sets. Which of the following is equivalent to $\bigcup_\alpha A_\alpha = \bigcap_\alpha A_\alpha^c$?

A. $\bigcap_\alpha A_\alpha = \varnothing$
B. $\bigcup_\alpha A_\alpha = \varnothing$
C. $\bigcup_\alpha A_\alpha^c = \varnothing$
D. $\bigcap_\alpha A_\alpha^c = \varnothing$
E. $\bigcup_\alpha A_\alpha = \mathbb{R}$

③ Let $A \subset \mathbb{R}$. Which of the following is true?

A. the closure of A is equal to A
B. the interior of A is a subset of A
C. the boundary of A is an open set
D. if A is closed, then the boundary of A is empty
E. if A is open, then the boundary of A is empty

④ In $\mathbb{R}$, which of the following statements is true?

A. every open set is connected
B. every closed set is compact
C. every bounded set is closed
D. every closed set is bounded
E. every compact set is closed

⑤ Which of the following statements is NOT true for a compact set in $\mathbb{R}$?

A. every compact set is limit point compact
B. every compact set is closed
C. every compact set is bounded
D. every bounded subset of a compact set is compact
E. every closed subset of a compact set is compact

⑥ Which of the following is true in $\mathbb{R}$?

A. any nonempty compact subset has a maximum element
B. any nonempty bounded subset has a maximum element
C. any nonempty closed subset has a maximum element
D. any nonempty connected subset has a maximum element
E. none of the above

⑦ Let A and B be subsets of $\mathbb{R}$. Which of the following statements is true?

A. if A is open and B is closed, then $A \cap B$ is closed
B. if A is closed and B is open, then $A \cup B$ is open
C. if A is compact and B is closed, then $A \cap B$ is compact
D. if A is open and B is compact, then $A \cap B$ is open
E. if A is dense and B is open, then $A \cap B$ is dense

⑧ Which of the following statements is true in $\mathbb{R}$?

A. every compact set is a perfect set
B. every perfect set is a compact set
C. every closed set is a perfect set
D. every perfect set is a closed set
E. every dense set is a perfect set

①E, ②B, ③B, ④E, ⑤D, ⑥A, ⑦C, ⑧D

Chapter 2: Exercises

Exercise 2.1 (I)

Show that the open interval $(0,1)$ and the closed interval $[0,1]$ have the same cardinality by constructing a bijective mapping between them.

Hint ① Take a countable set S in $(0,1)$ and construct a bijective mapping from S to $S \cup \{0,1\}$.

Exercise 2.2 (E)

Prove that $E \subset \mathbb{R}$ is open if and only if $E = E^\circ$.

Hint ① $\Rightarrow$: Assume that E is open. To show $E = E^\circ$, prove that $E \subset E^\circ$ and $E^\circ \subset E$.

Hint ② $\Leftarrow$: Assume that show $E = E^\circ$. To show E is open, prove that E° is open.

Exercise 2.3 (I)

Let $E \subset \mathbb{R}$. Prove that the interior E° of E is the largest open set contained in E.

Hint ① The interior E° is an open set contained in E.

Hint ② Show that if U is an open set contained in E, then $U \subset E^\circ$.

Hint ③ Let x be a point in U. Show that x is an interior point of E.

Exercise 2.4 (I)

Let $E_1, \ldots, E_n$ be a finite collection of subsets of real numbers. Prove that

$$\overline{\bigcup_{j=1}^{n} E_j} = \bigcup_{j=1}^{n} \overline{E}_j.$$

Hint ① Only need to prove the relation for two subsets A and B.

Hint ② Show $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$: Since $A \subset A \cup B$ and $B \subset A \cup B$, we have $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$.

Hint ③ Show $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$: Since $A \subset \overline{A}$ and $B \subset \overline{B}$, we have $A \cup B \subset \overline{A} \cup \overline{B}$.

Hint ④ The closures $\overline{A}$ and $\overline{B}$ are closed. The union of two closed sets is closed.

Exercise 2.5 (I)

Consider the set

$$S = \left\{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots, \right\}$$

1. For the open cover of S,

   $(-\varepsilon, \varepsilon), (1-\varepsilon, 1+\varepsilon), (\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon), \ldots,$

   where ε satisfies $0 < \varepsilon < \frac{1}{2}$, find a finite subcover of S.
2. Prove that S is compact.

Hint ① For part **1**, choose a positive integer N such $\varepsilon > \dfrac{1}{N}$. Show that the interval $(-\varepsilon, \varepsilon)$ contains $0, \dfrac{1}{N}, \dfrac{1}{N+1}, \dfrac{1}{N+2}, \ldots.$

Hint ② For part **2**, prove that S is bounded and closed.

Hint ③ Prove that S is closed, by showing that 0 is the only limit point of S and $0 \in S$.

Exercise 2.6 (I)

Suppose that $\{A_n\}$ is a sequence of nonempty bounded open subsets of $\mathbb{R}$ satisfying

$$A_n \supset \overline{A}_{n+1}, \quad n = 1, 2, \ldots.$$

Prove that $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$.

Hint ① For each n, $\overline{A}_n$ is nonempty and compact.

Hint ② Show that $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A}_n$.

Exercise 2.7 (I)

1. Prove that the union of a finite collection of perfect sets in $\mathbb{R}$ is perfect.
2. Construct a countable collection of perfect sets in $\mathbb{R}$ such that their union is not perfect.

Hint ① For part **1**, prove the conclusion for two perfect sets A and B.

Hint ② Suppose x is a point of $A \cup B$. Prove that x is a limit point of $A \cup B$.

Hint ③ For part **2**, consider the union of A_n, where

$$A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right], \quad n = 1, 2, \ldots.$$

Exercise 2.8 (D)

Prove that every open set in $\mathbb{R}$ can be expressed as the union of at most countably many disjoint open intervals.

Hint ① Let G be an open set in $\mathbb{R}$. For each $x \in G$, there are y and z, with $z < x < y$, such that $(z, y) \subset G$.

Hint ② Show that $a, b \notin G$, where

$$a = \inf\{z : (z, x) \subset G\}, \quad b = \sup\{y : (x, y) \subset G\}.$$

Hint ③ Show that G equals the union of such open intervals.

Hint ④ Show that these open intervals are disjoint.

Hint ⑤ Use rational numbers as representatives to show that the collection of these open intervals is countable.

CHAPTER

第 3 章

Sequences and Series 序列與級數

Overview of Chapter 3

We will study convergence of sequences and series in this chapter.

- For real sequences, we will prove that
  - Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.
  - A sequence converges if and only if it is a Cauchy sequence.
  - A monotonic sequence converges if and only if it is bounded.
- We will extend the concept of limit for sequences to the upper and the lower limits, whose values are in $\overline{\mathbb{R}}$ (the extended real numbers). We will prove that $\lim_{n\to\infty} x_n = x$ if and only if $\overline{\lim_{n\to\infty}}\, x_n = \underline{\lim_{n\to\infty}}\, x_n = x$, where $x \in \overline{\mathbb{R}}$.
- We will study convergence/divergence of series.
  - We will prove the Comparison Test, the Root Test, and the Ratio Test. These tests can be used to determine whether a series converges absolutely.
  - We will prove Dirichlet's Test, Abel's Test, and the Alternating Series Test. These tests are applicable for determining the conditional convergence.
- We will explain the differences between the absolute convergence and the conditional convergence.

§3.1 Convergent Sequences 收斂序列

3.1 Definition: convergent sequence (收斂序列)

Suppose $\{x_n\}$ is a sequence in $\mathbb{R}$. We say that $\{x_n\}$ is **convergent** (收斂) if there is a point $x \in \mathbb{R}$ with the following property:

for every $\varepsilon > 0$, there is an integer N such that $n \geq N$ implies that $|x_n - x| < \varepsilon$.

In this case, we also say that $\{x_n\}$ converges to x. The point x is called a **limit** (極限) of $\{x_n\}$, and write $x_n \to x$, or $\lim_{n\to\infty} x_n = x$.

If $\{x_n\}$ does not converge, it is said to be **divergent** (發散).

an equivalent statement

for every $\varepsilon^* > 0$, there is a number N such that $n \geq N$ implies that $|x_n - x| < f(\varepsilon^*)$, where f is some function such that $f(\varepsilon^*) < \varepsilon$.

- $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} (x_n - x) = 0$.
- $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} |x_n| = 0$.

3.2 Proposition: properties of convergent sequence

Let $\{x_n\}$ be a sequence in $\mathbb{R}$.

1. A convergent sequence has a unique limit: if $x_n \to x$ and $x_n \to x'$, then $x = x'$.

2. Sequence $\{x_n\}$ converges to $x \in \mathbb{R}$ if and only if every neighborhood of x contains x_n for all but finitely many n.

3. If $E \subset \mathbb{R}$ and if x is a limit point of E, then there is a sequence $\{x_n\}$ in E such that $x = \lim\limits_{n\to\infty} x_n$.

4. If $\{x_n\}$ converges, then $\{x_n\}$ is bounded. ▶

3.3 Proposition: operations on convergent sequences

Suppose $\{x_n\}$, $\{y_n\}$ are real sequences, and $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$. Thus

1. $\lim\limits_{n\to\infty} (x_n + y_n) = x + y$;

2. $\lim\limits_{n\to\infty} cx_n = cx$, $\lim\limits_{n\to\infty} (c + x_n) = c + x$, for any real number c;

3. $\lim\limits_{n\to\infty} x_n y_n = xy$;

4. $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = \dfrac{1}{x}$, provided $x_n \neq 0$ $(n = 1, 2, 3, \ldots)$, and $x \neq 0$. ▶

Subsequences 子序列

3.4 Definition: subsequence (子序列)

For a given sequence $\{x_n\}$ of real numbers, if some (possibly none) terms are omitted from the sequence, the remaining sequence is called a **subsequence** (子序列) of $\{x_n\}$. It is customary to write a subsequence as $\{x_{n_k}\}$, where $n_1 < n_2 < n_3 < \cdots$.

If a subsequence $\{x_{n_k}\}$ converges, its limit is called a ***subsequential limit*** of $\{x_n\}$.

Remark

- Any subsequence of the original sequence retains terms in their original order.
- The indices of any subsequence satisfy: $\boxed{n_k < n_{k+1}}$ and $\boxed{k \leq n_k}$.
- A sequence $\{x_n\}$ converges to x if and only if every subsequence of $\{x_n\}$ converges to x. ▶

3.5 Theorem: compact implies every sequence has a convergent subsequence

If $K \subset \mathbb{R}$ is compact, then every sequence of K has a subsequence that converges to a point in K. ▶

Corollary

Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence. ▶

3.6 Proposition: subsequential limits form a closed set

The subsequential limits of any sequence in $\mathbb{R}$ form a closed subset of $\mathbb{R}$. ▶

Cauchy Sequences 柯西序列

3.7 Definition: Cauchy sequence (柯西序列)

A sequence $\{x_n\}$ in $\mathbb{R}$ is a **Cauchy sequence** (柯西序列) if

for every $\varepsilon > 0$ there exists an integer N such that $|x_n - x_m| < \varepsilon$ if $n, m \geq N$.

3.8 Theorem: convergence of Cauchy sequences

A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence. ▶

3.9 Definition: monotonic sequence (單調序列)

Let $\{x_n\}$ be a sequence of real numbers.

If $x_n \leq x_{n+1}$ for $n = 1, 2, 3, \ldots$, we call the sequence **monotonically increasing** (單調上升).

If $x_n \geq x_{n+1}$ for $n = 1, 2, 3, \ldots$, we call the sequence **monotonically decreasing** (單調下降).

A sequence is **monotonic** (單調) if it is either monotonically increasing or monotonically decreasing.

3.10 Theorem: the monotone convergence theorem (單調收斂定理)

1. If a sequence is monotonically increasing and bounded above, then its supremum is the limit.
2. If a sequence is monotonically decreasing and bounded below, then its infimum is the limit.
3. A monotonic sequence converges if and only if it is bounded. ▶

Remark

- A monotonically increasing sequence is always bounded below, and if bounded above, then it is bounded. Similarly, a monotonically decreasing sequence is bounded if bounded below.

Upper and Lower Limits 上、下極限

3.11 Definition: infinite limits (無窮大極限)

Let $\{x_n\}$ be a sequence of real numbers.

We say that $\{x_n\}$ is ***divergent to*** $\boldsymbol{\infty}$ if for every real M there is an integer N such that $n \geq N$ implies $x_n \geq M$. We write $x_n \to \infty$, or $\lim\limits_{n\to\infty} x_n = \infty$.

We say that $\{x_n\}$ is ***divergent to*** $\boldsymbol{-\infty}$ if for every real M there is an integer N such that $n \geq N$ implies $x_n \leq M$. We write $x_n \to -\infty$, or $\lim\limits_{n\to\infty} x_n = -\infty$.

Remark

- A sequence that approaches a finite number has a convergent limit, while a sequence that does not approach a finite number, such as with infinite limits, is *divergent*.

Remark

- Some operational rules on limits can be easily extended to infinite limits, and some additional rules hold in $\overline{\mathbb{R}}$.
  - For any constant c, if $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty}(c+x_n) = \infty$; if $\lim_{n\to\infty} x_n = -\infty$, then $\lim_{n\to\infty}(c+x_n) = -\infty$.
  - if $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} cx_n = \infty$ if $c > 0$; $\lim_{n\to\infty} cx_n = -\infty$ if $c < 0$.

3.12 Definition: upper and lower limits (上極限、下極限)

Let $\{x_n\}$ be a sequence of real numbers.

In the *extended real number system*, let E be the set comprising all the numbers x such that $x_{n_i} \to x$ for some subsequence $\{x_{n_i}\}$. Define the numbers x^*, x_*:

$$\boxed{\textbf{upper limit}}\quad x^* = \sup E; \qquad \boxed{\textbf{lower limit}}\quad x_* = \inf E,$$

called the upper limit (上極限) and the lower limit (下極限) of the sequence $\{x_n\}$, respectively. We also write

$$x^* = \limsup x_n \text{ or } \varlimsup_{n\to\infty} x_n; \qquad x_* = \liminf x_n \text{ or } \varliminf_{n\to\infty} x_n.$$

Remark

- One advantage that upper and lower limits have over limits on sequences is their existence.

  Every sequence of real numbers must have the upper and the lower limits.

3.13 Proposition: characterization of the upper limit

Assume that $\{x_n\}$ is a sequence of real numbers. Let E be the set comprising all the numbers $x \in \overline{\mathbb{R}}$ (the extended real numbers) such that $x_{n_k} \to x$ for some subsequence $\{x_{n_k}\}$, and let x^* be the upper limit of the sequence $\{x_n\}$. Then x^* has the following two properties:

1. $x^* \in E$.

2. If $y > x^*$, there is an integer N such that $n \geq N$ implies $x_n < y$.

Moreover, x^* is the only number with these two properties.

An analogue result is also true for the lower limit x_*.

3.14 Proposition: order rules of upper and lower limits

Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. If $x_n \leq y_n$ for $n \geq N_0$, where N_0 is fixed, then

$$\varliminf_{n\to\infty} x_n \leq \varliminf_{n\to\infty} y_n, \qquad \varlimsup_{n\to\infty} x_n \leq \varlimsup_{n\to\infty} y_n.$$

Corollary

The limit $\lim_{n\to\infty} x_n = x$ if and only if

$$\varlimsup_{n\to\infty} x_n = \varliminf_{n\to\infty} x_n = x.$$

Remark

- The statement "$x_n < y_n$" does not imply "$\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$", even if both limits exist.

§3.2 Series 級數

3.15 Definition: convergent series (收斂級數)

Suppose $\{a_n\}$ is a sequence of real numbers. Consider the cumulative sum of $\{a_n\}$:

$$a_1 + a_2 + a_3 + \cdots ,$$

which is called an infinite series. In order to formally make sense of this infinite series, for each positive integer n, we call the number

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

the ***n-th partial sum*** (部分和) of the series. If the sequence $\{s_n\}$ has a limit $\lim_{n\to\infty} s_n = s$, where s is finite, then we say the infinite series is **convergent** (收斂), and denote $\sum_{k=1}^{\infty} a_k = s$. The value s is called the **sum** (級數和) of the series.

If the sequence $\{s_n\}$ is divergent, then we say the series is **divergent** (發散).

Remark

- $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n > 1$.
- Sometimes, one may consider $\sum_{n=0}^{\infty} a_n$, an infinite series begining with $n = 0$. In many cases, when there is no possible ambiguity, one may simply write $\sum a_n$.
- Unless dealing with some comparatively straightforward cases, determining the sum of a given series can be highly challenging.
- The challenge in deciding whether a series converges or diverges lies in the fact that there is no single test that can be used for all series. Rather, there exist numerous tests that are only applicable to specific types of series. The lack of a universal test for all series underscores the intricacy (錯綜複雜) of the subject and emphasizes the importance of scrutinizing and assessing each series on its own merit.

3.16 Theorem: the Cauchy criterion for convergence of series

A series $\sum a_n$ coverges if and only if the **Cauchy criterion** (柯西準則) holds:

for every $\varepsilon > 0$, there exists an integer N such that $\left|\sum_{k=n}^{m} a_k\right| < \varepsilon$ if $m \geq n \geq N$.

Corollary

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

the Divergence Test (發散判斷法)

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Remark

- The converse statement of the corollary is not true. In other words, one cannot conclude that if $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
- The Divergence Test is the contrapositive statement of the corollary.
- When the limit $\lim_{n\to\infty} a_n \neq 0$, it means that either the limit does not exist, or the limit exists but is not equal to 0.

3.17 Proposition: convergence of series with nonnegative terms (非負項級數收斂準則)

A series of real numbers with nonnegative terms converges if and only if its partial sums form a bounded sequence.

3.18 Proposition: addition and scalar multiplication of series

If $\sum a_n = A$, and $\sum b_n = B$, then

1. $\sum (a_n + b_n) = A + B$;

2. $\sum ca_n = cA$ for any finite number c.

Remark

- This statement demonstrates that treating infinite series as finite series is possible when it comes to addition and scalar multiplication.
- However, the same approach cannot be applied to the product of two convergent series, contrary to what one might assume.

Euler's Number e 歐拉數

3.19 Definition: the number e (歐拉數)

Define $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. This number e is known as **Euler's number**.

Remark

- The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges, and furthermore, the value falls between 2 and 3.

  In other words, the number e is well-defined and $2 < e < 3$.

Remark

- Euler's number e is an important universal constant, because it appears in many different areas of mathematics and science. One finds it is quite convenient to use the number e to describe exponential growth and decay, the solutions of differential equations, and the distribution of random variables.

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Remark

- This limit is one of the most important and fundamental limits in mathematics.

Convergence Tests 審斂法

3.20 Theorem: the Comparison Test (比較審斂法)

1. Suppose that $0 \le |a_n| \le b_n$ for $n \ge N_0$, where N_0 is some fixed integer. If $\sum b_n$ converges, then $\sum a_n$ converges.

2. Suppose that $0 \le a_n \le b_n$ for $n \ge N_0$, where N_0 is some fixed integer. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Remark

- In Item 1, under the same hypothesis, if $\sum b_n$ converges, then $\sum |a_n|$ and $\sum a_n$ both converge.
- The statement in Item 2 is the contrapositive statement in Item 1.

Remark

- The Comparison Test has a significant limitation in that it necessitates finding another series that is comparable to the series being analyzed and is known to converge or diverge. This can be challenging or unfeasible in certain cases, particularly for intricate series.
- The geometric series and the p-series are two commonly used series for comparing the convergence of other series.
- The limit version of the Comparison Test may be more convenient in some situations. ▶

3.21 Theorem: the Root Test and the Ratio Test

the Root Test (根值審斂法)

Given $\sum a_n$, put $\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$. Then

1. if $\alpha < 1$, $\sum a_n$ converges;
2. if $\alpha > 1$, $\sum a_n$ diverges;
3. if $\alpha = 1$, the test is inconclusive. ▶

the Ratio Test (比值審斂法)

Given $\sum a_n$ of nonzero terms,

1. if $\overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges;
2. if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq N_0$, where N_0 is fixed, then $\sum a_n$ diverges. ▶

▶

Remark

- When the Root Test or the Ratio Test determine that $\sum a_n$ converges, the series $\sum |a_n|$ also converges. This is because the conclusion relies on the Comparison Test.
- The Root Test is beneficial for series that contain terms with nth powers of the variable, whereas the Ratio Test is advantageous for series that contain terms with ratios of successive terms. ▶

Remark

- The Root Test and the Ratio Test are closely related as shown in the following result.

  upper-lower limit relations between the Root Test and the Ratio Test

  For any sequence $\{c_n\}$ of positive numbers,
  $$\underline{\lim_{n\to\infty}} \frac{c_{n+1}}{c_n} \leq \underline{\lim_{n\to\infty}} \sqrt[n]{c_n} \leq \overline{\lim_{n\to\infty}} \sqrt[n]{c_n} \leq \overline{\lim_{n\to\infty}} \frac{c_{n+1}}{c_n}.$$ ▶

  If $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = c$, then $\lim_{n\to\infty} \sqrt[n]{c_n} = c$. ▶

  Thus, it follows that $\boxed{\text{if } \overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < 1, \text{ then } \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} < 1.}$ Hence, the Root Test also shows convergence whenever the Ratio Test does. However, the converse is not true. ▶
- When the Root Test is inconclusive, it means that the test does not provide enough information to determine whether the series converges or diverges. In such cases, additional tests or techniques may be required to evaluate the series. Analogous to the Root Test, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then this gives an inconclusive case for the convergence or divergence of the series. The p-series with $p = 1, 2$ provide examples of inconclusive cases for both tests.

3.22 Theorem: Dirichlet's Test, Abel's Test, and the Alternating Series Test

Assume $\{a_n\}$ and $\{b_n\}$ are two sequence of real numbers and.

Dirichlet's Test (狄利克雷審斂法)

Suppose

(1) the partial sums of $\sum a_n$ form a bounded sequence;

(2) the sequence $\{b_n\}$ is monotonically decreasing and $\lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

the Alternating Series Test (交錯級數審斂法)

Suppose that the series $\sum c_n$ satisfies the following conditions:

(1) the terms are alternating: $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ for $m = 1, 2, 3, \ldots$;

(2) the sequence $\{|c_n|\}$ is monotonically decreasing and $\lim_{n\to\infty} c_n = 0$.

Then $\sum c_n$ converges.

Abel's Test (阿貝爾審斂法)

Suppose

(1) the series $\sum a_n$ converges;

(2) the sequence $\{b_n\}$ is monotonic and bounded.

Then $\sum a_n b_n$ converges.

Remark

- The Alternating Series Test is a special case of Dirichlet's Test.
- Despite the fact that Abel's Test can be derived from Dirichlet's Test, it is not simply a special case of the latter. There are situations where Abel's Test is applicable but Dirichlet's isn't, and *vice versa*.

Remark

- The comparison based tests, like the Root Test and the Ratio Test, suggest the convergence of $\sum a_n$ by showing that $\sum |a_n|$ converges. However, there are cases where a convergent series has a divergent $\sum |a_n|$. On the other hand, Dirichlet's Test and Abel's Test only guarantee the convergence of $\sum a_n$, not necessarily $\sum |a_n|$. Therefore, if the Root Test or Ratio Test are inconclusive, Dirichlet's Test and Abel's Test can be used instead.

Absolute Convergence 絕對收斂

3.23 Definition: absolute convergence and conditional convergence (絕對收斂與條件收斂)

Let $\sum a_n$ be a series of real numbers.

If $\sum |a_n|$ converges, the series $\sum a_n$ is said to be **absolutely convergent** (絕對收斂).

If $\sum a_n$ converges but $\sum |a_n|$ diverges, the series $\sum a_n$ is said to be **conditionally convergent** (條件收斂).

absolute convergence implies convergence

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Remark

- The alternating harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$ is an example of conditionally convergent series. The series is convergent, but the absolute value of the series, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, diverges.

3.24 Definition: Cauchy product of series (柯西積)

Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \qquad n = 1, 2, 3, \ldots$$

and call $\sum c_n$ the **Cauchy product** (柯西積) of the two given series.

Remark

- This definition may be motivated by a formal multiplication of two power series. If we take two power series $\sum a_n z^n$ and $\sum b_n z^n$, multiply them term by term, and collect terms containing the same power of z, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ &= a_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots \\ &= c_0 + c_1 z + c_2 z^2 + \cdots .\end{aligned}$$

  Setting $z = 1$, we arrive the above definition.

3.25 Proposition: convergence of the Cauchy product of two infinite series

Let $\sum a_n$ and $\sum b_n$ be real series, and let $\sum c_n$ be their Cauchy product. If $\sum a_n = A$, $\sum b_n = B$, and if at least one of them converges absolutely, then $\sum c_n = AB$. ▶

Remark

- The Cauchy product of two conditionally convergent series may not converge. ▶
- The Cauchy product of two absolutely convergent series must converge absolutely. ▶

Addendum 後記

Addendum of Chapter 3

- Infinite series can be used to define important functions, such as exponential functions e^x and trigonometric functions $\sin x$ and $\cos x$. By a natural extension to complex number field, one can prove the renowned Euler's formula: $e^{ix} = \cos x + i \sin x$ (歐拉公式), where x is real and $i = \sqrt{-1}$ is the imaginary unit satisfying $i^2 = -1$. The formula is a powerful tool in dealing with trigonometric functions and is widely used in many areas of mathematics and science, including signal processing, quantum mechanics, and electrical engineering.
- The difference between absolute convergence and conditional convergence is much deeper than the superficial differences given by their definitions.
  - If you are working with series that are absolutely convergent, you can treat them much like finite sums. This means you can rearrange the terms in the series without changing the total sum. When dealing with the product of two absolutely convergent series, you can multiply each term together and rearrange the order of the additions without altering the final sum.

Suppose that real series $\sum a_n$ converges absolutely. Then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

– However, for conditionally convergent series, one must exercise more caution as these rules no longer hold true. In fact, it is possible to rearrange the terms of a conditionally convergent series to obtain any desired sum, including infinity or negative infinity. This is known as the ***Riemann Rearrangement Theorem***.

the Riemann Rearrangement Theorem (黎曼重排定理)

Suppose that real series $\sum a_n$ converges conditionally. For $-\infty \leq \alpha \leq \beta \leq \infty$, there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that

$$\varliminf_{n\to\infty} s'_n = \alpha, \qquad \varlimsup_{n\to\infty} s'_n = \beta.$$

Therefore, it is important to exercise caution when dealing with conditionally convergent series, and to check for absolute convergence before rearranging the terms of an infinite series.

Exercises of Chapter 3 練習題

Chapter 3: Quiz **30 Minutes**

① Which of the following is NOT true?

A. every Cauchy sequence in $\mathbb{R}$ is convergent
B. every Cauchy sequence in $\mathbb{R}$ contains a convergent subsequence
C. every Cauchy sequence in $\mathbb{R}$ is bounded
D. every sequence in $\mathbb{R}$ contains a convergent subsequence
E. every convergent sequence in $\mathbb{R}$ is bounded

② Which of the following statements is true?

A. if a sequence is unbounded, then any of its subsequences diverges
B. if a sequence is bounded, then every subsequence converges
C. if a sequence converges, then every subsequence converges
D. if a sequence diverges, then any subsequence diverges
E. none of the above

③ Let K be a compact subset in $\mathbb{R}$. Which of the following statements is true?

A. every sequence in K contains a convergent subsequence
B. every sequence in K converges to a limit in K
C. every bounded sequence in K converges
D. every sequence in K is a Cauchy sequence
E. none of the above

④ Which of the following statements is true regarding the Cauchy criterion for infinite series?

A. the criterion is a sufficient condition for convergence
B. the criterion is a necessary condition for convergence
C. the criterion is a necessary and sufficient condition for convergence
D. the criterion requires that the terms of the series approach zero
E. the criterion can be applied to series only with nonnegative terms

⑤ Which of the following statements best describes the Monotone Convergence Theorem?

A. it applies only to finite sequences
B. it applies only to infinite sequences
C. it is for finding the limit of any sequence
D. it gives a necessary condition for a sequence to converge
E. it gives a sufficient condition for a sequence to converge

⑥ Let $\{x_n\}$ be a sequence of real numbers. If $\varliminf_{n\to\infty} x_n = -\infty$ and $\varlimsup_{n\to\infty} x_n$ is finite, which of the following statements is true?

A. $\{x_n\}$ must be bounded
B. $\{x_n\}$ must be bounded above but unbounded below
C. $\{x_n\}$ must be unbounded above but bounded below
D. $\{x_n\}$ must be unbounded above and unbounded below
E. none of the above

⑦ Which of the following tests can be used to determine conditional convergence?

A. the Ratio Test
B. the Root Test
C. the Comparison Test
D. Dirichlet's Test
E. the Divergence Test

⑧ If the series $\sum |a_n|$ converges, which of the following is true?

A. it is not apparent whether the series $\sum a_n$ converges or diverges
B. the series $\sum a_n$ converges, provided that all a_n are positive
C. the series $\sum a_n$ converges, provided that the sequence $\{a_n\}$ is monotonic
D. the series $\sum a_n$ converges, provided that the sequence $\{a_n\}$ is alternating
E. the series $\sum a_n$ converges

①D, ②C, ③A, ④C, ⑤E, ⑥B, ⑦D, ⑧E

Chapter 3: Exercises

Exercise 3.1 (I)

Prove directly by definition that if $\lim_{n\to\infty} x_n = \alpha$, then $\lim_{n\to\infty} x_n^2 = \alpha^2$.

Hint ① Show that the sequence $\{x_n\}$ is bounded.

Hint ② Use the inequality $\left|x_n^2 - \alpha^2\right| = |x_n+\alpha|\cdot|x_n-\alpha| \le |x_n-\alpha|\cdot(|x_n|+|\alpha|)$.

Exercise 3.2 (E)

Let $x_1 = \sqrt{2}$, and

$$x_{n+1} = \sqrt{2+\sqrt{x_n}}, \qquad n = 1, 2, 3, \ldots.$$

Prove that the sequence $\{x_n\}$ converges, and find its limit.

Hint ① Prove that the sequence $\{x_n\}$ is monotonically increasing and is bounded above.

Hint ② Prove by induction.

Exercise 3.3 (I)

Find the upper and lower limits of the sequence $\{x_n\}$ defined by

$$x_1 = 0; \quad x_{2m} = \tfrac{1}{2}x_{2m-1}; \qquad x_{2m+1} = \tfrac{1}{2} + x_{2m}.$$

Hint ① Show that $\{x_{2n-1}\}$ and $\{x_{2n}\}$ converge, respectively.

Hint ② Postulate (假設) expressions for x_{2n-1} and x_{2n} and prove them by induction.

Exercise 3.4 (I)

Prove that if a subsequence of a monotonic sequence converges, then the monotonic sequence itself is also convergent.

Hint ① Without loss of generality, assume that $\{x_n\}$ is monotonically increasing and $\lim_{n\to\infty} x_n = \alpha$. Prove by contradiction that $x_n \le \alpha$ for any integer n.

Hint ② By the hypothesis, for any $\varepsilon > 0$, there exists an integer K such that $\alpha - \varepsilon < x_{n_k} \le \alpha$ for $k > K$.

Hint ③ The sequence $\{x_n\}$ is monotonically increasing. Prove $|x_n - \alpha| < \varepsilon$ for $n > n_{K+1}$.

Exercise 3.5 (I)

Prove the $\boxed{\text{Squeeze Theorem}}$ (夾逼定理):

Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences of real numbers. Suppose the following conditions hold:

(1) $x_n \le y_n \le z_n$ for $n \ge N_0$, where N_0 is some fixed integer;

(2) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L \in \mathbb{R}\cup\{\infty, -\infty\}$.

Then $\lim_{n\to\infty} y_n = L$.

Apply the Squeeze Theorem to prove the following:

1. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

2. $\lim_{n\to\infty} \dfrac{n^{2023}}{(1+p)^n} = 0$ for $p > 0$.

Hint ① Apply Proposition 3.14 and its corollary.

Hint ② For part **1**, let $x_n = \sqrt[n]{n} - 1$. Show that

$$n = (1+x_n)^n \ge \frac{n(n-1)}{2}x_n^2,$$

Hint ③ For part **2**, choose a positive integer k such that $k > 2023$. Then, by the binomial formula (二項式公式), for $n > 2k$, we have

$$(1+p)^n > \binom{n}{k}p^k > \frac{n^k p^k}{2^k k!},$$

Hint ④ The binomial formula is

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k},$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Exercise 3.6 (E)

Determine whether the series $\sum a_n$ is convergent or divergent, if

1. $a_n = \sqrt{n+1} - \sqrt{n}$;
2. $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$;
3. $a_n = \left(\sqrt[n]{n} - 1\right)^n$;
4. $a_n = \dfrac{1}{1+z^n}$ for real values of z.

Hint ① For part **1**, evaluate the partial sums.

Hint ② For part **2**, show that $0 < a_n < \dfrac{1}{n^{3/2}}$.

Hint ③ For part **3**, by $\lim \sqrt[n]{n} = 1$, show that $0 < a_n < \left(\frac{1}{2}\right)^n$ for sufficiently large n.

Hint ④ For part **4**, show that $a_n \not\to 0$ if $|z| \le 1$; $|a_n| \le \dfrac{1}{|z|^n - 1}$ if $|z| > 1$.

Exercise 3.7 (E)

Suppose $\{a_n\}$ is nonnegative. Prove that if $\sum a_n$ converges, then $\sum \dfrac{\sqrt{a_n}}{n}$ converges.

Hint ① Use the inequality
$$\frac{\sqrt{a_n}}{n} = \sqrt{\frac{a_n}{n^2}} \le \frac{1}{2}a_n + \frac{1}{2}\cdot\frac{1}{n^2},$$
and the fact that $\sum \dfrac{1}{n^2}$ converges.

Exercise 3.8 (E)

Suppose that $\sum a_n$ converges. Does the series $\sum a_n^2$ converge?

Hint ① Consider the series $\sum a_n$, with
$$a_{2n-1} = \frac{1}{\sqrt{n}}, \quad a_{2n} = -\frac{1}{\sqrt{n}}, \quad n \ge 1.$$

Exercise 3.9 (I)

Prove the Limit Comparison Test (極限比較審斂法):

Suppose that $\sum a_n$ and $\sum b_n$ are two series with nonnegative terms. Denote
$$L^* = \varlimsup_{n\to\infty} \frac{a_n}{b_n}, \qquad L_* = \varliminf_{n\to\infty} \frac{a_n}{b_n}.$$

1. If $L^* < \infty$, and if $\sum b_n$ converges, then $\sum a_n$ converges.
2. If $L_* > 0$, and if $\sum b_n$ diverges, then $\sum a_n$ diverges.

Hint ① For part **1**, since $\varlimsup\limits_{n\to\infty} \dfrac{a_n}{b_n} = L^* < \infty$, there exists N such that $\dfrac{a_n}{b_n} < L^* + 1$ when $n \ge N$. Then apply the Comparison Test.

Hint ② For part **2**, since $\varliminf\limits_{n\to\infty} \dfrac{a_n}{b_n} = L_* > 0$, there exists N such that $\dfrac{a_n}{b_n} > \frac{1}{2}L_*$ when $n \ge N$. Then apply the Comparison Test.

Exercise 3.10 (E)

Let $\sum a_n$ and $\sum b_n$ be real series, and let $\sum c_n$ be their Cauchy product. If $\sum a_n = A$, $\sum b_n = B$, and both converge absolutely, then $\sum c_n = AB$ converges absolutely.

Hint ① Let $A_n = \sum\limits_{k=0}^{n} |a_k|$ and $B_n = \sum\limits_{k=0}^{n} |b_k|$. To show that $\sum c_n$ converges absolutely, prove that
$$\sum_{k=0}^{n} \left| \sum_{j=0}^{k} a_j b_{k-j} \right| \le A_n B_n.$$

CHAPTER

第 4 章

Continuity 連續性

Overview of Chapter 4

In this chapter, we will study continuity of real functions.

- Real continuous functions on $E \subset \mathbb{R}$ can be characterized by the fact that $f^{-1}(V)$ is open in E for every open set V in $\mathbb{R}$.
- We will prove that
  - Any real continuous function maps compact set to compact set. This implies that any real continuous function attains its extreme values (the Extreme Value Theorem).
  - Any real continuous function maps connected set to connected set. This implies that any real continuous function on a finite closed $[a, b]$ assumes all intermediate values between $f(a)$ and $f(b)$ (the Intermediate Value Theorem).
- We will prove that continuous functions on a compact set must be uniformly continuous. This result forms the basis for the Riemann integrability of continuous functions.
- Finally, we will discuss discontinuity and monotonic functions.

§4.1 Functions 函數

4.1 Definition: arithmetic operations of functions (函數的算術運算)

Suppose $f, g\colon E \subset \mathbb{R} \to \mathbb{R}$ are two functions. We define $f + g$, $f - g$, fg and f/g by

addition :	$(f + g)(x) = f(x) + g(x)$;	**multiplication** :	$(fg)(x) = f(x)g(x)$;
subtraction :	$(f - g)(x) = f(x) - g(x)$;	**division** :	$(f/g)(x) = f(x)/g(x)$.

For the division, it is defined only at those points x of E at which $g(x) \neq 0$.

4.2 Definition: composition (複合函數)

Suppose $f\colon E \subset \mathbb{R} \to \mathbb{R}$ and $g\colon f(E) \subset \mathbb{R} \to \mathbb{R}$. Define $h = g \circ f\colon E \subset \mathbb{R} \to \mathbb{R}$ by

$$h(x) = g(f(x)), \qquad x \in E.$$

This function h is called the **composition function** (複合函數) or ***composite*** of g and f.

Remark

- We note that the composition of functions is not commutative in general, that is, $g \circ f \neq f \circ g$. In fact, even if $g \circ f$ and $f \circ g$ are both defined, they may have different values as well as different domains and ranges.
- The function $g \circ f$ is called the "composite of g and f". We note that the order of symbol is consistent with the order of the written expression.

4.3 Definition: monotonic function (單調函數)

Let f be a real function on (a, b).

We call f to be **monotonically increasing** (單調上升) on (a, b) if $f(x) \leq f(y)$ for any x, y satisfying $a < x < y < b$. If the order $\leq$ is replaced by the strict order $<$, then we call f to be ***strictly increasing*** (嚴格上升).

We call f to be **monotonically decreasing** (單調下降) on (a, b) if $f(x) \geq f(y)$ for any x, y satisfying $a < x < y < b$. If the order $\geq$ is replaced by the strict order $>$, then we call f to be ***strictly decreasing*** (嚴格下降).

A function is **monotonic** (單調) if it is either increasing or decreasing.

4.4 Definition: even, odd and periodic functions (偶函數、奇函數與週期函數)

- Let $f\colon E \subset \mathbb{R} \to \mathbb{R}$, with E symmetric to the origin.

  If $f(-x) = f(x)$ for every $x \in E$, we say that f is an **even function** (偶函數) on E.

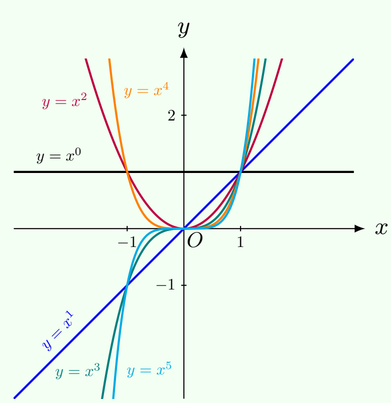
  If $f(-x) = -f(x)$ for every $x \in E$, we say that f is an **odd function** (奇函數) on E.

- Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real function. If there is a positive number p such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$, we say that f is a **periodic function** (週期函數). If there exists a least positive constant p with this property, it is called the basic period, or simply the ***period***.

Basic Elementary Functions 基本初等函數

4.5 Proposition: basic properties of power functions

For fixed $n \in \mathbb{N}$, the **power function** (冪函數) $f(x) = x^n$ has the following properties:

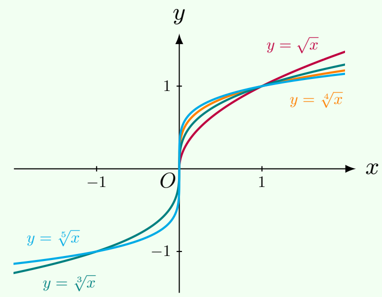


1. The domain is $(-\infty, \infty)$. When n is even, the range is $[0, \infty)$; when n is odd, the range is $(-\infty, \infty)$.
2. When n is even, f is monotonically decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$; when n is odd, f is monotonically increasing on $(-\infty, \infty)$.
3. When n is even, f is an even function; when n is odd, f is an odd function.

4.6 Proposition: basic properties of radical functions

For fixed $n \in \mathbb{N}_+$, the **radical function** (根函數) $f(x) = \sqrt[n]{x}$ has the following properties:

1. When n is even, the domain is $[0, \infty)$ and the range is $[0, \infty)$; when n is odd, the domain is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$.
2. The function f is strictly increasing in its domain;
3. When n is odd, f is an odd function.



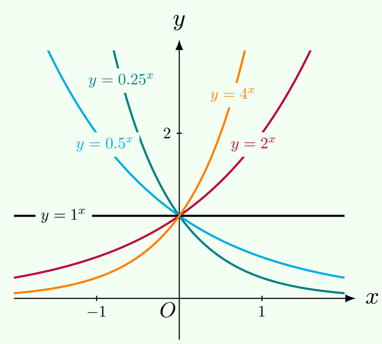
Remark

- The radical function f is well-defined, by Proposition 1.11.

4.7 Proposition: basic properties of exponential functions

For fixed a $(a > 0)$, the **exponential function** (指數函數) $f(x) = a^x$ has the following properties:

1. The domain is $(-\infty, \infty)$. When $a \neq 1$, the range is $(0, \infty)$; when $a = 1$, the range is a one-point set $\{1\}$.
2. When $0 < a < 1$, f is strictly decreasing in its domain; when $a > 1$, f is strictly increasing in its domain; when $a = 1$, f is constant.
3. Operations on exponential functions follow the laws of exponents.



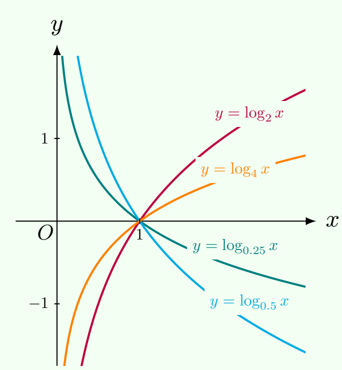
Remark

- According to the remark in the Addendum of Chapter 1, the exponential function f is well-defined for $a > 0$.

4.8 Proposition: basic properties of logarithmic functions

For fixed a $(a > 0$ and $a \neq 1)$, the **logarithmic function** (對數函數) $f(x) = \log_a x$ has the following properties:

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. When $0 < a < 1$, f is strictly decreasing in its domain; when $a > 1$, f is strictly increasing in its domain.
3. Operations on logarithmic functions follow the laws of logarithms.



Remark

- Based on the remark in the Addendum of Chapter 1, the logarithmic function f is well-defined for $a > 0$ and $a \neq 1$.

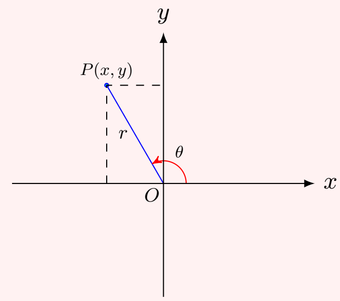
4.9 Proposition: basic properties of sine and cosine functions

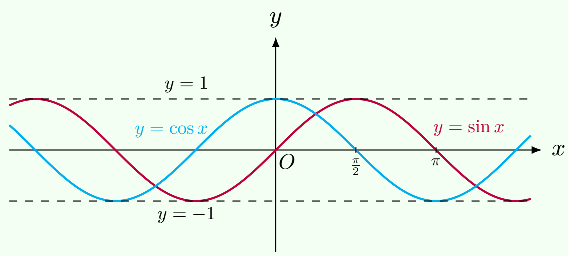
For an angle θ, with $\theta \in [0, 2\pi)$, if we let $P(x, y)$ be the point on the terminal side of θ and if let r be the distance $|OP|$, we define the values of the sine and the cosine as

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}.$$

For θ outside $[0, 2\pi)$, the values of the sine and the cosine are defined in terms of the following periodic property:

$$\sin(\theta + 2\pi) = \sin\theta, \qquad \cos(\theta + 2\pi) = \cos\theta.$$





The **sine** function (正弦函數) and the **cosine** function (餘弦函數) have the following properties:

1. The domain of both functions is $(-\infty, \infty)$. The range of both functions is $[-1, 1]$.

2. Both are 2π-periodic functions.

3. The values of these two functions for special angles in $[0, \frac{\pi}{2}]$ are listed as in the table:

degrees (度)	radians (弧度)	$\sin x$	$\cos x$
$0°$	0	0	1
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$90°$	$\frac{\pi}{2}$	1	0

4. For other special angles, the corresponding values of the functions can be obtained by using the following formulas:

$$\sin(x + \tfrac{\pi}{2}) = \cos x, \qquad \cos(x + \tfrac{\pi}{2}) = -\sin x$$
$$\sin(x + \pi) = -\sin x, \qquad \cos(x + \pi) = -\cos x$$

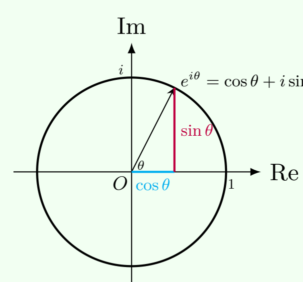
5. Two key equalities hold:

Pythagorean Identity (畢氏恆等式)

$$\sin^2 x + \cos^2 x = 1.$$

Euler's Formula (歐拉公式)

$$e^{ix} = \cos x + i\sin x, \qquad i = \sqrt{-1}.$$



Remark

- It is a remarkable fact that these two equalities can be used to derive almost all elementary trigonometric identities.
- Both will be proved in Chapter 8.

For other trigonometric functions, they are defined by using the two basic trigonometric functions:

tangent (正切): $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$; **cotangent** (餘切): $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$;

secant (正割): $\sec\theta = \dfrac{1}{\cos\theta}$; **cosecant** (餘割): $\csc\theta = \dfrac{1}{\sin\theta}$.

4.10 Proposition: basic properties of the inverse sine and the inverse cosine functions

- For each $x \in [-1,1]$, the equation $x = \sin y$ has a unique solution y in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Denote $y = \arcsin x$, and call it the **arcsine** function (反正弦函數);
- For each $x \in [-1,1]$, the equation $x = \cos y$ has a unique solution y in $[0,\pi]$. Denote $y = \arccos x$, and call it the **arccosine** function (反餘弦函數).
- For each $x \in (-\infty,\infty)$, the equation $x = \tan y$ has a unique solution y in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Denote $y = \arctan x$, and call it the **arctangent** function (反正切函數).

They have the following properties:

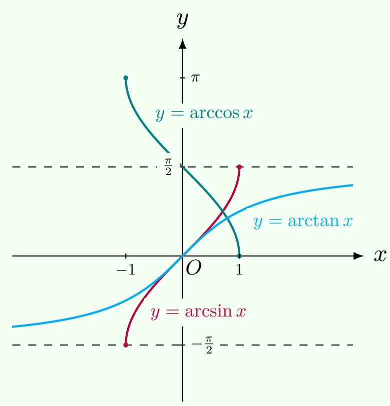
1. The domain of both the arcsine and arccosine functions is $[-1,1]$. The range of $\arcsin x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, while the range of $\arccos x$ is $[0,\pi]$. On the other hand, the domain of the arctangent function is $(-\infty,\infty)$, and its range is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

2. In their respective domains, the functions $\arcsin x$ and $\arctan x$ are monotonically increasing, while the function $\arccos x$ is monotonically decreasing.

3. The functions $\arcsin x$ and $\arctan x$ are odd, while the function $\arccos x$ is neither even nor odd.

4. The following equalities hold:

$$\sin(\arcsin x) = x, \quad \text{for all } x \in [-1,1];$$
$$\cos(\arccos x) = x, \quad \text{for all } x \in [-1,1];$$
$$\tan(\arctan x) = x, \quad \text{for all } x \in (-\infty,\infty).$$



Remark

- When it does not cause ambiguity, one often uses the notations $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$ for these three inverse trigonometric function, respectively.
- There are three additional inverse trigonometric functions, $\cot^{-1}(x)$, $\sec^{-1}(x)$, $\csc^{-1}(x)$, that can be defined similarly.

§4.2 Limits of Functions 函數極限

4.11 Definition: limit of function (函數極限) ▶

Suppose $f\colon E \subset \mathbb{R} \to \mathbb{R}$ and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or $\lim_{x\to p} f(x) = q$, if there is a point $q \in \mathbb{R}$ with the following property:

for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - q| < \varepsilon$ for all points $x \in E$ for which $0 < |x - p| < \delta$.

In this case, we say that the **limit** (極限) of function f at p equals q.

Remark

- Here are some synonyms for "the limit of function f at p equals q":
  - $f(x)$ has a limit of q at p
  - the limit of $f(x)$ as x approaches p equals q
  - $f(x)$ approaches q as x approaches p
  - $f(x)$ converges to q as x approaches p
- The limit $\lim_{x\to p} f(x)$ can be defined without requiring $p \in E$, and even if it does, it is usually not equal to $f(p)$.
- $\boxed{0 < |x-p| < \delta \text{ for } x \in E \iff |x-p| < \delta \text{ for } x \in E\backslash\{p\}.}$

4.12 Proposition: limit of function in term of sequential limits

Suppose $f\colon E \subset \mathbb{R} \to \mathbb{R}$. The limit $\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in $E\backslash\{p\}$ converging to p. ▶

Corollary

If a function has a limit, then the limit is unique. ▶

4.13 Proposition: arithmetic operations on limits of functions

Suppose $E \subset \mathbb{R}$, p is a limit point of E, f and g are two real functions on E, and

$$\lim_{x\to p} f(x) = A, \qquad \lim_{x\to p} g(x) = B.$$

Then

1. $\lim_{x\to p}(f+g)(x) = A+B$;
2. $\lim_{x\to p}(f-g)(x) = A-B$;
3. $\lim_{x\to p}(fg)(x) = AB$;
4. $\lim_{x\to p}(f/g)(x) = A/B$, if $B \neq 0$. ▶

4.14 Definition: limits of function in the extended real number system (在擴張實數系中的函數極限)

For any real c, the set of real numbers x such that $x > c$ is called a ***neighborhood*** of ∞ and is written (c, ∞). Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

Let f be a real function defined on $E \subset \mathbb{R}$. We write $f(x) \to q$ as $x \to p$, or $\lim_{x\to p} f(x) = q$, where p and q are in the extended real number system, if

> for every neighborhood U of q, there is a neighborhood V of p satisfying $V \cap E \neq \varnothing$ such that $f(x) \in U$ for all $x \in V \cap E$, $x \neq p$.

Remark

- When p is finite while q is infinite, the above defines two types of limits: $\lim_{x\to p} f(x) = \infty$ and $\lim_{x\to p} f(x) = -\infty$. They are called **infinite limits**.
- An equivalent definition for the infinite limit $\lim_{x\to p} f(x) = \infty$ is:

> for every M, there exists a $\delta > 0$ such that $f(x) > M$ for all points $x \in E$ for which $0 < |x-p| < \delta$.

Remark

- When p is infinite while q is finite, the above defines two types of limits: $\lim_{x\to\infty} f(x) = q$ and $\lim_{x\to-\infty} f(x) = q$. They are called **limits at infinity**.
- An equivalent definition for the limit at infinity $\lim_{x\to\infty} f(x) = q$ is:

  for every $\varepsilon > 0$ there exists an N such that $|f(x) - q| < \varepsilon$ for all points $x \in E$ for which $x > N$.

Remark

- When both p and q are infinite, the definition gives another four types of limits:

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = -\infty, \quad \lim_{x\to-\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = -\infty.$$

They are called **infinite limits at infinity**.

uniqueness of limit

Suppose $f\colon E \subset \mathbb{R} \to \mathbb{R}$. If $\lim_{x\to p} f(x) = q$ and $\lim_{x\to p} f(x) = q'$, where $p, q, q' \in \overline{\mathbb{R}}$, then $q = q'$.

Remark

- When both p and q are finite, the above definition coincides with Definition 4.11.

4.15 Proposition: arithmetic operations on limits of functions in the extended real number system

Suppose $E \subset \mathbb{R}$, f and g are two real functions on E, and $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$, where p, A, and B are in $\overline{\mathbb{R}}$. Then

1. $\lim_{x\to p}(f+g)(x) = A + B$;
2. $\lim_{x\to p}(f-g)(x) = A - B$;
3. $\lim_{x\to p}(fg)(x) = AB$;
4. $\lim_{x\to p}(f/g)(x) = A/B$, if $B \neq 0$,

provided that the right-hand sides of the above equalities are well-defined in $\mathbb{R} \cup \{\infty, -\infty\}$.

Remark

- Note that $\infty - \infty$, $0 \cdot \infty$, ∞/∞, $A/0$ are not well-defined in $\overline{\mathbb{R}}$.

§4.3 Continuous Function 連續函數

4.16 Definition: continuous function (連續函數)

Suppose $E \subset \mathbb{R}$, $p \in E$, and f maps E into $\mathbb{R}$. Then f is said to be **continuous** (連續) at p if

for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ for all points $x \in E$ for which $|x - p| < \delta$.

If f is continuous at every point of E, then f is said to be ***continuous on E***.

continuity at a limit point

Assume that p is a limit point of $E \subset \mathbb{R}$. Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Remark

- To be continuous at a point, it is necessary for the function to be defined at that point.
- If p is an isolated point of E, every functions with domain E is continuous at p. ▶

Remark

- The basic elementary functions (such as power, radical, exponential, logarithmic, trigonometric functions) are continuous in their domains.

4.17 Theorem: characterization of continuity

Let $f \colon E \subset \mathbb{R} \to \mathbb{R}$.

1. The function f is continuous if and only if $f^{-1}(V)$ is open in E for every open set V in $\mathbb{R}$.

2. The function f is continuous if and only if $f^{-1}(C)$ is closed in E for every closed set C in $\mathbb{R}$. ▶

Remark

- A continuous mapping
  - does not necessarily map from closed subsets to closed subsets;
  - does not necessarily map from open subsets to open subsets;
  - does not necessarily map from bounded subsets to bounded subsets. ▶
  - does map from compact subsets to compact subsets;
  - does map from connected subsets to connected subsets.

4.18 Proposition: continuity of operations of functions

1. Suppose f and g are real continuous functions on $E \subset \mathbb{R}$. Then $f+g$, $f-g$, fg, and f/g are continuous on E (for quotient, the denominator is not zero). ▶ ▶

2. Suppose f and g are real functions such that the composite of g and f, $g \circ f$, is well-defined on $E \subset \mathbb{R}$. If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then $g \circ f$ is continuous at p. ▶

3. Suppose f is a continuous bijective mapping of a compact set $K \subset \mathbb{R}$ onto $f(K) \subset \mathbb{R}$. Then the inverse mapping f^{-1} is a continuous mapping of $f(K)$ onto K. ▶

Remark

- A continuous bijective mapping on *noncompact* set may not have continuous inverse. ▶

Remark

- Elementary functions are functions that are defined as taking sums, products, roots and compositions of finitely many polynomial, rational, trigonometric, and exponential functions, including possibly their inverse functions. All elementary functions are continuous in their domains.

§4.4 Continuity and Compactness 連續性與緊致性

4.19 Theorem: continuous mapping maps compact set to compact set

Suppose f is a continuous mapping of a compact set K into $\mathbb{R}$. Then $f(K)$ is compact, that is, closed and bounded. ▶

A mapping f of a set $E \subset \mathbb{R}$ into $\mathbb{R}$ is said to be **bounded** if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.

Remark

- For any *noncompact* subset of $\mathbb{R}$, there exists an unbounded real continuous function. ▶

4.20 Theorem: the Extreme Value Theorem (極值定理)

Suppose f is a real continuous function on a compact set K. Then there exist points $p, q \in K$ such that

$$f(p) = \sup_{x \in K} f(x), \qquad f(q) = \inf_{x \in K} f(x).$$

That is, f attains its maximum and its minimum. ▶

Corollary

Any real continuous function on $[a, b]$ attains its maximum and minimum.

Remark

- For any *noncompact* subset of $\mathbb{R}$, there exists a continuous function that is bounded but has no maximum value. ▶

4.21 Definition: uniformly continuous (一致連續)

Suppose $f \colon E \subset \mathbb{R} \to \mathbb{R}$. We say that f is **uniformly continuous** (一致連續) on E if

for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(p) - f(q)| < \varepsilon$ for all points p and q in E for which $|p - q| < \delta$.

Remark

- Continuity and uniform continuity differ significantly. Continuity is defined at a point, while uniform continuity applies to a set. Hence, asking whether a given function is uniformly continuous at a certain point is meaningless.
- For continuity, the value of δ depends on the point being considered, while for uniform continuity, a single value of δ applies to all points in E. ▶

4.22 Theorem: the relationship between continuity and uniform continuity

Suppose $f \colon E \subset \mathbb{R} \to \mathbb{R}$.

1. If f is uniformly continuous on E, then f is continuous on E.
2. If E is compact, and if f is continuous on E, then f is uniformly continuous on E. ▶

Remark

- For any *noncompact* subset of $\mathbb{R}$, there exists a function that is continuous but not uniformly continuous. ▶

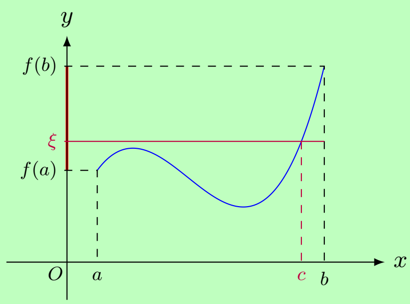
Continuity and Connectedness 連續性與連通性

4.23 Theorem: continuous mapping maps connected set to connected set

Suppose $f\colon E \subset \mathbb{R} \to \mathbb{R}$ is continuous. If E is connected, then $f(E)$ is connected.

the Intermediate Value Theorem (介值定理)

Any real continuous function f on $[a,b]$ assumes all intermediate values between $f(a)$ and $f(b)$. That is, for any value ξ between $f(a)$ and $f(b)$, there exists at least one point $c \in [a,b]$ such that $f(c) = \xi$.



Bolzano's Theorem (零点定理)

Any real continuous function f on $[a,b]$ has at least one root if $f(a) \cdot f(b) < 0$.

Remark

- One can modify the Intermediate Value Theorem to have that

  Any real continuous function f on $[a,b]$ assumes all intermediate values between $\min\limits_{x\in[a,b]} f(x)$ and $\max\limits_{x\in[a,b]} f(x)$.

Remark

- The converse statement of the Intermediate Value Theorem is not true. In fact, there is a *discontinuous* function f satisfies that for any two points x_1 and x_2, with $x_1 < x_2$, the function assumes every value between $f(x_1)$ and $f(x_2)$.

§4.5 Discontinuity and Monotonicity 間斷與單調性

4.24 Definition: one-sided limits (單邊極限) and discontinuities (不連續點)

Let f be defined on open interval (a,b). Suppose $x \in (a,b)$ is fixed.

If $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x,b) such that $t_n \to x$, we write $f(x+) = q$ or $\lim\limits_{t\to x+} f(t) = q$ and call the **right-hand limit** (右極限) of f at x.

If $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (a,x) such that $t_n \to x$, we write $f(x-) = q$ or $\lim\limits_{t\to x-} f(t) = q$ and call the **left-hand limit** (左極限) of f at x.

The limit $\lim\limits_{t\to x} f(t)$ exists if and only if $f(x+) = f(x-) = \lim\limits_{t\to x} f(t)$.

If $f(x+)$ and $f(x-)$ exist, the function f is said to have a **discontinuity of first kind** (第一類不連續點) at x. It is also called a ***simple discontinuity*** (簡單不連續點).

If at least one of $f(x+)$ and $f(x-)$ does not exist, the function f is said to have a **discontinuity of second kind** (第二類不連續點) at x. It is also called an ***essential discontinuity*** (本性不連續點).

Remark

- There are two ways in which a function can have a simple discontinuity:
  ① ***removable discontinuity*** (可去不連續點): $f(x+) = f(x-) \neq f(x)$;
  ② ***jump discontinuity*** (跳躍不連續點): $f(x+) \neq f(x-)$.
- A function is called **piecewise continuous** (分段連續) on a given interval $[a, b]$ if the interval can be broken into a finite number of open intervals on which the function is continuous on each open interval and it has a finite limit at the endpoints of each open interval. The term *piecewise* refers "a piece in a **finite** number of open intervals".

4.25 Proposition: monotonic function alway has one-sided limits

Let f be monotonically increasing on open interval (a, b). Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b). More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \leq f(y-).$$

An analogue result is also true for monotonically decreasing functions.

Corollary

Monotonic functions have no discontinuities of the second kind.

Corollary

Any monotonic function on an open interval has at most countably many discontinuities.

Remark

- Although the discontinuities of a monotonic function on an open interval form an at most countable set, they need not be isolated. In fact, it is possible to construct a function f on the interval (a, b) that is monotonic and discontinuous at every point of a given countable subset E of the interval, even if E is dense.

Remark

- In summary, the following properties are true for a monotonic function $f : \mathbb{R} \to \mathbb{R}$:
  - for every point $x \in \mathbb{R}$, both $f(x+)$ and $f(x-)$ exist;
  - the limits at infinity, $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$, equal either a finite number, or ∞, or $-\infty$;
  - the function f has only simple discontinuities;
  - the discontinuities of f form an at most countable set.

Addendum 後記

Addendum of Chapter 4

- Continuity is a fundamental concept in mathematics and has many significant applications in various fields such as calculus, analysis, topology, and geometry.
  - In calculus and analysis, continuity is essential for defining limits, derivatives, and integrals. The concept of continuity allows us to determine the behavior of a function at any point within its

domain, and it helps us to understand the relationship between different functions.

  - In topology, continuity is used to define the properties of spaces and the relationships between them. It is a fundamental concept that helps us to understand the topological structure of the universe and the behavior of objects within it.
  - In geometry, continuity is used to describe the smoothness and regularity of shapes and surfaces. It is a crucial concept in differential geometry, where it is used to study the curvature and other geometric properties of surfaces and curves.

- Uniform continuity ensures that a function changes gradually over an interval, enabling the Riemann sum (as defined in Chapter 6) to converge to the true value as the subintervals' width approaches zero. In the absence of uniform continuity, the function may oscillate excessively or have sharp spikes, rendering it impossible to accurately approximate the integral using Riemann sums.
- Monotonic functions are a large class of functions, besides continuous functions, that we often encounter in applications. Here are a few examples:
  - In optimization problems, monotonic functions are often used to model constraints or objective functions. This is because they have clear and predictable behavior, making them easier to work with and analyze.
  - In probability theory, monotonic functions are used to transform probability distributions to other distributions with desirable properties. For example, the cumulative distribution function (CDF) of a random variable is a monotonic function, and it can be used to transform any distribution to a uniform distribution.
  - In economics, monotonic functions are used to model utility functions, which describe how individuals make choices based on their preferences. Monotonicity ensures that more of a good is always preferred to less, and that preferences are consistent across different levels of consumption.
  - In computer science and machine learning, monotonic functions are used to model relationships between variables in regression and classification problems. Monotonicity ensures that the relationship between the input and output variables is consistent and predictable, making it easier to interpret and use the model.

Exercises of Chapter 4 練習題

Chapter 4: Quiz **30 Minutes**

① Let $x_0 \in E$ be a fixed point, and let $f\colon E \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Which of the following statements is true?

A. for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x_1 \in E$ and $|x_1 - x_0| < \delta$ imply $|f(x_1) - f(x_0)| < \varepsilon$
B. for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x_1 \in E$ and $|x_1 - x_0| < \varepsilon$ imply $|f(x_1) - f(x_0)| < \delta$
C. for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x_1, x_2 \in E$ and $|x_1 - x_2| < \delta$ imply $|f(x_1) - f(x_2)| < \varepsilon$
D. for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x_1, x_2 \in E$ and $|x_1 - x_2| < \varepsilon$ imply $|f(x_1) - f(x_2)| < \delta$
E. none of the above

① Which of the following statements is true for a continuous function f on an interval $[a, b]$?

A. f assumes every value in $[a, b]$
B. f is increasing on (a, b)
C. f has a maximum value on $[a, b]$
D. f has a minimum value on (a, b)
E. f is unbounded on $[a, b]$

② Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Which of the following sets is open?

A. $\{x \in \mathbb{R} : f(x) \geq 0\}$
B. $\{x \in \mathbb{R} : f(x) \leq 0\}$
C. $\{x \in \mathbb{R} : f(x) = 0\}$
D. $\{x \in \mathbb{R} : f(x) \neq 0\}$
E. none of the above

③ Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and E is a subset of $\mathbb{R}$. Which of the following must be true?

A. $f(\overline{E})$ is a closed subset in $\mathbb{R}$
B. $f((\overline{E})^c)$ is an open subset in $\mathbb{R}$
C. $f(\overline{E^c})$ is a closed subset in $\mathbb{R}$
D. $f^{-1}((\overline{E})^c)$ is a close subset in $\mathbb{R}$
E. $f^{-1}((\overline{E})^c)$ is an open subset in $\mathbb{R}$

④ Suppose f is continuous on $E = (a, b)$. Which of the following is true?

A. $f(E)$ is bounded
B. $f(E)$ is open
C. $f(E)$ is compact
D. $f(E)$ is connected
E. none of the above

⑥ Let $g\colon [a, b] \to \mathbb{R}$ be a continuous function. Which of the following statements is FALSE?

A. g takes every value between a and b
B. g takes every value between $g(a)$ and $g(b)$
C. g takes every value between $\sup\limits_{x\in[a,b]} g(x)$ and $\inf\limits_{x\in[a,b]} g(x)$
D. $g([a, b])$ is a closed subset of $\mathbb{R}$
E. $g([a, b])$ is a bounded subset of $\mathbb{R}$

⑦ Suppose $f(x) = 1/x$ on $(0, 1)$. Which of the following statements is true?

A. f is uniformly continuous on $(0, 1)$
B. f is continuous on $(0, 1)$ but not uniformly continuous
C. f is not continuous on $(0, 1)$.
D. f is uniformly continuous on $[0, 1]$.
E. none of the above.

⑧ Let $f(x) = \begin{cases} x^2 - 1, & x \leq 0, \\ x^2 + 1, & x > 0 \end{cases}$ and $g(x) = \begin{cases} 0, & x = 0, \\ 1/x, & x \neq 0. \end{cases}$
Which of the following statements is true?

A. at 0, both f and g have jump discontinuities
B. at 0, both f and g have removable discontinuities
C. at 0, both f and g have essential discontinuities
D. at 0, f has a jump discontinuity and g has an essential discontinuity
E. at 0, f has a removable discontinuity and g has an essential discontinuity

①A, ②C, ③D, ④E, ⑤D, ⑥A, ⑦B, ⑧D

Chapter 4: Exercises

Exercise 4.1 (E)

Show that the definition of limit can be modified as

"for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - q| \le \varepsilon^2$ for all points $x \in E$ for which $0 < |x - p| < \delta$."

Hint ① Show that the modified statement holds if and only if to the original statement holds.

Exercise 4.2 (I)

Prove directly by definition that the function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Hint ① Prove that the limits $\lim_{x\to 0+} \sqrt{x} = 0$ and $\lim_{x\to\alpha} \sqrt{x} = \sqrt{\alpha}$ for any $\alpha > 0$, respectively.

Hint ② When $\alpha > 0$, use the inequality

$$\left|\sqrt{x} - \sqrt{\alpha}\right| = \frac{|x-\alpha|}{\sqrt{x}+\sqrt{\alpha}} \le \frac{|x-\alpha|}{\sqrt{\alpha}}.$$

Exercise 4.3 (I)

Prove directly by definition that the function $f(x) = \dfrac{1}{\sqrt{x}}$ is continuous on $(0, \infty)$.

Hint ① Use the definition of limit to prove $\lim_{x\to\alpha} \dfrac{1}{\sqrt{x}} = \dfrac{1}{\sqrt{\alpha}}$ for any $\alpha > 0$.

Hint ② Use the inequality

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{\alpha}}\right| = \frac{|x-\alpha|}{\left|\sqrt{x}+\sqrt{\alpha}\right| \cdot \sqrt{x}\sqrt{\alpha}} \le \frac{|x-\alpha|}{\sqrt{\alpha}\cdot\sqrt{x}\sqrt{\alpha}}.$$

Hint ③ There exists $\delta_1 > 0$ such that $x > \frac{1}{2}\alpha$ for all points $x \in (0, \infty)$ for which $|x - \alpha| < \delta_1$.

Exercise 4.4 (E)

Suppose f and g are two real functions on $[a, b]$. Let

$$H(x) = \max\{f(x), g(x)\},$$
$$h(x) = \min\{f(x), g(x)\}.$$

Prove the following:

1. If f and g are monotonically increasing, so are H and h.
2. If f and g are continuous, so are H and h.

Hint ① For part **1**, use the definition to show that H is also monotonically increasing.

Hint ② For part **2**, prove that

$$H(x) = \tfrac{1}{2}\left[f(x) + g(x) + |f(x) - g(x)|\right],$$
$$h(x) = \tfrac{1}{2}\left[f(x) + g(x) - |f(x) - g(x)|\right].$$

Exercise 4.5 (E)

Let $f\colon E \subset \mathbb{R} \to \mathbb{R}$ be continuous. Prove that the zero set of f, $Z(f) = \{x \in E : f(x) = 0\}$, is closed.

Hint ① Prove that every limit point of $Z(f)$ is in $Z(f)$.

Exercise 4.6 (I)

Suppose $f\colon [a, b] \to [a, b]$ is continuous. Prove that the function f has at least one **fixed point** (不動點) in $[a, b]$, that is, there is at least one real number $x \in [a, b]$ such that $f(x) = x$.

Hint ① Apply Bolzano's Theorem to the function $F(x) = f(x) - x$.

Exercise 4.7 (I)

Suppose that $f\colon (a, b) \to \mathbb{R}$ is continuous. For any n values $x_1, \ldots, x_n \in (a, b)$, prove that there exits a value $\xi \in (a, b)$ such that

$$f(\xi) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Hint ① Apply the Intermediate Value Theorem.

Hint ② Denote

$$m = \min_{x\in[x_1,x_n]} f(x), \quad M = \max_{x\in[x_1,x_n]} f(x).$$

Show that

$$m \le \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \le M.$$

Exercise 4.8 (I)

Suppose that $f\colon [a,\infty)$ is continuous for some $a \in \mathbb{R}$, and suppose that $\lim\limits_{x\to\infty} f(x) = A \in \mathbb{R}$. Prove that f is bounded on $[a,\infty)$.

Hint ①: ① Use the hypothesis $\lim\limits_{x\to\infty} f(x) = A$ to show that there is a number $b > a$, the function f is bounded on (b,∞).

Hint ②: ② The function f is continuous on the compact set $[a, M]$. Apply Theorem 4.19 to obtain the boundedness of f on $[a,b]$.

Exercise 4.9 (I)

Suppose $f\colon E \subset \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Prove that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$ for every Cauchy sequence $\{x_n\}$ in E.

Hint ①: ① Directly use the definitions of uniform continuous and Cauchy sequence.

Exercise 4.10 (I)

Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}$. Prove that f is bounded on E.

Hint ①: ① Since E is bounded, it is contained in a bounded closed interval.

Hint ②: ② For any $\delta > 0$, E is covered by a finite collection of open intervals of length 2δ. Keep only the open intervals which intersect with E.

Hint ③: ③ Use the uniform continuity and the triangle inequality to show that f is bounded on E.

CHAPTER

第 5 章

Differentiation 微分

Overview of Chapter 5

In this chapter, we will study differentiation of real functions and its applications in analysis.

- We will prove the Mean Value Theorem, which is a fundamental theorem in calculus that relates the derivative of a function to its values at the endpoints of an interval. We will then use the Mean Value Theorem to prove the Monotone Test, that allows us to determine when a function is increasing or decreasing on an interval. This opens the gate for geometric applications.
- We will generalize the Mean Value Theorem to Cauchy's Mean Value Theorem and use it to prove l'Hôpital's Rule, which is a powerful tool in calculus for evaluating limits of indeterminate forms.
- Finally, we will prove Taylor's Theorem, that allows to represent functions using power series, so we can approximate functions with polynomials.

§5.1 Derivative 導數

5.1 Definition: derivative (導數)

Let f be a real-valued function defined on $[a, b]$. For any $x \in [a, b]$, denote

$$\phi(t) = \frac{f(t) - f(x)}{t - x}, \qquad a < t < b, t \neq x,$$

and define $\boxed{f'(x) = \lim_{t \to x} \phi(t)}$, that is,

> for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\phi(t) - f'(x)| < \varepsilon$ if $t \in [a, b]$ and $|t - x| < \delta$.

We thus associate with the function f a function f' whose domain is the set of points x at which the above limit exists; f' is called the **derivative** (導函數) of f. If f' is defined at a point x, we say that f is **differentiable** (可微分) at x. If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E.

Remark

- The concept of the derivative of a function was originated from two motivations:
  - Rate of change: measuring how fast the function is changing at a given point;
  - Slope of tangent line: computing the slope of the tangent line to the curve at a given point.
- In $\mathbb{R}$, "having derivative" and "differentiable" are equivalent. Generally, "differentiable" means *linearizable*. These two concepts are not the same in $\mathbb{R}^k$.

The right-hand limit $\phi(x+)$ is called the **right-hand derivative** (右導數) of f, denoted as $f'_+(x)$.

If the right-hand derivative exists, then f is said to be **right-hand differentiable** (右方可微) at x.

The left-hand limit $\phi(x-)$ is called the **left-hand derivative** (左導數) of f, denoted as $f'_-(x)$.

If the left-hand derivative exists, then f is said to be **left-hand differentiable** (左方可微) at x.

For $x \in (a,b)$, the derivative $f'(x)$ exists at x if and only if $f'_+(x) = f'_-(x) = \lim_{t\to x} \phi(t)$.

Remark

- We note that the definition of f' is for functions defined on $[a,b]$, including $f'(a)$ and $f'(b)$. For functions defined on (a,b), the derivatives at a and b are not defined altogether.

Remark

- Do not confuse $f'_+(x)$ (right-hand derivative at x) with $f'(x+)$ (the right-hand limit of the derivative at x). They have completely different meanings. For instance, for a differentiable function f defined on (a,b), it is meaningful to investigate $f'(a+)$, but it is meaningless to study $f'_+(a)$. ▶
- Let $x \in (a,b)$. If f' exists in a neighborhood of x, and if $f'(x+) = f'(x-)$, then the function f' is continuous at x. For comparison, if $f'_+(x) = f'_-(x)$, then f' exists at x.

5.2 Proposition: differentiability implies continuity

Let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x. ▶

Remark

- The converse of this theorem is not true. For instance, the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is continuous but not differentiable at 0. In fact, there is a function which is continuous on the whole line without being differentiable at any point.

5.3 Proposition: arithmetic operations on differentiation

Suppose f and g are defined on $[a,b]$ and are differentiable at a point $x \in [a,b]$. Then $f+g$, $f-g$, fg, and f/g are differentiable at x, and

1. $(f+g)'(x) = f'(x) + g'(x)$;
2. $(f-g)'(x) = f'(x) - g'(x)$;
3. $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$;
4. $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$, where $g(x) \neq 0$.

▶ ▶

5.4 Proposition: chain rule

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$, $a \leq t \leq b$, then h is differentiable at x, and

$$h'(x) = g'(f(x)) \cdot f'(x).$$

▶ ▶

5.5 Proposition: derivative of the inverse function

Suppose f is bijective and continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$. Denote $y = f(x)$. Then f^{-1} is differentiable at y, and

$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

▶ ▶

§5.2 Mean Value Theorems 中值定理

5.6 Definition: local extremum (局部極值)

Let $f \colon E \subset \mathbb{R} \to \mathbb{R}$.

We say that f has a **local maximum** (局部極大值) at a point $p \in E$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in E$ with $|p-q| < \delta$.

We say that f has a **local minimum** (局部極小值) at a point $p \in E$ if there exists $\delta > 0$ such that $f(q) \geq f(p)$ for all $q \in E$ with $|p-q| < \delta$.

We say that f has a **local extremum** (局部極值) at a point p if a local maximum or minimum is obtained at p.

5.7 Proposition: necessary condition for an extremum

Let f be defined on $[a,b]$. If f has a local extremum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$. ▶

The only locations where a continuous function attain extrema are critical points.

- A point x in the domain of f is called a **critical point** (臨界點) of f if $f'(x) = 0$ or $f'(x)$ is undefined.
- A point x in the domain of f is called a **stationary point** (駐點) of f if $f'(x) = 0$.

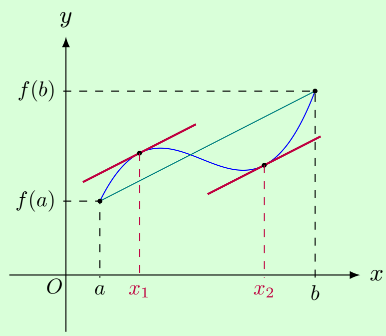
Remark

- This is a necessary but not sufficient condition for an extremum. ▶

5.8 Theorem: the Mean Value Theorem (中值定理)

If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(x).$$



Remark

- When we rewrite the result as $\dfrac{f(b) - f(a)}{b - a} = f'(x)$, we can interpret it as stating that the average rate of change of a function over an interval is equal to its instantaneous rate of change at some point within the interval.
- The theorem applies only to real-valued functions.

the Monotone Test (單調性測試)

Suppose f is differentiable in (a, b).

1. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.

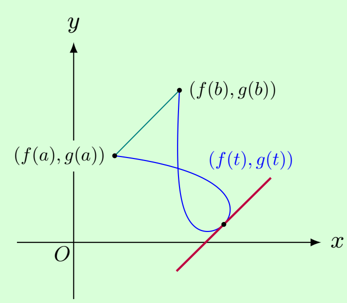
2. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

3. If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

Cauchy's Mean Value Theorem (柯西中值定理)

If f and g are real continuous functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$



Remark

- This is also known as the ***generalized mean value theorem***.

l'Hôpital's Rule 洛必達法則

5.9 Theorem: l'Hôpital's Rule (洛必達法則)

Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq \infty$. Assume that either ① ($*/\infty$ case) $\lim\limits_{x \to a+} g(x) = \infty$ or ② ($0/0$ case) $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a+} g(x) = 0$. Then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)},$$

provided the latter limit exists or infinite.

The analogous statement is also true if $x \to b-$, or if $g(x) \to -\infty$.

Remark

- The conclusion of l'Hôpital's rule is that if $\lim_{x\to a+}\frac{f'(x)}{g'(x)}=A$, then $\lim_{x\to a+}\frac{f(x)}{g(x)}=A$, not the other way around.
- l'Hôpital's rule holds generally for the limits in the extended real number system.
- l'Hôpital's rule cannot be used for complex-valued functions because it relies on the Mean Value Theorem, which only applies to real-valued functions.

Remark

- When $\lim_{x\to a}f(x)=0$ and $\lim_{x\to a}g(x)=0$, the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ is a so-called **indeterminate form** (不定式), and is referred by the symbol $\frac{0}{0}$. In general, an expression is called an indeterminate form if it cannot be evaluated directly, as they take on ambiguous or undefined values when certain variables approach certain limits. There are seven types of indeterminate forms that commonly arise in analysis:
  - two *basic* forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$;
  - one product form: $0\cdot\infty$;
  - one difference form: $\infty-\infty$;
  - three power forms: 0^0, ∞^0, 1^∞.

  We understand these symbols in a similar way as described above for $\frac{0}{0}$.
- In practice, in order to apply l'Hôpital's rule to evaluate a limit of an indeterminate form, one first needs to convert it to two basic forms mentioned above.

§5.3 Derivatives of Higher Order 高階導數

5.10 Definition: derivative of higher order (高階導數)

If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the second derivative of f. Continuing in this way, we obtain functions $f, f', f'', f^{(3)}, \dots, f^{(k)}$, where each function is the derivative of the preceding one. The function $f^{(k)}$ is called the **kth derivative** (k 階導數), or the derivative of order k, of f.

Remark

- For existence of $f^{(k)}$ at a point x, the function $f^{(k-1)}$ must exist in a neighborhood of x including at x (or in a one-sided neighborhood, if x is an endpoint of the interval on which f is defined). In fact, all derivatives of lower orders $f, f', \dots, f^{(k-1)}$ must exist in a neighborhood of x.

Taylor's Theorem 泰勒定理

5.11 Theorem: Taylor's Theorem (泰勒定理) ▶

Suppose f is a real function on $[a,b]$ such that $f^{(n)}$ is continuous on $[a,b]$ and $f^{(n+1)}$ exists on (a,b). Then, for any given points $x_0, x\in[a,b]$, there exists a point ξ between x_0 and x such that

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$ ▶

In particular, when $n = 0$, Taylor's Theorem is just the Mean Value Theorem.

- The function
$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$
is called the **Taylor polynomial** (泰勒多項式) of degree n.

- The difference
$$R_n(x) = f(x) - T_n(x),$$
is called the **Taylor remainder** (泰勒餘項).

Remark

- Taylor's Theorem shows that the f can be approximated by the Taylor polynomial, with the remainder being $\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$. This expression is called the **Lagrange remainder** (拉格朗日餘項). There exist other formulations for the remainder.
- If $\lim_{n\to\infty} R_n(x) = 0$, one obtains the so-called **Taylor expansion** (泰勒展開):
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Remark

- The significance of Taylor's theorem lies in its applications in many areas of mathematics, science, and engineering. It is a fundamental result in analysis and a powerful tool in numerous applications, including in
  - numerical analysis;
  - optimization;
  - control theory;
  - probability and statistics.

Addendum 後記

Addendum of Chapter 5

- Based on the Monotone Test, one can easily establish the so-called First Derivative Test for local extrema.

the First Derivative Test (第一階導數測試)

Suppose that $f\colon (a, b) \to \mathbb{R}$ is continuous at $x_0 \in (a, b)$, and that r is a positive number.

1. If $f'(x) \geq 0$ on $(x_0 - r, x_0]$ and $f'(x) \leq 0$ on $[x_0, x_0 + r)$, then f has a local maximum at x_0.

2. If $f'(x) \leq 0$ on $(x_0 - r, x_0]$ and $f'(x) \geq 0$ on $[x_0, x_0 + r)$, then f has a local minimum at x_0.

3. If f does not change sign at x_0, then f has no local extreme value at x_0.

- By applying the Mean Value Theorem and the First Derivative Test, one can further establish the so-called Second Derivative Test for local extrema.

the Second Derivative Test (第二階導數測試)

Suppose $f''(x_0)$ exists.

1. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a local minimum at x_0.

2. If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a local maximum at x_0.

3. If $f'(x_0) = 0$ and $f''(x_0) = 0$, it is inconclusive about local extreme of f at x_0.

- There are commonly two ways to generalize the derivative tests. One is to examine higher-order derivatives, and the other is to have derivative tests for functions of more than one variable. Both of them can be put into a framework of general Taylor's theorem.

Exercises of Chapter 5 練習題

Chapter 5: Quiz **30 Minutes**

① Suppose that the derivative $f'(x)$ exists at every $x \in (a, b)$. Which of the following must be true?

A. $f'(x-) = f'(x+) = \lim\limits_{t\to x} \dfrac{f(t) - f(x)}{t - x}$

B. $f'_+(x) = f'_-(x) = \lim\limits_{t\to x} \dfrac{f(t) - f(x)}{t - x}$

C. $f'_+(x) = f'_-(x) = f'(x-) = f'(x+) = \lim\limits_{t\to x} \dfrac{f(t) - f(x)}{t - x}$

D. $\lim\limits_{t\to x-} f'(x) = \lim\limits_{t\to x+} f'(x) = \lim\limits_{t\to x} \dfrac{f(t) - f(x)}{t - x}$

E. none of the above

② Let $f\colon [a, b] \to \mathbb{R}$ be a differentiable function. Suppose that $f(a) = f(b)$. Which of the following must be true?

A. there exists $x \in [a, b]$ such that $f'(x) = 0$

B. there exists $x \in [a, b]$ such that $f'(x) = 1$

C. there exists $x \in [a, b]$ such that $f'(x) = -1$

D. there exists $x \in [a, b]$ such that $|f'(x)| \geq \frac{1}{2}$

E. none of the above

③ Which of the following statements is true regarding a critical point x of a function f?

A. at which $f'(x)$ is undefined

B. at which $f'(x) = 0$

C. at which $f'(x) = 0$ or $f'(x)$ does not exist

D. at which $f'(x) = 0$ or $f'(x)$ is unbounded

E. at which $f(x) = 0$ and $f'(x) = 0$

⑥ Which of the following statements is true about the Mean Value Theorem?

A. it applies only to continuous functions on closed intervals

B. it guarantees that a function has a local maximum or minimum on an open interval

C. it requires that the function be differentiable on the entire interval

D. it states that the average rate of change of a function over an interval equals its instantaneous rate of change at some point in the interval

E. it can be applied to any function, regardless of its properties

④ Suppose a real function f is differentiable on $\mathbb{R}$ and there is a number M such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Which of the following statements is true?

A. there is a number B such that $|f(x)| < B$ for all $x \in \mathbb{R}$

B. f attain its supremum and infimum on $\mathbb{R}$

C. f' attain its supremum and infimum on $\mathbb{R}$

D. f is differentiable but not uniformly continuous on $\mathbb{R}$

E. f is uniformly continuous on $\mathbb{R}$

⑤ As x approaches a, if $f(x)/g(x)$ is an indeterminate form $0/0$ or ∞/∞, which of the following statements is a correct version of l'Hôpital's rule?

A. if $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(x)}{g'(x)}$

B. if $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ exists, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(x)}{g'(x)}$

C. if $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ exists, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

D. if $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

E. none of the above

⑦ Which of the following statements best describes Taylor's theorem?

A. it is for evaluating the limit of a function at a point

B. it is for approximating a function with a polynomial

C. it is for computing the derivative of a function at a point

D. it is for determining the discontinuities of a function over an interval

E. it is for finding the roots of a function

⑧ Suppose that f is a function such that $|f^{(n+1)}(x)| \leq M$ for all $x \in [a, b]$. What is the maximum possible error of the nth-degree Taylor polynomial of f about the point $x = a$?

A. $M/n!$

B. $M/(n + 1)!$

C. $M(b - a)^n/n!$

D. $M(b - a)^{n+1}/(n + 1)!$

E. M

①B, ②A, ③C, ④D, ⑤E, ⑥D, ⑦B, ⑧D

Chapter 5: Exercises

Exercise 5.1 (E)

Suppose that $f\colon \mathbb{R}\to\mathbb{R}$ satisfies
$$|f(x)-f(y)|\le C|x-y|^2$$
fors all $x,y\in\mathbb{R}$. Prove that f is a constant.

Hint ① Show that $f'(x)=0$.

Hint ② Apply the Mean Value Theorem or the Monotone Test.

Exercise 5.2 (I)

Suppose that f is a real continuous function on $[a,\infty)$ which is differentiable in (a,∞). If $\lim\limits_{x\to\infty} f(x)=f(a)$, then there exists $\xi\in(a,\infty)$ such that $f'(\xi)=0$.

Hint ① If there is $x\in(a,\infty)$ such that $f(x)=f(a)$, then apply the Mean Value Theorem immediately.

Hint ② If for all $x\in(a,\infty)$ such that $f(x)\ne f(a)$, then assume that there is a number $c\in(a,\infty)$ such that $f(c)>f(a)$. Since $\lim\limits_{x\to\infty} f(x)=f(a)$, prove that there is $X>c$ such that $f(X+1)<f(c)$.

Hint ③ Apply the Extreme Value Theorem on the closed interval $[a,X+1]$ that the function f attains its maximum value at some point $\xi\in(a,X+1)$.

Exercise 5.3 (I)

Suppose g is a real function on $\mathbb{R}$, with bounded derivative (say $|g'|<M$). Fix $\varepsilon>0$, and define $f(x)=x+\varepsilon g(x)$. Prove that f is one-to-one if ε is sufficiently small.

Hint ① Use the Mean Value Theorem to show that f is strictly increasing for sufficiently small ε.

Exercise 5.4 (E)

Assume that
$$C_0+\frac{C_1}{2}+\cdots+\frac{C_{n-1}}{n}+\frac{C_n}{n+1}=0,$$
where $C_0,\ldots,C_n$ are real constants. Prove that the equation
$$C_0+C_1x+\cdots+C_{n-1}x^{n-1}+C_nx^n=0$$
has at least one real root between 0 and 1.

Hint ① Apply the Mean Value Theorem to the function
$$P(x)=C_0x+\frac{C_1}{2}x^2+\cdots+\frac{C_{n-1}}{n}x^n+\frac{C_n}{n+1}x^{n+1}.$$

Exercise 5.5 (I)

Suppose that $f\colon[a,b]\to\mathbb{R}$ is continuous, $f(a)=f(b)=0$, and $f'_+(a)\cdot f'_-(b)>0$. Prove that there exists $\xi\in(a,b)$ such that $f(\xi)=0$.

Hint ① Without loss of generality, assume that $f'_+(a)>0$. Prove that there is a number x_1 in (a,b) such that $f(x_1)>0$.

Hint ② Similarly, the inequality $f'_-(b)>0$ implies that there is a number $x_2\in(a,b)$ such that $f(x_2)<0$.

Hint ③ Apply the Intermediate Value Theorem.

Exercise 5.6 (E)

Suppose $f'(x)$, $g'(x)$ exist, $g'(x)\ne 0$, and $f(x)=g(x)=0$. Prove that $\lim\limits_{t\to x}\dfrac{f(t)}{g(t)}=\dfrac{f'(x)}{g'(x)}$.

Hint ① Apply the definition of derivative and Proposition 4.13.

Exercise 5.7 (I)

Suppose $f\colon\mathbb{R}\to\mathbb{R}$. Suppose $f''(x)>0$ for all $x\in\mathbb{R}$. Prove that the inequality
$$\frac{f(x_1)+f(x_2)}{2}>f\big(\tfrac{1}{2}(x_1+x_2)\big)$$
holds for all distinct $x_1,x_2\in\mathbb{R}$.

Hint ① Apply Taylor's Theorem.

Hint ② In the equality
$$f(y)=f(x)+f'(x)(y-x)+\frac{f''(\xi)}{2}(y-x)^2,$$
put $y=x+h,\ x-h$ respectively.

Exercise 5.8 (I)

Let $f\colon \mathbb{R} \to \mathbb{R}$. Suppose $f'''(x) > 0$ for all $x \in \mathbb{R}$. Prove that the inequality

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > f'\big(\tfrac{1}{2}(x_1 + x_2)\big)$$

holds for all $x_1, x_2 \in \mathbb{R}$ with $x_2 > x_1$.

Hint ① Apply Taylor's Theorem.

Hint ② Denote $\bar{x} = \frac{1}{2}(x_1 + x_2)$. In the equality

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{1}{2}f''(\bar{x})(x - \bar{x})^2 + \frac{1}{6}f'''(\xi)(x - \bar{x})^3,$$

put $x = x_1, x_2$, respectively.

Exercise 5.9 (D)

Suppose f is a twice-differentiable real function on (a, ∞), and $M_0, M_1, M_2 \in \mathbb{R}$ are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞). Prove that $M_1^2 \le 4M_0 M_2$.

Hint ① When either $M_0 = 0$ or $M_2 = 0$, it is easy to prove the conclusion.

Hint ② For $M_0 > 0$ and $M_2 > 0$, obtain

$$f(x + 2h) = f(x) + \frac{1}{1!}f'(x)(2h) + \frac{1}{2!}f''(\xi)(2h)^2,$$

by Taylor's Theorem. Then show that

$$|f'(x)| \le \frac{1}{h}M_0 + hM_2.$$

Hint ③ Take $h = \sqrt{M_0/M_2}$.

Exercise 5.10 (D)

Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function with $|f'(x)| \le A < 1$ for all $x \in \mathbb{R}$, where A is a constant. Prove that

1. The function f has a unique fixed point in $\mathbb{R}$.
2. For any real number x_1, define
$$x_{n+1} = f(x_n), \qquad n = 1, 2, 3, \ldots.$$
The sequence $\{x_n\}$ converges to the unique fixed point of f.

This result is known as the $\boxed{\text{contraction principle}}$ (壓縮映像原理).

Hint ① For part **1**, let x_1 and x_2 be two fixed points of f. Show that $x_1 = x_2$ by applying the Mean Value Theorem.

Hint ② For part **2**, show that $\{x_n\}$ is a Cauchy sequence.

Hint ③ Apply the Mean Value Theorem.

Hint ④ Show that

$$|x_{n+1} - x_n| \le A^{n-1} \cdot |x_2 - x_1|, \qquad n = 1, 2, 3, \ldots.$$

CHAPTER

第 6 章

The Riemann Integral 黎曼積分

Overview of Chapter 6

In this chapter, our focus will be on studying the Riemann integral of real functions.

- We will give a rigorous definition of the Riemann integral and prove integrability criterion, as well as two necessary and sufficient conditions.
- We will prove that Riemann integrability applies to continuous functions, monotonic functions, and functions with finitely many discontinuities. We will also demonstrate that the composition of a continuous function and a Riemann integrable function is also Riemann integrable.
- We will show that Riemann integrals satisfy linearity, additivity, and monotonicity.
- We will prove the Fundamental Theorem of Calculus and demonstrate how differentiation and integration are inverse operations of each other.

§6.1 Integrability 可積性

6.1 Definition: Riemann integral (黎曼積分) ▶

Let $[a, b]$ be a given interval. Let P be a partition (劃分) of $[a, b]$:

$$P: \quad a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b.$$

Suppose f is a bounded real function defined on $[a, b]$. On each subinterval $[x_{i-1}, x_i]$, put

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x), \qquad m_i = \inf_{x_{i-1} \le x \le x_i} f(x).$$

Since f is bounded, there exists m and M such that $m \le f(x) \le M$ for $x \in [a, b]$.

Write $\Delta x_i = x_i - x_{i-1}$, $1 \le i \le n$. Form

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

For each fixed partition P, $\quad m(b-a) \le L(P, f) \le U(P, f) \le M(b-a),$

so that the numbers $\{L(P, f)\}$ and $\{U(P, f)\}$ form bounded sets. Denote the ***upper*** integral (上積分) and the ***lower*** integral (下積分), respectively:

$$\overline{\int_a^b} f \,\mathrm{d}x = U(f) = \inf_P U(P, f), \qquad \underline{\int_a^b} f \,\mathrm{d}x = L(f) = \sup_P L(P, f).$$

When two values are equal, we represent the common value using the notation

$$\int_a^b f \, dx \quad \text{or} \quad \int_a^b f(x) \, dx,$$

which is referred to as the **Riemann integral** (黎曼積分) of f over the interval $[a, b]$. The function f is referred to as the **integrand** (被積函數),

If the integral exists, we say that f is **Riemann integrable** (黎曼可積) on $[a, b]$, and write $f \in \mathscr{R}[a, b]$. Here $\mathscr{R}[a, b]$ denotes the set of Riemann integrable functions on $[a, b]$.

6.2 Proposition: inequalities on integral sums

The partition P^* is a **refinement** (細化) of P if every point of P is a point of P^*, that is, $P^* \supset P$. Given two partitions, P_1 and P_2, we say that $P^* = P_1 \vee P_2$ is their ***common refinement*** (共同細化).

Suppose P^* is *any* refinement of partition P of $[a, b]$. Then

$$L(P, f) \leq L(P^*, f), \qquad U(P^*, f) \leq U(P, f).$$

Corollary

For any two partitions P_1 and P_2 of $[a, b]$, denote P^* their common refinement. Then

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f).$$

lower integral $\leq$ upper integral

$$\underline{\int_a^b} f \, dx \leq \overline{\int_a^b} f \, dx.$$

6.3 Theorem: integrability criterion (可積性判據)

A function $f \in \mathscr{R}[a, b]$ if and only if the **integrability criterion** (可積性判據) holds:

for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Necessary and Sufficient Conditions for Integrability

A function $f \in \mathscr{R}[a, b]$ if and only if one of the following conditions holds:

Condition 1

- For every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that
$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta x_i < \varepsilon$$
for any $s_i, t_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$.

Condition 2

- For every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that, for some real number I,
$$\left| \sum_{i=1}^{n} f(t_i) \Delta x_i - I \right| < \varepsilon$$
for any $t_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$.

Remark

- By Proposition 6.2, we know that if the integrability criterion holds for some partition P, then it also holds for any refinement of P. This is also true for Condition 1.
- If the integrability criterion holds for a partition P, then Condition 1 and Condition 2 hold for the same P.
- If Condition 2 holds, then $I = \int_a^b f \, dx$.
- If $f \in \mathscr{R}[a, b]$, then f is bounded on $[a, b]$.

Riemann Integrable Functions 黎曼可積函數

6.4 Theorem: Riemann integrable functions

1. If f is continuous on $[a, b]$, then $f \in \mathscr{R}[a, b]$. ▶
2. If f is bounded on $[a, b]$ and has only finitely many discontinuities, then $f \in \mathscr{R}[a, b]$. ▶
3. If f is monotonic on $[a, b]$, then $f \in \mathscr{R}[a, b]$. ▶

Remark

- One key result in Lebesgue Theory states that a function defined on an interval is Riemann integrable if and only if the function is bounded and continuous almost everywhere.

6.5 Theorem: Riemann integrability of composition

Suppose $f \in \mathscr{R}[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}[a, b]$. ▶

Remark

- A variety of integrable functions can be generated from different functions φ.
  - $\boxed{\text{If } f \in \mathscr{R}[a, b], \text{ then } f^2 \in \mathscr{R}[a, b].}$ Here we take $\varphi(x) = x^2$.
  - $\boxed{\text{If } f \in \mathscr{R}[a, b], \text{ then } |f| \in \mathscr{R}[a, b].}$ Here we take $\varphi(x) = |x|$.
- This theorem can be applied further to demonstrate that many binary operations of functions, including arithmetic operations, preserve integrability. ▶

§6.2 Properties of the Riemann Integral 積分的性質

6.6 Proposition: properties of the Riemann integral

1. ***Linearity*** (線性性): If $f, g \in \mathscr{R}[a, b]$ and α, β are real numbers, then $\alpha f + \beta g \in \mathscr{R}[a, b]$ and
$$\int_a^b (\alpha f + \beta g)\,\mathrm{d}x = \alpha \int_a^b f\,\mathrm{d}x + \beta \int_a^b g\,\mathrm{d}x.$$ ▶
2. ***Additivity*** (可加性): If $f \in \mathscr{R}[a, b]$ and if $a < c < b$, then $f \in \mathscr{R}[a, c] \cap \mathscr{R}[c, b]$, and
$$\int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x.$$ ▶
3. ***Monotonicity*** (單調性): If $f, g \in \mathscr{R}[a, b]$, and if $f(x) \le g(x)$, then
$$\int_a^b f\,\mathrm{d}x \le \int_a^b g\,\mathrm{d}x.$$ ▶

Corollary

If $f \in \mathscr{R}[a, b]$, then $|f| \in \mathscr{R}[a, b]$ and
$$\left|\int_a^b f\,\mathrm{d}x\right| \le \int_a^b |f|\,\mathrm{d}x.$$

Furthermore, if $|f(x)| \le M$ on $[a, b]$, then
$$\left|\int_a^b f\,\mathrm{d}x\right| \le M(b-a).$$ ▶

§6.3 Integration and Differentiation 積分與微分的關係

6.7 Theorem: the Fundamental Theorem of Calculus (微積分基本定理)

(Part 1) Suppose $f \in \mathscr{R}[a,b]$. For $a \le x \le b$, put

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Then F is continuous on $[a,b]$. Furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

(Part 2) If $f \in \mathscr{R}[a,b]$, and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then the **Newton–Leibniz formula** (牛頓-萊布尼茲公式) holds:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Remark

- This theorem show that differentiation and integration are inverse operations of each other.
- In general, integration operation makes a function smoother.

For a given function f, a differentiable function F is called an **antiderivative** (反導函數) of f if $F'(x) = f(x)$.

6.8 Theorem: change of variable for the Riemann integral

Suppose $\varphi\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative $\varphi' \in \mathscr{R}[a,b]$. For any real continuous function f on $\varphi([a,b])$, the change of variable formula holds:

$$\int_a^b f(\varphi(x))\,\varphi'(x)\,\mathrm{d}x = \int_{\varphi(a)}^{\varphi(b)} f(t)\,\mathrm{d}t.$$

6.9 Theorem: integration by parts

Suppose F and G are differentiable functions on $[a,b]$, $F' = f \in \mathscr{R}[a,b]$, and $G' = g \in \mathscr{R}[a,b]$. Then

$$\int_a^b F(x)g(x)\,\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,\mathrm{d}x.$$

Addendum 後記

Addendum of Chapter 6

- Riemann integral is a fundamental concept in integration theory that provides a way to calculate the area under a curve. While the Riemann integral is a powerful tool for solving many mathematical problems, there are several directions of generalization that can be made to extend its applicability. Some of these directions include:
  - Lebesgue integration: This is a more general form of integration that extends the Riemann integral to a wider class of functions, including those that are not continuous or have infinite discontinuities. Lebesgue integration also allows for the integration of functions defined on more general spaces, such as infinite-dimensional spaces.
  - Improper integrals: These are integrals that involve functions that are not defined or are not continuous at one or more points within the integration interval. Improper integrals can be evaluated

using limits, and they are used to calculate integrals that would otherwise be impossible to solve using the Riemann integral.

– Vector-valued integrals: These are integrals that involve functions that take values in a vector space, rather than real numbers. Vector-valued integrals are used in many areas of mathematics and physics, including calculus of variations, differential geometry, and quantum mechanics.
– Stochastic integrals: These are integrals that involve random variables or stochastic processes. Stochastic integrals are used in probability theory and mathematical finance to model the behavior of random systems and to calculate probabilities of events.

Exercises of Chapter 6 練習題

Chapter 6: Quiz **30 Minutes**

① Which of the following is a necessary condition for a function to be Riemann integrable on an interval?

A. the function must be continuous on the interval
B. the function must be differentiable on the interval
C. the function must be bounded on the interval
D. the function must be monotonic on the interval
E. none of the above

② Which of the following is a sufficient condition for a function to be Riemann integrable on $[a,b]$?

A. the function is continuous on the interval $[a,b]$
B. the function is bounded on the interval $[a,b]$
C. the function is monotonic on the interval (a,b)
D. the function is differentiable on the interval (a,b)
E. the function has finitely many discontinuities on $[a,b]$

③ Suppose $f\colon [a,b] \to \mathbb{R}$. Which of the following statements is true?

A. if $\inf_P L(P,f) = \sup_P U(P,f)$, then $f \in \mathscr{R}[a,b]$

B. if $\overline{\int_a^b} f\,\mathrm{d}x - \underline{\int_a^b} f\,\mathrm{d}x < \varepsilon$ for every $\varepsilon > 0$, then $f \in \mathscr{R}[a,b]$

C. if for some positive integer n, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \dfrac{1}{n}$, then $f \in \mathscr{R}[a,b]$

D. if for some $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$, then $f \in \mathscr{R}[a,b]$

E. none of the above

④ Suppose that $f_1, f_2 \in \mathscr{R}[a,b]$ and g_1, g_2, g are continuous on $[a,b]$. Which of the following statements is true?

A. on $[a,b]$ if $f_1 \le f_2$, then $\int_a^b f_1 g\,\mathrm{d}x \le \int_a^b f_2 g\,\mathrm{d}x$

B. on $[a,b]$ if $f_1 \le f_2$, then $\int_a^b |f_1 g|\,\mathrm{d}x \le \int_a^b |f_2 g|\,\mathrm{d}x$

C. on $[a,b]$ if $f_1 \le f_2$ and $g_1 \le g_2$, then $\int_a^b f_1 g_1\,\mathrm{d}x \le \int_a^b f_2 g_2\,\mathrm{d}x$

D. on $[a,b]$ if $f_1 \le f_2$ and $g_1 \le g_2$, then
$$\int_a^b (f_1 g_2 + f_2 g_1)\,\mathrm{d}x \le \int_a^b (f_1 g_1 + f_2 g_2)\,\mathrm{d}x$$

E. none of the above

⑤ Which of the following statements is NOT true?

A. if $f \in \mathscr{R}[a,b]$, then $\dfrac{1}{1+f^2} \in \mathscr{R}[a,b]$
B. if $f \in \mathscr{R}[a,b]$, then $f^2 + f^3 \in \mathscr{R}[a,b]$
C. if $f \in \mathscr{R}[a,b]$, then $|f| \in \mathscr{R}[a,b]$
D. if $f \in \mathscr{R}[a,b]$, then $e^f \in \mathscr{R}[a,b]$
E. none of the above

⑥ Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and α is a differentiable function on $\mathbb{R}$. Which of the following equalities is true?

A. $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^{\alpha(x)} f(t)\,\mathrm{d}t = f\big(\alpha(x)\big) - f(0)$

B. $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^{\alpha(x)} f(t)\,\mathrm{d}t = f\big(\alpha'(x)\big) - f(0)$

C. $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^{\alpha(x)} f(t)\,\mathrm{d}t = f\big(\alpha(x)\big) \cdot \alpha'(x)$

D. $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^{\alpha(x)} f(t)\,\mathrm{d}t = f\big(\alpha'(x)\big) \cdot \alpha(x)$

E. $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^{\alpha(x)} f(t)\,\mathrm{d}t = f\big(\alpha'(x)\big) \cdot \alpha'(x)$

⑦ Suppose that on $[a,b]$, f is continuous and α' is continuous. Assume that $F'(x) = f(x)$, $F_\alpha'(x) = G(x) = f(x)\alpha'(x)$. Which of the following statements is true?

A. $\int_a^b f\alpha'\,\mathrm{d}x = G(b) - G(a)$

B. $\int_a^b f\alpha'\,\mathrm{d}x = F_\alpha(b) - F_\alpha(a)$

C. $\int_a^b f\alpha'\,\mathrm{d}x = F(b)\alpha(b) - F(a)\alpha(a)$

D. all of the above

E. none of the above

⑧ Which of the following is a consequence of the Fundamental Theorem of Calculus for a function f on an interval $[a,b]$?

A. if $f = 0$ on $[a,b]$, then $\int_a^b f\,\mathrm{d}x = 0$

B. if $f > 0$ on $[a,b]$, then $\int_a^b f\,\mathrm{d}x > 0$

C. if $f \ge 0$ on $[a,b]$, then $\int_a^b f\,\mathrm{d}x \ge 0$

D. if f is continuous on $[a,b]$, then, for any $c \in [a,b]$,
$$\int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x$$

E. none of the above

①C, ②A, ③B, ④D, ⑤E, ⑥C, ⑦B, ⑧A

Chapter 6: Exercises

Exercise 6.1 (I)

Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f\,\mathrm{d}x = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Hint ① If $f(x^*) > 0$ for some $x^* \in [a,b]$, then, by the continuity of f at x^*, show that there is an interval with positive length, on which $f(x) > \frac{1}{2} f(x^*)$.

Hint ② Apply the monotonicity of the Riemann integral.

Exercise 6.2 (E)

The **Dirichlet function** $I_{\mathbb{Q}} \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$I_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $I_{\mathbb{Q}} \notin \mathscr{R}[a,b]$ for any $-\infty < a < b < \infty$.

Hint ① Show that for any partition P of $[a,b]$,

$$U(P, I_{\mathbb{Q}}) = b - a, \qquad L(P, I_{\mathbb{Q}}) = 0.$$

Exercise 6.3 (E)

Suppose f is a bounded real function on $[a,b]$, and $f^2 \in \mathscr{R}[a,b]$. Does it follow that $f \in \mathscr{R}[a,b]$? Does the answer change if we assume that $f^3 \in \mathscr{R}[a,b]$?

Hint ① Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ -1, & \text{if } x \text{ is irrational.} \end{cases}$$

Hint ② Apply item 3 in Proposition 4.18 to show that the function $\phi(x) = \sqrt[3]{x}$ is continuous.

Hint ③ Apply Theorem 6.5.

Exercise 6.4 (I)

Suppose $f \in \mathscr{R}[a,b]$ and $g \in \mathscr{R}[a,b]$.

1. Prove that $fg \in \mathscr{R}[a,b]$.
2. If $|g(x)| \geq c > 0$ for some constant c, prove that $f/g \in \mathscr{R}[a,b]$.
3. Prove that both $\max\{f,g\} \in \mathscr{R}[a,b]$ and $\min\{f,g\} \in \mathscr{R}[a,b]$.

Hint ① For part **1**, use the equality

$$fg = \tfrac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

Hint ② Apply the linearity of the Riemann integral and Theorem 6.5.

Hint ③ For part **2**, first prove that $1/g \in \mathscr{R}[a,b]$, and then apply part **1**.

Hint ④ Show that $\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| \leq c^{-2}|g(x) - g(y)|$ to get

$$\sup_{x\in I}\frac{1}{g(x)} - \inf_{x\in I}\frac{1}{g(x)} \leq c^{-2}\left[\sup_{x\in I} g(x) - \inf_{x\in I} g(x)\right]$$

for any interval $I \subset [a,b]$.

Hint ⑤ For part **3**, use the equalities

$$\max\{f,g\} = \tfrac{1}{2}\left[f + g + |f-g|\right],$$

$$\min\{f,g\} = \tfrac{1}{2}\left[f + g - |f-g|\right].$$

Hint ⑥ The absolute value function $\phi(x) = |x|$ is continuous. Apply Theorem 6.5.

Exercise 6.5 (I)

Suppose $f \in \mathscr{R}[a,b]$. If $g(x) = f(x)$ on $[a,b]$ except possibly at $x^* \in [a,b]$, prove that $g \in \mathscr{R}[a,b]$, and

$$\int_a^b g\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x.$$

Hint ① Let $h = g - f$. Then h is continuous on $[a,b]$ except possibly at x^*, so that $h \in \mathscr{R}[a,b]$ by item 2 of Theorem 6.4.

Hint ② Apply the linearity of the Riemann integral.

Hint ③ To show $\int_a^b h\,\mathrm{d}x = 0$, show that $\overline{\int_a^b} h\,\mathrm{d}x = 0$ if $h(x^*) \geq 0$.

Exercise 6.6 (I)

Suppose f is a *continuous*, *positive*, *decreasing* function on $[1, \infty)$ and let $a_k = f(k)$. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the ***improper integral*** $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges, i.e., the limit $\lim_{n\to\infty} \int_1^n f(x)\,\mathrm{d}x$ exists and is finite.

This result is known as the Integral Test (積分審斂法) for infinite series.

Hint ①: ① $\Leftarrow$: Show that for any positive integer n,

$$\sum_{k=1}^{n} f(k) \le f(1) + \int_1^{\infty} f(x)\,\mathrm{d}x.$$

Hint ②: ② $\Rightarrow$: Show that for any positive b,

$$\int_1^b f(x)\,\mathrm{d}x \le \sum_{k=1}^{\infty} f(k).$$

Exercise 6.7 (I)

Prove the Mean Value Theorem for Integrals (積分中值定理) as stated below:

Let f be a real continuous function on $[a, b]$. If $g \in \mathscr{R}[a, b]$ and $g(x)$ does not change its sign on $[a, b]$, then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(\xi)\int_a^b g(x)\,\mathrm{d}x.$$

In particular, when $g \equiv 1$, then the result deduces the equality

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = f(\xi).$$

Hint ①: ① Since both f and g are Riemann integrable, their product fg is integrable by Ex. 6.4.

Hint ②: ② Show that

$$m\int_a^b g(x)\mathrm{d}x \le \int_a^b f(x)g(x)\mathrm{d}x \le M\int_a^b g(x)\mathrm{d}x,$$

where

$$m = \min_{x\in[a,b]} f(x), \qquad M = \max_{x\in[a,b]} f(x).$$

Hint ③: ③ If $\int_a^b g(x)\,\mathrm{d}x > 0$, then

$$m \le \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \le M.$$

Apply the Intermediate Value Theorem.

Exercise 6.8 (I)

Suppose $f \in \mathscr{R}[a, b]$. Prove that for every $\varepsilon > 0$, there exists a continuous function g on $[a, b]$ such that

$$\int_a^b |f - g|\,\mathrm{d}x < \varepsilon.$$

Hint ①: ① Since $f \in \mathscr{R}[a, b]$, there are m and M such that

$$m \le f(x) \le M, \quad x \in [a, b].$$

Show that the problem is trivial if $M = m$.

Hint ②: ② Suppose $M \ne m$. By the integrability criterion, there exists a partition P of $[a, b]$:

$$P: a = x_0 < x_1 < \cdots < x_n = b,$$

such that $U(P, f) - L(P, f) < \varepsilon$. Define a function g_P to be piecewise linear on $[x_{i-1}, x_i]$, with $g_P(x_i) = f(x_i)$. Show that g_P is one of such function that satisfy the requirements in the problem.

Hint ③: ③ Let $m_i = \inf_{x_{i-1}\le x\le x_i} f(x)$, $M_i = \sup_{x_{i-1}\le x\le x_i} f(x)$. On $[x_{i-1}, x_i]$, we have $m_i \le g_P(x) \le M_i$, so that

$$|f(x) - g_P(x)| \le M_i - m_i \le M - m.$$

Show that $\int_a^b |f - g_P|\,\mathrm{d}x < \varepsilon$.

Exercise 6.9 (I)

Suppose that f is a real function such that $f^{(n+1)}$ is continuous on $[a, b]$. Prove that for any given points $x_0, x \in [a, b]$, the Taylor remainder

$$R_n(x) = f(x) - T_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is given by

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,\mathrm{d}t.$$

This is called the ***integral form*** of the Taylor remainder.

Hint ①: ① Prove by induction.

Hint ②: ② Integrate by parts.

Exercise 6.10 (D)

Suppose that $f\colon [a,b] \to \mathbb{R}$ be a differentiable function such that its derivative f' is continuous. If $f(a)+f(b)=0$, prove that

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \frac{b-a}{2}\int_a^b \left|f'(x)\right|\,\mathrm{d}x.$$

Hint ①

① Apply the Newton–Leibniz formula to have

$$2f(x) = \int_a^x f'(t)\,\mathrm{d}t - \int_x^b f'(t)\,\mathrm{d}t.$$

Hint ②

② Express the integral $\int_a^b f(x)\,\mathrm{d}x$ as an iterative integral. Apply the properties of the Riemann integral (Proposition 6.6) to have the desired inequality.

CHAPTER

第 7 章

Sequences and Series of Functions 函數序列與級數

Overview of Chapter 7

The main focus of this chapter is on the *uniform convergence* of functions.

- We will explore the significant role uniform convergence plays in limit processes involving continuity, differentiation, and integration.
- We will prove Dini's theorem, which gives a sufficient condition for uniform convergence of a piecewise convergent sequence of functions.
- We will prove the Weierstrass approximate theorem, which states that any continuous function on a finite closed interval can be uniformly approximated by polynomials.

§7.1 Interchange of Limit Processes 極限的交換順序

7.1 Definition: pointwise convergence (逐點收斂)

Suppose $\{f_n\}$ is a sequence of real-valued functions defined on a set $E \subset \mathbb{R}$, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in E,$$

and say that $\{f_n\}$ **converges pointwise** (逐點收斂) to f on E. That is, a sequence $\{f_n\}$ of functions converges pointwise to f on E if

for every point $x \in E$ and every $\varepsilon > 0$ there exists an integer N such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$.

In this case, we call the function f is the ***limit*** of $\{f_n\}$.

A series $\sum f_n(x)$ converges pointwise on E if the sequence $\{s_n\}$ of partial sums, $s_n(x) = \sum_{i=1}^{n} f_i(x)$, converges pointwise on E. If f is the limit of $\{s_n\}$, we denote

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \qquad x \in E,$$

and call the function f the ***sum*** of the series $\sum f_n$.

Remark

- Pointwise convergence only requires that the limit holds for each point in the domain, not *uniformly*. The value of N depends on the point being considered.
- For the sake of simplicity, we restrict our studies in this chapter to real-valued functions, although most of the results can be extended to complex-valued functions.

Remark: on continuity of the limit

- Suppose $\{f_n\}$ converges pointwise to f on $E \subset \mathbb{R}$. If each f_n is continuous, is the limit f continuous? Or equivalently, does $\lim_{t \to x} f(t) = f(x)$ hold? Re-write last equality as
$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t),$$
so, the question is to ask whether changing the order of two limit processes above is permissible.

Example: $\lim_{t \to x} \lim_{n \to \infty} f_n(t) \neq \lim_{n \to \infty} \lim_{t \to x} f_n(t)$

For real x, let
$$f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1+x^2)^k}, \qquad n = 0, 1, 2, \ldots.$$
Then
$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \begin{cases} 0, & \text{if } x = 0, \\ 1+x^2, & \text{if } x \neq 0. \end{cases}$$
It follows that $\lim_{t \to 0} f(t) \neq f(0)$, that is,
$$\lim_{t \to 0} \lim_{n \to \infty} f_n(t) \neq \lim_{n \to \infty} \lim_{t \to 0} f_n(t).$$
It demonstrates that the limit of continuous functions may not be a continuous function, and also a convergent series of continuous functions may have a discontinuous sum.

Remark: on convergence of sequence of derivatives

- Suppose $\{f_n\}$ converges pointwise to f on $E \subset \mathbb{R}$. If each f_n is differentiable, is the limit f differentiable? If the answer is yes, does $\lim_{n \to \infty} f_n'(x) = f'(x)$ holds for every $x \in E$? In a general setting, the last limit can be re-written as
$$\lim_{n \to \infty} \lim_{t \to x} \frac{f_n(t) - f_n(x)}{t - x} = \lim_{t \to x} \lim_{n \to \infty} \frac{f_n(t) - f_n(x)}{t - x}.$$
so, the question is again to ask whether changing the order of two limit processes is permissible.

Example: $\lim_{n \to \infty} \lim_{t \to x} \frac{f_n(t) - f_n(x)}{t - x} \neq \lim_{t \to x} \lim_{n \to \infty} \frac{f_n(t) - f_n(x)}{t - x}$

For $-1 \leq x \leq 1$, let
$$f_n(x) = \frac{x}{1+n^2x^2}, \qquad n = 1, 2, 3, \ldots.$$
Then
$$f(x) = \lim_{n \to \infty} f_n(x) = 0.$$
Since $f_n'(x) = \dfrac{1-n^2x^2}{(1+n^2x^2)^2}$, we have
$$\lim_{n \to \infty} f_n'(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$
so that $\lim_{n \to \infty} f_n'(x) \neq f'(x)$. Hence,
$$\lim_{n \to \infty} \lim_{t \to x} \frac{f_n(t) - f_n(x)}{t - x} \neq \lim_{t \to x} \lim_{n \to \infty} \frac{f_n(t) - f_n(x)}{t - x}.$$
This shows that the limit of the derivatives may not be the derivative of the limit.

Remark: on convergence of sequence of integrals

- Suppose $\{f_n\}$ converges pointwise to f on $[a,b]$. If each f_n is integrable, is the limit f integrable? If the answer is yes, does $\lim_{n\to\infty}\int_a^b f_n\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x$ hold? This again involves changing the order of two operations:
$$\lim_{n\to\infty}\int_a^b f_n\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n\,\mathrm{d}x.$$

Example: $\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x \neq \int_a^b \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x$

For $0\le x\le 1$, let $f_n(x) = nx(1-x^2)^n$, $n=1,2,3,\ldots$. Then
$$\lim_{n\to\infty} f_n(x) = 0, \qquad 0\le x\le 1.$$
It is easy to calculate
$$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{n}{2n+2}.$$
Thus,
$$\lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x = \tfrac{1}{2} \neq 0 = \int_0^1 \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x.$$
It shows that the limit of the integrals may not be the integral of the limit.

Remark

- From the examples above, we see that under the assumption of pointwise convergence, some of the important properties of functions, namely continuity, differentiation, and integration, cannot be carried over from the sequence to its limit.

§7.2 Uniform Convergence 一致收斂

7.2 Definition: uniform convergence (一致收斂)

Suppose $\{f_n\}$ is a sequence of real-valued functions defined on a set $E\subset\mathbb{R}$. We say that $\{f_n\}$ **converges uniformly** (一致收斂) to f on E if

for every $\varepsilon>0$ there exists an integer N such that $n\ge N$ implies $|f_n(x)-f(x)|<\varepsilon$ for *any* $x\in E$.

In this case, we also write $f_n\to f$ uniformly on E.

A series $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums, $s_n(x) = \sum_{k=1}^{n} f_k(x)$, converges uniformly on E.

equivalent definition

Suppose $\{f_n\}$ is a sequence of functions defined on $E\subset\mathbb{R}$. Then $\{f_n\}$ converges uniformly to f on E if and only if
$$\lim_{n\to\infty}\sup_{x\in E}|f_n(x)-f(x)| = 0.$$

Remark

- For pointwise convergence, the value of N depends on the point being considered, while for uniform convergence, a single value of N applies to all points in E.
- If a sequence converges uniformly on a set, then it converges pointwise on the set. However, the converse is not true, even it is on a compact set.

7.3 Theorem: Cauchy Criterion for uniform convergence

The sequence of functions $\{f_n\}$, defined on $E \subset \mathbb{R}$, converges uniformly on E if and only if it satisfies the **Cauchy Criterion** for uniform convergence:

for every $\varepsilon > 0$, there exists an integer N such that $m, n \geq N$ implies $|f_n(x) - f_m(x)| < \varepsilon$ for all $x \in E$.

7.4 Theorem: the Weierstrass M-Test (魏爾斯特拉斯 M 檢驗)

Suppose $\{f_n\}$ is a sequence of functions defined on $E \subset \mathbb{R}$, and suppose there is a sequence of nonnegative numbers $\{M_n\}$ such that

$$|f_n(x)| \leq M_n, \qquad n = 1, 2, 3, \ldots,$$

for all $x \in E$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

7.5 Theorem: sufficient condition for changing order of limits

Suppose $f_n \to f$ uniformly on a set $E \subset \mathbb{R}$. Let x be a limit point of E, and suppose that

$$\lim_{t \to x} f_n(t) = A_n, \qquad n = 1, 2, 3, \ldots.$$

Then $\{A_n\}$ converges, and $\lim\limits_{t \to x} f(t) = \lim\limits_{n \to \infty} A_n$, or equivalently,

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

Remark

- This theorem is the foundation for changing order of limits.

Uniform Convergence and Continuity 一致收斂性與連續性

7.6 Theorem: continuity of the limit function

If $f_n \to f$ uniformly on $E \subset \mathbb{R}$ and if $\{f_n\}$ is a sequence of continuous functions on E, then f is continuous on E.

Remark

- In short, with uniform convergence, the limit of continuous functions is continuous.

7.7 Theorem: Dini's theorem (迪尼定理)

Suppose $K \subset \mathbb{R}$ is compact, and

(1) $\{f_n\}$ is a sequence of continuous functions on K,

(2) $\{f_n\}$ converges pointwise to a continuous function f on K,

(3) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

Then $f_n \to f$ uniformly on K.

Remark

- In the theorem, if any of the assumptions
  - K is compact;
  - f is continuous;
  - $\{f_n(x)\}$ decreases with respect to n

  is removed, then the conclusion of the theorem may not hold true.

Uniform Convergence and Integration 一致收斂性與積分

7.8 Theorem: interchange the limit and the integration

Suppose $f_n \to f$ uniformly on $[a,b]$. If $f_n \in \mathscr{R}[a,b]$ for $n = 1, 2, 3, \ldots$, then $f \in \mathscr{R}[a,b]$, and

$$\int_a^b f\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n\,\mathrm{d}x,$$

that is,

$$\int_a^b \Big[\lim_{n\to\infty} f_n\Big]\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n\,\mathrm{d}x.$$

Corollary

If $f_n \in \mathscr{R}[a,b]$ and

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

uniformly on $[a,b]$. Then

$$\int_a^b f\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b f_n\,\mathrm{d}x.$$

Remark

- In short, with uniform convergence, the limit of the integrals equals the integral of the limit.

Uniform Convergence and Differentiation 一致收斂性與微分

7.9 Theorem: interchange the limit and the differentiation

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a,b]$ such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

that is,

$$\lim_{t\to x}\lim_{n\to\infty} \frac{f_n(t) - f_n(x)}{t - x} = \lim_{n\to\infty}\lim_{t\to x} \frac{f_n(t) - f_n(x)}{t - x}.$$

Remark

- In the theorem, one assumption is that $\{f_n'\}$, not $\{f_n\}$, converges uniformly on $[a,b]$. In fact, uniform convergence of $\{f_n\}$ implies nothing about the sequence $\{f_n'\}$, for instance, on $[0, 2\pi]$,

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad n = 1, 2, \ldots.$$

Remark

- Without assuming that $f_n(x_0)$ converges for some $x_0 \in [a,b]$, the theorem may fail.
- In short, with uniform convergence of $\{f_n\}$ and $\{f_n'\}$, the limit of the derivatives equals the derivative of the limit.

§7.3 Approximation in Function Space 逼近定理

7.10 Definition: the Cartesian product of sets

The **Cartesian product** (笛卡兒乘集) of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a,b) where a is in A and b is in B, that is,

$$A \times B = \{(a,b) : a \in A,\ b \in B\}.$$

It is also simply called the ***product*** of A and B.

Remark

- $A \times B \neq B \times A$, unless $A = B$.
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

  $(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$

7.11 Definition: metric spaces (度量空間)

A set X, whose elements are called ***points***, is said to be a **metric space** (度量空間) if there exists an associated function $d\colon X \times X \to \mathbb{R}$, called the ***distance*** from point p to point q, such that

(1) $d(p,q) \geq 0$ and $d(p,q) = 0$ if and only if $p = q$;
(2) $d(p,q) = d(q,p)$;
(3) $d(p,q) \leq d(p,r) + d(r,q)$, for any $r \in X$.

We call d a ***metric*** or ***distance function*** on X.

Remark

- For any given X, there trivially exists a so-called *discrete metric*, defined by
$$d(p,q) = \begin{cases} 1, & \text{if } p \neq q, \\ 0, & \text{if } p = q. \end{cases}$$
- Every subset Y of a metric space X is a metric space in its own right, with the same distance function.

We use the notation (X,d) for the metric space with the distance function d, or sometimes simply X.

Remark

- On $\mathbb{R}$, the absolute function, $|\cdot|$, induces a distance function $d(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$. Thus, $\mathbb{R}$ is a metric space. Similarly, the complex set $\mathbb{C}$ is also a metric space.
- On $\mathbb{R}^k$, there is a metric induced by the norm: $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^k$.

Remark

- The topological concepts of neighborhood, open set, closed set, dense set, etc, defined in Chapter 2 on the real line, can be extended to metric spaces, where the value $|x-y|$ between two real numbers x and y will be replaced by the distance $d(p,q)$ between two points p and q in the set X. To avoid complexity, we will not repeat these definitions again. ▶
- The metric determines which sets are open. This means that distinct metrics will result in distinct open sets.

If every Cauchy sequence in a metric space (X,d) has a limit that is also in X, then (X,d) is said to be **complete** (完備).

Remark

- The set of all real numbers $\mathbb{R}$ is a complete metric space. ▶
- The set of complex numbers $\mathbb{C}$ and the euclidean spaces $\mathbb{R}^k$ are complete.
- Any closed subset of a complete metric space is complete. In particular, every closed interval is complete. Any bounded open interval is *not* complete.

7.12 Definition: the metric space $\mathscr{C}(X)$

Let X be a metric space. Define $\mathscr{C}(X)$ to be the set of all real-valued, continuous, bounded functions with domain X. We associate with each $f \in \mathscr{C}(X)$ its ***supremum norm***
$$\|f\| = \sup_{x\in X} |f(x)|,$$
which gives an induced metric on $\mathscr{C}(X)$:
$$d(f,g) = \|f-g\| = \sup_{x\in X} |f(x)-g(x)|, \quad f, g \in \mathscr{C}(X).$$

$\mathscr{C}(X)$ is a complete metric space

The set $\mathscr{C}(X)$ together with the metric
$$d(f,g) = \sup_{x\in X} |f(x) - g(x)|$$
form a complete metric space. Moreover, a sequence $\{f_n\}$ converges to f with respect to the metric if and only if $f_n \to f$ uniformly on X. ▶

The Weierstrass Approximation Theorem 魏尔施特拉斯逼近定理

7.13 Theorem: the Weierstrass approximation theorem (魏尔施特拉斯逼近定理)

If f is a real continuous function on $[a, b]$, there exists a sequence of polynomials $\{P_n\}$, such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$.

equivalent statement

Suppose that the real function f is continuous on $[a, b]$. Then, for any $\varepsilon > 0$, there is a polynomial P such that for all $x \in [a, b]$,

$$|f(x) - P(x)| < \varepsilon.$$

Remark

- The theorem means that the set of all polynomials is dense in $\mathscr{C}([a, b])$
- The trigonometric version of the Weierstrass approximation theorem will be given in Theorem 8.13.

Addendum 後記

Addendum of Chapter 7

- The compactness of the function space $\mathscr{C}(X)$ is characterized by the renowned Arzelá-Ascoli theorem.

the Arzelá-Ascoli theorem (阿爾澤拉-阿斯科利定理)

Let K be a compact metric space and $\{f_n\}$ be a sequence of functions in $\mathscr{C}(K)$.

If $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

1. $\{f_n\}$ is uniformly bounded on K;
2. $\{f_n\}$ contains a uniformly convergent subsequence.

Conversely, if every subsequence of $\{f_n\}$ itself has a uniformly convergent subsequence, then

1. $\{f_n\}$ is uniformly bounded on K;
2. $\{f_n\}$ is equicontinuous on K.

- The Weierstrass approximation theorem is a powerful tool in mathematics with a wide range of applications, including:
  - Numerical analysis: The theorem can be used to approximate functions by polynomials, which can be easier to compute and manipulate.
  - Signal processing: The theorem can be used to approximate signals by polynomials, which can be useful in filtering and noise reduction.
  - Computer graphics: The theorem can be used to approximate curves and surfaces by polynomial functions, which can be rendered and displayed on a computer screen.
  - Probability theory: The theorem can be used to prove the existence of continuous stochastic processes, which are important in modeling and analyzing random phenomena.

Exercises of Chapter 7 練習題

Chapter 7: Quiz **30 Minutes**

① Let $\{f_n\}$ be a sequence of functions on $[a, b]$. Which of the following statements is true about uniform convergence?

A. if $\{f_n\}$ converges pointwise, then it converges uniformly
B. if $\{f_n\}$ converges uniformly, then the limit function is continuous
C. if $\{f_n\}$ converges uniformly, then it satisfies the Cauchy criterion
D. if $\{f_n\}$ converges uniformly, then $\{f_n'\}$ converges uniformly
E. none of the above

② Which of the following is equivalent to uniform convergence of $\{f_n\}$ on a set E?

A. for every $\varepsilon > 0$ and every $x \in E$, there exists an integer N such that $m, n \geq N$ implies $|f_n(x) - f_m(x)| < \varepsilon$
B. for every integer $k > 0$ and every $x \in E$, there exists an integer N such that $m, n \geq N$ implies $|f_n(x) - f_m(x)| < 1/k$
C. for every integer $k > 0$ and every $x \in E$, there exists an integer N such that $m, n \geq N$ implies $|f_n(x) - f_m(x)| < 1/k$
D. for every integer $k > 0$, every $\varepsilon > 0$, and every $x \in E$, there exists an integer N such that $m, n \geq N$ implies $|f_n(x) - f_m(x)| < \varepsilon/k$
E. none of the above

③ Suppose that the series $\sum f_n(x)$ of functions passes the Weierstrass M-Test on E. Which of the following statements is NOT true?

A. the series $\sum f_n(x)$ converges pointwise on E
B. the series $\sum f_n(x)$ converges uniformly on E
C. the sequence $\{f_n\}$ converges pointwise on E
D. the sequence $\{f_n\}$ converges uniformly on E
E. none of the above

④ Suppose a sequence of continuous functions pointwise converges on a closed and bounded interval. Which of the following is a sufficient condition for the sequence converging uniformly?

A. the limit function is continuous at every point
B. the sequence of functions is differentiable at every point
C. the sequence of functions is monotonically increasing at every point
D. the sequence of functions is monotonically decreasing at every point
E. none of the above

⑤ Which of the following is NOT true in the function space $\mathscr{C}(X)$?

A. every sequence $\{f_n\}$ in the space is pointwise bounded, that is, $|f_n(x)| \leq M(x)$ for all $x \in X$
B. every sequence that satisfies the Cauchy Criterion in the space converges to a continuous function on X
C. every convergent sequence in the space satisfies the Cauchy Criterion
D. every sequence that satisfies the Cauchy Criterion in the space is uniformly convergent
E. none of the above

⑥ Which of the following is a consequence of the Weierstrass approximation theorem?

A. every polynomial function on a bounded closed interval can be approximated uniformly by continuous functions
B. every continuous function on a bounded closed interval can be approximated uniformly by polynomial functions
C. every differentiable function on a bounded closed interval can be approximated uniformly by continuous functions
D. every integrable function on a bounded closed interval can be approximated uniformly by continuous functions
E. every bounded function on a bounded closed interval can be approximated uniformly by polynomial functions

①C, ②E, ③E, ④D, ⑤A, ⑥B

Chapter 7: Exercises

Exercise 7.1 (E)

Show that the sequence

$$f_n(x) = x^n, \quad n = 1, 2, 3, \ldots,$$

does not uniformly converge on $(0, 1)$.

Hint ① Determine the pointwise limit of the sequence.

Hint ② Use the definition to show that the sequence does not uniformly converge on $(0, 1)$.

Hint ③ Consider the sequence $f(x_n)$, where $x_n = 1 - \frac{1}{n}$, $n = 1, 2, 3, \ldots$.

Exercise 7.2 (E)

Prove that every uniformly convergent sequence $\{f_n\}$ of bounded functions on E is uniformly bounded, that is, there exists a number M such that

$$|f_n(x)| < M, \qquad x \in E, \; n = 1, 2, 3, \ldots.$$

Hint ① If $\{f_n\}$ is a sequence of bounded functions, then for each n and any $x \in E$, $|f_n(x)| \le M_n$ for some M_n.

Hint ② If $f_n \to f$ uniformly on E, then for $\varepsilon = 1$, there is N such that $|f_n(x) - f(x)| < 1$ if $n \ge N$ and $x \in E$.

Exercise 7.3 (E)

Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n$ converges and $\{x_n\}$ be a sequence of distinct points of (a, b), and

$$I(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

be the unit step function. Prove that the cumulative distribution function defined by the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n), \qquad a \le x \le b$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Hint ① Apply the Weierstrass M-Test to show the uniform convergence.

Hint ② Show that the partial sums of the series are continuous at $x \notin \{x_n\}$.

Hint ③ Put $g_m(x) = \sum_{n=1}^{m} c_n I(x - x_n)$. Show that for any fixed $x \notin \{x_n\}$, if $|t - x| < \min\{|x - x_1|, \ldots, |x - x_m|\}$, then $|g_m(t) - g_m(x)| = 0 < \varepsilon$.

Exercise 7.4 (I)

Consider the positive series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}, \qquad x > 0.$$

1. Show that the series converges on $(0, \infty)$.
2. Show that for any $a > 0$, the series converges uniformly on $[a, \infty)$, but fails to converge uniformly on $(0, a]$.
3. Show that the function f is continuous on $(0, \infty)$.
4. Show that for any $a > 0$, f is bounded on $[a, \infty)$, but unbounded on $(0, a]$.

Hint ① For part 1, show that for sufficiently large n,

$$0 < \frac{1}{1 + n^2 x} \le x^{-1} \cdot \frac{1}{n^2}, \quad x > 0.$$

Apply the Comparison Test.

Hint ② For part 2, to show that the series converges uniformly on $[a, \infty)$, apply the Weierstrass M-Test.

Hint ③ To show that the series fails to converge uniformly on $(0, a]$, examine the Cauchy Criterion.

Hint ④ Let $s_n(x)$ be the partial sum of the series. Show that $s_{2n}(n^{-2}) > \frac{1}{5}n$.

Hint ⑤ For part 3, use part 2 and apply Theorem 7.6.

Hint ⑥ For part 4, take a sequence $\{x_n\}$ in $(0, \infty)$ such that $f(x_n)$ is unbounded. On the other hand, show that $0 < f(x) \le M$ for $x \in [a, \infty)$ with $a > 0$, where M is a finite number.

Exercise 7.5 (I)

Let

$$f_n(x) = \begin{cases} 0, & \text{if } x \le \dfrac{1}{n+1}, \\ \sin^2 \dfrac{\pi}{x}, & \text{if } \dfrac{1}{n+1} < x \le \dfrac{1}{n}, \\ 0, & \text{if } \dfrac{1}{n} < x. \end{cases}$$

1. Show that $\{f_n\}$ converges to a continuous function, but not uniformly on $\mathbb{R}$.

2. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Hint ① For part **1**, show $\{f_n\}$ converges pointwise to $f = 0$.

Hint ② For $x_n = \dfrac{1}{n+\frac{1}{2}}$, get $|f_n(x_n) - 0| = 1$, $n = 1, 2, 3, \ldots$. This implies that $\{f_n\}$ does not converge to $f = 0$ uniformly on $\mathbb{R}$.

Hint ③ For part **2**, use the fact that only one term in $\sum f_n$ is not zero.

Hint ④ Show that the Cauchy Criterion for uniform convergence fails.

Exercise 7.6 (E)

Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}.$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Hint ① Show that the Cauchy Criterion for uniform convergence holds.

Hint ② The given series is a sum of two uniformly convergent series.

Hint ③ Apply the Comparison Test to show that the series is not absolutely convergent.

Exercise 7.7 (I)

Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Hint ① Show that for sufficiently large n, $|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon$.

Hint ② The term $|f_n(x_n) - f(x_n)|$ is small since $\{f_n\}$ converges to f uniformly. The term $|f(x_n) - f(x)|$ is small by Theorem 7.6.

Hint ③ For the converse, consider the sequence $\{f_n\}$ on $(0,1)$, where $f_n(x) = \dfrac{1}{nx}$, $n = 1, 2, 3, \ldots$.

Exercise 7.8 (I)

Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and

(1) for all $x \in E$,

$$\left| \sum_{k=1}^{n} f_k(x) \right| \le M, \quad n = 1, 2, 3, \ldots;$$

(2) $g_n \to 0$ uniformly on E;

(3) $g_n(x) \ge g_{n+1}(x)$ for $n \ge 1$ and $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E.

This result is known as Dirichelet's Test for uniform convergence.

Hint ① Apply the summation by parts formula.

Hint ② Verify the Cauchy Criterion for uniform convergence.

Exercise 7.9 (I)

If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)\,x^n\,\mathrm{d}x = 0, \qquad n = 0, 1, 2, \ldots,$$

prove that $f(x) = 0$ on $[0,1]$.

Hint ①: ① Apple the Weierstrass approximation theorem to have a sequence of polynomials $\{P_n\}$ that converges uniformly to f.

Hint ②: ② For any $\varepsilon > 0$, there exists a polynomial P such that

$$\left|f(x)P(x) - [f(x)]^2\right| < \varepsilon, \qquad x \in [0,1].$$

Hint ③: ③ Show that $\int_0^1 [f(x)]^2\,\mathrm{d}x = 0$, which implies $f(x) \equiv 0$ by Exercise 6.1.

Exercise 7.10 (I)

Assume f is Riemann integrable on $[a,b]$. Prove that there is a sequence of polynomials $\{P_n\}$ such that

$$\lim_{n\to\infty} \int_a^b |f - P_n|\,\mathrm{d}x = 0.$$

Hint ①: ① Apply the result of Exercise 6.8.

Hint ②: ② Apply the Weierstrass approximation theorem.

CHAPTER

第 8 章

Power Series and Fourier Series 冪級數與傅立葉級數

Overview of Chapter 8

In this chapter, we will discuss two common types of infinite series of functions, the power series and the Fourier series.

- For power series, we show that
  - there is an associated so-called disk of convergence in which the series is convergent;
  - within the circle of convergence, the function defined by the power series is analytic;
  - when the series is real, the power series is uniformly convergent on any interior of the interval of convergence.
- For Fourier series, we show that
  - Bessel's inequality holds for any integrable function;
  - the series converges if the function is integrable and periodic, and uniformly converges if the function is continuous, periodic, and piecewise continuously differentiable;
  - Parseval's identity holds for any integrable function under a complete orthogonal basis.
- Finally, we prove the trigonometric version of the Weierstrass approximation theorem.

§8.1 Power Series 冪級數

8.1 Definition: power series

Given a fixed complex number z_0 and a sequence $\{c_n\}$ of complex numbers, the series $\sum\limits_{n=0}^{\infty} c_n(z-z_0)^n$ is called a $\boxed{\textbf{power series}}$ (冪級數) centered at z_0, where $z \in \mathbb{C}$. The numbers c_n are called the ***coefficients*** (係數) of the series.

A function f is called $\boxed{\textbf{analytic}}$ (解析) on an open set D in the complex plane if for *any* $z_0 \in D$ one can write

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n,$$

in which the coefficients $\{c_n\}$ are complex numbers and the series is convergent to $f(z)$ for z in a neighborhood of z_0.

- In the definition above, if replacing "complex" with "real" and "complex plane" with "real line", then the function is called ***real analytic*** (實解析).

Remark

- When we study power series, for the sake of convenience, we will denote complex numbers by the letter z, such as z_0, z_1, and so on, while real numbers will be denoted by the letter x, such as x_0, x_1, and so forth.

8.2 Theorem: disk of convergence of power series

Given the power series $\sum_{k=0}^{\infty} c_n(z-z_0)^n$, put

$$\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}, \qquad R = \frac{1}{\alpha} \in [0,\infty) \cup \{\infty\}.$$

Then $\sum_{k=0}^{\infty} c_n(z-z_0)^n$ (absolutely) converges if $|z-z_0| < R$, and diverges if $|z-z_0| > R$. ▶ ▶

Remark

- The number R in the theorem, which is a nonnegative extended real number, is called the **radius of convergence** (收斂半徑) of the power series $\sum c_n(z-z_0)^n$. While the value of R is typically determined by computing the root limit as stated, there are situations where it may be more convenient to use the ratio limit instead.
- The inequality $|z-z_0| < R$ geometrically represents an open disk centered at z_0 on the complex plane called the **disk of convergence** (收斂圓盤) (when the series is real, the disk of convergence degenerates to an open interval). The theorem asserts that the power series converges if z lies within the disk's interior, and diverges if it lies outside. When the disk has an infinite radius (i.e., $R = \infty$), the power series converges for all complex z.

Remark

- It is clear that when the disk has a radius of zero (i.e., $R = 0$), the power series converges solely at $z = z_0$. However, when $R > 0$, the theorem does not offer any insight into the convergence of the power series $\sum c_n(z-z_0)^n$ on the circle $|z-z_0| = R$, which is often referred to as the ***boundary of convergence***. In fact, determining the behavior of the power series on the boundary is more challenging than its behavior inside the disk. The power series may converge at certain points on the boundary and diverge at others, and whether it converges or diverges may rely on the specific terms of the series.

Abel's Test on the boundary of convergence of power series

Let $\sum c_n(z-z_0)^n$ be a power series, where $z \in \mathbb{C}$. Suppose

(1) the radius of convergence is R;

(2) the sequence $\{c_n R^n\}$ is monotonically decreasing and $\lim_{n\to\infty} c_n R^n = 0$.

Then $\sum c_n(z-z_0)^n$ converges at every point on the circle $|z-z_0| = R$, except possibly at $z - z_0 = R$. ▶ ▶

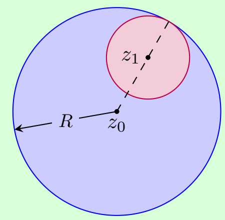
Remark

- The result provides a sufficient condition, but not necessary, for the convergence of power series on the boundary of convergence.

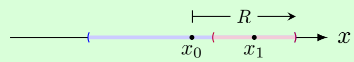
8.3 Theorem: analyticity of power series in the disk of convergence

1. Suppose $f(z)=\sum\limits_{n=0}^{\infty} c_n(z-z_0)^n$ for $|z-z_0|<R$, where $\{c_n\}$, z and z_0 are complex numbers. Then f is analytic in $|z-z_0|<R$. More explicitly, if $|z_1-z_0|<R$, then for some complex numbers $\{d_n\}$,

$$f(z)=\sum_{n=0}^{\infty} d_n(z-z_1)^n, \qquad |z-z_1|<R-|z_1-z_0|.$$



2. Suppose $f(x)=\sum\limits_{n=0}^{\infty} c_n(x-x_0)^n$ for $|x-x_0|<R$, where $\{c_n\}$, x and x_0 are real numbers. Then f is real analytic in $|x-x_0|<R$. More explicitly, if $|x_1-x_0|<R$, then for some real numbers $\{d_n\}$,



$$f(x)=\sum_{n=0}^{\infty} d_n(x-x_1)^n, \qquad |x-x_1|<R-|x_1-x_0|.$$ ▶

Remark

- The series $\sum\limits_{n=0}^{\infty} d_n(z-z_1)^n$ may actually converge in a larger interval than the one given by $|z-z_1|<R-|z_1-z_0|$.
- In part 2, by the corollary of Theorem 8.5, we know that $d_n=\dfrac{f^{(n)}(x_1)}{n!}$.

8.4 Proposition: uniqueness of power series

Suppose $\sum a_n(z-z_0)^n$ and $\sum b_n(z-z_0)^n$ converge in $S=\{z:\ |z-z_0|<R\}$ for some $R>0$. Let E be the set of all $z\in S$ at which

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n=\sum_{n=0}^{\infty} b_n(z-z_0)^n.$$

If E has a limit point in S, then $a_n=b_n$ for $n=0,1,2,\ldots$. ▶

Remark

- Typically, regardless of the method used, the power series obtained are the same according to this theorem.

8.5 Theorem: uniform convergence of power series

Suppose the *real* series $\sum\limits_{n=0}^{\infty} c_n(x-x_0)^n$ converges for $|x-x_0|<R$, where $\{c_n\}$, x, and x_0 are all real. Then for any $\varepsilon>0$, the series converges uniformly on the interval $[x_0-(R-\varepsilon), x_0+(R-\varepsilon)]$. Moreover, if we define

$$f(x)=\sum_{n=0}^{\infty} c_n(x-x_0)^n, \qquad |x-x_0|<R,$$

then f is continuous and differentiable in (x_0-R, x_0+R), and

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-x_0)^{n-1}, \qquad |x-x_0|<R.$$ ▶

Corollary

Under the hypotheses of the theorem, f has derivatives of all orders in (x_0-R, x_0+R), and

$$f^{(k)}(x)=\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n(x-x_0)^{n-k}.$$

In particular, $f^{(k)}(x_0)=k!c_k$, $k=0,1,2,\ldots$.

Remark

- It is important to note that the corollary states that the coefficients of a power series representation of a function f must be expressed in terms of its derivatives at the center of the series. On the other hand, if the coefficients are given by $c_k = \frac{f^{(k)}(x_0)}{k!}$, it does not necessarily imply that the resulting power series (the Taylor expansion of f)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

will converge, and even if it does converge, it may not necessarily converge to $f(x)$ for any $x \neq x_0$.

continuity of power series at the endpoints

Suppose $\sum c_n R^n$ converges. For $|x-x_0| < R$, put $f(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$. Then

$$\lim_{x \to x_0 + R^-} f(x) = \sum_{n=0}^{\infty} c_n R^n.$$

Remark

- Therefore, power series must be continuous at the points of convergence, including endpoints.

The Exponential and Logarithmic Functions 指數與對數函數

8.6 Theorem: properties of the exponential function

For complex z, denote $\boxed{E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}}$. Then for any real x,

$$E(x) = e^x,$$

where $e^x = \sup e^p$ with the sup being taken over all rational p such that $p \leq x$. Equivalently,

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad -\infty < x < \infty.}$$

Moreover, the following properties hold:

1. The equality $e^{x+y} = e^x e^y$ holds, as well as other laws of exponents.
2. e^x is strictly increasing function of x, and $e^x > 0$;
3. e^x is continuous and differentiable for all real x with $\boxed{(e^x)' = e^x}$;
4. $\lim_{x\to\infty} x^k e^x = \infty$, $\lim_{x\to-\infty} x^k e^x = 0$ for any integer k.

Remark

- Power series are often more convenient than finding a supremum.

Remark

- Because of this theorem, it is customary to write, for complex z, $\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}}$.
- For complex z, the exponential function e^z satisfies the laws of exponents.
- The complex function e^z is *complex* differentiable, with $\boxed{(e^z)' = e^z}$, where

$$(e^z)' = \lim_{h\to 0} \frac{e^{z+h} - e^z}{h}, \qquad z, h \in \mathbb{C}.$$

8.7 Theorem: properties of the logarithmic function

For positive x, let $\boxed{L(x) = \ln x}$ be the inverse function of $E(x) = e^x$. Then for any $x > 0$ and any real α,

$$E(\alpha L(x)) = x^\alpha,$$

where the right-hand side is defined as

$$x^\alpha = \sup x^p, \qquad x > 1,$$

with the sup being taken over all rational p such that $p \le \alpha$; for $0 < x < 1$, define $x^\alpha = (x^{-1})^{-\alpha}$. Equivalently,

$$\boxed{x^\alpha = e^{\alpha \ln x}, \qquad 0 < x < \infty.}$$

Moreover, the following properties hold:

1. The equality $\ln(xy) = \ln x + \ln y$ holds, as well as other laws of logarithms;
2. $\ln x$ is strictly increasing function of x, and $\ln x < 0$ for $0 < x < 1$ and $\ln x > 0$ for $x > 1$;
3. $\ln x$ is continuous and differentiable for all positive x with $\boxed{(\ln x)' = 1/x}$;
4. $\lim\limits_{x\to\infty} \ln x = \infty$, $\lim\limits_{x\to 0^+} \ln x = -\infty$; and $\lim\limits_{x\to\infty} x^{-k} \ln x = 0$, $\lim\limits_{x\to 0^+} x^k \ln x = 0$ for any positive k.

Remark

- For all $x > 0$ and any real α, $\boxed{(x^\alpha)' = \alpha x^{\alpha-1}}$, by the chain rule.

The Trigonometric Functions 三角函數

8.8 Theorem: properties of the trigonometric functions

For real x, denote

$$\boxed{C(x) = \frac{1}{2}[E(ix) + E(-ix)]},$$

$$\boxed{S(x) = \frac{1}{2i}[E(ix) - E(-ix)]}.$$

Then, for real x,

$$C(x) = \cos x, \qquad S(x) = \sin x,$$

and Euler's formula holds:

$$E(ix) = C(x) + iS(x),$$

or equivalently,

$$\boxed{e^{ix} = \cos x + i \sin x.}$$

Moreover, the following properties hold:

1. the functions $\cos x$ and $\sin x$ are periodic, with period 2π;
2. $\boxed{(\cos x)' = -\sin x}$, $\boxed{(\sin x)' = \cos x}$;
3. $E(it) = C(t) + iS(t)$ is a bijective mapping between $[0, 2\pi)$ and $\{z \in \mathbb{C} : \ |z| = 1\}$.

Remark

- We have the power series representations of the sine and cosine functions:

$$\boxed{\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}}, \qquad \boxed{\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}}.$$

Remark

- The complex exponential function e^z is periodic, with period $2\pi i$.

Remark

- One of the key significances of Euler's formula is that it provides a connection between exponential functions and trigonometric functions. This relationship is useful in many areas of mathematics, including complex analysis, Fourier analysis, and differential equations. For example, the formula can be used to simplify complex trigonometric expressions, to derive trigonometric identities, and to solve differential equations with trigonometric functions.

§8.2 Fourier Series 傅立葉級數

8.9 Definition: Fourier series

Let $\{\phi_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of complex functions on $[a, b]$, such that

$$\int_a^b \phi_n(x) \cdot \overline{\phi_m(x)}\, \mathrm{d}x = 0, \qquad n \neq m.$$

Then $\{\phi_n\}$ is said to be an **orthogonal system** (正交系) of functions on $[a, b]$.

In addition, if

$$\int_a^b |\phi_n(x)|^2\, \mathrm{d}x = 1, \quad n = 1, 2, 3, \ldots,$$

then $\{\phi_n\}$ is said to be **orthonormal** (標準正交).

Remark

- If V is a linear space over $\mathbb{C}$, an ***inner product*** on V is a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ with the following three properties:
  1. ***Linearity*** (線性性): $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$ for all $u, v, w \in V$ and $a, b \in \mathbb{C}$;
  2. ***Positive Definiteness*** (正定性): $\langle v, v \rangle \geq 0$ for all $v \in V$, and $\langle v, v \rangle = 0$ if and only if $v = 0$;
  3. ***Conjugate Symmetry*** (共軛對稱性): $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$.

  A linear space together with an inner product on it is called an ***inner product space***.
- An inner product space $(V, \langle \cdot, \cdot \rangle)$ induces a norm
  $$\|v\| = \sqrt{\langle v, v \rangle}, \quad \text{for all } v \in V.$$
- In the space of (complex or real) continuous functions on $[a, b]$, define the mapping
  $$\langle f, g \rangle = \int_a^b f(x)\, \overline{g(x)}\, \mathrm{d}x.$$
  It is easy to check that it is an inner product on this space. It induces the so-called ***2-norm***,
  $$\|f\|_2 = \left(\int_a^b |f|^2 \right)^{1/2}.$$
- One can also consider a function space consisted by polynomials.
- The Gram-Schmidt process in finite dimensional vector spaces can be generalized to the space of continuous functions on $[a, b]$ to allow us to convert a set of linearly independent functions into a set of orthonormal functions.

Let $\{\phi_n\}$ be orthonormal on $[a, b]$. Put $c_n = \displaystyle\int_a^b f(t) \cdot \overline{\phi_n(t)}\, \mathrm{d}t$, $n = 1, 2, 3, \ldots$, and write

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x).$$

We call this series the **Fourier series** (傅立葉級數) of f relative to $\{\phi_n\}$, and c_n the nth ***Fourier coefficient*** of f (relative to $\{\phi_n\}$).

Remark

- We shall assume that the functions f and $\{\phi_n\}$ are Riemann-integrable on $[a, b]$.
- We do not write $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ for two reasons: first, the series may not converge; and second, even if the series does converge, it may not converge to $f(x)$.

8.10 Theorem: Bessel's inequality

Lemma

Let $\{\phi_n\}$ be orthonormal on $[a,b]$. Suppose

$$s_n = \sum_{m=1}^{n} c_m \phi_m(x)$$

is the nth partial sum of the Fourier series of f. Then

$$\int_a^b |f - s_n|^2 \, \mathrm{d}x \le \int_a^b |f - t_n|^2 \, \mathrm{d}x,$$

for any $t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$. The equality holds if and only if $\gamma_m = c_m$, $m = 1, \ldots, n$. ▶

Suppose $\{\phi_n\}$ is orthonormal on $[a,b]$. If $f \in \mathscr{R}$ on $[a,b]$ and if

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

then Bessel's inequality holds:

$$\sum_{n=1}^{\infty} |c_n|^2 \le \int_a^b |f(x)|^2 \, \mathrm{d}x.$$

In particular, $\lim_{n\to\infty} c_n = 0$. ▶

Remark

- The lemma shows that among all "polynomials" of the same degree, s_n gives the best possible mean square approximation to f.

8.11 Definition: trigonometric series

Consider the orthonormal system on $[-\pi,\pi]$:

$$\{(2\pi)^{-1/2} e^{inx}\}, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Assume that f is a Riemann-integrable function on $[-\pi,\pi]$. The Fourier series of f is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x.$$

Let

$$s_N(x) = s_N(f;x) = \sum_{n=-N}^{N} c_n e^{inx}$$

be the Nth partial sum of the Fourier series of f.

Define the **Dirichlet kernel** (狄利克雷核):

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}.$$

Then

$$D_N(x) = \frac{\sin(N+\frac{1}{2})x}{\sin(\frac{1}{2}x)}.$$

Furthermore, if $f \in \mathscr{R}$ on $[-\pi,\pi]$ and 2π-periodic, then

$$\begin{aligned} s_N(f;x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t) \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) \, \mathrm{d}t. \end{aligned}$$ ▶

Bessel's inequality of trigonometric series

Let $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x$. Then Bessel's inequality holds

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x,$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x = 0.$$

Remark

- We observe that when $|z| = 1$, we can express z as e^{ix}, and therefore, $\sum_{-N}^{N} c_n e^{inx} = \sum_{-N}^{N} c_n z^n$. As a result, we often refer to finite series $\sum_{-N}^{N} c_n e^{inx}$, as well as the expression $a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$, as a **trigonometric polynomial** (三角多項式) (which is essentially a rational function in z).

Remark

- Traditionally, a trigonometric series refers an expression of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

which is generated by using the orthonormal system

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots.$$

This is equivalent to the exponential form in the current definition, thanks to Euler's formula.

This special type of Fourier series is referred to as a trigonometric series.

Convergence of Trigonometric Series

8.12 Theorem: pointwise convergence of Fourier series

Suppose $f \in \mathscr{R}$ on $[-\pi, \pi]$ and 2π-periodic. If, for some x, both $f(x-)$ and $f(x+)$ exist, and there exists a constants $\delta > 0$ and $M < \infty$ such that for all $0 < t < \delta$,

$$|f(x+t) - f(x+)| \leq Mt,$$
$$|f(x-t) - f(x-)| \leq Mt,$$

then

$$\lim_{N\to\infty} s_N(f;x) = \frac{f(x+) + f(x-)}{2}.$$

Corollary

Suppose $f \in \mathscr{R}$ on $[-\pi, \pi]$ and 2π-periodic. If, for some x, there are constants $\delta > 0$ and $M < \infty$ such that

$$|f(x+t) - f(x)| \leq M|t|$$

for all $t \in (-\delta, \delta)$, then $\lim_{N\to\infty} s_N(f;x) = f(x)$.

localization theorem

If $f(t) = g(t)$ for all t in some neighborhood of x, then, as $N \to \infty$,

$$s_N(f;x) - s_N(g;x) = s_N(f-g;x) \to 0.$$

Remark

- It shows that the convergence of the sequence $\{s_N(f;x)\}$ depends only on the values of f in some (arbitrarily small) neighborhood of x.

8.13 Theorem: uniform approximation by trigonometric polynomials

Suppose that f is continuous and 2π-periodic. Then, for any $\varepsilon > 0$, there is a trigonometric polynomial P such that for all real x,

$$|f(x) - P(x)| < \varepsilon.$$

Remark

- The trigonometric polynomial P in the theorem is not necessary to be the partial sum of the Fourier series of f.

8.14 Theorem: uniform convergence of Fourier series

If f is continuous, 2π-periodic, and piecewise continuously differentiable, Then $s_N \to f$ uniformly.

Remark

- The term "piecewise continuously differentiable" is defined similarly as "piecewise continuous".
- Thus, with some natural additional conditions, integrating and differentiating Fourier series are permissible, and this makes the Fourier series a powerful tool for analyzing various functions.

8.15 Theorem: Parseval's identity

Suppose $f \in \mathscr{R}$ on $[-\pi, \pi]$ and 2π-periodic. Let

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Then

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_N(f;x)|^2 \,\mathrm{d}x = 0,$$

which gives Parseval's identity:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Remark

- Parseval's identity can be thought of as a generalized Pythagorean theorem for inner-product spaces from a geometrical perspective. In the field of physics, it is commonly viewed as a representation of energy conservation between the time domain and frequency domain.

real version

Suppose f is a real function, $f \in \mathscr{R}$ on $[-\pi, \pi]$, and 2π-periodic. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \,\mathrm{d}t, \quad n \geq 0; \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \,\mathrm{d}t, \quad n \geq 1.$$

Then Parseval's identity holds:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Remark

- While Bessel's inequality applies to any orthogonal system, Parseval's identity requires that the system be *complete*.

Addendum 後記

Addendum of Chapter 8

- Two versions (Theorems 7.13 and 8.13) of the Weierstrass approximation theorem state that any continuous function on a compact interval can be uniformly approximated by a polynomial or a trigonometric polynomial. These results can be unified into the following general result.

  the Stone-Weierstrass approximation theorem

  Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.

- Many applications require the determination of Fourier series representations for functions that are defined on intervals other than $[-\pi, \pi]$, including but not limited to $[0, 2\pi]$, $[0, L]$, or $[-\frac{1}{2}L, \frac{1}{2}L]$, and even more general intervals like $[a, b]$.

Exercises of Chapter 8 練習題

Chapter 8: Quiz **30 Minutes**

① Given the power series $\sum\limits_{k=0}^{\infty} c_n(z-z_0)^n$. Which of the following statements is true?

A. the radius of convergence of the power series does not exist when $\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|c_n|}$ does not exist
B. the radius of convergence is always positive
C. the power series is divergent everywhere if the disk of convergence is empty
D. the disk of convergence is always symmetric about its center
E. none of the above

② Suppose the power series $\sum\limits_{k=0}^{\infty} c_n(z-z_0)^n$ has a positive radius of convergence, $R>0$. Which of the following statements is true?

A. there are infinitely many points on the boundary of convergence $|z-z_0|=R$ such that the series diverges
B. there are finitely many points on the boundary of convergence $|z-z_0|=R$ such that the series diverges
C. the series always converges on the boundary of convergence $|z-z_0|=R$
D. the series always diverges on the boundary of convergence $|z-z_0|=R$
E. none of the above

③ Suppose f is real analytic in $|x-x_0|<R$ with $R>0$. Which of the following is true ?

A. the power series representation of f is unique
B. the power series representation of f has positive radius of convergence at every point
C. the function f is always differentiable in $|x-x_0|<R$
D. the function f must be a polynomial
E. the function f must be a rational function

④ Which of the following is Not an example of an orthogonal system on $[-1,1]$?

A. $\sin\pi x, \sin 2\pi x, \sin 3\pi x, \ldots$
B. $x, \sin\pi x, \sin 2\pi x, \sin 3\pi x, \ldots$
C. $\cos\pi x, \cos 2\pi x, \cos 3\pi x, \ldots$
D. $1, \cos\pi x, \cos 2\pi x, \cos 3\pi x, \ldots$
E. $1, e^{i\pi x}, e^{2i\pi x}, e^{3i\pi x}, \ldots$

⑤ Suppose $f\in\mathscr{R}[a,b]$ and $\{\phi_n\}$ is a sequence of orthogonal system of Riemann-integrable functions on $[a,b]$. Which of the following is true?

A. the series $\sum\limits_{n=1}^{\infty}|c_n|^2$ is convergent
B. $\sum\limits_{n=1}^{\infty} c_n\phi_n(x) = \dfrac{f(x+)+f(x-)}{2}$
C. $\lim\limits_{n\to\infty}\displaystyle\int_a^b \left| f(x) - \sum_{k=1}^{n} c_k\phi_k(x)\right|^2 \mathrm{d}x = 0$
D. Parseval's identity holds
E. none of the above

⑥ Which of the following is true about the convergence of the Fourier series?

A. the Fourier series always converges uniformly
B. the Fourier series always converges pointwise
C. the Fourier series may converge uniformly or pointwise, depending on the function
D. the Fourier series never converges
E. the convergence of the Fourier series is random

①D, ②E, ③C, ④B, ⑤E, ⑥C

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Appendix A: the justifications and the examples

Example: direct proof 1

Claim: the square of an odd number is also odd.

Proof Let p be the statement that n is an odd integer and q be the statement that n^2 is an odd integer. We need to prove that $p \Rightarrow q$.

Assume that n is an odd integer, then by definition $n = 2k + 1$ for some integer k. We will now use this to show that n^2 is also an odd integer. In fact, since

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1, \end{aligned}$$

we know that n^2 has the form of an odd integer since $(2k^2 + 2k)$ is an integer. Therefore, we have shown that $p \Rightarrow q$ and so we have completed our proof. ◀

Example: direct proof 2

Let l, m, and n be integers. Claim: if l divides m and l divides n then l also divides $m+n$.

Proof Let p be the statement that l divides m and l divides n, and q be the statement that l divides $m+n$. We need to prove that $p \Rightarrow q$.

Since l divides m, by definition, there is some integer k_1 such that $m = lk_1$. Also as l divides n, there is some integer k_2 such that $n = lk_2$. Thus,

$$m + n = lk_1 + lk_2 = l(k_1 + k_2).$$

Hence, l divides $m+n$ since $(k_1 + k_2)$ is an integer. Therefore, we have shown that $p \Rightarrow q$ and so we have completed our proof. ◀

Example: a false proof by induction

Claim: for any positive integer n, the following formula holds:

$$\sum_{j=1}^{n} j = \tfrac{1}{2}(n + \tfrac{1}{2})^2.$$

Proof We prove the claim by induction.

Base Step:

It is obvious the claim holds when $n = 1$.

Inductive Step:

Assume that the claim holds for $n = k$. Then

$$\begin{aligned}
\sum_{j=1}^{k+1} j &= \sum_{j=1}^{k} j + (k+1) \\
&= \tfrac{1}{2}(k + \tfrac{1}{2})^2 + (k+1) \\
&= \tfrac{1}{2}(k^2 + k + \tfrac{1}{4} + 2k + 2) \\
&= \tfrac{1}{2}[(k^2 + 2k + 1) + 2 \cdot (k+1) \cdot \tfrac{1}{2} + \tfrac{1}{4}] \\
&= \tfrac{1}{2}[(k+1) + \tfrac{1}{2}]^2.
\end{aligned}$$

Thus, the claim holds for $n = k + 1$, so the induction step is complete.

Consequently, the claim holds for all positive integers n by the principle of induction.

Remark

- The false induction proof illustrated in this example is due to the mistake made in the base step.

Example: proof by induction 1

Claim: for any positive integer n, the following formula holds:

$$\sum_{j=1}^{n} j = \tfrac{1}{2}n(n+1).$$

Proof We prove the claim by induction.

Base Step:

When $n = 1$, we have

$$\sum_{j=1}^{n} j = \sum_{j=1}^{1} j = 1 = \tfrac{1}{2}1(1+1) = \tfrac{1}{2}n(n+1),$$

so the claim holds .

Inductive Step:

Assume that the claim holds for $n = k$. Then

$$\begin{aligned}\sum_{j=1}^{k+1} j &= \sum_{j=1}^{k} j + (k+1)\\ &= \tfrac{1}{2}k(k+1) + (k+1)\\ &= \tfrac{1}{2}(k+1)(k+2)\\ &= \tfrac{1}{2}(k+1)[(k+1)+1].\end{aligned}$$

Thus, the claim holds for $n = k+1$, so the induction step is complete.

Consequently, the claim holds for all positive integers n by the principle of induction. ◀

Example: proof by induction 2

Claim: for any positive integer n, $11^n - 6$ is divisible by 5.

Proof We prove the claim by induction.

Base Step:

When $n = 1$, we have

$$11^n - 6 = 11^1 - 6 = 5 = 5 \cdot 1,$$

so the claim holds.

Inductive Step:

Assume that the claim holds for $n = k$. That means $11^k - 6$ is divisible by 5 and hence $11^k - 6 = 5m$ for some integer m. So $11^k = 5m + 6$. Then

$$\begin{aligned} 11^{k+1} - 6 &= 11 \cdot 11^k - 6 \\ &= 11 \cdot (5m + 6) - 6 \\ &= 5 \cdot (11m + 12). \end{aligned}$$

As $(11m+12)$ is an integer we have that $11^{k+1}6$ is divisible by 5. Thus, the claim holds for $n = k+1$, so the induction step is complete.

Consequently, the claim holds for all positive integers n by the principle of induction. ◀

Example: strong induction

Given $n \in \mathbb{N}$, define a_n recursively as follows:

$$a_0 = 1; \quad a_1 = 3; \quad a_n = 2a_{n-1} - a_{n-2} \quad \text{for } n \geq 2.$$

Claim: $a_n = 2n + 1$, for all $n \geq 1$.

Proof We prove the claim by strong induction.

Base Step:

When $n = 1, 2$, we have

$$\begin{aligned} a_0 &= 1 = 2(0) + 1; \\ a_1 &= 3 = 2(1) + 1. \end{aligned}$$

So the claim holds for $n = 1, 2$.

Inductive Step:

Assume that the claim holds for $1 < 2 \leq n \leq k$. That means $a_n = 2n + 1$ for $1 < 2 \leq n \leq k$. Then

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} \\ &= 2(2k+1) - [2(k-1)+1] \\ &= 2(k+1) + 1. \end{aligned}$$

Thus, the claim holds for $n = k + 1$, so the induction step is complete.

Consequently, the claim holds for all positive integers n by the principle of induction. ◀

Example: proof by contraposition 1

Claim: for integers m and n, if $m+n$ is odd, then m is odd or n is odd.

Proof We prove the claim by contraposition.

Suppose neither m is odd nor n is odd. Then m and n are both even. So we have $m = 2k_1$ and $n = 2k_2$ for some integers k_1 and k_2. Now $m+n = 2k_1 + 2k_2 = 2(k_1+k_2)$. Since (k_1+k_2) is an integer, we see that $m+n$ must be even.

We have therefore proven that for all integers m and n, if $m+n$ is odd, then m is odd or n is odd, by contraposition. ◀

Example: proof by contraposition 2

Claim: for three integers l, m, and n, if their sum is not less than 15, then at least one of these three integers must be greater than or equal to 5.

Proof We prove the claim by contraposition.

Suppose none of these three integers, l, m, and n, is greater than or equal to 5. Then every one of them are less than 5, So we have $l + m + n < 15$.

Therefore, at least one of these three integers must be greater than or equal to 5, by contraposition. ◀

Example: proof by contraposition 3

Claim: for any integers m, if $m^2 + 4m + 1$ is even, then m is odd.

Proof We prove the claim by contraposition.

Suppose m is even. Then we have $m = 2k$ for some integer k. Now

$$m^2 + 4m + 1 = (2k)^2 + 4m + 1 = 2(2k^2 + m) + 1.$$

Since $(2k^2 + m)$ is an integer, we see that $m^2 + 4m + 1$ must be odd.

We have therefore proven that for any integers m, if m^2+4m+1 is even, then m is odd, by contraposition. ◀

Example: proof by contradiction 1

Claim: for any integers m and n, $m^2 - 4n \neq 2$.

Proof We prove the claim by contradiction.

Assume there exist integers m and n such that $m^2 - 4n = 2$. Then, we have $m^2 = 4n + 2 = 2(2n + 1)$. This means that m^2 is an even number so that m must be even. Thus there is an integer k such that $m = 2k$. Hence, we have $(2k)^2 - 4n = 2$. Dividing by 2, we get $2k^2 - 2n = 1$, or $2(k^2 - 2n) = 1$. Since $k^2 - 2n$ is an integer, the equality implies that 1 is an even number. So we have a contradiction.

Hence, our assumption $m^2 - 4n = 2$ is false.

Therefore, for any integers m and n, $m^2 - 4n \neq 2$. ◀

Example: proof by contradiction 2

Claim: there are infinitely many prime numbers.

Proof We prove the claim by contradiction.

Assume there are only finitely many prime numbers. We list them as $p_1, \ldots, p_n$. Let $P = p_1 \cdot p_2 \cdots p_n$ be the product of all the listed primes and p is a prime factor of $P+1$. Since p must be among the listed primes, we see that p is also a factor of P. Then, p divides both P and $P+1$, therefore also their difference, which is 1. So we have a contradiction.

Therefore, there are infinitely many prime numbers. ◀

Example: proof by exhaustion 1

Claim: for any integer n, the number n^2 must be of the form $4k$ or $4k+1$ for some $k \in \mathbb{Z}$.

Proof In order to prove the claim, we consider the following four cases.

Case (i): $n = 4m$. It follows that
$$n^2 = (4m)^2 = 16m^2 = 4(4m^2),$$
which is of the form $4k$, where $k = 4m^2 \in \mathbb{Z}$.

Case (ii): $n = 4m+1$. It follows that
$$n^2 = (4m+1)^2 = 16m^2 + 8m + 1 = 4(4m^2 + 2m) + 1,$$
which is of the form $4k+1$, where $k = 4m^2 + 2m \in \mathbb{Z}$.

Case (iii): $n = 4m+2$. It follows that
$$n^2 = (4m+2)^2 = 16m^2 + 16m + 4 = 4(4m^2 + 4m + 1),$$
which is of the form $4k$, where $k = 4m^2 + 4m + 1 \in \mathbb{Z}$.

Case (iv): $n = 4m+3$. It follows that
$$n^2 = (4m+3)^2 = 16m^2 + 24m + 9 = 4(4m^2 + 6m + 2) + 1,$$
which is of the form $4k+1$, where $k = 4m^2 + 6m + 2 \in \mathbb{Z}$.

Since these four cases exhaust the possibilities and since the desired result holds in each case, our proof is complete. ◀

Example: proof by exhaustion 2

Claim: for any integer n, the number $2n^2 + n + 1$ is not divisible by 3.

Proof In order to prove the claim, we consider the following three cases.

Case (i): $n = 3m$. It follows that

$$2n^2 + n + 1 = 2(3m)^2 + 3m + 1 = 3(6m^2 + m) + 1.$$

Since $(6m^2 + m)$ is an integer, we know that when $2n^2 + n + 1$ is divided by 3, the remainder is 1. Hence, $2n^2 + n + 1$ is not divisible by 3.

Case (ii): $n = 3m + 1$. It follows that

$$2n^2 + n + 1 = 2(3m + 1)^2 + (3m + 1) + 1 = 3(6m^2 + 5m + 1) + 1.$$

Since $(6m^2 + 5m + 1)$ is an integer, we know that when $2n^2 + n + 1$ is divided by 3, the remainder is 1. Hence, $2n^2 + n + 1$ is not divisible by 3.

Case (iii): $n = 3m + 2$. It follows that

$$2n^2 + n + 1 = 2(3m + 2)^2 + (3m + 2) + 1 = 3(6m^2 + 9m + 3) + 2.$$

Since $(6m^2 + 9m + 3)$ is an integer, we know that when $2n^2 + n + 1$ is divided by 3, the remainder is 2. Hence, $2n^2 + n + 1$ is not divisible by 3.

Since these three cases exhaust the possibilities and since the desired result holds in each case, our proof is complete. ◀

Example: proof by counterexample 1

Claim: disprove that all prime numbers are odd numbers.

Proof A counterexample to the statement “all prime numbers are odd numbers” is the number 2, as it is a prime number but is not an odd number. ◀

Example: proof by counterexample 2

Claim: disprove the equality $(m+1)^2 = m^2 + 1$, where m is an integer.

Proof A counterexample to the equality $(m+1)^2 = m^2+1$ is $m = 1$, as 1 is an integer but $(1+1)^2 \neq 1^2+1$. ◀

Example: equation $p^2 = 2$ has no rational solution

Show that the equation

$$p^2 = 2$$

has no rational solution. In other words, there exists no rational number p such that $p^2 = 2$.

Proof We prove by contradiction. Suppose there exists a rational number p such that $p^2 = 2$. Put $p = m/n$, with m, n not both being even. This gives $m^2 = 2n^2$, which implies that m is even. Put $m = 2k$. Then we have $n^2 = 2k^2$. This implies that n is also even. This contradicts to the hypothesis that m and n are not both even. Thus, the equation $p^2 = 2$ has no rational solution. ◀

Justification: field properties

Properties on addition:

Item 1 Assume $x+y=x+z$. By the axioms for addition (A),

$$\begin{aligned} y &= 0+y=(-x+x)+y=-x+(x+y) \\ &= -x+(x+z)=(-x+x)+z=0+z=z. \end{aligned}$$

Item 2 Assume $x+y=x$. Then $x+y=x+0$. By Item 1, we get $y=0$.

Item 3 Assume $x+y=0$. Then $x+y=x+(-x)$. By Item 1, we get $y=-x$.

Item 4 Since $(-x)+x=0$, by Item 3, we get $x=-(-x)$.

Properties on multiplication:

The proof is similar to that for Properties on addition. We omit the details.

Properties on the zero and the negative elements:

Item 1 Assume $0x=0$. By the distributive law and the commutativity, we get

$$0x+0x=(0+0)x=0x.$$

Thus, by Item 1 of Properties on addition, we have $0x=0$.

Item 2 Assume $x\neq 0$, $y\neq 0$, but $xy=0$. Thus, by Item 1, we get

$$1=y^{-1}x^{-1}xy=y^{-1}x^{-1}0=0,$$

a contradiction. Hence, $xy\neq 0$.

Item 3 Since

$$xy+(-x)y=[x+(-x)]y=0y=0,$$

we get $(-x)y=-(xy)$, which is the first equality in Item 3. Since

$$xy+x(-y)=x[y+(-y)]=x0=0,$$

we get $x(-y)=-(xy)$, which is the second equality in Item 3.

Item 4 Finally, by Item 3 and Item 4 of Properties on addition, we get

$$(-x)(-y)=-[x(-y)]=-[-(xy)]=xy.$$

◀

Justification: properties on ordered field

Item 1 If $x > 0$, then $0 = -x + x > -x + 0$, so that $-x < 0$. Similarly, if $x > 0$, then $0 = -x + x < -x + 0$, so that $-x > 0$. This proves Item 1.

Item 2 Since $y < z$, we have $z - y > y - y = 0$. This, $x(z - y) > 0$. Hence

$$xz = x(z - y) + xy > 0 + xy = xy.$$

Item 3 If $x < 0$, by Item 1, we know that $-x > 0$. By Item 2, we get $(-x)y < (-x)z$. Thus,

$$-[x(z - y)] = (-x)(z - y) > 0,$$

so tha $x(z - y) < 0$. Hence

$$xz = x(z - y) + xy < 0 + xy = xy.$$

Item 4 If $x > 0$, then, by the definition of ordered field, we have $x^2 > 0$. If $x < 0$, we have $-x > 0$, so that $(-x)^2 > 0$. By one of the field properties, we have $x^2 = (-x)^2$. Thus, for any $x \neq 0$, we always have $x^2 > 0$.

In particular, since $1 \neq 0$ and $1 = 1^2$, we have $1 = 1^2 > 0$.

Item 5 From Item 2, it is easy to see that if $y > 0$ and $w \leq 0$, then $yw \leq 0$. Since $y \cdot y^{-1} = 1 > 0$, we know that $y^{-1} > 0$.

Similarly, since $x > 0$, we know that $x^{-1} > 0$.

Furthermore, by multiplying both sides of the inequality $x < y$ by the positive quantity $x^{-1}y^{-1}$, by Item 2, we have

$$y^{-1} < x^{-1}.$$

Hence, we get $0 < y^{-1} < x^{-1}$. So, Item 5 holds. ◀

Justification: $\mathbb{Q}$ does not have the least-upper-bound property

With the usual addition, multiplication, and the order, it is easy to verify the hypotheses in the definition of *ordered field*. We only show that $\mathbb{Q}$ does not have the least-upper-bound property.

Consider the sets

$$A = \left\{p : \ p \text{ is a positive rational number such that } p^2 < 2\right\},$$
$$B = \left\{p : \ p \text{ is a positive rational number such that } p^2 > 2\right\}.$$

We will show that

1. the set A contains no largest member and the set B contains no smallest member;
2. every member in B is an upper bound of A, and every member in A is a lower bound of B.

If fact, we associate with each positive rational number p the rational number

$$q = p - \frac{p^2-2}{p+2} = \frac{2p+2}{p+2} > 0.$$

Then

$$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}.$$

Item 1 If $p \in A$ then $p^2 < 2$, and $q^2 < 2$. Thus, $q \in A$. Since $q > p$, we show that A contains no largest member. Similarly, if $p \in B$ then $p^2 > 2$, and $q^2 > 2$. Thus, $q \in B$. Since $q < p$, we show that B contains no smallest member.

Item 2 To see that every member in B is an upper bound of A, we take $p_\alpha \in A$ and $p_\beta \in B$. Then $P_\alpha^2 < 2 < p_\beta^2$. It gives $P_\alpha < P_\beta$, which shows that every member in B is an upper bound of A, and every member in A is a lower bound of B.

Hence, the set A is bounded above. Since A contains no largest member, any member in A cannot be the least upper bound of A. Furthermore, since B contains no smallest member, any member of B cannot be the least upper bound of A either. Therefore, A has no least upper bound in $\mathbb{Q}$. Consequently, $\mathbb{Q}$ does not have the least-upper-bound property. ◀

Justification: $\mathbb{R}$ has the least-upper-bound property

The statement that $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield means that

1. $\mathbb{Q} \subset \mathbb{R}$;
2. the usual addition, multiplication, and the order in $\mathbb{R}$, when applied to the members of $\mathbb{Q}$, coincide with the usual addition, multiplication, and the order in $\mathbb{Q}$.

It is easy to check these two items.

The remaining is to prove that $\mathbb{R}$ having the least-upper-bound property. Since the proof is rather long and tedious. We omit it here. ◀

Example: supremum and infimum

Find the supremum and the infimum of the set

$$A = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \right\}.$$

Solution For any $n \in \mathbb{N}_+$,

$$\frac{1}{2} \le \frac{n}{n+1} < 1,$$

so that $\frac{1}{2}$ is a lower bound of A, and 1 an upper bound.

Since, for any given $\varepsilon > 0$, $\frac{1}{2} < \frac{1}{2} + \varepsilon$, we know that $\frac{1}{2} + \varepsilon$ cannot be a lower bound of A. Thus, $\frac{1}{2}$ is the largest lower bound of A, that is, $\inf A = \frac{1}{2}$.

To prove that $\sup A = 1$, we only need to show that for any given $\varepsilon > 0$, $1 - \varepsilon$ cannot be an upper bound of A. In fact, if we take any positive integer n such that $n > \varepsilon^{-1}$, then

$$\frac{n}{n+1} = 1 - \frac{1}{1+n} > 1 - \frac{1}{n} = 1 - \varepsilon.$$

Hence, 1 is the least upper bound of A, that is, $\sup A = 1$. ◀

Justification: equivalence of the least-upper-bound property and the greatest-lower-bound property

Suppose that S be an ordered set that possesses the least-upper-bound property. Let B be a nonempty subset of S that is bounded below. Denote

$$L(B) = \{x \in S : \ x \leq b \text{ for all } b \in B\},$$

which is the set of lower bounds for set B. The set $L(B)$ is nonempty by assumption.

Then for any fixed $b_0 \in B$ and any $l \in L(B)$, we have $l \leq b_0$. This means that $L(B)$ is bounded above. Since S possesses the lower-upper-bound property, we know $l_0 = \sup L(B)$ exists.

Claim: $l_0 = \inf B$.

For this, we will show that

1. l_0 is a lower bound for B;
2. l_0 is greater than any lower bound of B.

Item 1 For any $b \in B$, as above, b is an upper bound of $L(B)$. Because l_0 is the least upper bound of $L(B)$, we have $l_0 \leq b$. Thus, l_0 is a lower bound for B.

Item 2 Suppose l is any lower bound of B. By definition, $l \in L(B)$. As l_0 is the least upper bound of $L(B)$, we have $l \leq l_0$. So, l_0 is greater than any lower bound of B. ◂

Justification: the archimedean property of $\mathbb{R}$ and the rational density theorem

Item 1 We prove the statement by contradiction.

Suppose the statement is false. Denote $A = \{nx : n \in \mathbb{N}_+\}$. Then y would be an upper bound of A. Thus, A has a least upper bound, say $\alpha = \sup A$. Since $x > 0$, we know that $\alpha - x < \alpha$. Because $\alpha - x$ is not an upper bound of A, $\alpha - x < mx$ for some positive integer m. It follows that $\alpha < (m+1)x \in A$, which contradicts that α is an upper bound of A.

Item 2 We will show that there are two integers m and n ($n \neq 0$) such that $x < m/n < y$.

In fact, since $x < y$, we have $y - x > 0$. By Item 1, there is a positive integer n such that $n(y-x) > 1$. Again, by Item 1, there are two positive integers m_1 and m_2, such that

$$m_1 > nx, \qquad m_2 > -nx.$$

Thus, $-m_2 < nx < m_1$. By progressively decreasing m_1 and increasing $-m_2$, we see that there is an integer m, with $-m_2 \leq m \leq m_1$, such that $m - 1 \leq nx < m$. Hence, we have two integers m and n, such that

$$nx < m \leq 1 + nx < ny.$$

Since $n > 0$, it follows that

$$x < \frac{m}{n} < y.$$

◀

Justification: existence of nth root of positive real numbers

Uniqueness:

This is clear, since, if $0 < y_1 < y_2$, then $y_1^n < y_2^n$.

Existence:

For any given positive real number x and positive integer n, let

$$E = \{t : \ t > 0 \text{ and } t^n < x\}.$$

We prove the existence of the nth root of x by showing the following

1. E is nonempty.
2. E has an upper bound.
3. $y = \sup E$ satisfies $y^n = x$.

Item 1: Take $t = \dfrac{x}{1+x}$. It is clear that $0 < t < 1$. Thus, $t^n < t < x$. This show that $t \in E$, so that E is nonempty.

Item 2: Take any $w > 1 + x$. Then we have $w^n > w > x$ so that $w \notin E$. This indicates that if $t \in E$, then $t < 1 + x < w$. Thus, w is an upper bound of E.

Item 3: We show that, for $y = \sup E$, each of the inequalities $y^n < x$ and $y^n > x$ leads to a contradiction, so that $y^n = x$.

Since $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$, we have

$$b^n - a^n < (b-a)nb^{n-1}$$

when $0 < a < b$.

Assume $y^n < x$. Choose h so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$

Then, for $a = y$ and $b = y + h$, we have

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus $(y+h)^n < x$, and $y + h \in E$. Since $y + h > y$, this contradicts to the fact that y is an upper bound of E.

Assume $y^n > x$. Write

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y$. If $t \geq y - k$, then

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x,$$

which implies $t^n > x$, and $t \notin E$. In other words, if $t \in E$, then $t < y - k$. Hence $y - k$ is an upper bound of E. This contradicts to the hypothesis that y is the least upper bound of E.

Hence $y^n = x$. ◀

Justification: law of radicals

Denote $\alpha = a^{1/n}$, $\beta = b^{1/n}$. Then

$$ab = \alpha^n \beta^n = (\alpha\beta)^n.$$

The latter equality holds since multiplication is commutative. Hence, by the uniqueness of nth root of positive real numbers, we get $\alpha\beta = (ab)^{1/n}$, so that

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

◀

Justification: the complex field $\mathbb{C}$

Item 1 For $\mathbb{C}$, we verify the field axioms as listed in the definition of field.

Let $x=(a,b), y=(c,d), z=(e,f)\in\mathbb{C}$.

(A1) $x+y=(a+c,b+d)\in\mathbb{C}$.

(A2) $x+y=(a+c,b+d)=(c+a,d+b)=y+x$.

(A3)
$$\begin{aligned}(x+y)+z&=(a+c,b+d)+(e,f)\\&=(a+c+e,b+d+f)\\&=(a,b)+(c+e,d+f)\\&=x+(y+z).\end{aligned}$$

(A4) $x+0_{\mathbb{C}}=(a,b)+(0,0)=(a,b)=x$.

(A5) Put $(-x)=(-a,-b)$. Then $x+(-x)=(0,0)=0_{\mathbb{C}}$.

(M1) $x\times y=(ac-bd,ad+bc)\in\mathbb{C}$.

(M2) $x\times y=(ac-bd,ad+bc)=(ca-db,da+cb)=y\times x$.

(M3)
$$\begin{aligned}(x\times y)\times z&=(ac-bd,ad+bc)\times(e,f)\\&=(ace-bde-adf-bcf,acf-bdf+ade+bce)\\&=(a,b)\times(ce-df,cf+de)\\&=x\times(y\times z).\end{aligned}$$

(M4) $1_{\mathbb{C}}\times x=(1,0)\times(a,b)=(a,b)=x$.

(M5) If $x\neq 0_{\mathbb{C}}$, that is, $(a,b)\neq(0,0)$, so that at least one of the real numbers a and b is not 0. Thus, $a^2+b^2\neq 0$. Denote
$$x^{-1}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right).$$
Then
$$x\times x^{-1}=(a,b)\times\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=(1,0)=1_{\mathbb{C}}.$$

(D)
$$\begin{aligned}x\times(y+z)&=(a,b)\times(c_e,d+f)\\&=(ac+ae-bd-df,ad+af+bc+be)\\&=(ac-bd,ad+bc)+(ae-df,af+be)\\&=x\times y+x\times z.\end{aligned}$$

Item 2 By the definition, clearly we have
$$(a,0)+(b,0)=(a+b,0),\qquad (a,0)\times(b,0)=(ab,0).$$

Justification: square root of -1

Item 1 $i^2 = (0,1) \times (0,1) = (-1,0) = -1$.

Item 2 $a + bi = (a,0) + (b,0) \times (0,1) = (a,0) + (0,b) = (a,b)$. ◀

Justification: properties of complex conjugates

Denote $z = a + bi$, $w = c + di$, with a, b, c, d real.

Item 1 $\overline{z+w} = (a+c) - (b+d)i$

$$= (a - bi) + (c - di) = \overline{z} + \overline{w}.$$

Item 2 $\overline{zw} = \overline{(a+bi)(c+di)}$

$$= \overline{(ac-bd)+(ad+bc)i}$$

$$= (ac-bd) - (ad+bc)i$$

$$= (a-bi)(c-di) = \overline{z} \cdot \overline{w}.$$

Item 3 $z + \overline{z} = (a+bi) + (a-bi) = 2a = 2\,\mathrm{Re}(z)$. Similarly, $z - \overline{z} = (a+bi) - (a-bi) = 2bi = 2i\,\mathrm{Im}(z)$.

Item 4 If $z \neq 0$, then at least one of a and b is not zero, so that $a^2 + b^2 > 0$. Thus, from the equality

$$z\overline{z} = (a+bi)(a-bi) = a^2 + b^2,$$

we know that $z\overline{z}$ is real and positive. ◀

Justification: properties of the absolute value

Denote $z = a + bi$, $w = c + di$, with a, b, c, d real.

Item 1 Since

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2,$$

we have $|z| = (z\overline{z})^{1/2} = \sqrt{a^2 + b^2} \geq 0$.

If $|z| = 0$, then $\sqrt{a^2 + b^2} = 0$, so that $a = b = 0$. Hence $z = 0$. Conversely, if $z = 0$, then $z\overline{z} = 0$, so that $|z| = 0$.

Item 2 $|\overline{z}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$.

Item 3 $$\begin{aligned} |zw| &= |(a + bi)(c + di)| \\ &= |(ac - bd) + (ad + bc)i| \\ &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} = |z| \cdot |w|. \end{aligned}$$

Item 4 Since $a^2 \leq a^2 + b^2$, we get $|a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$, that is, $|\operatorname{Re}(z)| \leq |z|$. We can similarly show $|\operatorname{Im}(z)| \leq |z|$.

Item 5 It is easy to see that $\overline{z}w$ is the conjugate of $z\overline{w}$, so that $z\overline{w} + \overline{z}w = 2\operatorname{Re}(z\overline{w})$. Hence,

$$\begin{aligned} |z + w|^2 = (z + w)(\overline{z} + \overline{w}) &= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\ &= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2|z\overline{w}| + |w|^2 \\ &\leq |z|^2 + 2|z| \cdot |\overline{w}| + |w|^2 \\ &= |z|^2 + 2|z| \cdot |w| + |w|^2 = (|z| + |w|)^2, \end{aligned}$$

so that $|z + w| \leq |z| + |w|$. ◀

Justification: the Cauchy-Schwarz inequality

Denote $A = \sum_{j=1}^{n} |a_j|^2$, $B = \sum_{j=1}^{n} |b_j|^2$, $C = \sum_{j=1}^{n} a_j \bar{b}_j$.

If $B = 0$, then $b_1 = \cdots = b_n = 0$. In this case, the conclusion holds trivially.

If $B \neq 0$, then $B > 0$. By the properties of complex numbers, we have

$$\begin{aligned}
0 \leq \sum_{j=1}^{n} |Ba_j - Cb_j|^2 &= \sum_{j=1}^{n} (Ba_j - Cb_j)\left(B\overline{a}_j - \overline{C}\overline{b}_j\right) \\
&= B^2 \sum_{j=1}^{n} |a_j|^2 - B\overline{C} \sum_{j=1}^{n} a_j \bar{b}_j - BC \sum_{j=1}^{n} \overline{a}_j b_j + |C|^2 \sum_{j=1}^{n} |b_j|^2 \\
&= B^2 A - B\overline{C}C - BC\overline{C} + B|C|^2 \\
&= B(AB - |C|^2).
\end{aligned}$$

Thus, we have $AB - |C|^2 \geq 0$, which gives the desired inequality. ◀

Justification: properties of euclidean space $\mathbb{R}^k$

Denote $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k)$, with $x_1, \ldots, x_k, y_1, \ldots, y_k$ real.

Item 1 Since

$$\mathbf{x} \cdot \mathbf{x} = x_1^2 + \cdots + x_k^2,$$

we have $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \sqrt{x_1^2 + \cdots + x_k^2} \geq 0$.

If $\|\mathbf{x}\| = 0$, then $\sqrt{x_1^2 + \cdots + x_k^2} = 0$, so that $x_1 = \cdots = x_k = 0$. Hence $\mathbf{x} = 0$. Conversely, if $\mathbf{x} = \mathbf{0}$, then $\mathbf{x} \cdot \mathbf{x} = 0$, so that $\|\mathbf{x}\| = 0$.

Item 2 $\|\alpha\mathbf{x}\| = \sqrt{(\alpha x_1)^2 + \cdots + (\alpha x_k)^2}$

$= |\alpha| \cdot \sqrt{x_1^2 + \cdots + x_k^2} = |\alpha| \, \|\mathbf{x}\|$.

Item 4 We first prove Item 4 then Item 3.

Denote $A = \sum_{j=1}^{k} x_j^2$, $B = \sum_{j=1}^{k} y_j^2$, $C = \sum_{j=1}^{k} x_j y_j$.

If $B = 0$, then $y_1 = \cdots = y_k = 0$. In this case, the conclusion holds trivially.

If $B \neq 0$, then $B > 0$. By the properties of real numbers, we have

$$\begin{aligned}
0 \leq \sum_{j=1}^{k} \left(Bx_j - Cy_j\right)^2 &= \sum_{j=1}^{k} \left(B^2 x_j^2 - 2BCx_j y_j + C^2 y_j^2\right) \\
&= B^2 \sum_{j=1}^{k} x_j^2 - 2BC \cdot \sum_{j=1}^{k} x_j y_j + C^2 \sum_{j=1}^{k} y_j^2 \\
&= B^2 A - 2BC^2 + BC^2 \\
&= B(AB - C^2).
\end{aligned}$$

Thus, we have $AB - C^2 \geq 0$, so that

$$|\mathbf{x} \cdot \mathbf{y}| = |C| \leq \sqrt{A} \cdot \sqrt{B} = \|\mathbf{x}\| \, \|\mathbf{y}\|.$$

Item 3 We use the conclusion of Item 4 to prove Item 3. In fact,

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \, \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2,
\end{aligned}$$

so that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Remark

- As shown above, we prove the triangle inequality by the Cauchy-Schwarz inequality. In fact, we can also prove the Cauchy-Schwarz inequality by the triangle inequality.

  Suppose $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. Since

  $$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{x}\|^2,$$
  $$(\|\mathbf{x}\| + \|\mathbf{y}\|)^2 = \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \, \|\mathbf{y}\| + \|\mathbf{x}\|^2,$$

  we get $\mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$. Replacing $\mathbf{x}$ by $-\mathbf{x}$, we further have $-\mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$. Combining the last two inequalities, we have the Cauchy-Schwarz inequality

  $$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|.$$

Example: infinite set is equivalent to one of its proper subsets

Show that

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\} \sim \mathbb{N}_+ = \{1, 2, 3, \dots\}.$$

That is, $\mathbb{Z}$ is equivalent to one of its proper subsets.

Proof Define the mapping $f\colon \mathbb{N}_+ \to \mathbb{Z}$ as following:

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ -\dfrac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

It suffices to show that f is bijective, i.e., both surjective and injective.

Surjective:

Suppose $k \in \mathbb{Z}$. We need to show that there exists a number $l \in \mathbb{N}_+$ such that $f(l) = k$.

In fact, if $k = 0 \in \mathbb{Z}$, then $l = 1 \in \mathbb{N}_+$, and $f(l) = f(1) = -\dfrac{1-1}{2} = 0 = k$. If $k \in \mathbb{Z}$ is a positive integer, then $l = 2k \in \mathbb{N}_+$, and $f(l) = f(2k) = \dfrac{2k}{2} = k$. If $k \in \mathbb{Z}$ is a negative integer, then then $l = -2k + 1 \in \mathbb{N}_+$, and $f(l) = f(-2k+1) = -\dfrac{(-2k+1)-1}{2} = k$. Hence, f is surjective.

Injective:

Suppose $f(n_1) = f(n_2)$, with $n_1, n_2 \in \mathbb{N}_+$. We need to show that $n_1 = n_2$.

If n_1 is even, then n_2 must be also even, otherwise $f(n_1) = \dfrac{n_1}{2} > 0 > -\dfrac{n_2 - 1}{2} = f(n_2)$ which is impossible. It follows that $\dfrac{n_1}{2} = f(n_1) = f(n_2) = \dfrac{n_2}{2}$, so that $n_1 = n_2$.

If n_1 is odd, then n_2 must be also odd, otherwise $f(n_1) = -\dfrac{n_1 - 1}{2} < 0 < \dfrac{n_2}{2} = f(n_2)$ which is impossible. It follows that $-\dfrac{n_1 - 1}{2} = f(n_1) = f(n_2) = -\dfrac{n_2 - 1}{2}$, so that $n_1 = n_2$. Hence, f is injective. ◀

Justification: infinite subset of a countable set is countable

Let A be a countable set, and $E \subset A$ an infinite subset.

First arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements.

Next, construct a subsequence $\{x_{n_k}\}$ as follows:

1. Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, \ldots, n_{k-1}$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.
2. Put $f(k) = x_{n_k}$, $k = 1, 2, 3, \ldots$.

It is obvious that f is a bijective mapping from $\mathbb{N}_+$ to E. Consequently, E is countable. ◀

Justification: the distributive law for union and intersection of sets

The distributive law:

Denote $S = A \cap (B \cup C)$, $T = (A \cap B) \cup (A \cap C)$. To show that $S = T$, it suffices to show that both $S \subset T$ and $T \subset S$ hold.

Suppose $x \in S$. Then $x \in A$ and $x \in B \cup C$. The latter implies $x \in B$ or $x \in C$ (possibly both). Hence, $x \in A \cap B$ or $x \in A \cap C$, so that $x \in (A \cap B) \cup (A \cap C)$. That is, $x \in T$. Thus, $S \subset T$.

Suppose $x \in T$. Then $x \in A \cap B$ or $x \in A \cap C$. That is, $x \in A$, and $x \in B \cup C$. Hence $x \in A \cap (B \cup C) = S$, so that $T \subset S$. ◀

Justification: countable union of countable sets is coutable

Denote $S = \bigcup_{n=1}^{\infty} E_n$.

Let E_n be arranged in a sequence $\{x_{n,k}\}$, $k = 1, 2, 3, \ldots$, and consider the infinite array

$$\begin{array}{cccccccc}
E_1 & \to & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & \ldots \\
E_2 & \to & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & \ldots \\
E_3 & \to & x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & \ldots \\
E_4 & \to & x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & \ldots \\
\vdots & & & & & & \\
E_n & \to & x_{n,1} & x_{n,2} & x_{n,3} & x_{n,4} & \ldots \\
\vdots & & & & & &
\end{array}$$

These elements can be arranged in one single sequence:

$$x_{1,1}; x_{2,1}, x_{1,2}; x_{3,1}, x_{2,2}, x_{1,3}; x_{4,1}, x_{3,2}, x_{2,3}, x_{1,4}; \ldots .$$

By this arrangement as a sequence, we know that if any two sets E_n have no elements in common, then $S \sim \mathbb{N}_+$. Otherwise, there is a subset $T \subset \mathbb{N}_+$ such that $S \sim T$. Thus S is at most countable. Since $E_1 \subset S$, and E_1 is infinite, so S is infinite. Hence S is countable. ◀

Justification: n-tuples of a countable set

We prove the proposition by mathematical induction.

Since $B_1 = A$, B_1 is countable.

Suppose B_{n-1} is countable $(n = 2, 3, 4, \dots)$. To show that B_n is countable, we notice that the elements of B_n are of the form (b, a) with $b \in B_n$ and $a \in A$. For every fixed b, the set of pairs (b, a) is equivalent to A, and hence countable. Thus B_n is the union of a countable set of countable sets, which is countable. ◀

Justification: the set of all rational numbers is countable

Consider the set S comprising all ordered pairs (m, n), with $m, n \in \mathbb{N}$. We know that S is countable. Since every rational number is of the form m/n, the set of all rational numbers is equivalent to an infinite subset of S. Hence, the set of all rational numbers is countable. ◀

Justification: the set of 0-1 sequences

We prove the proposition by a contradiction. Suppose A is countable. We can arrange the elements of A as a sequence $s_1, s_2, s_3, \ldots$. Here each s_k is a 0-1 sequence, like $1, 0, 1, 1, 1, 0, \ldots$.

We construct a 0-1 sequence s as follows. If the first digit of s_1 is 1, we let the first digit of s be 0, and vice versa. In general, if the nth digit in s_n is 1, we let the n digit of s be 0, and vice versa. Then the sequence s differs from every member of A, so that $s \notin A$. This contradicts to the definition of A, the set of all 0-1 sequences.

Consequently, the elements of A cannot be arranged as a sequence. Therefore, A is uncountable. ◂

Example: a limit point of a set is not necessarily a point in the set

Consider $E = (0, 1) \subset \mathbb{R}$. It is obvious that 1 is a limit point of E, but $1 \notin E$. The point 1 is also a boundary point of E.

Example - various subsets in $\mathbb{R}$

Consider the following subsets of $\mathbb{R}$:

(a) the set of all real x such that $|x| < 1$;
(b) the set of all real x such that $|x| \leq 1$;
(c) a finite set;
(d) the set of all integers;
(e) the set consists of the numbers $1/n$ $(n = 1, 2, 3, \ldots)$;
(f) the set of all real numbers, $\mathbb{R}$;
(g) the interal (a, b).

Determine whether they are closed, open, perfect, and/or bounded subsets in $\mathbb{R}$.

Solution

	closed	open	perfect	bounded
(a)	no	yes	no	yes
(b)	yes	no	yes	yes
(c)	yes	no	no	yes
(d)	yes	no	no	no
(e)	no	no	no	yes
(f)	yes	yes	yes	no
(g)	no	yes	no	yes

◀

Justification: both open and closed in $\mathbb{R}$

We will show that the empty set and whole Suppose $A \subsetneq \mathbb{R}$ be non-empty. Since $A \neq \mathbb{R}$ and A is both open and closed, the complement A^{c} is not empty and is also both open and closed. Pick $x \in A$ and $y \in A^{\mathsf{c}}$. Clearly, $x \neq y$, so suppose, without loss of generality that $x < y$.

Let $S = \{c : \ c \in A, \, c < y\}$. The set S is not empty as $x \in S$; it is bounded above as $c < y$ for all $c \in S$. Thus, $a = \sup S$ is well-defined. In addition, $x \leq a \leq y$. Since A and A^c partition $\mathbb{R}$, a must be contained in exactly one of them.

Case 1: $a \in A$. Clearly $a \neq y$ since $y \in A^{\mathsf{c}}$. Hence, $a < y$. Since A is open, there exists some $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subset A$. Thus, we have $a + \frac{1}{2}\varepsilon \in A$ and $a + \frac{1}{2}\varepsilon < y$. But this means that a is not an upper bound of the set S, a contradiction.

Case 2: $a \notin A$. Note that $a \neq x$ since $x \in A$. Hence, $a > x$. Since A^{c} is open, there exists some $\varepsilon > 0$ such that $a - \varepsilon > x$ and the interval $(a - \varepsilon, a + \varepsilon) \subset A^{\mathsf{c}}$. Thus, we have $(a - \varepsilon, a] \in A^{\mathsf{c}}$. Therefore, $a - \varepsilon$ is also an upper bound of the set S. But this means that a is not the *least* upper bound of the set S, contradiction again.

◀

Justification: neighborhoods are open sets

Suppose $N_r(p)$ is a neighborhood. Let q be any point in $N_r(p)$. Then there is a positive number h such that

$$|p - q| = r - h.$$

To show that q is an interior point of $N_r(p)$, we just need to show that the neighborhood $N_h(q)$ of q is in $N_r(p)$. In fact, for $z \in N_h(q)$, since

$$|p - z| \leq |p - q| + |q - z| < r - h + h = r,$$

we see that $z \in N_r(p)$. Thus $N_h(q) \subset N_r(p)$. ◀

Justification: neighborhood of a limit point contains infinitely many points

We prove the result by contradiction.

Suppose there is a neighborhood N of p which contains only a finite number of points of E. Denote them as $q_1, \ldots, q_n \in N \cap E$, which are distinct from p, and put

$$r = \min_{1 \le m \le n} |q_m - p|.$$

It is obvious that $r > 0$.

Now the neighborhood $N_r(p)$ contains no point in E except p. This contradicts to the hypothesis that p is a limit of E. ◀

Justification: the complement of a union equals the intersection of complements

The first equality:

If $x \in \big(\bigcup_{\alpha} E_\alpha\big)^{\mathsf{c}}$, then $x \notin \bigcup_{\alpha} E_\alpha$. Thus, $x \notin E_\alpha$ for any α so that $x \in E_\alpha^{\mathsf{c}}$ for every α. Hence, $x \in \bigcap_{\alpha} E_\alpha^{\mathsf{c}}$. So, we know that $\big(\bigcup_{\alpha} E_\alpha\big)^{\mathsf{c}} \subset \bigcap_{\alpha} E_\alpha^{\mathsf{c}}$.

Conversely, if $x \in \bigcap_{\alpha} E_\alpha^{\mathsf{c}}$, then $x \in E_\alpha^{\mathsf{c}}$ for every α. Thus $x \notin E_\alpha$ for any α so that $x \notin \bigcup_{\alpha} E_\alpha$. Hence $x \in \big(\bigcup_{\alpha} E_\alpha\big)^{\mathsf{c}}$. So, we know that $\bigcap_{\alpha} E_\alpha^{\mathsf{c}} \subset \big(\bigcup_{\alpha} E_\alpha\big)^{\mathsf{c}}$.

Therefore $\big(\bigcup_{\alpha} E_\alpha\big)^{\mathsf{c}} = \bigcap_{\alpha} E_\alpha^{\mathsf{c}}$.

The second equality:

Denote $F_\alpha = E_\alpha^{\mathsf{c}}$. Then $F_\alpha^{\mathsf{c}} = E_\alpha$. Thus, by the first equality, we have $\big(\bigcup_{\alpha} F_\alpha^{\mathsf{c}}\big)^{\mathsf{c}} = \bigcap_{\alpha} F_\alpha$, so that $\bigcup_{\alpha} F_\alpha^{\mathsf{c}} = \big(\bigcap_{\alpha} F_\alpha\big)^{\mathsf{c}}$. Hence, if writing E_α in place of F_α, we get the second equality $\big(\bigcap_{\alpha} E_\alpha\big)^{\mathsf{c}} = \bigcup_{\alpha} E_\alpha^{\mathsf{c}}$. ◀

Justification: a set is open if and only if its complement is closed

First, suppose that E^c is closed. We need to show that every point in E is an interior point of E. In fact, let $x \in E$. Since $x \notin E^c$, we know that x is not a limit point of E^c. Hence there is a neighborhood N of x such that $N \cap E^c = \varnothing$, or $N \subset E$. Thus, x is an interior point of E. So E is open.

Next, suppose E is open. To show that E^c is closed, we need to show that every limit point of E^c must be still in E^c. In fact, if x is a limit point of E^c. Then every neighborhood of x contains a point $p \neq x$, $p \in E^c$. Hence x cannot be an interior point of E. Since E is open, this means that $x \notin E$, or $x \in E^c$. Thus E^c is closed. ◀

Justification: the union of open sets is an open set

Item 1 Suppose $x \in \bigcup_{\alpha} G_\alpha$. Then $x \in G_\alpha$ for some α. Since G_α is open, there is a neighborhood N of x such that $N \subset G_\alpha$. It follows that $N \subset \bigcup_{\alpha} G_\alpha$. Thus x is an interior point of $\bigcup_{\alpha} G_\alpha$. Therefore $\bigcup_{\alpha} G_\alpha$ is open.

Item 2 By Proposition 2.12, the complement of a closed set is open, so that F_α^{c} is open for any α. By Item 1, we know that $\bigcup_{\alpha} F_\alpha^{\mathsf{c}}$ is open. By Proposition 2.11, we have

$$\Big(\bigcap_{\alpha} F_\alpha\Big)^{\mathsf{c}} = \bigcup_{\alpha} F_\alpha^{\mathsf{c}}.$$

Hence $\big(\bigcap_{\alpha} F_\alpha\big)^{\mathsf{c}}$ is open. Again by Proposition 2.12, the complement of an open set is closed, so that $\bigcap_{\alpha} F_\alpha$ is closed.

Item 3 Suppose $x \in \bigcap_{j=1}^{n} G_i$. We know that $x \in G_j$ for every j, $1 \le j \le n$. For each j, since G_j is open, there is a neighborhood N_j of x, with radius r_j, such that $N_j \subset G_j$. Put

$$r = \min\{r_1, \ldots, r_n\}.$$

Then $r > 0$. For the neighborhood $N_r(x)$, we have $N_r(x) \subset N_j$ for $1 \le j \le n$. Thus $N_r(x) \subset \bigcap_{j=1}^{n} G_j$. This means that x is an interior point of $\bigcap_{j=1}^{n} G_j$. Hence, $\bigcap_{j=1}^{n} G_j$ is open.

Item 4 The proof is similar to that of Item 2. It follows from Item 3, the equality $\big(\bigcap_{j=1}^{n} F_j\big)^{\mathsf{c}} = \bigcup_{j=1}^{n} F_j^{\mathsf{c}}$ (Proposition 2.11), and the fact that the complement of a closed (open) set is open (closed) (Proposition 2.12). ◀

Example - a limit point of a set is not necessarily a point in the set

(1) Consider a collection of open sets $G_j = \left(-\frac{1}{j}, \frac{1}{j}\right)$, $j = 1, 2, 3, \ldots$, in $\mathbb{R}$. It is easy to see that $\bigcap_{j=1}^{\infty} G_j = \{0\}$, that is not open in $\mathbb{R}^1$.

(2) Consider a collection of closed sets $F_j = \left[\frac{1}{j+1}, 1 - \frac{1}{j+2}\right]$, $j = 1, 2, 3, \ldots$, in $\mathbb{R}$. It is easy to see that $\bigcup_{j=1}^{\infty} F_j = (0, 1)$, that is not closed in $\mathbb{R}^1$.

◀

Justification: properties of closure of a set

Item 1 To show $\overline{E}$ is closed, we only need to show that $\overline{E}^{\mathsf{c}}$ is open. Let $p \in \overline{E}^{\mathsf{c}} \subset \mathbb{R}$. By Proposition 2.11,

$$\overline{E}^{\mathsf{c}} = (E \cup E')^{\mathsf{c}} = E^{\mathsf{c}} \cap (E')^{\mathsf{c}},$$

so we have $p \notin E$ and p is not a limit point of E. Hence, there is a neighborhood N of p such that $N \cap (E \cup E') = \varnothing$. This means that $N \subset \overline{E}^{\mathsf{c}}$.

Item 2 If $E = \overline{E}$, by Item 1, we know that E is closed. Conversely, if E is closed, then $E' \subset E$. So $\overline{E} = E \cup E' = E$.

Item 3 If F is closed and $E \subset F$, then $E' \subset F' \subset F$. Thus $\overline{E} = E \cup E' \subset F$. ◀

Justification: the supremum of bounded-above set of real numbers is in the closure

If $y \in E$, then $y \in \overline{E}$. If $y \notin E$, then for every $h > 0$, there exists a point $x \in E$ such that $y - h < x < y$, for otherwise $y - h$ would be an upper bound of E. By the definition, y is a limit point of E, or $y \in E' \subset \overline{E}$. ◀

Justification: nonempty perfect set is uncountable

Since P has limit points, P must be infinite.

Suppose P is countable, and denote the points of P as $x_1, x_2, x_3, \ldots$. We now construct a sequence $\{V_n\}$ of neighborhoods, as follows, to have a contradiction eventually.

Let V_1 be a neighborhood $N_r(x_1)$ for some $r > 0$.

Suppose V_n has been constructed, so that $V_n \cap P \neq \varnothing$. Since every point of P is a limit point of P, we take V_{n+1} to be a neighborhood such that

(i) $\overline{V}_{n+1} \subset V_n$, (ii) $x_n \notin \overline{V}_{n+1}$, (iii) $V_{n+1} \cap P \neq \varnothing$.

It is obvious that (iii) is the induction hypothesis to proceed the construction.

Put $K_n = \overline{V}_n \cap P$. Since $\overline{V}_n$ is closed and bounded, and P is closed, we know that K_n is compact. Since $K_n \subset P$, we have $\bigcap_{n=1}^{\infty} K_n \subset P$. The fact $x_n \notin K_{n+1}$ implies that $\bigcap_{n=1}^{\infty} K_n$ contains no point of P. Hence $\bigcap_{n=1}^{\infty} K_n = \varnothing$. But (iii) implies that each K_n is nonempty, and (i) implies that $K_n \supset K_{n+1}$. These imply, by Cantor's intersection theorem, that $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$, a contradiction.

◀

Example - two sets disjoint but not separated

Suppose A and B are separated:

$$A \cap \overline{B} = \overline{A} \cap B = \varnothing.$$

Then,

$$A \cap B \supset A \cap \overline{B} = \varnothing,$$

so that $A \cap B = \varnothing$, i.e. A and B are disjoint.

However, two disjoint sets are not necessarily separated. In fact, the interval $[0,1]$ and the open interval $(1,2)$ are not separated, since

$$[0,1] \cap \overline{(1,2)} = [0,1] \cap [1,2] = \{1\} \neq \varnothing,$$

but they are disjoint: $[0,1] \cap (1,2) = \varnothing$. ◀

Justification: characterization of connected set

(⇒) Suppose E is connected, and suppose $x \in E$, $y \in E$, and $x < z < y$, but $z \notin E$. We will deduce a contradiction. In fact, if we put

$$A_z = E \cap (-\infty, z), \qquad B_z = E \cap (z, \infty),$$

then $E = A_z \cup B_z$. Since $x \in A_z$ and $y \in B_z$, A and B both are nonempty. Since

$$\begin{aligned} A_z \cap \overline{B_z} &\subset (-\infty, z) \cap [z, \infty) = \varnothing, \\ \overline{A_z} \cap B_z &\subset (-\infty, z] \cap (z, \infty) = \varnothing, \end{aligned}$$

we get $A_z \cap \overline{B_z} = \overline{A_z} \cap B_z = \varnothing$. Thus E is not connected, a contradiction to the hypothesis.

(⇐) Suppose that for any $x \in E, y \in E$, and $x < z < y$, we always have $z \in E$. In this case, if E is not connected, we will deduce a contradiction. In fact, for E being not connected, there are nonempty separated sets A and B such that $A \cup B = E$. Pick $x \in A$, $y \in B$. Without loss of generality, we assume that $x < y$. Define

$$z = \sup(A \cap [x, y]).$$

By Proposition 2.15, $z \in \overline{A \cap [x, y]} \subset \overline{A}$. Since A and B are separated, $z \notin B$. Hence $x \leq z < y$.

If $z \notin A$, then $z \notin A \cup B = E$. Since in this case we have $x < z < y$, by the hypothesis, $z \in E$, a contradiction.

If $z \in A$, then $z \notin \overline{B} = B \cup B'$. By the fact that $z \notin B'$, there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Hence $x < z_1 < y$ and $z_1 \notin A \cup B = E$, again a contradiction to the hypothesis. ◀

Example - open interval is not compact

For any $n \in \mathbb{N}_+$, the subset $U_n = (1/n, 1) \subset (0, 1)$ is open. Since

$$(0,1) = \bigcup_{n=1}^{\infty} U_n,$$

the colloection $\{U_n\}$ is an open cover of $(0,1)$. It is obvious that there are no finite subcovers, so that $(0,1)$ is not compact. ◀

Justification: the relationship between compact sets and closed sets

Item 1 Let K be a compact subset of $\mathbb{R}$. To show that K is closed, we only need to show that K^{c} is open.

Suppose $p \in K^{\mathsf{c}} \subset \mathbb{R}$. For each $q \in K$, we define a neighborhood W_q of q, with the radius less than $\frac{1}{2}|p-q|$. Then $\{W_q\}$ is an open cover of K. Since K is compact, there are finitely many points $q_1, \ldots, q_n$ in K such that

$$K \subset W_{q_1} \cup \cdots \cup W_{q_n}.$$

For each of these q_i, $1 \le i \le n$, we define a neighborhood V_{q_i} of p, with the radius also less than $\frac{1}{2}|p-q|$. Then $V_{q_i} \cap W_{q_i} = \varnothing$. Set $V = V_{q_1} \cap \cdots \cap V_{q_n}$. Then V is a neighborhood of p. The construction of $\{W_{q_i}\}$ and $\{V_{q_i}\}$ imply $p \in V$. We also have $V \cap K = \varnothing$, since

$$V \cap K \subset V \cap (W_{q_1} \cup \cdots \cup W_{q_n}) = (V \cap W_{q_1}) \cup \cdots \cup (V \cap W_{q_n}) = \varnothing \cup \cdots \cup \varnothing = \varnothing.$$

We conclude that p is an interior point of K^{c}. Therefore K^{c} is open.

Item 2 Suppose $F \subset K \subset \mathbb{R}$, with F being closed, and K compact. Let $\{V_\alpha\}$ be an open cover of F. Notice F^{c} is open. So, if we add F^{c} into the open cover to obtain a collection $\{V_\alpha\} \cup \{F^{\mathsf{c}}\}$, it is still an open cover of F. Since K is compact, we have a finite subcollection of $\{V_\alpha\} \cup \{F^{\mathsf{c}}\}$ to cover K. This implies that

$$K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n} \cup F^{\mathsf{c}},$$

for finitely many indices $\alpha_1, \ldots, \alpha_n$. Since F is a subset of K, we have

$$F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n} \cup F^{\mathsf{c}},$$

From $F \cap F^{\mathsf{c}} = \varnothing$, we have

$$F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Thus, F is compact. ◀

Justification: finite closed intervals are compact

Let $I = [a, b]$ be a finite closed interval. Put $\delta = |b - a|$. Then $|x - y| \le \delta$ for $x, y \in I$.

If I is not compact, then there is an open cover $\{G_\alpha\}$ of I which contains no finite subcover of I. Put $c = (a + b)/2$. Then at least one of the finite closed intervals $[a, c]$ and $[c, b]$, call it I_1, cannot be covered by any finite subcollection of $\{G_\alpha\}$. Next we subdivide I_1 and continue the process. Then we obtain a sequence $\{I_n\}$ with the following properties:

(a) $I_1 \supset I_2 \supset I_3 \supset \cdots$;

(b) I_n is not covered by any finite subcollection of $\{G_\alpha\}$;

(c) $|x - y| \le 2^{-n}\delta$ for any $x, y \in I_n$.

Write $I_n = [a_n, b_n]$ and $E = \{a_n\}$. Since $I_n \supset I_{n+1}$, we know that

$$a_n \le a_{n+1} \le b_{n+1} \le b_n \le b_1.$$

Hence E is nonempty and bounded above. Put $x^* = \sup E$. The inequalities imply $x^* \le b_m$ for each m. It is obvious that $a_m \le x^*$ for each m. Hence $x^* \in I_m$ for $m = 1, 2, 3, \ldots$. This implies that $\bigcap_{n=1}^{\infty} I_n \ne \varnothing$.

Since $\{G_\alpha\}$ covers I, there is some α such that $x^* \in G_\alpha$. Sine G_α is open, there exists $r > 0$ such that $|y - x^*| < r$ implies that $y \in G_\alpha$. Let n be a positive integer so that $2^{-n}\delta < r$. Then item (c) implies that $I_n \subset G_\alpha$, which contradicts to item (b). ◀

Justification: Cantor's intersection theorem

Fix a member K_1 of $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^{\mathsf{c}}$. Assume that $\bigcap_\alpha K_\alpha = \varnothing$. Then there is no point of K_1 that belongs to every K_α. This implies that $\{G_\alpha\}$ forms an open cover of K_1. Since K_1 is compact, there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K_1 \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

This relation means $K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \varnothing$. This contradicts to the hypothesis. ◀

Justification: charaterizations of compact sets in $\mathbb{R}$ and the Heine-Borel theorem

Item 1 $\Rightarrow$ Item 2:

Since K is bounded, there is a finite closed interval I such that $K \subset I$. By Theorem 2.19, I is compact. By item 2 of Proposition 2.18, we know that $K = K \cap I$ is compact.

Item 2 $\Rightarrow$ Item 3:

We prove this by contradiction.

Assume that an infinite subset E of a compact set K has no limit point in K. Then for each $q \in K$, there is a neighborhood V_q of q which contains at most one point of E (this point is actually q if $q \in E$). Since E is an infinite set, any finite subcollection of the collection $\{V_q\}$ cannot cover E. The same is true for K since $E \subset K$. This contradicts to the facts that $\{V_q\}$ is an open cover of K and K is compact.

Item 3 $\Rightarrow$ Item 1:

If K is not bounded, then for each integer $n > 0$, there exists a point $x_n \in K$ such that

$$|x_n| > n.$$

The set S consisting of these points $\{x_n\}$ is infinite and obviously has no limit point in $\mathbb{R}$. Hence S has no limit point in K. This contradicts to Item 3.

If K is not closed, then there is a point $x_0 \in \mathbb{R}$ which is a limit point of K but not a point in K. Hence, for each integer $n > 0$, there is a point $x_n \in K$ such that $|x_n - x_0| < 1/n$. By Proposition 2.10, we can further assume that any two points in $\{x_n\}$ are distinct. Hence, if let S be the set of $\{x_n\}$, then S is infinite, with x_0 being one of its limit points. However, S has no other limit point in $\mathbb{R}$, since for any $y \in \mathbb{R}$ with $y \neq x_0$,

$$\begin{aligned} |x_n - y| &\geq |x_0 - y| - |x_n - x_0| \\ &\geq |x_0 - y| - 1/n \geq \tfrac{1}{2}|x_0 - y| \end{aligned}$$

for all but finitely many n. This implies that x_0 is the only limit of S. Thus, S has no limit point in K, a contradiction to Item 3. ◀

Justification: the Bolzano-Weierstrass theorem

Let E be a bounded subset in $\mathbb{R}$. Then there is a finite closed interval I such that $E \subset I$. By Theorem 2.19, I is compact, and so E has a limit point in I, by Theorem 2.21.

Example: a simple limit

For any positive rational number p, the limit $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Solution Let $p = l/m$, where l, m are positive integers. For any $\varepsilon > 0$, take an integer N such that $N > \frac{1}{\varepsilon^{m/l}}$, then $n \geq N$ implies that

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^{l/m}} \leq \frac{1}{N^{l/m}} < \varepsilon.$$

Hence, by the definition of limit, we conclude that $\lim_{n\to\infty} \frac{1}{n^p} = 0$. ◀

Example: divergent sequences

(1) The sequence $\{(-1)^n\}$ is divergent.

This can be proved by contradiction. If $\lim_{n\to\infty}(-1)^n = \alpha$. Then, for $\varepsilon = 1$, there is an integer N such that $n \ge N$ implies that $|(-1)^n - \alpha| < 1$. It follows that, for any even integer n satisfying $n \ge N$,

$$|\alpha - 1| < 1 \Longrightarrow 0 < \alpha < 2.$$

Similarly, for any odd integer n satisfying $n \ge N$,

$$|\alpha + 1| < 1 \Longrightarrow -2 < \alpha < 0,$$

There is a contradiction because the two statements "$0 < \alpha < 2$" and "$-2 < \alpha < 0$" cannot both be true at the same time.

(2) The sequence $\{n\}$ is divergent.

This can also be proved by contradiction. If $\lim_{n\to\infty} n = \alpha$. Then, for $\varepsilon = 1$, there is an integer N such that $n \ge N$ implies that $|n - \alpha| < 1$. Thus, for any even integer n satisfying $n \ge N$, we have

$$|n - \alpha| < 1 \Longrightarrow -1 + \alpha < n < \alpha + 1.$$

There is a contradiction because the statement "$-1+\alpha < n < \alpha+1$" implies that there are infinitely many integers located between two numbers $-1+\alpha$ and $\alpha + 1$, which violates the archimedean property. ◀

Example: apply an equivalent statement of the definition

Assume $\lim_{n\to\infty} x_n = \alpha$, where $x_n \geq 0$, $n = 1, 2, \ldots$. Prove that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\alpha}$.

Solution Let $\varepsilon > 0$ be given.

Case 1: when $\lim_{n\to\infty} x_n = 0$.

Use the original definition:

There exists an integer N such that $n \geq N$ implies $|x_n - 0| < \varepsilon^2$. Thus, whenever $n \geq N$, we have

$$\left|\sqrt{x_n} - \sqrt{0}\right| = \sqrt{x_n} < \sqrt{\varepsilon^2} = \varepsilon,$$

so that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{0}$ by definition.

Use the equivalent statement:

For every $\varepsilon^* > 0$, there exists an integer N such that $n \geq N$ implies $|x_n - 0| < \varepsilon^*$. Thus, whenever $n \geq N$, we have

$$\left|\sqrt{x_n} - \sqrt{0}\right| = \sqrt{x_n} < \sqrt{\varepsilon^*}.$$

Hence, if we choose ε^* to satisfy $\varepsilon^* < \varepsilon^2$, then, whenever $n \geq N$, we have

$$\left|\sqrt{x_n} - \sqrt{0}\right| < \sqrt{\varepsilon^2} = \varepsilon,$$

so that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{0}$ by definition.

Case 2: when $\lim_{n\to\infty} x_n = \alpha > 0$.

Use the original definition:

There exists an integer N such that $n \geq N$ implies $|x_n - \alpha| < \sqrt{\alpha} \cdot \varepsilon$. Thus, whenever $n \geq N$, we have

$$\begin{aligned}\left|\sqrt{x_n} - \sqrt{\alpha}\right| &= \frac{|x_n - \alpha|}{\sqrt{x_n} + \sqrt{\alpha}} \\ &\leq \frac{|x_n - \alpha|}{\sqrt{\alpha}} < \frac{\sqrt{\alpha} \cdot \varepsilon}{\sqrt{\alpha}} = \varepsilon.\end{aligned}$$

Hence, by the definition of limit, we conclude that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\alpha}$.

Use the equivalent statement:

For every $\varepsilon^* > 0$, there exists an integer N such that $n \geq N$ implies $|x_n - \alpha| < \varepsilon^*$. Thus, whenever $n \geq N$, we have

$$\left|\sqrt{x_n} - \sqrt{\alpha}\right| = \frac{|x_n - \alpha|}{\sqrt{x_n} + \sqrt{\alpha}} \leq \frac{|x_n - \alpha|}{\sqrt{\alpha}} < \frac{\varepsilon^*}{\sqrt{\alpha}}.$$

Hence, if we choose ε^* to satisfy $\dfrac{\varepsilon^*}{\sqrt{\alpha}} < \varepsilon$, then, whenever $n \geq N$, we have

$$\left|\sqrt{x_n} - \sqrt{\alpha}\right| < \varepsilon,$$

so that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\alpha}$ by the definition of limit. ◀

Justification: properties of convergent sequence

Item 1 For any given $\varepsilon > 0$, there exist integers N, N' such that

$$\begin{aligned} n \geq N \quad &\text{implies} \quad |x_n - x| < \varepsilon/2, \\ n \geq N' \quad &\text{implies} \quad |x_n - x'| < \varepsilon/2. \end{aligned}$$

Hence, if $n \geq \max\{N, N'\}$, then

$$|x - x'| \leq |x_n - x| + |x_n - x'| \leq \varepsilon.$$

Since ε is arbitrary, we have $x = x'$.

Item 2 ($\Rightarrow$) Let $V = N_\varepsilon(x)$ be any neighborhood of x for some $\varepsilon > 0$. Since $x = \lim\limits_{n\to\infty} x_n$, there exists N such that $n \geq N$ implies $|x_n - x| < \varepsilon$. Thus $n \geq N$ implies $x_n \in V$.

($\Leftarrow$) Suppose every neighborhood of x contains all but finitely many of x_n. For any $\varepsilon > 0$, for this neighborhood $V = N_\varepsilon(x)$, there exists N such that $x_n \in V$ if $n \geq N$. Thus $|x_n - x| < \varepsilon$ if $n \geq N$. Hence $x_n \to x$ by the definition.

Item 3 For each positive integer n, there is a point $x_n \in E$ such that $|x_n - x| < 1/n$. Now we show that the sequence $\{x_n\}$ converges to x. In fact, for any $\varepsilon > 0$, we choose an integer N such that $N \cdot \varepsilon > 1$. Then for $n \geq N$, $|x_n - x| < 1/n \leq 1/N < \varepsilon$. Thus, by the definition, we conclude that $x_n \to x$.

Item 4 For $\varepsilon = 1$, there exists an integer N such that $n \geq N$ implies $|x_n - x| < 1$. Put

$$r = \max\{1, |x_1 - x|, \ldots, |x_{N-1} - x|\}.$$

Then $|x_n - x| \leq r$ for $n = 1, 2, 3, \ldots$. Hence, $\{x_n\}$ is bounded. ◀

Justification: operations on convergent sequences

Item 1 For any given $\varepsilon > 0$, there exist integers N_1, N_2 such that

$$\begin{aligned} n \geq N_1 \quad &\text{implies} \quad |x_n - x| < \varepsilon/2, \\ n \geq N_2 \quad &\text{implies} \quad |y_n - y| < \varepsilon/2. \end{aligned}$$

Denote $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves Item 1.

Item 2 The proof is trivial.

Item 3 For any given $\varepsilon > 0$, there exist integers N_1, N_2 such that

$$\begin{aligned} n \geq N_1 \quad &\text{implies} \quad |x_n - x| < \sqrt{\varepsilon}, \\ n \geq N_2 \quad &\text{implies} \quad |y_n - y| < \sqrt{\varepsilon}. \end{aligned}$$

Denote $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|(x_n - x)(y_n - y)| = |x_n - x| \cdot |y_n - y| < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon.$$

Thus, $\lim\limits_{n\to\infty} (x_n - x)(y_n - y) = 0$. Applying Items 1 and 2 to the identity

$$x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)$$

gives

$$\begin{aligned} \lim_{n\to\infty} (x_n y_n - xy) &= \lim_{n\to\infty} (x_n - x)(y_n - y) + \lim_{n\to\infty} x(y_n - y) + \lim_{n\to\infty} y(x_n - x) \\ &= 0 + x \cdot 0 + y \cdot 0 = 0. \end{aligned}$$

This proves Item 3.

Item 4 Since $x \neq 0$ and $\lim\limits_{n\to\infty} x_n = x$, we can choose an integer N_1 such that $|x_n - x| < \frac{1}{2}|x|$ when $n \geq N_1$. Thus, if $n \geq N_1$, then, by the triangle inequality

$$|x_n| \geq |x| - |x_n - x| > |x| - \tfrac{1}{2}|x| = \tfrac{1}{2}|x|.$$

For any given $\varepsilon > 0$, there is an integer N_2 such that

$$n \geq N_2 \quad \text{implies} \quad |x_n - x| < \tfrac{1}{2}|x|^2\varepsilon.$$

Denote $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \left|\frac{x_n - x}{x_n x}\right| < \frac{\frac{1}{2}|x|^2\varepsilon}{\frac{1}{2}|x| \cdot |x|} = \varepsilon.$$

This proves Item 4. ◀

Justification: a sequence converges iff every subsequence converges to the same limit

($\Rightarrow$) Suppose that the sequence $\{x_n\}$ converges to x and $\{x_{n_k}\}$ is any subsequence. Then, for every $\varepsilon > 0$, there exists N such that $|x_n - x| < \varepsilon$ whenever $n \geq N$. Thus, if $k \geq N$, then $n_k \geq n_N \geq N$ so that $|x_{n_k} - x| < \varepsilon$. This shows that every subsequence converges to x.

($\Leftarrow$) Suppose that every subsequence converges to x. Then the sequence $\{x_k\}$ converges x since $\{x_k\}$ is itself a subsequence of itself. ◀

Justification: compact implies every sequence has a convergent subsequence

Assume that $\{x_n\}$ is a sequence in K. If $\{x_n\}$ contains a subsequence $\{x_{n_i}\}$ such that $x_{n_1} = x_{n_2} = \cdots = x$, then it is clear that this subsequence converges to x, a point in K. Otherwise, the set E comprising all the terms in the sequence $\{x_n\}$ is infinite. By Theorem 2.21, the set E has a limit point $x \in K$. To construct a subsequence which converges to x, we first choose $x_{n_1} \in E$ such that $|x_{n_1} - x| < 1$. Having chosen $n_1, \ldots, n_{i-1}$, by Proposition 2.10, any neighborhood of x contains infinitely many points of $\{x_n\}$, we can choose an integer $n_i > n_{i-1}$ such that $|x_{n_i} - x| < 1/i$. Then $\{x_{n_i}\}$ converges to x. ◀

Justification: bounded sequence in $\mathbb{R}$ contains a convergent subsequence

Any bounded sequence in $\mathbb{R}$ lies in some finite closed interval, and every finite closed interval is compact by Theorem 2.19. The conclusion follows immediately from the theorem that every sequence of a compact set has a subsequence that converges to a point in the compact set. ◀

Justification: subsequential limits form a closed set

Assume that $\{x_n\}$ is a sequence in $\mathbb{R}$. Let E^* be the set of all subsequential limits of $\{x_n\}$ and let y be a limit point of E^*. We will show that $y \in E^*$, so that E^* is a closed subset in $\mathbb{R}$.

If E^* contains only one point, then E^* is obviously closed. Otherwise, choose n_1 so that $x_{n_1} \neq y$, and put $\delta = |x_{n_1} - y|$. Suppose $n_1, \ldots, n_{i-1}$ are chosen. Since y is a limit point of E^*, there is an $z \in E^*$ such that $|y - z| < 2^{-i}\delta$. Because $z \in E^*$, there is an $n_i > n_{i-1}$ such that $|x_{n_i} - z| < 2^{-i}\delta$. Thus,

$$|x_{n_i} - y| \leq |y - z| + |x_{n_i} - z| < 2^{1-i}\delta.$$

Hence we obtain a subsequence $\{x_{n_i}\}$ that converges to y. Thus, $y \in E^*$. ◀

Justification: convergence of Cauchy sequences

($\Rightarrow$) Suppose that $x_n \to x$. Then for every $\varepsilon > 0$, there is an integer N such that $|x_n - x| < \varepsilon/2$ if $n \geq N$. Hence, if $n, m \geq N$, then
$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \varepsilon.$$
Thus $\{x_n\}$ is a Cauchy sequence.

($\Leftarrow$) Suppose $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$. We prove that $\{x_n\}$ converges to some $x \in \mathbb{R}$ by completing the following steps:

1. The sequence $\{x_n\}$ is bounded.
2. The sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ that converges to some $x \in \mathbb{R}$.
3. The sequence $\{x_n\}$ converges to x.

Step 1 Since $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$, for $\varepsilon = 1$, there is an integer N such that $|x_n - x_m| < 1$ if $n, m \geq N$. Thus, for $n \geq N$,
$$|x_n| \leq |x_n - x_N| + |x_N| < 1 + |x_N|.$$
Let $B = \max\{|x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\}$. Then, for $n = 1, 2, \ldots,$
$$|x_n| \leq B,$$
that is, $\{x_n\}$ is bounded.

Step 2 We know that $\{x_n\}$ is bounded from Step 1. By the corollary of Theorem 3.5, we know that there is a subsequence $\{x_{n_k}\}$ that converges to some point x in $\mathbb{R}$.

Step 3 From Step 2, for every $\varepsilon > 0$, there is an integer K such that $|x_{n_k} - x| < \varepsilon/2$ if $k \geq K$. For the same ε, there is an integer M such that $|x_n - x_m| < \varepsilon/2$ if $n, m \geq M$. Put $N = \max\{M, K\}$. If we take an index n_k with $k \geq N$, then we have $n_k \geq n_N \geq N$. Hence, for $n \geq N$, we have
$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \varepsilon.$$
that is, $x_n \to x$. ◀

Justification: the monotone convergence theorem

Item 1 Let $\{x_n\}$ be a monotonically increasing sequence. Assume that $\{x_n\}$ is nonempty and bounded above. By the least-upper-bound property of real numbers, $x = \sup\{x_n\}$ exists and is finite. For any given $\varepsilon > 0$, there exists an integer N such that $x - \varepsilon < x_N$, since otherwise $x - \varepsilon$ is an upper bound of $\{x_n\}$, which contradicts the definition of x. Because $\{x_n\}$ is monotonically increasing, and x is its upper bound, for every $n > N$, we have $x - \varepsilon < x_n \leq x < x + \varepsilon$. Thus, for every $n > N$, $|x_n - x| < \varepsilon$. Hence, by definition, the limit of $\{x_n\}$ is $x = \sup\{x_n\}$.

Item 2 The proof of Item 2 is similar to that of Item 1.

Item 3 ($\Rightarrow$) Suppose that a monotonic sequence converges. Then, by Item 1 of Proposition 3.2, the sequence is bounded.

($\Leftarrow$) Suppose that a sequence $\{x_n\}$ is bounded. If the sequence is monotonically increasing, then, by Item 1, $\{x_n\}$ converges to the supremum of the sequence. If the sequence is monotonically decreasing, then, by Item 2, $\{x_n\}$ converges to the infimum of the sequence. ◀

Example: upper and lower limits

Let $x_n = \dfrac{(-1)^n n}{n+1}$. Find the set E comprising all the subsequential limits (in $\overline{\mathbb{R}}$) of $\{x_n\}$, $\varlimsup\limits_{n\to\infty} x_n$ and $\varliminf\limits_{n\to\infty} x_n$.

Solution Since

$$\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} \frac{2k}{2k+1} = 1,$$
$$\lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} \frac{-(2k+1)}{2k+2} = -1,$$

we know that $E = \{-1, 1\}$. So,

$$\varlimsup_{n\to\infty} x_n = 1, \qquad \varliminf_{n\to\infty} x_n = -1.$$

◀

Example: upper and lower limits

Let $\{x_n\}$ be a sequence containing all rational numbers. Find the set E comprising all the subsequential limits in $\overline{\mathbb{R}}$ (the extended real numbers), $\overline{\lim_{n\to\infty}} x_n$ and $\underline{\lim_{n\to\infty}} x_n$.

Solution By Theorem 1.10, we know that every real number is a subsequential limit of $\{x_n\}$. The infinities, ∞ and $-\infty$, are also in E. Hence, $E = \mathbb{R} \cup \{\infty, -\infty\}$. It follows that

$$\overline{\lim_{n\to\infty}} x_n = \infty, \qquad \underline{\lim_{n\to\infty}} x_n = -\infty.$$

Justification: existence of the upper and the lower limits

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Denote E to be the set comprising all the subsequential limits in $\overline{\mathbb{R}}$ (the extended real numbers) of the sequence $\{x_n\}$. To show the existence of the upper and lower limits of the sequence, we consider the following three cases:

1. The sequence $\{x_n\}$ is bounded above, but unbounded below.
2. The sequence $\{x_n\}$ is bounded below, but unbounded above.
3. The sequence $\{x_n\}$ is bounded above and below.

Case 1 If the sequence is unbounded below, then it has a subsequence converging to $-\infty$, so that $\varliminf_{n\to\infty} x_n = -\infty$. Since the sequence is bounded above, by Proposition 2.15, we know that $\sup E$ is a finite number, and $\varlimsup_{n\to\infty} x_n = \sup E$.

Case 2 The proof is similar to that of Case 1.

Case 3 If $\{x_n\}$ is bounded, by Proposition 2.15, we know that both $\sup E$ and $\inf E$ are finite, and

$$\varlimsup_{n\to\infty} x_n = \sup E, \qquad \varliminf_{n\to\infty} x_n = \inf E.$$

Justification: characterization of the upper limit

Item 1 If $x^* = \infty$, then E is not bounded above. Hence $\{x_n\}$ is not bounded above. This implies that there is a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to \infty$. Thus, $x^* = \infty \in E$.

If x^* is a finite number, then E is bounded above. By Proposition 2.15, $x^* \in \overline{E}$. Proposition 3.6 says that the subsequential limits form a closed subset of $\mathbb{R}$, so that $\overline{E} = E$. Thus, $x^* \in E$.

If $x^* = -\infty$, then $x_* = -\infty$. Thus, E contains only one element $-\infty$. Hence, for any real number M, $x_n > M$ for at most finitely many n. This means that $\lim\limits_{n\to\infty} x_n = -\infty$, so that $-\infty = x^* \in E$.

Item 2 If $y > x^*$, and if there are infinitely many n such that $x_n \geq y$, then there exists $z \in E$ such that $z \geq y > x^*$. This contradicts to the definition of x^*.

Uniqueness:

Suppose that there are two numbers, p and q, which satisfy both Items 1 and 2, and suppose $p < q$. We choose a number x such that $p < x < q$. Since p satisfies Item 2, we have an integer N such that $n \geq N$ implies $x_n < p$. Hence $x_n < x$ for $n \geq N$. This implies that q cannot satisfy Item 1. ◀

Justification: order rules of upper and lower limits - Corollary

Let E be the set comprising all the subsequential limits of $\{x_n\}$ in $\overline{\mathbb{R}}$ (the extended real numbers), and let $x^* = \overline{\lim\limits_{n\to\infty}}\, x_n$, $x_* = \underline{\lim\limits_{n\to\infty}}\, x_n$.

(⇒) Assume that $\lim\limits_{n\to\infty} x_n = x$. Then $E = \{x\}$. Thus, $\overline{\lim\limits_{n\to\infty}}\, x_n = \sup E = x = \inf E = \underline{\lim\limits_{n\to\infty}}\, x_n$.

(⇐) Assume that $\overline{\lim\limits_{n\to\infty}}\, x_n = \underline{\lim\limits_{n\to\infty}}\, x_n = x$. To show $\lim\limits_{n\to\infty} x_n = x$, we consider the following three cases:

1. $x = \infty$.
2. $x = -\infty$.
3. x is finite.

Case 1 In this case, if $\lim\limits_{n\to\infty} x_n \neq \infty$, then for some $M > 0$, no matter how large N is, there is an integer $K \geq N$ such that $x_K < M$. Thus, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ satisfying $x_{n_i} < M$. It follows that $\underline{\lim\limits_{n\to\infty}}\, x_n \leq M$. This contradicts to the hypothesis $\underline{\lim\limits_{n\to\infty}}\, x_n = \infty$. Therefore, $\lim\limits_{n\to\infty} x_n = \infty = x$.

Case 2 The proof is similar to that of Case 1.

Case 3 In this case, for any given $\varepsilon > 0$, by Proposition 3.13, for $x + \varepsilon > x = x^*$, there is an integer N_1 such that $n \geq N_1$ implies $x_n < x + \varepsilon$. Similarly, for $x - \varepsilon < x = x_*$, there is an integer N_2 such that $n \geq N_2$ implies $x_n > x - \varepsilon$. Denote $N = \max\{N_1, N_2\}$. Then, we have that $n \geq N$ implies

$$x - \varepsilon < x_n < x + \varepsilon,$$

or equivalently, $|x_n - x| < \varepsilon$. Therefore, $\lim\limits_{n\to\infty} x_n = x$.

Consequently, under the assumption that $\overline{\lim\limits_{n\to\infty}}\, x_n = \underline{\lim\limits_{n\to\infty}}\, x_n = x$, we always have $\lim\limits_{n\to\infty} x_n = x$. ◀

Justification: order rules of upper and lower limits

We only show that $\overline{\lim_{n\to\infty}} x_n \leq \overline{\lim_{n\to\infty}} y_n$. The other inequality can be proved similarly.

Let $\overline{\lim_{n\to\infty}} x_n = x$. Then there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$. To prove the desired inequality, we consider the following three cases:

1. $x = \infty$.
2. $x = -\infty$.
3. x is finite.

Case 1 In this case, for every M, there exists an integer K such that $k \geq K$ implies $x_{n_k} \geq M$. Put $N = \max\{N_0, K\}$. Then, for $k \geq N$, we have $y_{n_k} \geq x_{n_k} \geq M$. This implies that $y_{n_k} \to \infty$. Thus, in this case, $\overline{\lim_{n\to\infty}} x_n = \overline{\lim_{n\to\infty}} y_n$.

Case 2 In this case, $\overline{\lim_{n\to\infty}} x_n = -\infty$. Apparently, we have

$$\overline{\lim_{n\to\infty}} x_n = -\infty \leq \overline{\lim_{n\to\infty}} y_n.$$

Case 3 In this case, since x is a finite real number, then for every $\varepsilon > 0$, there is an integer K such that $k \geq K$ implies $|x_{n_k} - x| < \varepsilon$. Put $N = \max\{N_0, K\}$. Then, for $k \geq N$, we have $x < x_{n_k} + \varepsilon < y_{n_k} + \varepsilon$. Thus, $\overline{\lim_{k\to\infty}} y_{n_k} \geq x - \varepsilon$. Since ε is arbitrary, we have $\overline{\lim_{k\to\infty}} y_{n_k} \geq x$. Hence,

$$\overline{\lim_{n\to\infty}} x_n = x \leq \overline{\lim_{k\to\infty}} y_{n_k} \leq \overline{\lim_{n\to\infty}} y_n.$$

Hence, the desired inequality holds in all three cases. ◀

Example: geometric series

Let a and r be complex, with $a \neq 0$. Show that the **geometric series** (幾何級數) $\sum_{k=1}^{\infty} ar^{k-1}$ is convergent if and only if $|r| < 1$.

Proof When $r \neq 1$, a partial sum of the series $\sum ar^{n-1}$ is

$$s_n = \sum_{k=1}^{n} ar^{k-1} = a \cdot \frac{1-r^n}{1-r}.$$

It is easy to see that the sequence $\{r^n\}$ converges if $|r| < 1$; diverges otherwise. Hence, when $r \neq 1$, the sequence $\{s_n\}$ is convergent if and only if $|r| < 1$.

When $r = 1$, a partial sum of the series $\sum ar^{n-1}$ is

$$s_n = \sum_{k=1}^{n} ar^{k-1} = a \cdot n.$$

Obviously, when $r = 1$, the sequence $\{s_n\}$ is divergent.

Therefore, the geometric infinite series $\sum_{k=1}^{\infty} ar^{k-1}$ is convergent if and only if $|r| < 1$. ◀

Example: telescoping series

(1) Show that the $\boxed{\textbf{telescoping series}}$ (裂項級數) $\sum_{k=1}^{\infty}(u_k - u_{k+1})$ converges if and only if $\{u_k\}$ converges.

(2) Find the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

Solution

(1) A partial sum of the series is

$$\begin{aligned} s_n &= \sum_{k=1}^{n}(u_k - u_{k+1}) \\ &= (u_1 - u_2) + (u_2 - u_3) + \cdots + (u_n - u_{n+1}) = u_1 - u_{n+1}. \end{aligned}$$

Obviously, the sequence $\{s_n\}$ is converges if and only if $\{u_n\}$ converges.

(2) The given series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is a telescoping series, since

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

It is easy to obtain a partial sum of the series

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} \\ &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1}. \end{aligned}$$

Hence, the sum of the series is $s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$ ◀

Example: p series

Show that the p**-series** $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Remark

- The most common way to study the convergence of the p-series is to use the so-called ***integral test***. However, since the justification of the test cannot be done before we formally introduce the Riemann integral, here we use the Cauchy's condensation test.

Cauchy's Condensation Test

Suppose that $\{a_n\}$ is a monotonically decreasing nonnegative sequence, i.e. $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum a_n$ converges if and only if the condensed series $\sum 2^n a_{2^n}$ converges. ▶

Proof If $p \leq 0$, by the Divergence Test, the p-series is divergent. If $p > 0$, one can apply the Cauchy's Condensation Test to the p-series. In fact, we have

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

It is obvious that $2^{1-p} < 1$ if and only if $p > 1$. The desired result on the p-series follows by that of the geometric series. ◀

Justification: Cauchy's condensation test

Let s_n and σ_n denote the partial sums of $\sum a_n$ and $\sum 2^n a_{2^n}$, respectively. Since $\{a_n\}$ is a nonnegative sequence, we know that both $\{s_n\}$ and $\{\sigma_n\}$ are monotonically increasing.

($\Rightarrow$) Suppose $\sum a_n$ converges. By Proposition 3.17, $\{s_n\}$ is bounded above, that is, there is some number M such that $s_n \leq M$ for all integer n. Since $\{a_n\}$ is monotonically decreasing, we have, for all integer n,

$$\begin{aligned}
\sigma_n &= a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \\
&= 2 \cdot (\tfrac{1}{2}a_1) + 2 \cdot (a_2) + 2 \cdot (2a_4) + \cdots + 2 \cdot (2^{n-1} a_{2^n}) \\
&\leq 2 \cdot (a_1) + 2 \cdot (a_2) + 2 \cdot (a_3 + a_4) + \cdots + 2 \cdot (a_{2^{n-1}+1} + \cdots + a_{2^n}) \\
&\leq 2 \cdot (a_1 + a_2 + a_3 + \cdots + a_{2^n}) = 2s_{2^n} \leq 2M.
\end{aligned}$$

Thus, the monotonically increasing sequence $\{\sigma_n\}$ is bounded above. By Theorem 3.10 (the monotone convergence theorem), the sequence $\{\sigma_n\}$ converges. Hence, the series $\sum 2^n a_{2^n}$ converges.

($\Leftarrow$) Suppose $\sum 2^n a_{2^n}$ converges. By Proposition 3.17, $\{\sigma_n\}$ is bounded above, that is, there is some number M such that $\sigma_n \leq M$ for all integer n. For every integer n, take k such that $n < 2^k$. Since $\{a_n\}$ is monotonically decreasing, we have

$$\begin{aligned}
s_n &= a_1 + a_2 + a_3 + \cdots + a_n \\
&\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\
&\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = \sigma_k \leq M.
\end{aligned}$$

Thus, the monotonically increasing sequence $\{s_n\}$ is bounded above. By Theorem 3.10, the sequence $\{s_n\}$ converges. Hence, the series $\sum a_n$ converges. ◀

Justification: the Cauchy criterion for convergence of series

($\Rightarrow$) Suppose that $\sum a_n$ is convergent. By item 1 of Theorem 3.8, $\{s_n\}$ is a Cauchy sequence. Thus, for every $\varepsilon > 0$ there is an integer K such that $|s_m - s_n| < \varepsilon$ if $n, m \geq K$. Notice that $\sum\limits_{k=n}^{m} a_k = s_m - s_{n-1}$. If we take $N = K + 1$, then we get that $\left|\sum\limits_{k=n}^{m} a_k\right| < \varepsilon$ if $m \geq n \geq N$.

($\Leftarrow$) Suppose that for every $\varepsilon > 0$, there is an integer N such that $\left|\sum\limits_{k=n}^{m} a_k\right| < \varepsilon$ if $m \geq n \geq N$. Since $\sum\limits_{k=n}^{m} a_k = s_m - s_{n-1}$, the sequence $\{s_n\}$ is a Cauchy sequence. By item 3 of Theorem 3.8, we know that $\{s_n\}$ is convergent, so that the series $\sum a_n$ is convergent. ◀

Example: divergence test

Show that the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+1}$ diverges.

Solution For the given series $\sum a_n$, we have $a_n = \frac{(-1)^n n}{n+1}$. Since, as $m \to \infty$,

$$a_{2m} = \frac{(-1)^{2m} 2m}{2m+1} = \frac{1}{1+1/(2m)} \to 1,$$

we know that the sequence $\{a_n\}$ has a subsequence whose limit is not 0. Thus, $\lim_{n\to\infty} a_n \neq 0$. Hence, by the Divergence Test, the given series is divergent. ◀

Example: harmonic series is divergent

Show that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Proof We can show that the partial sums $s_2, s_4, s_8, s_{16}, s_{32}, \dots$ become unbounded, so the sequence $\{s_n\}$ does not have a finite limit. For the first few terms, we see that

$$\begin{aligned}
s_2 &= 1 + \tfrac{1}{2}, \\
s_4 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2}, \\
s_8 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) = 1 + \tfrac{3}{2}, \\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \tfrac{1}{10} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) = 1 + \tfrac{4}{2}.
\end{aligned}$$

In general, we can prove by induction that for all positive integer n,

$$s_{2^n} \geq 1 + \frac{n}{2}.$$

In fact, we have already seen that the inequality holds for $n = 1$. Suppose that the inequality holds for some positive integer $n = k$. Then,

$$\begin{aligned}
s_{2^{k+1}} &= s_{2^k} + \left(\frac{1}{2^k + 1} + \frac{1}{2^k + 2} + \cdots + \frac{1}{2^k + 2^k}\right) \\
&\geq 1 + \frac{k}{2} + \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}\right) \\
&= 1 + \frac{k}{2} + 2^k \cdot \frac{1}{2^{k+1}} = 1 + \frac{k+1}{2}.
\end{aligned}$$

This shows that the desired inequality also holds for $n = k + 1$. Hence, the desired inequality holds for all positive integer n.

Consequently, $\{s_n\}$ does not have a finite limit. That is, the harmonic series diverges. ◀

Justification: convergence of series with nonnegative terms

Suppose that $\sum a_n$ is a series of real numbers with nonnegative terms, i.e. $a_n \geq 0$ for all n. Let $s_n = \sum_{k=1}^{n} a_k$ be the n-th partial sum of the series. Since $a_n \geq 0$ for all n, the sequence $\{s_n\}$ is monotonically increasing.

($\Rightarrow$) Suppose that $\sum a_n$ converges. If $\{s_n\}$ is not bounded, then it is not bounded above since it is monotonically increasing. Thus, the sequence $\{s_n\}$ must tend to ∞, so the series does not have a finite limit. Hence, $\{s_n\}$ must be bounded.

($\Leftarrow$) Suppose that $\{s_n\}$ is bounded. Then it is bounded above. Since it is monotonically increasing, by Theorem 3.10 (the Monotone Convergence Theorem), the sequence $\{s_n\}$ converges, so the series $\sum a_n$ converges.

◀

Justification: e is well-defined

The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ has all positive terms. Thus, by Theorem 3.10 (the Monotone Convergence Theorem), it suffices to show that the sequence $\{s_n\}$ of the partial sums, $s_n = \sum_{k=0}^{n} \frac{1}{k!}$, is bounded above. Actually, when $n \geq 2$, we have

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 + \frac{1}{1} - \frac{1}{n} < 3. \end{aligned}$$

To show that the sum of the series falls between 2 and 3, in the same manner, for $n \geq 5$, we have

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5 \cdot 4} + \frac{1}{6 \cdot 5} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{4} - \frac{1}{n} = \frac{71}{24} - \frac{1}{n}. \end{aligned}$$

Obviously, $s_n \geq 1 + 1 + \frac{1}{2!}$. Thus, for $n \geq 5$,

$$\tfrac{5}{2} \leq s_n \leq \tfrac{71}{24}.$$

By Propositions 3.14 and its corollary, we get

$$\tfrac{5}{2} \leq s \leq \tfrac{71}{24}.$$

Therefore, $2 < s < 3$. ◀

Justification: Euler's number as the limit of a sequence

Let
$$s_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad t_n = \left(1 + \frac{1}{n}\right)^n.$$
By the binomial formula, for every positive integer n,
$$\begin{aligned} t_n &= 1 + \binom{n}{1} \cdot \left(\frac{1}{n}\right) + \binom{n}{2} \cdot \left(\frac{1}{n}\right)^2 + \binom{n}{3} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \binom{n}{n} \cdot \left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{n(n-1)}{2!} \cdot \left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \frac{n(n-1)(n-2)\cdots 2 \cdot 1}{n!} \cdot \left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$
Hence, $t_n \le s_n$ for every positive integer n, so that
$$\overline{\lim_{n\to\infty}}\, t_n \le \overline{\lim_{n\to\infty}}\, s_n = \lim_{n\to\infty} s_n = e,$$
by Propositions 3.14 and its corollary.

Next, if $n \ge m$, we see that
$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$
If letting $n \to \infty$ while keeping m fixed, by Propositions 3.14 and its corollary, then we get
$$\underline{\lim_{n\to\infty}}\, t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!},$$
that is,
$$s_m \le \underline{\lim_{n\to\infty}}\, t_n.$$
Letting $m \to \infty$, we finally get
$$e \le \underline{\lim_{n\to\infty}}\, t_n.$$
Hence, we get
$$e \le \underline{\lim_{n\to\infty}}\, t_n \le \overline{\lim_{n\to\infty}}\, t_n \le e.$$
It follows that $\lim_{n\to\infty} t_n = e$. ◀

Justification: Comparison Test

Item 1 Since $\sum b_n$ converges, by the Cauchy criterion, for any given $\varepsilon > 0$, there exists an integer N_1 such that

$$\left|\sum_{k=n}^{m} b_k\right| < \varepsilon,$$

if $m \geq n \geq N_1$. Denote $N = \max\{N_0, N_1\}$. Thus, if $m \geq n \geq N$, then

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} b_k < \varepsilon.$$

Hence, by the Cauchy criterion again, $\sum a_n$ converges.

Item 2 Item 2 is the contrapositive statement of Item 1 when $\{a_n\}$ is a sequence of real numbers. ◄

Justification: Root Test

Item 1 If $\alpha < 1$, we can choose β such that $\alpha < \beta < 1$. By Item 2 of Theorem 3.13, there is an integer N such that $n \geq N$ implies $\sqrt[n]{|a_n|} < \beta$, or $|a_n| < \beta^n$. Since $0 < \beta < 1$, $\sum \beta^n$ converges. By Theorem 3.20 (the Comparison Test), $\sum a_n$ converges.

Item 2 If $\alpha > 1$, then, by Theorem 3.13, there is a sequence $\{n_k\}$ such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Hence $|a_n| > 1$ for infinitely many values of n, so that $\lim a_n \neq 0$. By the Divergence Test, $\sum a_n$ diverges.

Item 3 Consider two series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. It is easy to see that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{1}{n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1,$$

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{1}{n^2}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^2} = 1.$$

We know that $\sum \frac{1}{n}$ is a divergent p-series ($p = 1$), while $\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2$). These two series demonstrate that the test is inconclusive about the convergence of $\sum a_n$ if $\alpha = 1$.

Justification: Ratio Test

Item 1 If $\overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < 1$, we choose β such that $\overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$. By Proposition 3.13, there is an integer N, such that $n \geq N$ implies

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta.$$

Hence, for $n \geq N$ and any positive integer p, we have

$$|a_{n+p}| < \beta |a_{n+p-1}| < \beta^2 |a_{n+p-2}| < \cdots < \beta^{n+p-N} |a_N|.$$

That is,

$$|a_n| < |a_N| \beta^{-N} \cdot \beta^n, \qquad n \geq N.$$

This implies that $\sum a_n$ converges by Theorem 3.20 (the Comparison Test).

Item 2 If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq N_0$, then $|a_{n+1}| \geq |a_n|$ for $n \geq N_0$, so that $\lim_{n\to\infty} a_n \neq 0$. By the Divergence Test, $\sum a_n$ diverges. ◀

Example: the Root Test and the Ratio Test

Consider the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots.$$

Direct calculations give

$$\begin{aligned}
\varliminf_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0,\\
\varlimsup_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{1}{2}\left(\frac{3}{2}\right)^n = \infty,\\
\varlimsup_{n\to\infty}\sqrt[n]{a_n} &= \lim_{n\to\infty}\left(\frac{1}{3^n}\right)^{1/(2n)} = \frac{1}{\sqrt{3}},\\
\varlimsup_{n\to\infty}\sqrt[n]{a_n} &= \lim_{n\to\infty}\left(\frac{1}{2^n}\right)^{1/(2n)} = \frac{1}{\sqrt{2}}.
\end{aligned}$$

We see that the root test indicates convergence, but the ratio test does not apply. ◀

Example - the Root Test and the Ratio Test

For each of the following series, determine whether converges or diverges.

(1) $\sum\limits_{n=1}^{\infty}\left(\dfrac{3n-1}{4n+1}\right)^n$.

(2) $\sum\limits_{n=1}^{\infty}\dfrac{b^n}{n!}$, where $b>0$.

Solution

(1) The given series $\sum\limits_{n=1}^{\infty}\left(\dfrac{3n-1}{4n+1}\right)^n$ has all positive terms. It is easy to have

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{3n-1}{4n+1}\right)^n}=\lim_{n\to\infty}\frac{3n-1}{4n+1}=\frac{3}{4}<1.$$

By the Root Test, the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{3n-1}{4n+1}\right)^n$ is convergent.

(2) For any $b>0$, the given series $\sum\limits_{n=1}^{\infty}\dfrac{b^n}{n!}$ has all positive terms. Since,

$$\lim_{n\to\infty}\frac{\dfrac{b^{n+1}}{(n+1)!}}{\dfrac{b^n}{n!}}=\lim_{n\to\infty}\frac{b}{n+1}=0,$$

by the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{b^n}{n!}$ converges for any positive b.

Justification: upper-lower limit relations between the Root Test and the Ratio Test

In the sequence of the inequalities, the second is trivial. We only show the third, since the first can be shown similarly.

Put $\alpha = \overline{\lim_{n\to\infty}} \dfrac{c_{n+1}}{c_n}$.

If $\alpha = \infty$, there is nothing to prove.

If α is finite or $-\infty$, we fix a number $\beta > \alpha$. By Proposition 3.13, there is an integer N such that $n \geq N$ implies

$$\frac{c_{n+1}}{c_n} \leq \beta.$$

The inequality implies that for $n \geq N$ and any positive integer p

$$c_{n+p} \leq \beta c_{n+p-1} \leq \beta^2 c_{n+p-2} \leq \cdots \leq \beta^{n+p-N} c_N.$$

Hence, for $n \geq N$,

$$c_n \leq c_N \beta^{-N} \cdot \beta^n,$$

or

$$\sqrt[n]{c_n} \leq \sqrt[n]{c_N \beta^{-N}} \cdot \beta.$$

Thus,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{c_n} \leq \beta.$$

Since the last inequality holds for every $\beta > \alpha$, thus, we have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{c_n} \leq \alpha = \overline{\lim_{n\to\infty}} \frac{c_{n+1}}{c_n}.$$

◀

Example: compute the root by using the ratio

Compute the limit $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}$.

Solution Let $c_n = \frac{n^n}{n!}$. Then

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Thus, we get

$$\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = e,$$

that is, $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

◀

Example: the Root Test vs the Ratio Test

For each of the following series, determine whether converges or diverges.

(1) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$.

(2) $\sum_{n=1}^{\infty}\frac{n!}{n^n}$.

Solution

(1) The given series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ has all positive terms. Since

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}}=\frac{1}{e}<1.$$

By the Root Test, the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges.

(2) All terms in the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ are positive as well. Since

$$\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}}=\frac{1}{e}<1.$$

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges.

Remark

- For each of the above series, you might as well try a different test!

Justification: Dirichlet's Test, Abel's Test, and Alternating Series Test

Dirichlet's Test:

Remark

- A key step to prove Dirichlet's Test is to convert a finite sum by a so-called "summation-by-parts formula".

summation-by-parts formula

Given two sequence $\{a_n\}$, $\{b_n\}$, put

$$A_n = \begin{cases} \sum_{k=0}^{n} a_k, & \text{if } n \geq 0, \\ 0, & \text{if } n = -1. \end{cases}$$

Then, for $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Since the partial sums A_n of $\sum a_n$ form a bounded sequence, there is a positive number M such that $|A_n| \leq M$ for all n. Because the sequence $\{b_n\}$ is monotonically decreasing and $\lim_{n\to\infty} b_n = 0$, we know that $\{b_n\}$ is a nonnegative sequence, and for any given $\varepsilon > 0$, there is a positive integer N such that $b_N < (\varepsilon/2M)$. Thus, for $p, q \geq N$, we have

$$\begin{aligned} \left| \sum_{n=p}^{q} a_n b_n \right| &= \left| \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right| \\ &\leq \sum_{n=p}^{q-1} |A_n| \, |b_n - b_{n+1}| + |A_q| \, |b_q| + |A_{p-1}| \, |b_p| \\ &\leq M \left(\sum_{n=p}^{q-1} |b_n - b_{n+1}| + |b_q| + |b_p| \right) \\ &= M \left(\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p \right) \\ &= 2M b_p \leq 2M b_N < \varepsilon. \end{aligned}$$

Hence, the series $\sum a_n b_n$ satisfies the Cauchy criterion, so that the series $\sum a_n b_n$ converges. ◀

Abel's Test:

Suppose $\{b_n\}$ is monotonically decreasing, otherwise we use $-b_n$ to replace b_n in the proof. Since $\{b_n\}$ is also bounded, by Theorem 3.10, it converges. Denote the limit to be b. Then the series $\sum a_n(b_n - b)$ satisfies the conditions in Dirichlet's Test. Hence $\sum a_n(b_n - b)$ converges. The series $\sum a_n b_n$ converges follows from $\sum a_n b_n = \sum a_n(b_n - b) + b \sum a_n$. ◀

Alternating Series Test:

Put $a_n = (-1)^{n+1}$, $b_n = |c_n|$. Then the series $\sum c_n$ is $\sum a_n b_n$, and Dirichlet's Test applies. ◀

Justification: summation-by-parts formula

The elementary calculations give

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n \\ &= \left(\sum_{n=p}^{q-1} A_n b_n + A_q b_q\right) - \left(\sum_{n=p}^{q-1} A_n b_{n+1} - A_{p-1} b_p\right) \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.\end{aligned}$$

◀

Example: the alternating harmonic series

Determine the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ whether converges or diverges.

Solution For the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, denote

$$a_n = (-1)^{n-1}, \quad b_n = \frac{1}{n}, \quad n = 1, 2, 3, \ldots.$$

Then we have the following:

(1) Since the terms a_n are ± 1, the partial sums A_n of $\sum a_n$ satisfy $|A_n| \le 1$, so bounded.

(2) The sequence $\{b_n\}$ is monotonically decreasing and $\lim_{n\to\infty} b_n = 0$.

Thus, we can apply Dirichlet's Test to conclude that the series converges.

Remark

- Although one can use the Alternating Series Test to prove the convergence of the alternating harmonic series, the application of Abel's Test to the series may not be straightforward.

Example: Abel's Test

Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(1+\frac{1}{n}\right)^n$.

Solution In the given series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(1+\frac{1}{n}\right)^n}{n}$, denote

$$a_n = \frac{(-1)^{n-1}}{n}, \quad b_n = \left(1+\frac{1}{n}\right)^n, n = 1, 2, 3, \ldots.$$

Then we have the following:

(1) The series $\sum a_n$ is the alternating harmonic series that is convergent.

(2) The sequence $\{b_n\}$ is bounded, since $\lim_{n\to\infty} b_n = e$ exists. To show that the sequence $\{b_n\}$ is monotonic, we apply the binomial formula to have, for every positive integer n,

$$\begin{aligned} b_n &= 1 + \binom{n}{1}\cdot\left(\frac{1}{n}\right) + \binom{n}{2}\cdot\left(\frac{1}{n}\right)^2 + \binom{n}{3}\cdot\left(\frac{1}{n}\right)^3 + \cdots + \binom{n}{n}\cdot\left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{n(n-1)}{2!}\cdot\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\cdot\left(\frac{1}{n}\right)^3 + \cdots + \frac{n(n-1)(n-2)\cdots 2\cdot 1}{n!}\cdot\left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \end{aligned}$$

The last expression is a summation of $n+1$ positive terms. Obviously, each of these terms gets larger if replacing n by $n+1$, while b_{n+1} yields a summation of $n+2$ positive terms. Thus, $b_n \le b_{n+1}$, so the sequence $\{b_n\}$ is monotonically increasing.

Hence, we can apply Abel's Test to conclude that the series converges.

Remark

- It is not a straightforward task to apply Dirichlet's Test to this series.

Justification: absolute convergence implies convergence

The result follows from

$$\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k|$$

and the Cauchy criterion. ◀

Justification: convergence of Cauchy product of two infinite series

Without loss of generality, we assume that $\sum a_n$ converges absolutely. Put

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k,$$

and denote $\beta_n = B_n - B$. We re-write the partial sum of $\sum c_n$ as follows:

$$\begin{aligned} C_n &= a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0) \\ &= a_0B_n + a_1B_{n-1} + \cdots + a_nB_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_nB + a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0. \end{aligned}$$

Since $\lim_{n\to\infty} A_n = A$, if we can prove that, as $n \to \infty$,

$$\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0 \to 0,$$

then we have $\lim_{n\to\infty} C_n = AB$.

Put $\alpha = \sum_{n=0}^{\infty} |a_n|$. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} \beta_n = 0$, there is an integer N such that $n \geq N$ implies $|\beta_n| < \varepsilon$. Thus, for $n \geq N$, we have

$$\begin{aligned} |\gamma_n| &\leq |\beta_0a_n + \cdots + \beta_Na_{n-N}| + |\beta_{N+1}a_{n-N-1} + \cdots + \beta_na_0| \\ &\leq |\beta_0a_n + \cdots + \beta_Na_{n-N}| + |\beta_{N+1}|\,|a_{n-N-1}| + \cdots + |\beta_n|\,|a_0| \\ &< |\beta_0a_n + \cdots + \beta_Na_{n-N}| + \varepsilon\alpha. \end{aligned}$$

Keeping N fixed and letting $n \to \infty$, since $\lim_{k\to\infty} a_k = 0$, we get

$$\varlimsup_{n\to\infty} |\gamma_n| \leq \varepsilon\alpha.$$

Because ε is arbitrary, it follows that $\varlimsup_{n\to\infty} |\gamma_n| = 0$, so that $\lim_{n\to\infty} \gamma_n = 0$. ◀

Example: divergent Cauchy product

The series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$
converges by Dirichlet's Test. Its absolute value series
$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$
diverges since it is a p-series with $p = \frac{1}{2}$. Thus, the series $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{\sqrt{n+1}}$ converges conditionally.

Consider the Cauchy product of this series with itself:
$$\sum_{n=0}^{\infty} c_n = 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{4}} + \frac{1}{\sqrt{4}}\right) + \cdots.$$
We have that, for $n = 0, 1, 2, \ldots,$
$$c_n = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \textstyle\sum_{k=0}^{n} \displaystyle\frac{1}{\sqrt{(n-k+1)(k+1)}}.$$
Since
$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2,$$
we have
$$|c_n| \geq \textstyle\sum_{k=0}^{n} \displaystyle\frac{2}{n+2} = \frac{2(n+1)}{n+2},$$
so that $\lim\limits_{n\to\infty} c_n \neq 0$. Hence, by the Divergence Test, the series $\sum c_n$ diverges. ◀

Justification: limit of function in term of sequential limits

(⇒) Suppose $\lim_{x\to p} f(x) = q$, and suppose a sequence $\{p_n\}$ is in $E\setminus\{p\}$ such that $\lim_{n\to\infty} p_n = p$. Then for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - q| < \varepsilon$ if $x \in E$ and $0 < |x - p| < \delta$. Also, there exists an integer N such that $n \geq N$ implies $0 < |p_n - p| < \delta$. Thus, for $n \geq N$, we have $|f(p_n) - q| < \varepsilon$. This means that $\lim_{n\to\infty} f(p_n) = q$.

(⇐) Suppose $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in $E\setminus\{p\}$ such that $\lim_{n\to\infty} p_n = p$. If $\lim_{x\to p} f(x) \neq q$, then there exists some $\varepsilon > 0$ such that for every $\delta > 0$, there exists a point $x \in E$ for which $|f(x) - q| \geq \varepsilon$ but $0 < |x - p| < \delta$. Take $\delta_n = 1/n$, $n = 1, 2, 3, \ldots$, we obtain a sequence $\{x_n\}$ in $E\setminus\{p\}$, with $\lim_{n\to\infty} x_n = p$, such that $|f(x_n) - q| \geq \varepsilon$. Thus $\lim_{n\to\infty} f(x_n) \neq q$. This contradicts to the hypothesis. ◀

Justification: uniqueness of limit of function

The uniqueness follows from the uniqueness of limit for sequences (item 1 of Proposition 3.2) and Proposition 4.12. ◀

Justification: arithmetic operations on limits of functions

In view of Proposition 4.12, these assertions follow immediately from the analogous properties of sequences (Proposition 3.3). ◀

Justification: every function is continuous at isolated points

If p is an isolated point of E, then there exists a number $\delta > 0$ such that the only point $x \in E$ for which $|x - p| < \delta$ is $x = p$. Thus, for every $\varepsilon > 0$, if we take this δ, then

$$|f(x) - f(p)| = 0 < \varepsilon,$$

for all points $x \in E$ for which $|x - p| < \delta$. ◀

Example: continuity of the absolute value function on $\mathbb{R}$

Let $f(x) = |x|$. The function $f\colon \mathbb{R} \to \mathbb{R}$ is the so-called **absolute value function** on $\mathbb{R}$. Show that f is continuous at every point.

Solution The continuity of the norm function f follows from the triangle inequality

$$|f(x) - f(y)| = \big||x| - |y|\big| \leq |x - y|.$$

◀

Justification: continuity of arithmetic operations

At isolated points of E there is nothing to prove. At limit points, the result follows from Proposition 4.13 and the equalities $\lim_{x\to p} f(x) = f(p)$ and $\lim_{y\to q} g(y) = g(q)$.

◀

Example: continuity of polynomials and rational functions on $\mathbb{R}$

(1) Show that every polynomial
$$P(x) = \sum_{k=1}^{n} c_k x^k, \qquad x \in \mathbb{R},$$
is continuous on $\mathbb{R}$.

(2) Show that every rational function, that is, every quotient of two polynomials, is continuous on $\mathbb{R}$ wherever the denominator is different from zero.

Proof

(1) If $x \in \mathbb{R}$, the *identity* functions ϕ defined by
$$\phi(x) = x$$
is continuous on $\mathbb{R}$, since
$$|\phi(x) - \phi(y)| \leq |x - y|.$$
By applying Proposition 4.18 repeatedly, we know that the power function x^k is continuous on $\mathbb{R}$. The same is true of constant multiples to the power function, since constants are evidently continuous. Hence, for real coefficients c_k with k being nonnegative integers, every polynomial P, given by
$$P(x) = \sum_{k=0}^{n} c_k x^k$$
is continuous on $\mathbb{R}$.

(2) The result follows from part (1) and Proposition 4.18. ◀

Justification: continuity of composition

Denote $h = g \circ f$. Since g is continuous at $f(p)$, for every $\varepsilon > 0$, there exists $\eta > 0$ such that $|y - f(p)| < \eta$ and $y \in f(E)$ imply $|g(y) - g(f(p))| < \varepsilon$. Since f is continuous at p, there exists $\delta > 0$ such that $|x - p| < \delta$ and $x \in E$ implies $|f(x) - f(p)| < \eta$. Hence, if $|x - p| < \delta$ and $x \in E$, then we have

$$|h(x) - h(p)| = |g(f(x)) - g(f(p))| < \varepsilon.$$

Therefore, h is continuous at p. ◀

Justification: characterization of continuity

Item 1 ($\Rightarrow$) Suppose f is continuous on E and V is an open set in $\mathbb{R}$. We have to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. Let $p \in f^{-1}(V)$. Then $f(p) \in V$. Since V is open in $\mathbb{R}$, there exists $\varepsilon > 0$ such that $y \in V$ if $|f(p) - y| < \varepsilon$. Since f is continuous at p, there exists $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ if $|x - p| < \delta$. Thus, if $|x - p| < \delta$, then $f(x) \in V$, or $x \in f^{-1}(V)$. This means that p is an interior point of $f^{-1}(V)$.

($\Leftarrow$) Suppose $f^{-1}(V)$ is open in E for every open set V in $\mathbb{R}$. Fix $p \in E$. For $\varepsilon > 0$, let V be the set of all $y \in \mathbb{R}$ such that $|y - f(p)| < \varepsilon$. For this open set V, $f^{-1}(V)$ is open in E. Since $p \in f^{-1}(V)$, there exists $\delta > 0$ such that $x \in f^{-1}(V)$ if $|x - p| < \delta$. But $x \in f^{-1}(V)$ implies $f(x) \in V$, or $|f(x) - f(p)| < \varepsilon$. Thus, if $|x - p| < \delta$, we have $|f(x) - f(p)| < \varepsilon$.

Item 2 Let us show that for every $E \subset \mathbb{R}$,

$$f^{-1}(E^c) = [f^{-1}(E)]^c.$$

In fact, if $x \in f^{-1}(E^c)$, then $f(x) \in E^c$, or $f(x) \notin E$. This implies $x \notin f^{-1}(E)$, or $x \in [f^{-1}(E)]^c$. Thus, $f^{-1}(E^c) \subset [f^{-1}(E)]^c$. Conversely, if $x \in [f^{-1}(E)]^c$, then $x \notin f^{-1}(E)$. Hence $f(x) \notin E$, or $f(x) \in E^c$. This indicates that $x \in f^{-1}(E^c)$. Thus, $[f^{-1}(E)]^c \subset f^{-1}(E^c)$.

Now the result follows from Item 1 and the fact that E is closed if and only if E^c is open. ◀

Example: images of continuous function

(1) Consider the function $f\colon (0,1) \to \mathbb{R}$ defined by $f(x) = x$. It is easy to see that $E = (0,1)$ is a closed subset in $(0,1)$, but $f(E) = (0,1)$ is not a closed subset in $\mathbb{R}$.

(2) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. It is easy to see that $E = (-1,\infty)$ is an open subset in $\mathbb{R}$, but $f(E) = [0,\infty)$ is not an open subset in $\mathbb{R}$.

(3) Consider the function $f\colon (0,1) \to \mathbb{R}$ defined by $f(x) = 1/x$. It is easy to see that $E = (0,1)$ is a bounded subset in $(0,1)$, but $f(E) = (1,\infty)$ is not a bounded subset in $\mathbb{R}$. ◀

Justification: continuous mapping maps compact space to compact space

Let $\{V_\alpha\}$ be an open cover of $f(K)$. Since f is continuous, by Theorem 4.17, each $f^{-1}(V_\alpha)$ is open. By

$$f(K) \subset \bigcup_\alpha V_\alpha,$$

we know that

$$K \subset f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha f^{-1}(V_\alpha).$$

In other words, $\{f^{-1}(V_\alpha)\}$ is an open cover of K. Since K is compact, there are finitely many indices, say $\alpha_1, \ldots, \alpha_n$, such that

$$K \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}).$$

It is known that $f(f^{-1}(E)) \subset E$. Thus, we have

$$f(K) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Hence, $f(K)$ is compact.

Remark

- For any $E \subset \mathbb{R}$, the set relation holds: $f(f^{-1}(E)) \subset E$.

  Let $y \in f(f^{-1}(E))$. By the definition of $f(f^{-1}(E))$, there exists an $x \in f^{-1}(E)$ such that $y = f(x)$. Since $x \in f^{-1}(E)$ implies $f(x) \in E$, it follows that $y = f(x) \in E$.
- For any $E \subset \mathbb{R}$, the set relation holds: $f^{-1}(f(E)) \supset E$.

  Let $x \in E$. Then $f(x) \in f(E)$. By the definition of $f^{-1}(f(E))$, $x \in f^{-1}(f(E))$.

Example: continuous function not bounded on noncompact set

Suppose that E is a noncompact subset in $\mathbb{R}$.

Case 1: The subset E is unbounded. In this case, consider the function f defined by $f(x) = x$. It is easy to see that f is continuous on E, but f is unbounded.

Case 2. The subset E is bounded. In this case, since E is noncompact, it is not closed. Thus, there is a limit point x_0 of E which is not a point of E. Consider the function f defined by $f(x) = \dfrac{1}{x - x_0}$. Then f is continuous on E, but it is unbounded. ◀

Justification: Extreme Value Theorem

By Theorem 4.19, $f(K)$ is compact, so it is closed and bounded. By Proposition 2.15, we know that $\sup_{x\in K} f(x) \in \overline{f(K)} = f(K)$.

The proof is analogous to the infimum.

Example: bounded continuous function has no maximum on noncompact set

Suppose that E is a noncompact subset in $\mathbb{R}$.

Case 1: The subset E is unbounded. In this case, consider the function g defined by $g(x) = \dfrac{x^2}{1+x^2}$. It is easy to see that g is bounded continuous on E, and $\sup\limits_{x \in E} g(x) = 1$. Since $g(x) < 1$ for all $x \in E$, the function g has no maximum on E.

Case 2. The subset E is bounded. In this case, since E is noncompact, it is not closed. Thus, there is a limit point x_0 of E which is not a point of E. Consider the function g defined by $g(x) = \dfrac{1}{1+(x-x_0)^2}$. Then g is bounded continuous on E, and $\sup\limits_{x \in E} g(x) = 1$. Since $g(x) < 1$ for all $x \in E$, the function g has no maximum on E. ◀

Justification: continuity of bijection

To show that f^{-1} is continuous, by Theorem 4.17, we show that $f(V)$ is open in Y if V is open in K.

In fact, for V^c being closed in K, we know V^c is compact by item 2 of Proposition 2.18, since K is compact. Hence $f(V^c)$ is compact by Theorem 4.19, since f is continuous. By item 1 of Proposition 2.18, $f(V^c)$ is closed. Since f is one-to-one and onto, the complement of $f(V)$ is $f(V^c)$. Thus $f(V)$ is open.

● Although it is generally recommended to state and prove results in a logical order, we have cited Theorem 4.19 before proving it and placed item 3 of Proposition 4.18 after it. This was done to group relevant results together. ◀

Example: continuous bijective mapping does not have continuous inverse

Let $E = (-1, 0] \cup [1, 2]$. Consider the function $f\colon E \to [0, 4]$, where $f(x) = x^2$ for $x \in E$. It is easy to check that f is continuous and bijective. However, the inverse is not continuous, since $[0, 4]$ is connected but E is not. ◀

Example: delta dependence on epsilon

The function $f\colon (0,1) \to \mathbb{R}$, defined by $f(x) = \dfrac{1}{x}$, is continuous at every point in $(0,1)$, so it is continuous on $(0,1)$. For any $x, y \in (0,1)$, since

$$|f(x) - f(y)| = \frac{|x-y|}{xy}.$$

It is easy to see that there is not single value δ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in (0,1)$ with $|x-y| < \delta$. This is because one can choose x, y sufficiently close to 0 such that $|f(x) - f(y)|$ is larger than any finite number.

In fact, for any $\varepsilon > 0$, with $\varepsilon < 1$, no matter how small $\delta > 0$ is, we can choose n sufficient large such that $n > \max\{1, 1/(2\delta)\}$. Then, for $x = 1/(2n)$, $y = 1/n$, we have $|x-y| = 1/(2n) < \delta$, but

$$|f(x) - f(y)| = |2n - n| = n > \varepsilon,$$

so that f is not uniformly continuous on $(0,1)$. ◂

Justification: the relationship between continuity and uniform continuity

Item 1 This is evident from the definitions of continuity and uniform continuity.

Item 2 Let $\varepsilon > 0$ be given. Since f is continuous on E, for each $p \in E$, there is an associated $\phi(p) > 0$, such that $q \in E$ and $|p - q| < \phi(p)$ imply $|f(p) - f(q)| < \frac{1}{2}\varepsilon$. Let $J(p)$ be the set of all $q \in E$ for which $|p - q| < \frac{1}{2}\phi(p)$. Thus, the collection of all sets $J(p)$ forms an open cover of E. Since E is compact, there is a finite sets of points $p_1, \ldots, p_n$ in E, such that

$$E \subset J(p_1) \cup \cdots \cup J(p_n).$$

Put $\delta = \frac{1}{2}\min\{\phi(p_1), \ldots, \phi(p_n)\}$. Then $\delta > 0$.

For any two points $p, q \in E$, with $|p - q| < \delta$, we claim that $|f(p) - f(q)| < \varepsilon$. In fact, for the above finite cover, there is a point p_m, $1 \leq m \leq n$, such that $p \in J(p_m)$. Thus, $|p - p_m| < \frac{1}{2}\phi(p_m)$. This gives

$$|q - p_m| \leq |p - q| + |p - p_m| \leq \delta + \tfrac{1}{2}\phi(p_m) < \phi(p_m),$$

which implies

$$|f(p) - f(q)| \leq |f(p) - f(p_m)| + |f(q) - f(p_m)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

By the definition, f is uniformly continuous on E. ◀

Example: continuous function is not uniformly continuous on noncompact set

Suppose that E is a noncompact subset in $\mathbb{R}$.

Case 1: The subset E is unbounded. In this case, consider the function h defined by $h(x) = x^2$. It is easy to see that h is continuous on E.

To see that the function h is not uniformly continuous on E. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary. Taking t close enough to x, we can then make the difference $|h(t) - h(x)| = |t + x| \cdot |t - x|$ greater than ε, although $|t - x| < \delta$. Since this is true for every $\delta > 0$, consequently, the function h is not uniformly continuous on E.

Case 2. The subset E is bounded.

In this case, since E is noncompact, it is not closed. Thus, there is a limit point x_0 of E which is not a point of E. Consider the function h defined by $h(x) = \dfrac{1}{x - x_0}$. Then h is continuous on E.

To see that the function h is not uniformly continuous on E. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, and choose a point $x \in E$ such that $|x - x_0| < \delta$. Taking t close enough to x_0, we can then make the difference $|h(t) - h(x)| = \dfrac{|t - x|}{|t - x_0| \cdot |x - x_0|}$ greater than ε, although $|t - x| < \delta$. Since this is true for every $\delta > 0$, consequently, the function h is not uniformly continuous on E. ◀

Justification: continuous mapping maps connected subset to connected subset

Assume, on the contrary, that $f(E) = A \cup B$, where A and B are nonempty separated subsets of Y. Put $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

It is easy to see that

$$\begin{aligned} G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\ &= E \cap (f^{-1}(A) \cup f^{-1}(B)) \\ &= E \cap f^{-1}(A \cup B) \\ &\supset E \cap E = E. \end{aligned}$$

On the other hands, both G and H are subsets of E. Hence $E = G \cup H$.

Since $A \neq \varnothing$, by the definition of A, we know that there are points p of E such that $f(p) \in A$. This implies $G \neq \varnothing$. Similarly, $H \neq \varnothing$.

Since $A \subset \overline{A}$, we have $G \subset f^{-1}(\overline{A})$. Since f is continuous, we know that $f^{-1}(\overline{A})$ is closed. Hence, $\overline{G} \subset f^{-1}(\overline{A})$. It follows that $f(\overline{G}) \subset \overline{A}$. Since $f(H) \subset f(f^{-1}(B)) \subset B$ and $\overline{A} \cap B = \varnothing$, we have $f(\overline{G} \cap H) = f(\overline{G}) \cap f(H) \subset \overline{A} \cap B = \varnothing$. Thus, $\overline{G} \cap H = \varnothing$. Similarly, we also have $G \cap \overline{H} = \varnothing$. Thus, G and H are separated. This contradicts to the hypothesis that E is connected. ◀

Justification: Intermediate Value Theorem

Without loss of generality, we assume $f(a) < f(b)$, otherwise we only need to consider the function $-f$.

We know that the interval $[a, b]$ is connected. Hence, we know that $f([a, b])$ is a connected subset of $\mathbb{R}$, by Theorem 4.23, and the assertion follows.

Example: discontinuous functions

(1) The **Dirichlet function** $D\colon \mathbb{R} \to \mathbb{R}$ defined by

$$D(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

It has a discontinuity of the second kind at every point, since neither $D(x+)$ nor $D(x-)$ exists.

(2) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = xD(x)$, where D is the Dirichlet function. The function f is continuous at $x = 0$ and has a discontinuity of the second kind at every other point.

(3) The **sign function** $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

It has a simple discontinuity at $x = 0$ and is continuous at every other point. ◀

Justification: monotonic function alway has one-sided limits

Let x be a given point in (a,b). Since f is monotonically increasing, the set of numbers $f(t)$, where $a<t<x$, is bounded above by $f(x)$. Let $A=\sup\limits_{a<t<x} f(t)$ be the least upper bounded. Then $A\le f(x)$. We prove that $A=f(x-)$.

For any fixed $\varepsilon>0$, by the definition of A, there is a number $\tilde{x}\in(a,x)$ such that

$$A-\varepsilon<f(\tilde{x})\le A.$$

(Note: $\tilde{x}\ne x$ since the least upper bound of f is taken in the open interval (a,x).) Denote $\delta=x-\tilde{x}$. Then $\delta>0$. Since f is monotonically increasing, for $x-\delta<t<x$, we have

$$f(x-\delta)\le f(t)\le f(x).$$

Thus, for $x-\delta<t<x$,

$$A-\varepsilon<f(t)<A.$$

This implies that $f(x-)=A$. Therefore, we have

$$\sup_{a<t<x} f(t)=f(x-)\le f(x).$$

It is similar to prove

$$f(x)\le f(x+)=\inf_{x<t<b} f(t).$$

Next, if $a<x<y<b$, by the first part proved, we have

$$f(x+)=\inf_{x<t<b} f(t)=\inf_{x<t<y} f(t).$$

Similarly,

$$f(y-)=\sup_{a<t<y} f(t)=\sup_{x<t<y} f(t).$$

Thus

$$f(x+)\le f(y-).$$

◀

Justification: monotonic function has at most countably many discontinuities

We assume that f is increasing. It will be similar to prove if f is decreasing. Let E be the set of points at which f is discontinuous. With every point x of E we associate a rational number $r(x)$ such that

$$f(x-) < r(x) < f(x+).$$

Notice that $x_1 < x_2$ implies $f(x_1+) < f(x_2-)$. Hence, if x_1 and x_2 are two distinct discontinuities with $x_1 < x_2$, then

$$f(x_1-) < r(x_1) < f(x_1+) < f(x_2-) < r(x_2) < f(x_2+),$$

which implies $r(x_1) \neq r(x_2)$.

Therefore, we have shown that there is a one-to-one correspondence between the elements in set E and a subset of the rational numbers, which is a known countable set. ◀

Example: one-sided derivatives vs one-sided limits of derivative

Consider the unit step function $I\colon \mathbb{R} \to \mathbb{R}$ defined by

$$I(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Direct calculation gives

$$I'(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ \text{not exist}, & \text{if } x = 0. \end{cases}$$

Moreover, we have

$$\begin{aligned} I'(0+) &= \lim_{h\to 0^+} I'(h) = \lim_{h\to 0^+} 0 = 0, \\ I'(0-) &= \lim_{h\to 0^+} I'(-h) = \lim_{h\to 0^+} 0 = 0, \\ I'_+(0) &= \lim_{h\to 0^+} \frac{I(0+h) - I(0)}{h} = \lim_{h\to 0^+} \frac{1-1}{h} = 0, \\ I'_-(0) &= \lim_{h\to 0^+} \frac{I(0-h) - I(0)}{-h} = \lim_{h\to 0^+} \frac{0-1}{-h} = \text{not exist}. \end{aligned}$$

The function I' is discontinuous at 0 because $I'(0)$ does not exist. ◂

Justification: differentiability implies continuity

As $t \to x$, we have, by Proposition 4.13,

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \to f'(x) \cdot 0 = 0.$$

◀

Justification: arithmetic operations on differentiation

Item 1 Let $h = f + g$. Then

$$\frac{h(t)-h(x)}{t-x} = \frac{[f(t)+g(t)]-[f(x)+g(t)]}{t-x} = \frac{f(t)-f(x)}{t-x} + \frac{g(t)-g(x)}{t-x}.$$

Taking $t \to x$, by Proposition 4.13, we have

$$h'(x) = f'(x) + g'(x).$$

Item 2 The proof is similar to that of Item 1.

Item 3 Let $h = fg$. Then

$$h(t) - h(x) = [f(t) - f(x)]g(t) + [g(t) - g(x)]f(x).$$

If we divide the equality by $t - x$ and note that $g(t) \to g(x)$ as $t \to x$, we have

$$h'(x) = f'(x)g(x) + g'(x)f(x).$$

Item 4 Let $h = f/g$. Then

$$h(t) - h(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)} = \frac{1}{g(t)g(x)} \left\{[f(t) - f(x)]g(x) - [g(t) - g(x)]f(x)\right\}.$$

Dividing the equality by $t - x$ gives

$$\frac{h(t)-h(x)}{t-x} = \frac{1}{g(t)g(x)} \left[\frac{f(t)-f(x)}{t-x} \cdot g(x) - \frac{g(t)-g(x)}{t-x} \cdot f(x)\right].$$

Taking the limit $t \to x$ and noting that $g(t) \to g(x)$ as $t \to x$, we get

$$h'(x) = \frac{1}{[g(x)]^2}[f'(x)g(x) - g'(x)f(x)].$$

◀

Example: arithmetic operations on differentiation

(1) By definition, it is clear that the derivative of any constant is zero: $(c)' = 0$.

(2) For the function $f(x) = x$, we have $f'(x) = 1$. In fact,

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{t - x}{t - x} = 1.$$

(3) For the function $f(x) = x^n$, we can have $f'(x) = nx^{n-1}$ by induction. In fact, this is true when $n = 1$. Assume $(x^k)' = kx^{k-1}$ for some integer $k \geq 1$. Then, by Item 3 of Proposition 5.3, we have

$$\begin{aligned}(x^{k+1})' &= (x \cdot x^k)' \\ &= (x)' \cdot (x^k) + (x^k)' \cdot x \\ &= 1 \cdot (x^k) + kx^{k-1} \cdot x = (k+1)x^k.\end{aligned}$$

Hence, the formula $(x^n)' = nx^{n-1}$ for all positive integers n.

(4) Le $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. Then, by Items 1 and 3 of Proposition 5.3, we have

$$\begin{aligned}f'(x) &= (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)' \\ &= (a_n x^n)' + (a_{n-1}x^{n-1})' + \cdots + (a_1 x)' + (a_0)' \\ &= a_n \cdot nx^{n-1} + a_{n-1} \cdot (n-1)x^{n-2} + \cdots + a_1 \cdot 1 + 0 \\ &= na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.\end{aligned}$$

(5) The above has shown that every polynomial is differentiable, and so is every rational function, except at the points where the denominator is zero. ◀

Justification: arithmetic operations on differentiation

Let $y = f(x)$. By the definition of the derivative, we have

$$f(t) - f(x) = (t - x)[f'(x) + u(t)],$$
$$g(s) - g(y) = (s - y)[g'(y) + v(s)],$$

where $t \in [a, b]$, $s \in I$, and $u(t) \to 0$ as $t \to x$, $v(s) \to 0$ as $s \to y$. Let $s = f(t)$. The two formulas above give

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= [f(t) - f(x)] \cdot [g'(y) + v(s)] \\ &= (t - x) \cdot [f'(x) + u(t)] \cdot [g'(y) + v(s)]. \end{aligned}$$

For $t \neq x$, we divide the both sides above by $t - x$, and then let $t \to x$. Since $s \to y$ by the continuity of f, we obtain

$$h'(x) = g'(y)f'(x),$$

so that

$$h'(x) = g'(f(x))f'(x).$$

◀

Example: chain rule

Consider the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by $h(x) = (2x^2+1)^{3000}$. It is the composite $g \circ f$, with $f(x) = 2x^2 + 1$ and $g(y) = y^{3000}$. Thus,

$$h'(x) = (g \circ f)'(x) = g'(f(x))f'(x) = 3000[f(x)]^{2999} \cdot (4x) = 12000x \cdot (2x^2+1)^{2999}.$$

Justification: derivative of the inverse function

By the definition of the inverse function, we have

$$(f^{-1})(f(x)) = x.$$

Differentiating with respect to x, by the Chain Rule, we get

$$(f^{-1})'(f(x))f'(x) = 1.$$

Thus, we have

$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

Example: derivative of the inverse function

It is known that for nonnegative integer n, we have

$$(x^n)' = nx^{n-1}.$$

We can further show that, for any rational number r,

$$(x^r)' = rx^{r-1}, \qquad x > 0.$$

First, we show that the equality holds for any rational number $r = \dfrac{1}{n}$, where n is a positive integer. In fact, for any positive integer n, the function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f(x) = x^n$ is a bijection, and its inverse f^{-1} is $f^{-1}(x) = x^{1/n}$. Denote $y = x^n$. Then $x = y^{1/n}$. By Proposition 5.5, we have

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{n}y^{1/n-1}.$$

This gives $(x^{1/n})' = \dfrac{1}{n}x^{1/n-1}$.

Next, for any positive rational number $r = m/n > 0$ (both m, n are positive integers), by the Chain Rule, we have

$$(x^r)' = \left[(x^{1/n})^m\right]' = m(x^{1/n})^{m-1} \cdot (x^{1/n})' = m(x^{1/n})^{m-1} \cdot \frac{1}{n}x^{1/n-1} = rx^{r-1}.$$

For any negative rational number $r = -m/n$ (both m, n are positive integers), we have

$$(x^r)' = \left(\frac{1}{x^{m/n}}\right)' = \frac{-(x^{m/n})'}{[x^{m/n}]^2} = -\frac{(m/n)x^{m/n-1}}{x^{2m/n}} = (-m/n)x^{-m/n-1} = rx^{r-1}.$$

◀

Justification: necessary condition for an extremum

With loss of generality, we assume that the function f has a local maximum at $x \in (a, b)$.

By the definition of the local maximum, there is $\delta > 0$ such that $a < x - \delta < x < x + \delta < b$ and $f(t) \leq f(x)$ for any $t \in (x - \delta, x + \delta)$. This implies that for $x - \delta < t < x$,

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \to x$, we have $f'(x) \geq 0$. Similarly, for $x < t < x + \delta$,

$$\frac{f(t) - f(x)}{t - x} \leq 0,$$

which yields $f'(x) \leq 0$. Hence $f'(x) = 0$. ◀

Example: necessary but not sufficient condition for an extremum

Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. It is clear that $x = 0$ is a critical point, since $f'(0) = 0$. However, since $x^3 < 0$ for $x < 0$ and $x^3 > 0$ for $x > 0$, the function f does not have an extremum at 0. ◀

Justification: Mean Value Theorem

Cauchy's Mean Value Theorem:

Put

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t).$$

Then h is continuous on $[a, b]$, h is differentiable in (a, b), and $h(a) = f(b)g(a) - f(a)g(b) = h(b)$.

If h is constant on $[a, b]$, then $h'(x) = 0$ for any $x \in [a, b]$, which implies the desired equality. If h is not constant on $[a, b]$, without loss of generality, we assume that $h(t) > h(a) = h(b)$ for some $t \in (a, b)$. By Theorem 4.20, since $[a, b]$ is compact, there is a point $x \in [a, b]$ at which h attains its maximum. Certainly, by the hypothesis, $x \in (a, b)$. Thus, Proposition 5.7 implies that $h'(x) = 0$, and the conclusion follows. ◀

Mean Value Theorem:

This is a special case of Cauchy's Mean Value Theorem, with $g(x) = x$. ◀

Example: MVT fails for complex functions

Consider the complex function $f(x) = \cos x + i \sin x$. It is easy to see that

$$f(2\pi) - f(0) = 1 - 1 = 0.$$

However, there is not point c such that

$$f(2\pi) - f(0) = f'(c)(2\pi - 0),$$

since $|f'(c)| = \cos^2 c + \sin^2 c = 1$. ◀

Justification: Monotone Test

By Theorem 5.8, for any two points $x_1, x_2 \in (a, b)$, there is a point x between x_1 and x_2, such that

$$f(x_1) - f(x_2) = (x_1 - x_2) f'(x).$$

The conclusions of the theorem follow from this equation. ◀

Justification: l'Hôpital's Rule

We first show that the hypothesis $g'(x) \neq 0$ for all $x \in (a,b)$ implies that $g(x) \neq 0$ for $x \in (a,b)$. In fact, if $g(x_0) = 0$ for some $x_0 \in (a,b)$, then for any $x \in (a,b)$, by the Mean Value Theorem, there is a number c between x and x_0 such that
$$g(x) - g(x_0) = g'(c)(x - x_0).$$
It follows that $g(x) \neq 0$.

Now we prove l'Hôpital's rule in three steps.

Step 1 We prove that if $-\infty \le A < \infty$, and if $A < q$ for some q, then there exists a real number C_1 such that $\dfrac{f(x)}{g(x)} < q$ for all $x \in (a, C_1)$.

We choose a number r, such that $A < r < q$. By the condition $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a+$, we know that there exists c such that $x \in (a,c)$ implies $\dfrac{f'(x)}{g'(x)} < r$. For $a < x < y < c$, by Cauchy's Mean Value Theorem, there is a number $t \in (x,y) \subset (a,c)$ such that
$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

② If $f(x) \to 0$, $g(x) \to 0$ as $x \to a+$, then by taking $x \to a+$ in the last inequality, we obtain
$$\frac{f(y)}{g(y)} \le r < q, \qquad y \in (a,c).$$

① If $g(x) \to \infty$ as $x \to a+$, then there is a number $c_1 \in (a,y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $x \in (a, c_1)$. This gives
$$\frac{f(x)}{g(x)} = \frac{f(x) - f(y)}{g(x) - g(y)} \cdot \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)} < r \cdot \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)}.$$
Since the right hand side of the last inequality approaches r as $x \to a+$, there exists $c_2 \in (a, c_1)$ such that $x \in (a, c_2)$ implies
$$\frac{f(x)}{g(x)} < q.$$

Step 2 We prove that if $-\infty < A \le \infty$, and if $p < A$ for some number p, then there exists a real number C_2 such that $p < \dfrac{f(x)}{g(x)}$ for all $x \in (a, C_2)$.

The proof is similar to Step 1.

Step 3 We prove that $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = A$.

In fact, if $A = -\infty$, Step 1 implies the desired limit by the definition of infinity limit.

Similarly, if $A = \infty$, Step 2 implies the desired limit.

In $-\infty < A < \infty$, for any $\varepsilon > 0$, Step 1 implies there is a number $\delta_1 = C_1 - a > 0$, such that $\dfrac{f(x)}{g(x)} < A + \varepsilon$ for $x \in (a, a + \delta_1)$. On the other hand, Step 2 implies that there is a number $\delta_2 = C_2 - a > 0$, such that $A - \varepsilon < \dfrac{f(x)}{g(x)}$ for $x \in (a, a + \delta_2)$. Hence, $x \in (a, a + \min\{\delta_1, \delta_2\})$ implies
$$A - \varepsilon < \frac{f(x)}{g(x)} < A + \varepsilon,$$
so that $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = A$.

◀

Justification: Taylor's Theorem

Denote
$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k,$$
and put
$$g(t) = f(t) - P(t) - \frac{f(x) - P(x)}{(x-x_0)^n}(t-x_0)^n.$$
If we can show that there exists ξ between x_0 and x such that $g^{(n)}(\xi) = 0$, then, by $P^{(n)}(\xi) = 0$, we have
$$f^{(n)}(\xi) - \frac{f(x) - P(x)}{(x-x_0)^n} \cdot n! = 0,$$
which gives the Taylor's formula.

In fact, it is easy to check that $P^{(k)}(x_0) = f^{(k)}(x_0)$, $k = 0, 1, \ldots, n-1$. Hence, we have
$$g(x_0) = g'(x_0) = \cdots = g^{(n-1)}(x_0) = 0.$$
By the fact that $g(x_0) = 0$, by the Mean Value Theorem, there is x_1 between x_0 and x such that $g'(x_1) = 0$. Since $g'(x_0) = 0$, again by the Mean Value Theorem, there is x_2 between x_0 and x_1 such that $g''(x_2) = 0$. Repeated application of the Mean Value Theorem, we conclude that there is x_n between x_0 and x_{n-1} such that $g^{(n)}(x_n) = 0$. All x_i, $1 \le i \le n$, are between x_0 and x. ◀

Justification: inequalities on integral sums

We prove the first inequality in the case that P^* contains just one point more than P. If P^* contains k points more than P, we repeat the reasoning k times.

Denote this extra point to be x^*, and suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P. Put

$$w_1 = \inf_{x_{i-1} \le x \le x^*} f(x), \qquad w_2 = \inf_{x^* \le x \le x_i} f(x).$$

It is clear that $w_1 \ge m_i$, $w_2 \ge m_i$, where $m_i = \inf\limits_{x_{i-1} \le x \le x_i} f(x)$. Thus,

$$\begin{aligned}
&L(P^*, f) - L(P, f) \\
= {}& w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i(x_i - x_{i-1}) \\
= {}& (w_1 - m_i)(x^* - x_{i-1}) + (w_2 - m_i)(x_i - x^*) \ge 0.
\end{aligned}$$

The second inequality can be proved as above in a similar manner. ◀

Justification: lower integral $\leq$ upper integral

Let P_1 and P_2 be two arbitrary partitions of $[a, b]$ and P^* be the common refinement of P_1 and P_2. By Proposition 6.2,

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f).$$

Fixing P_2 and taking the supremum over all P_1, the inequality

$$L(P_1, f) \leq U(P_2, f)$$

implies

$$\underline{\int_a^b} f \, \mathrm{d}\alpha \leq U(P_2, f).$$

The conclusion follows by taking the infimum over all P_2 in the last inequality. ◀

Justification: the integrability criterion

(⇒) Suppose $f \in \mathscr{R}[a,b]$. For $\varepsilon > 0$, by the definition of the upper and lower integrals, there exist partitions P_1 and P_2 such that

$$\underline{\int_a^b} f\,\mathrm{d}x - L(P_1, f) < \varepsilon/2,$$

and

$$U(P_2, f) - \overline{\int_a^b} f\,\mathrm{d}x < \varepsilon/2.$$

Let P be the common refinement of P_1 and P_2. Since $\underline{\int_a^b} f\,\mathrm{d}x = \overline{\int_a^b} f\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x$, we have

$$U(P,f) \le U(P_2,f) < \int_a^b f\,\mathrm{d}x + \varepsilon/2 < L(P_1,f) + \varepsilon \le L(P,f) + \varepsilon,$$

so that

$$U(P,f) - L(P,f) < \varepsilon.$$

(⇐) Suppose the integrability criterion holds. For every $\varepsilon > 0$, there exists a partition P such that $U(P,f) - L(P,f) < \varepsilon$. Since

$$L(P,f) \le \underline{\int_a^b} f\,\mathrm{d}x \le \overline{\int_a^b} f\,\mathrm{d}x \le U(P,f),$$

we obtain

$$0 \le \overline{\int_a^b} f\,\mathrm{d}x - \underline{\int_a^b} f\,\mathrm{d}x < \varepsilon.$$

Because ε is arbitrary, we have

$$\overline{\int_a^b} f\,\mathrm{d}x - \underline{\int_a^b} f\,\mathrm{d}x = 0,$$

so that $f \in \mathscr{R}[a,b]$. ◀

Justification: necessary and sufficient conditions for integrability - Condition 1

Condition 1

($\Rightarrow$) Suppose $f \in \mathscr{R}[a,b]$. By the integrability criterion, there exists a partition P of $[a,b]$, such that $U(P,f) - L(P,f) < \varepsilon$. For any $s_i, t_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, since

$$|f(s_i) - f(t_i)| \leq M_i - m_i,$$

we have

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta x_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta x_i = U(P,f) - L(P,f) < \varepsilon.$$

That is, for the same partition P, Condition 1 for integrability holds.

($\Leftarrow$) Suppose that Condition 1 for integrability holds. Thus, for every $\varepsilon > 0$ there exists a partition $P\colon a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$, such that

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta x_i < \tfrac{1}{3}\varepsilon.$$

for any $s_i, t_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. For each i, choose $s_i^*, t_i^* \in [x_{i-1}, x_i]$, such that

$$f(s_i^*) > M_i - \frac{\frac{1}{3}\varepsilon}{b-a+1},$$
$$f(t_i^*) < m_i + \frac{\frac{1}{3}\varepsilon}{b-a+1}.$$

Thus,

$$\begin{aligned}
U(P,f) - L(P,f) &= \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i \\
&\leq \sum_{i=1}^{n} \left[f(s_i^*) + \frac{\frac{1}{3}\varepsilon}{b-a+1} \right] \Delta x_i - \sum_{i=1}^{n} \left[f(t_i^*) - \frac{\frac{1}{3}\varepsilon}{b-a+1} \right] \Delta x_i \\
&\leq \sum_{i=1}^{n} |f(s_i^*) - f(t_i^*)| \, \Delta x_i + 2 \cdot \frac{\frac{1}{3}\varepsilon}{b-a+1} \cdot \sum_{i=1}^{n} \Delta x_i \\
&< \tfrac{1}{3}\varepsilon + 2 \cdot \tfrac{1}{3}\varepsilon = \varepsilon.
\end{aligned}$$

Hence, Condition 1 for integrability holds, so that $f \in \mathscr{R}[a,b]$. ◀

Justification: necessary and sufficient conditions for integrability - Condition 2

Condition 2

(⇒) Suppose $f \in \mathscr{R}[a,b]$. By the integrability criterion, there exists a partition P of $[a,b]$, such that $U(P,f) - L(P,f) < \varepsilon$. By the definition of the Riemann integral, we know

$$L(P,f) \leq \underline{\int_a^b} f \, \mathrm{d}x = \int_a^b f \, \mathrm{d}x = \overline{\int_a^b} f \, \mathrm{d}x \leq U(P,f).$$

For any choice of $t_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, since $m_i \leq f(t_i) \leq M_i$, we have

$$L(P,f) = \sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n f(t_i)\Delta x_i \leq \sum_{i=1}^n M_i \Delta x_i = U(P,f).$$

Thus, we know that both $\sum_{i=1}^n f(t_i)\Delta x_i$ and $\int_a^b f \, \mathrm{d}x$ lie in the interval $[L(P,f), U(P,f)]$. It follows that if taking $I = \int_a^b f \, \mathrm{d}x$, then

$$\left| \sum_{i=1}^n f(t_i)\Delta x_i - I \right| \leq U(P,f) - L(P,f) < \varepsilon.$$

So, for the same partition P, Condition 2 for integrability holds.

(⇐) Suppose that Condition 2 for integrability holds. Thus, for every $\varepsilon > 0$ there exists a partition $P\colon a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$, such that

$$\left| \sum_{i=1}^n f(t_i)\Delta x_i - I \right| < \tfrac{1}{4}\varepsilon,$$

for some $I \in \mathbb{R}$ and any choice of $t_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. For each i, $i = 1, 2, \ldots, n$, choose $t_i^*, t_i^{**} \in [x_{i-1}, x_i]$, such that

$$f(t_i^*) > M_i - \frac{\frac{1}{4}\varepsilon}{b-a+1},$$
$$f(t_i^{**}) < m_i + \frac{\frac{1}{4}\varepsilon}{b-a+1}.$$

Thus,

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^n M_i \Delta x_i - \sum_{i=1}^n m_i \Delta x_i \\ &\leq \sum_{i=1}^n \left[f(t_i^*) + \frac{\frac{1}{4}\varepsilon}{b-a} \right] \Delta x_i - \sum_{i=1}^n \left[f(t_i^{**}) - \frac{\frac{1}{4}\varepsilon}{b-a} \right] \Delta x_i \\ &\leq \left| \sum_{i=1}^n f(t_i^*)\Delta x_i - I \right| + \left| \sum_{i=1}^n f(t_i^{**})\Delta x_i - I \right| + 2 \cdot \frac{\frac{1}{4}\varepsilon}{b-a} \cdot \sum_{i=1}^n \Delta x_i \\ &< \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + 2 \cdot \tfrac{1}{4}\varepsilon = \varepsilon. \end{aligned}$$

Hence, the integrability criterion holds, so that $f \in \mathscr{R}[a,b]$. ◂

Justification: unbounded function not integrable

Suppose f is unbounded on $[a, b]$. We will show that the Riemann integral $\int_a^b f \,\mathrm{d}x$ does not exists, by showing Condition 1 for integrability fails.

In fact, fix an arbitrary $\varepsilon > 0$ and an arbitrary partition $P\colon a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$ of $[a, b]$. Since f is unbounded on $[a, b]$, there must be some $j_0 \in \{1, \ldots, n\}$ such that f is unbounded on $[x_{j_0-1}, x_{j_0}]$. Thus, for any chosen $s_{j_0} \in [x_{j_0-1}, x_{j_0}]$, there exists $t_{j_0} \in [x_{j_0-1}, x_{j_0}]$ such that

$$|f(s_{j_0}) - f(t_{j_0})| \ge \frac{\varepsilon}{\Delta x_{j_0}},$$

otherwise it would contradict to the hypothesis of unboundedness. This gives

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta x_i \ge |f(s_{j_0}) - f(t_{j_0})| \, \Delta x_{j_0} \ge \varepsilon,$$

so that Condition 1 for integrability fails. ◀

Justification: Riemann integrability of continuous functions

Since f is continuous on $[a,b]$, by Theorem 4.22, f is uniformly continuous on $[a,b]$. For any $\varepsilon > 0$, there is $\delta > 0$ such that for $x, t \in [a,b]$ with $|x-t| < \delta$,

$$|f(x) - f(t)| < \frac{\varepsilon}{b-a+1}.$$

Let P be any partition of $[a,b]$ such that $\Delta x_i < \delta$. Then the above inequality implies

$$M_i - m_i \leq \frac{\varepsilon}{b-a+1}.$$

Hence,

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &\leq \frac{\varepsilon}{b-a+1} \cdot \sum_{i=1}^{n} \Delta x_i \\ &< \frac{\varepsilon}{b-a+1} \cdot (b-a) < \varepsilon. \end{aligned}$$

By the integrability criterion, we conclude that $f \in \mathscr{R}[a,b]$. ◀

Justification: Riemann integrability of functions with finitely many discontinuities

Let $\varepsilon > 0$ be given. Put $M = \sup_{x\in[a,b]} |f(x)|$. Denote

$$E = \{z_1, \ldots, z_m\}, \qquad z_i \text{ in increasing order}$$

the set of points at which f is discontinuous, where m is a positive integer. Since the set of points at which f is discontinuous at each discontinuity z_i, $i = 1, \ldots, m$, we can cover z_i by a small interval $[u_j, v_j] \subset [a, b]$ such that

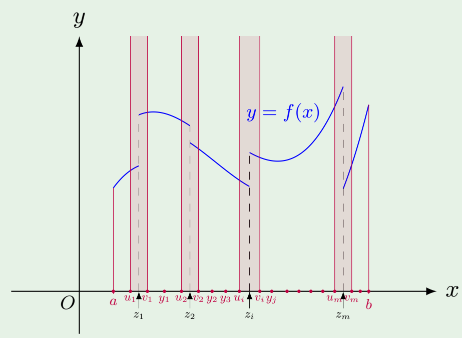
$$v_i - u_i < \frac{\varepsilon^*}{m},$$

where ε^* is any given positive number. We can require these intervals $\{[u_i, v_i]\}$ disjoint and every point of $E \cap (a, b)$ lies in the interior of some $[u_i, v_i]$.

Note that $K = [a, b] - \bigcup_{i=1}^{m}(u_i.v_i)$ is compact. The function f must be uniformly continuous on K, and there exists $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon^*$ if $s, t \in K$, with $|s - t| < \delta$.

To construct a partition P, we start with all the endpoints u_i and v_i, $i = 1, \ldots, m$. Then we add more points y_j's (in increasing order) outside of these $[u_i, v_i]$, such that

(1) none of the y_j's lies in $[u_i, v_i]$ for any i;
(2) if $[y_{j-1}, y_j] \cap [u_i, v_i] = \varnothing$, take $y_j - y_{j-1} < \delta$;
(3) if j is the largest index such that if $y_j < u_i$ for some i, then $u_i - y_j < \delta$; and similarly, if j is the smallest index such that $v_i < y_{j-1}$ for some i, then $y_{j-1} - v_i < \delta$.



Denote the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. It contains "good" (without discontinuities) and "bad" (with discontinuities) intervals I_i. For each of the good intervals, we have $M_i - m_i \le \varepsilon^*$, so that the total contribution from the good intervals to $U(P, f) - L(P, f)$ is not greater than $(b - a)\cdot\varepsilon^*$. The sum of the increments over the bad intervals is not greater than ε^*, with $M_i - m_i \le 2M$ for all these intervals. Explicitly, these give

$$\begin{aligned}
U(P, f) - L(P, f) &= \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \\
&= \sum_{x_{i-1} = u_j \text{ for some } j}(M_i - m_i)\Delta x_i + \sum_{x_{i-1} \ne u_j \text{ for any } j}(M_i - m_i)\Delta x_i \\
&\le 2M \cdot \sum_{x_{i-1} = u_j \text{ for some } j}\Delta x_i + \varepsilon^* \cdot \sum_{x_{i-1} \ne u_j \text{ for any } j}\Delta x_i \\
&\le 2M\varepsilon^* + \varepsilon^* \cdot (b - a).
\end{aligned}$$

If we take $\varepsilon^* < \dfrac{\varepsilon}{2M + (b - a)}$, then we see that the integrability criterion holds for the constructed partition P. Consequently, we know that $f \in \mathscr{R}[a, b]$. ◀

Justification: Riemann integrability of monotonic functions

Without loss of generality, we may assume that f is monotonically increasing on $[a, b]$.

Let $\varepsilon > 0$ be given. Since both f and α are monotonically increasing on $[a, b]$, we can choose a positive integer n such that

$$\frac{b-a}{n} \cdot [f(b) - f(a)] < \varepsilon.$$

We further take the equal-spaced partition

$$P\colon a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

such that, for $i = 0, 1, \ldots, n$,

$$\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}.$$

Now we show that the integrability criterion holds.

In fact, for $1 \le i \le n$, since f is monotonically increasing on $[a, b]$, we know

$$M_i = f(x_i), \qquad m_i = f(x_{i-1}).$$

Hence,

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \cdot \frac{b-a}{n} \\ &= [f(b) - f(a)] \cdot \frac{b-a}{n} < \varepsilon. \end{aligned}$$

Hence, $f \in \mathscr{R}[a, b]$. ◀

Justification: Riemann integrability of composition

Let $\varepsilon > 0$ be given.

Since ϕ is uniformly continuous on $[m, M]$, for any given positive number ε^*, there exists $\delta > 0$ such that $\delta < \varepsilon^*$ and $|\phi(s) - \phi(t)| < \varepsilon^*$ if $s, t \in [m, M]$, with $|s - t| < \delta$.

Since $f \in \mathscr{R}[a, b]$, there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, f) - L(P, f) < \delta^2.$$

Let M_i, m_i be the supremum and infimum of f on $[x_{i-1}, x_i]$, respectively, while M_i^*, m_i^* be the analogous numbers for h. Divide the set $\{1, 2, \ldots, n\}$ into two sets A and B: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \geq \delta$.

For $i \in A$, the choice of δ shows that $M_i^* - m_i^* \leq \varepsilon^*$.

For $i \in B$, $M_i^* - m_i^* \leq 2K$, where $K = \sup\limits_{m \leq t \leq M} |\phi(t)|$. From

$$\delta \sum_{i \in B} \Delta x_i \leq \sum_{i \in B} (M_i - m_i) \Delta x_i \leq U(P, f) - L(P, f) < \delta^2,$$

we have

$$\sum_{i \in B} \Delta x_i < \delta.$$

It follows that

$$\begin{aligned} U(P, h) - L(P, h) &= \sum_{i \in A} (M_i^* - m_i^*) \Delta x_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta x_i \\ &\leq \varepsilon^*(b - a) + 2K\delta \\ &= (b - a + 2K)\varepsilon^*. \end{aligned}$$

If we take $\varepsilon^* < \dfrac{\varepsilon}{b - a + 2K}$, then we see that integrability criterion holds for the partition P. Consequently, we know that $h \in \mathscr{R}[a, b]$.

◀

Justification: properties of the Riemann integral - Linearity

We will prove the linearity by completing the following:

1. If $f, g \in \mathscr{R}[a,b]$, then $f+g \in \mathscr{R}[a,b]$ and
$$\int_a^b (f+g)\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x + \int_a^b g\,\mathrm{d}x.$$
2. If $f \in \mathscr{R}[a,b]$ and α is a real number, then $\alpha f \in \mathscr{R}[a,b]$ and
$$\int_a^b \alpha f\,\mathrm{d}x = \alpha \int_a^b f\,\mathrm{d}x.$$

Item 1 Denote $h = f+g$. For any partition P of $[a,b]$, we have
$$L(P,f) + L(P,g) \le L(P,h) \le U(P,h) \le U(P,f) + U(P,g).$$
Since $f, g \in \mathscr{R}[a,b]$, by integrability criterion, for any given $\varepsilon > 0$, there are partitions P, Q of $[a,b]$ such that
$$U(P,f) - L(P,f) < \tfrac{1}{2}\varepsilon, \qquad U(Q,g) - L(Q,g) < \tfrac{1}{2}\varepsilon.$$
Thus, for their common refinement $P \vee Q$, we have
$$\begin{aligned} U(P\vee Q,h) - L(P\vee Q,h) &\le [U(P\vee Q,f) + U(P\vee Q,g)] - [L(P\vee Q,g) + L(P\vee Q,g)] \\ &\le [U(P,f) + U(Q,g)] - [L(P,f) + L(Q,g)] \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, \end{aligned}$$
so that integrability criterion holds for h. Hence, $h \in \mathscr{R}[a,b]$.

To show that the integral of the sum equals the sum of the integrals, we let P and Q be the partitions such that
$$U(P,f) \le \int_a^b f\,\mathrm{d}x + \varepsilon, \qquad U(Q,g) \le \int_a^b g\,\mathrm{d}x + \varepsilon.$$
Then for their common refinement $P \vee Q$, we have
$$\begin{aligned} \int_a^b h\,\mathrm{d}x &\le U(P\vee Q,h) \\ &\le U(P\vee Q,f) + U(P\vee Q,g) \\ &\le U(P,f) + U(Q,g) \\ &\le \int_a^b f\,\mathrm{d}x + \int_a^b g\,\mathrm{d}x + 2\varepsilon. \end{aligned}$$
Sine ε is arbitrary, we have
$$\int_a^b h\,\mathrm{d}x \le \int_a^b f\,\mathrm{d}x + \int_a^b g\,\mathrm{d}x.$$
If we replace f and g in the last inequality by $-f$ and $-g$, the inequality is reversed. Combining both we have the equality.

Item 2 The proof is in the same manner as in Item 1. ◀

Justification: properties of the Riemann integral - Additivity

Let $\varepsilon > 0$ be given. Since $f \in \mathscr{R}[a,b]$, by integrability criterion, there is a partition P of $[a,b]$ such that

$$U_{[a,b]}(P,f) - L_{[a,b]}(P,f) < \varepsilon,$$

where the subscripts in the upper and lower sums emphasize the intervals where the integrals are considered. We add one extra point $x = c$ into the partition P to obtain a refinement P^*. The partition P^* can be divided into two partitions P_1 and P_2 of $[a,c]$ and $[c,b]$, respectively. Then, by Proposition 6.2, we have

$$\begin{aligned}&[U_{[a,c]}(P_1,f) + U_{[c,b]}(P_2,f)] - [L_{[a,c]}(P_1,f) + L_{[c,b]}(P_2,f)]\\ =\ &U_{[a,b]}(P^*,f) - L_{[a,b]}(P^*,f) < \varepsilon.\end{aligned}$$

It follows that

$$U_{[a,c]}(P_1,f) - L_{[a,c]}(P_1,f) < \varepsilon,$$

and

$$U_{[c,b]}(P_2,f) - L_{[c,b]}(P_2,f) < \varepsilon,$$

so that we have $f \in \mathscr{R}[a,c] \cap \mathscr{R}[c,b]$.

To prove the identity of the integrals, we assume that the partition P is chosen so that

$$U_{[a,b]}(P,f) < \int_a^b f\,\mathrm{d}x + \varepsilon.$$

Then,

$$\begin{aligned}\int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x &\le U_{[a,c]}(P_1,f) + U_{[c,b]}(P_1,f)\\ &= U_{[a,b]}(P^*,f) \le U_{[a,b]}(P,f) < \int_a^b f\,\mathrm{d}x + \varepsilon.\end{aligned}$$

Since ε is arbitrary, it follows

$$\int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x \le \int_a^b f\,\mathrm{d}x.$$

On the other hand, if we consider the lower sums, we can similarly prove that

$$\int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x \ge \int_a^b f\,\mathrm{d}x.$$

Combining both we have the equality. ◀

Justification: properties of the Riemann integral - Monotonicity

The inequality is a direct consequence of the definition of the Riemann integral. ◀

Justification: properties of the Riemann integral - monotonicity - Corollary

Let $\phi(y) = |y|$. Then, Theorem 6.5 (Riemann integrability of composition) implies that $|f| \in \mathscr{R}[a,b]$ if $f \in \mathscr{R}[a,b]$. We also have $-|f| \in \mathscr{R}[a,b]$ by the *linearity* of the Riemann integral.

Since $-|f(x)| \leq f(x) \leq |f(x)|$, by the *monotonicity* of the Riemann integral, we have

$$\int_a^b -|f| \,\mathrm{d}x \leq \int_a^b f \,\mathrm{d}x \leq \int_a^b |f| \,\mathrm{d}x.$$

By the linearity again, we get

$$-\int_a^b |f| \,\mathrm{d}x \leq \int_a^b f \,\mathrm{d}x \leq \int_a^b |f| \,\mathrm{d}x,$$

so that

$$\left| \int_a^b f \,\mathrm{d}x \right| \leq \int_a^b |f| \,\mathrm{d}x.$$

By the definition of the integral, we have

$$\int_a^b M \,\mathrm{d}x = M(b-a).$$

Thus, if $|f(x)| \leq M$ on $[a,b]$, then, we get

$$\int_a^b |f| \,\mathrm{d}x \leq \int_a^b M \,\mathrm{d}x = M(b-a),$$

so that the second inequality holds. ◀

Justification: the Fundamental Theorem of Calculus - Part 1

Since $f \in \mathscr{R}[a, b]$, f is bounded. Suppose $|f(t)| \leq M$ for $a \leq t \leq b$. If $a \leq x < y \leq b$, then

$$|F(y) - F(x)| = \left| \int_x^y f(t)\, \mathrm{d}t \right| \leq M(y - x).$$

The continuity of F follows from the inequality.

If f is continuous at x_0, then, for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon,$$

if $|t - x_0| < \delta$, and $a \leq t \leq b$. Hence, if $x_0 - \delta < s \leq x_0 \leq t < x_0 + \delta$ and $a \leq s < t \leq b$,

$$\begin{aligned} \left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| &= \left| \frac{1}{t - s} \int_s^t [f(u) - f(x_0)]\, \mathrm{d}u \right| \\ &\leq \frac{1}{t - s} \int_s^t |f(u) - f(x_0)|\, \mathrm{d}u < \varepsilon. \end{aligned}$$

It follows that $F'(x_0) = f(x_0)$. ◀

Justification: the Fundamental Theorem of Calculus - Part 2

Let $\varepsilon > 0$ be given. As $f \in \mathscr{R}[a,b]$, by the integrability criterion, there exists a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon.$$

For each i, $i = 1, \ldots, n$, by the Mean Value Theorem, there is $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i.$$

It follows that

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a).$$

According to the remark of Theorem 6.3, if the integrability criterion holds for the partition P, then the same partition P is valid for Condition 2. Thus,

$$\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)\,\mathrm{d}x\right| < \varepsilon,$$

that is,

$$\left|F(b) - F(a) - \int_a^b f(x)\,\mathrm{d}x\right| < \varepsilon.$$

As ε is arbitrary, we have the desired identity. ◀

Justification: change of variable for the Riemann integral

Since f is continuous, by Part 1 of the Fundamental Theorem of Calculus, we can define

$$F(x) = \int_{\varphi(a)}^{x} f(t)\,\mathrm{d}t,$$

and have $F'(x) = f(x)$. By the hypothesis, the function φ is differentiable, so that, by the chain rule, we have

$$[F(\varphi(x))]' = F'(\varphi(x)) \cdot \varphi'(x) = f(\varphi(x)) \cdot \varphi'(x).$$

Since f and φ are continuous, the composition $f(\varphi(x))$ is continuous and therefore integrable. ◀

Justification: integration by parts

Put $H(x) = F(x)G(x)$. Since $H' = Fg + Gf$, we know that $H' \in \mathscr{R}[a, b]$. By Part 2 of the Fundamental Theorem of Calculus, we have

$$\int_a^b H'(x)\,\mathrm{d}x = H(b) - H(a).$$

Rewriting this identity, we get the formula of the integration by parts. ◀

Justification: Cauchy Criterion for uniform convergence

(⇒) Suppose $\{f_n\}$ converges uniformly to f on E. Then, for every $\varepsilon > 0$, there exists an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \varepsilon/2$$

for all $x \in E$. Hence, if $n, m \geq N$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \varepsilon$$

for all $x \in E$, so that the Cauchy Criterion for uniform convergence holds.

(⇐) Suppose the Cauchy Criterion for uniform convergence holds. Then $\{f_n(x)\}$ is a Cauchy sequence for each $x \in E$. By Theorem 3.8, the sequence $\{f_n(x)\}$ converges for each fixed x to a limit which we may call $f(x)$. Hence the sequence $\{f_n\}$ converges pointwise to f on E. Let $\varepsilon > 0$ be given. There exists an integer N such that $m, n \geq N$ implies

$$|f_n(x) - f_m(x)| < \varepsilon/2.$$

If we let $m \to \infty$ in the inequality, by Proposition 3.14 we have

$$|f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon, \qquad n \geq N,$$

for all $x \in E$. Thus $\{f_n\}$ converges uniformly to f on E. ◀

Justification: the Weierstrass M-test

If $\sum M_n$ converges, then, for every $\varepsilon > 0$, there exists an integer N such that $m \geq n \geq N$ implies

$$\sum_{i=n}^{m} M_i < \varepsilon.$$

Hence, if $m \geq n \geq N$,

$$\left|\sum_{i=n}^{m} f_i(x)\right| \leq \sum_{i=n}^{m} M_i < \varepsilon,$$

for all $x \in E$. It follows from the Cauchy Criterion for uniform convergence that $\sum f_n$ converges uniformly on E. ◀

Justification: sufficient condition for changing order of limits

Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges uniformly on E, there exists an integer N such that $n, m \geq N$ implies

$$|f_n(t) - f_m(t)| < \varepsilon,$$

for all $t \in E$. Letting $t \to x$, we have

$$|A_n - A_m| \leq \varepsilon,$$

for $n, m \geq N$. Hence, $\{A_n\}$ is a Cauchy sequence, and therefore converges, say to A.

From the inequality

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|,$$

we now give the estimates for the terms on the right hand side. In fact, since $f_n \to f$ uniformly on E, we can choose n sufficiently large such that

$$|f(t) - f_n(t)| < \varepsilon/3$$

for all $t \in E$, and such that

$$|A_n - A| < \varepsilon/3.$$

For this large n, since

$$\lim_{t \to x} f_n(t) = A_n,$$

we can choose a neighborhood V of x such that

$$|f_n(t) - A_n| < \varepsilon/3$$

if $t \in V \cap E$, $t \neq x$. Thus, we know that for $t \in V \cap E$, $t \neq x$,

$$|f(t) - A| < \varepsilon.$$

That is, $\lim_{t \to x} f(t) = A = \lim_{n \to \infty} A_n$. ◀

Justification: Dini's theorem

Put $g_n = f_n - f$. Then g_n is continuous, $g_n \to 0$, and $g_n \geq g_{n+1}$. It is sufficient to prove that $g_n \to 0$ uniformly on K.

Let $\varepsilon > 0$ be given. Write

$$K_n = \{x \in K \colon g_n(x) \geq \varepsilon\}, \qquad n = 1, 2, 3, \ldots.$$

Since g_n is continuous, by Theorem 4.17, $K_n \subset K$ is closed for each n. By Proposition 2.18, K_n is compact. Since $g_n \geq g_{n+1}$, we know that $K_n \supset K_{n+1}$. Fix $x \in K$, since $g_n(x) \to 0$, we see that $x \notin K_n$ if n is sufficiently large. Hence $x \notin \bigcap K_n$. In other words, $\bigcap K_n$ is empty. Hence K_N is empty for some N, by Theorem 2.20. It follows that $0 \leq g_n(x) < \varepsilon$ for all $x \in K$ and all $n \geq N$. This proves the theorem. ◀

Example: failures of Dini's theorem

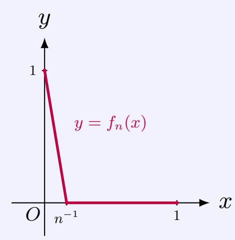
(1) Non-compact set K: Consider the sequence of functions $\{f_n\}$, $f_n(x) = x^n$, on $K = (0, 1)$. It is clear that $\{f_n\}$ converges pointwise to the zero function on K. Since, as $n \to \infty$,

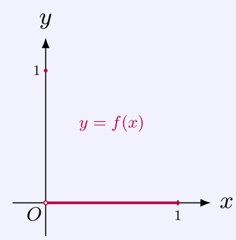
$$\sup_{x\in(0,1)} |f_n(x) - 0| = \sup_{x\in(0,1)} x^n = 1 \not\to 0,$$

by the equivalent definition for uniform convergence, we know that $\{f_n\}$ does not converge to the zero function uniformly on K.

(2) Discontinuous function f: Consider the sequence of continuous functions $\{f_n\}$, where $f_n\colon [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 1 - nx, & \text{if } 0 \le x \le n^{-1}, \\ 0, & \text{if } n^{-1} < x \le 1. \end{cases}$$



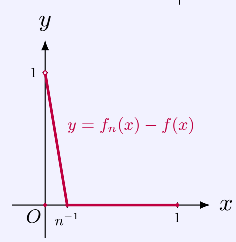


It is easy to see that the sequence $\{f_n\}$ converges to f on K, where

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } 0 < x \le 1, \end{cases}$$

which is discontinuous at 0. Because

$$f_n(x) - f(x) = \begin{cases} 1 - nx, & \text{if } 0 < x \le n^{-1}, \\ 0, & \text{if } x = 0 \text{ or } n^{-1} < x \le 1, \end{cases}$$



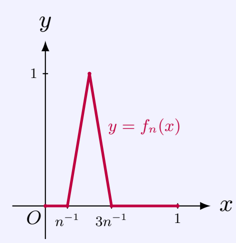
we have, as $n \to \infty$,

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \not\to 0.$$

By the equivalent definition for uniform convergence, we know that $\{f_n\}$ does not converge to f uniformly on K.

(3) The sequence $\{f_n(x)\}$ does not monotonically decrease as n increases: Consider the sequence of continuous functions $\{f_n\}$, where $f_n\colon [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} nx - 1, & \text{if } n^{-1} < x \le 2n^{-1}, \\ -nx + 3, & \text{if } 2n^{-1} < x \le 3n^{-1}, \\ 0, & \text{if } 0 \le x \le n^{-1} \text{ or } 3n^{-1} \le x \le 1. \end{cases}$$



It is easy to see that the sequence $\{f_n\}$ converges to the zero function on K. Because

$$\sup_{x\in[0,1]} |f_n(x) - 0| = 1 \not\to 0.$$

By the equivalent definition for uniform convergence, we know that $\{f_n\}$ does not converge to f uniformly on K. ◀

Justification: interchange the limit and the integration

Put
$$\varepsilon_n = \sup_{x \in [a,b]} |f_n(x) - f(x)|.$$
Then
$$f_n - \varepsilon_n \le f \le f_n + \varepsilon_n.$$
These inequalities give, for any partition P of $[a, b]$,
$$\int_a^b (f_n - \varepsilon_n)\,\mathrm{d}x \le L(P, f) \le U(P, f) \le \int_a^b (f_n + \varepsilon_n)\,\mathrm{d}x.$$
It follows that
$$0 \le U(P, f) - L(P, f) \le 2\varepsilon_n(b - a),$$
which implies $f \in \mathscr{R}$ on $[a, b]$ by the Integrability Criterion, since $\varepsilon_n \to 0$ by the hypothesis.

From the inequality
$$\left| \int_a^b f_n\,\mathrm{d}x - \int_a^b f\,\mathrm{d}x \right| \le \varepsilon_n(b - a),$$
we let $n \to \infty$ and obtain the desired limit. ◀

Justification: interchange the limit and the differentiation

Show that $\{f_n\}$ converges uniformly to some function f on $[a, b]$.

Consider the following estimates: for any $x \in [a, b]$,

$$\begin{aligned}|f_n(x) - f_m(x)| &\le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \\ &\le |f_n'(t) - f_m'(t)| \cdot |x - x_0| + |f_n(x_0) - f_m(x_0)|,\end{aligned}$$

where t is a number between x and x_0.

Let $\varepsilon > 0$ be given. Choose N such that $m, n \ge N$ implies

$$|f_n(x_0) - f_m(x_0)| < \varepsilon/2,$$

and

$$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)},$$

for all $t \in [a, b]$. Hence, for $n, m \ge N$, and $x \in [a, b]$,

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2(b-a)} \cdot |x - x_0| + \varepsilon/2 \le \varepsilon.$$

This shows that the Cauchy Criterion for uniform convergence holds, so that $\{f_n\}$ converges uniformly on $[a, b]$.

Prove the equality $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$.

For a fixed $x \in [a, b]$, put

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \qquad \phi(t) = \frac{f(t) - f(x)}{t - x},$$

where $t \in [a, b]$, $t \ne x$. It is clear that

$$\lim_{t\to x} \phi_n(t) = f_n'(x), \qquad n = 1, 2, 3, \ldots.$$

By Theorem 7.5, if we can show that $\phi_n \to \phi$ uniformly on $[a, b] \setminus \{x\}$, then $\lim\limits_{t\to x} \phi(t) = \lim\limits_{n\to\infty} f_n'(x)$, which is the desired limit since $f'(x) = \lim\limits_{t\to x} \phi(t)$.

To show that $\{\phi_n\}$ converges uniformly to ϕ on $[a, b] \setminus \{x\}$, we have the following estimates: by the Mean Value Theorem, there exists $\tilde{t}$ between t and x, such that

$$\begin{aligned}|\phi_n(t) - \phi_m(t)| &= \left| \frac{f_n(t) - f_n(x)}{t - x} - \frac{f_m(t) - f_m(x)}{t - x} \right| \\ &= \left| \frac{[f_n(t) - f_m(t)] - [f_n(x) - f_m(x)]}{t - x} \right| \\ &= \left| f_n'(\tilde{t}) - f_m'(\tilde{t}) \right| < \frac{\varepsilon}{2(b-a)},\end{aligned}$$

if $n, m \ge N$, $t \ne x$. Hence, the sequence $\{\phi_n\}$ satisfies the Cauchy Criterion for uniform convergence, so that it converges uniformly, for $t \ne x$. Since $\{f_n\}$ converges to f, we know that $\phi_n(t) \to \phi(t)$ pointwise for $t \ne x$. Thus $\{\phi_n\}$ converges uniformly to ϕ, for $t \ne x$.

Remark

- Based on the above, we observe that assuming convergence of $\{f_n(x_0)\}$ at some $x_0 \in [a, b]$, we can establish the uniform convergence of $\{f_n\}$ from the uniform convergence of $\{f_n'\}$. Consequently, incorporating the continuity of the functions f_n' into the existing assumptions of the theorem simplifies the proof significantly, as Theorem 7.8 and the Fundamental Theorem of Calculus can be employed to deduce the uniform convergence of $\{f_n\}$.

Justification: the metric space $\mathscr{C}(X)$ is complete

We will prove the following statements:

1. The so-called ***supremum norm***
$$\|f\| = \sup_{x\in X} |f(x)|$$
is indeed a norm on $\mathscr{C}(X)$. In other words, it satisfies the following
   (a) $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$;
   (b) $\|\alpha f\| = |\alpha|\,\|f\|$;
   (c) the triangle inequality $\|f+g\| \leq \|f\| + \|g\|$.
2. The set $\mathscr{C}(X)$ is a metric space, equipped with the distance function $d(f,g) = \|f-g\|$ for all $f, g \in \mathscr{C}(X)$.
3. The metric space $(\mathscr{C}(X), d)$ is complete.
4. If a sequence $\{f_n\}$ in $\mathscr{C}(X)$ converges to $f \in \mathscr{C}(X)$, that is, if $d(f_n, f) \to 0$ as $n \to \infty$, then $f_n \to f$ uniformly on X.

Item 1: Since every function in $\mathscr{C}(X)$ is bounded, $\|f\|$ is finite for all $f \in \mathscr{C}(X)$.

(a) Obviously, $\|f\| = \sup_{x\in X} |f(x)| \geq 0$. It is clear that $\|f\| = \sup_{x\in X} |f(x)| = 0$ only if $f(x) = 0$ for every $x \in X$, that is, only if $f = 0$.

(b) $\|\alpha f\| = \sup_{x\in X} |\alpha f(x)| = \sup_{x\in X} |\alpha| \cdot |f(x)| = |\alpha| \cdot \sup_{x\in X} |f(x)| = |\alpha|\,\|f\|$.

(c) If $h = f + g$, then, for all $x \in X$,
$$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|.$$
Hence,
$$\|f+g\| = \sup_{x\in X} |h(x)| \leq \|f\| + \|g\|.$$

Item 2: From Item 1, it is easy to verify that the distance function $d(f,g) = \|f-g\|$ satisfies

(1) $d(f,g) \geq 0$ and $d(f,g) = 0$ if and only if $f = g$;
(2) $d(f,g) = d(g,f)$;
(3) $d(f,g) \leq d(f,h) + d(h,g)$, for any $h \in \mathscr{C}(X)$.

Hence, $(\mathscr{C}(X), d)$ is a metric space.

Item 3: Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. For any $\varepsilon > 0$, there exists an integer N such that $\|f_n - f_m\| < \varepsilon$ if $n, m \geq N$. Since $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$ for all $x \in X$, by the Cauchy Criterion for uniform convergence, there is a function f with domain X to which $\{f_n\}$ converges uniformly. Since f_n is continuous for every n, by the corollary of Theorem 7.5, f is continuous. Moreover, since there is an n such that $|f(x) - f_n(x)| < 1$ for all $x \in X$, we know that $|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| < 1 + \|f_n\|$ which implies that $f(x)$ is bounded. Thus, $f \in \mathscr{C}(X)$. Since $f_n \to f$ uniformly on X, from the equivalent definition for uniform convergence, we know that $\|f_n - f\| \to 0$ as $n \to \infty$. Therefore, $\mathscr{C}(X)$ is a complete metric space.

Item 4: This is the rephrased statement of Theorem 7.6. ◀

Justification: the Weierstrass approximation theorem

Without loss of generality, we assume that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$, otherwise we use

$$g(x) = f(x) - f(0) - x[f(1) - f(0)]$$

to replace the original f in the theorem. Now we can define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

Put

$$Q_n(x) = c_n(1 - x^2)^n, \qquad n = 1, 2, 3, \ldots,$$

where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)\,\mathrm{d}x = 1, \qquad n = 1, 2, 3, \ldots.$$

Since

$$\begin{aligned}\int_{-1}^{1}(1 - x^2)^n\,\mathrm{d}x = 2\int_0^1 (1 - x^2)^n\,\mathrm{d}x &\geq 2\int_0^{1/\sqrt{n}} (1 - x^2)^n\,\mathrm{d}x \\ &\geq 2\int_0^{1/\sqrt{n}} (1 - nx^2)\,\mathrm{d}x \\ &= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}},\end{aligned}$$

we have $c_n < \sqrt{n}$, so that, for $0 < \delta \leq |x| \leq 1$,

$$Q_n(x) < \sqrt{n}(1 - x^2)^n \leq \sqrt{n}(1 - \delta^2)^n.$$

Hence $Q_n \to 0$ uniformly in $0 < \delta \leq |x| \leq 1$.

Put

$$P_n(x) = \int_{-x}^{1-x} f(x + t)Q_n(t)\,\mathrm{d}t = \int_0^1 f(t)Q_n(t - x)\,\mathrm{d}t.$$

It is clear that P_n is a polynomial in x. If f is a real function, then all P_n, $n = 1, 2, 3, \ldots$, are real functions. It remains to prove that $\{P_n\}$ converges uniformly to f on $[0, 1]$.

Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x)| < \varepsilon/2.$$

Let $M = \sup|f(x)|$. Then, for $0 \leq x \leq 1$,

$$\begin{aligned}|P_n(x) - f(x)| &= \left|\int_{-1}^{1}[f(x + t) - f(x)]Q_n(t)\,\mathrm{d}t\right| \\ &\leq \int_{-1}^{1}|f(x + t) - f(x)|Q_n(t)\,\mathrm{d}t \\ &= \left(\int_{-1}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{1}\right)|f(x + t) - f(x)|Q_n(t)\,\mathrm{d}t \\ &\leq 2M\int_{-1}^{-\delta} Q_n(t)\,\mathrm{d}t + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,\mathrm{d}t + 2M\int_{\delta}^{1} Q_n(t)\,\mathrm{d}t \\ &\leq 4M\sqrt{n}(1 - \delta^2)^n + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

for sufficiently large n. Thus, $\{P_n\}$ converges uniformly to f on $[0, 1]$. ◀

Justification: disk of convergence of power series

For the series $\sum a_n$, where $a_n = c_n(z - z_0)^n$, we see that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = |z - z_0| \cdot \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = \frac{|z - z_0|}{R}.$$

Thus, by applying the Root Test, the series $\sum c_n(z - z_0)^n$ converges if $\dfrac{|z - z_0|}{R} < 1$, and diverges if $\dfrac{|z - z_0|}{R} > 1$. The conclusion follows. ◀

Example: power series and radius of convergence

(1) The power series $\sum n^n z^n$ has $R = 0$ and converges only when $z = 0$.

(2) The power series $\sum \dfrac{z^n}{n!}$ has $R = \infty$ and converges for all complex z. In this case the ratio test is easier to apply than the root test.

(3) The power series $\sum z^n$ has $R = 1$. If $|z| = 1$, the series diverges by the Divergence Test, since $\lim z^n \neq 0$.

Justification: Abel's Test on the boundary of convergence of power series

Put $a_n = \dfrac{(z-z_0)^n}{R^n}$ and $b_n = c_n R^n$. If $|z-z_0| = R$, $z - z_0 \neq R$, then the partial sums A_n of $\sum a_n$ are bounded:

$$|A_n| = \left|\sum_{k=0}^{n} a_k\right| = \left|\sum_{k=0}^{n} \frac{(z-z_0)^k}{R^k}\right| = \left|\frac{1-\dfrac{(z-z_0)^{n+1}}{R^{n+1}}}{1-\dfrac{z-z_0}{R}}\right| \leq \frac{2|R|}{|R-(z-z_0)|}.$$

Hence, the hypotheses of Dirichlet's Test are satisfied for the series $\sum c_n(z-z_0)^n$ and the conclusion follows. ◀

Example: convergence of power series on the boundary of convergence

(1) The power series $\sum z^n$ has $R = 1$ and diverges at every point of $z = 1$.

(2) The power series $\sum \dfrac{z^n}{n}$ has $R = 1$ and diverges at $z = 1$ (harmonic series) while converges at $z = -1$ (alternating harmonic series).

(3) The power series $\sum \dfrac{z^n}{n^2}$ has $R = 1$. It converges for all z with $|z| = 1$, since $|z^2/n^2| = 1/n^2$ that gives a convergent p-series $(p = 2)$. ◀

Example: convergence of power series on the circle of convergence

(1) The power series $\sum \dfrac{z^n}{n}$ has $R = 1$. It diverges if $z = 1$ since it is the harmonic series, and converges for all other z with $|z| = 1$, by Abel's Test.

(2) The power series $\sum \dfrac{z^n}{n^2}$ has $R = 1$. It converges for all z with $|z| = 1$, since $|z^2/n^2| = 1/n^2$ that gives a convergent p-series ($p = 2$).

Justification: uniform convergence of power series

Let $\varepsilon > 0$ be given. By the Root Test, we know that if the series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges for $|x-x_0| < R$, then its radius of convergence is not smaller than R, and the series converges absolutely in the interior of its interval of convergence. Hence $\sum c_n(R-\varepsilon)^n$ converges absolutely. For $|x-x_0| \le R-\varepsilon$, since

$$|c_n(x-x_0)^n| \le |c_n(R-\varepsilon)^n|,$$

the Weierstrass M-Test implies that $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$.

By

$$\varlimsup_{n\to\infty} \sqrt[n]{n|c_n|} = \varlimsup \sqrt[n]{|c_n|},$$

the series $\sum c_n(x-x_0)^n$ and $\sum nc_n(x-x_0)^{n-1}$ have the same interval of convergence. By what we have just proved, we know that $\sum nc_n(x-x_0)^{n-1}$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ for $\varepsilon > 0$. By Theorem 7.9, we know that

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-x_0)^{n-1},$$

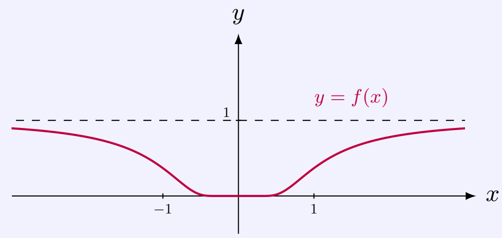
for $|x-x_0| \le R-\varepsilon$. But, for any given x such that $|x-x_0| < R$, we can find an $\varepsilon > 0$ such that $|x-x_0| < R-\varepsilon$. Thus, the above derivative holds for $|x-x_0| < R$.

Since f is differentiable, so it is continuous. ◀

Example: Taylor expansion that does not converge to the function

Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$



Prove that

$$f^{(n)}(x) = \begin{cases} e^{-\frac{1}{x^2}} P_{3n}(x^{-1}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where P_{3n} is a polynomial of degree $3n$.

Consequently, for any $x \neq 0$,

$$f(x) \neq 0 = 0 + 0 \cdot x + 0 \cdot x^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Proof We prove the expression of $f^{(n)}$ by induction.

When $n = 0$, the expression for f holds trivially. When $n = 1$, for $x \neq 0$, direct computation gives $f'(x) = \dfrac{2}{x^3} e^{-\frac{1}{x^2}}$, and

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{y \to \infty} \frac{y}{e^{y^2}} = 0,$$

so that the expression for f' also holds.

Suppose that the expression for $f^{(k)}$ holds, that is,

$$f^{(k)}(x) = \begin{cases} e^{-\frac{1}{x^2}} P_{3k}(x^{-1}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $P_{3k}(t)$ is a polynomial of degree $3k$ of t. Thus, for $x \neq 0$,

$$\begin{aligned} f^{(k+1)}(x) &= \left[e^{-\frac{1}{x^2}} P_{3k}(x^{-1}) \right]' \\ &= \frac{2}{x^3} e^{-\frac{1}{x^2}} P_{3k}(x^{-1}) + e^{-\frac{1}{x^2}} P'_{3k}(x^{-1}) \cdot (-\frac{1}{x^2}) \\ &= e^{-\frac{1}{x^2}} \left[2(x^{-1})^3 P_{3k}(x^{-1}) - (x^{-1})^2 P'_{3k}(x^{-1}) \right]. \end{aligned}$$

Since $P'_{3k}(t)$ is a polynomial of degree $3k - 1$ of t, if we put

$$Q(x^{-1}) = 2(x^{-1})^3 P_{3k}(x^{-1}) - (x^{-1})^2 P'_{3k}(x^{-1})$$

then $Q(t)$ is a polynomial of t, and the first product on the right-hand side produces the highest term of t for $Q(t)$, with the degree being $3k + 3 = 3(k+1)$. Hence, we may write $Q(t)$ as $P_{3(k+1)}(t)$, which is a polynomial of degree $3(k+1)$ of t. Thus, it follows that, for $x \neq 0$,

$$f^{(k+1)}(x) = e^{-\frac{1}{x^2}} P_{3(k+1)}(x^{-1}).$$

Moreover,

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}} P_{3k}(x^{-1})}{x} \\ &= \lim_{y \to \infty} \frac{y P_{3k}(y)}{e^{y^2}} = 0. \end{aligned}$$

This shows that the expression for $f^{(k+1)}$ holds.

Therefore, the expression for $f^{(n)}$ holds for all integer $n \geq 0$. ◀

Justification: continuity of power series at the endpoints

For the sake of simplicity, we prove the result in the case $x_0 = 0$.

Let $s_n = c_0 + c_1 + \cdots + c_n$, $s_{-1} = 0$. Then

$$\sum_{n=0}^{m} c_n x^n = \sum_{n=0}^{m} (s_n - s_{n-1}) x^n = (1-x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m.$$

We let $m \to \infty$ and obtain, for $|x| < 1$,

$$f(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n.$$

Let $\varepsilon > 0$ be given. Suppose $s = \lim\limits_{n\to\infty} s_n$. Choose N so that $n \geq N$ implies

$$|s_n - s| < \varepsilon/2.$$

Since $(1-x) \sum\limits_{n=0}^{\infty} x^n = 1$ for $|x| < 1$, we have

$$\begin{aligned} |f(x) - s| &= \left| (1-x) \sum_{n=0}^{\infty} (s_n - s) x^n \right| \\ &\leq (1-x) \sum_{n=0}^{N} |s_n - s|\, |x|^n + (1-x) \sum_{n=N+1}^{\infty} |s_n - s|\, |x|^n \\ &\leq (1-x) \sum_{n=0}^{N} |s_n - s| + (1-x) \cdot \frac{\varepsilon}{2} \sum_{n=N+1}^{\infty} |x|^n \\ &\leq (1-x) \sum_{n=0}^{N} |s_n - s| + \frac{\varepsilon}{2}. \end{aligned}$$

Since for fixed N, $\lim\limits_{x\to 1} (1-x) \sum\limits_{n=0}^{N} |s_n - s| = 0$, there is $\delta > 0$ such that, for $0 < 1 - x < \delta$,

$$(1-x) \sum_{n=0}^{N} |s_n - s| < \frac{\varepsilon}{2}.$$

Thus, if $0 < 1 - x < \delta$,

$$|f(x) - s| < \varepsilon,$$

which implies that $\lim\limits_{x\to 1^-} f(x) = s = \sum\limits_{n=0}^{\infty} c_n$. ◀

Justification: analyticity of power series in the disk of convergence

By

$$f(z) = \sum_{n=0}^{\infty} c_n[(z-z_1)+(z_1-z_0)]^n = \sum_{n=0}^{\infty} c_n \sum_{m=0}^{n} \binom{n}{m}(z_1-z_0)^{n-m}(z-z_1)^m,$$

if we can change the order of the summations, then we obtain a power series for the function about the point $x = a$:

$$f(z) = \sum_{m=0}^{\infty}\left[\sum_{n=m}^{\infty}\binom{n}{m}c_n(z_1-z_0)^{n-m}\right](z-z_1)^m.$$

Lemma

Given a double sequence $\{a_{ij}\}$, $i = 1,2,3,\dots$, $j = 1,2,3,\dots$, suppose that

$$\sum_{j=1}^{\infty}|a_{ij}| = b_j, \qquad i = 1,2,3,\dots,$$

and $\sum b_i$ converges. Then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

By the lemma above, the desired changing order of summation holds if

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left|c_n\binom{n}{m}(z_1-z_0)^{n-m}(z-z_1)^m\right|$$

converges. The latter series is the same as

$$\sum_{n=0}^{\infty}|c_n|\cdot(|z-z_1|+|z_1-z_0|)^n$$

which indeed converges if $|z-z_1|+|z_1-z_0| < R$, by the hypotheses.

Finally, since f has a convergent power series about $z = z_1$, the expression of the coefficients is given in the Corollary of Theorem 8.5.

Justification: change order of summation for double series

Take a countable set $E = \{x_0, x_1, x_2, \dots\}$ and assume $x_n \to x_0$ as $n \to \infty$. Define

$$f_i(x_0) = \sum_{j=1}^{\infty} a_{ij}, \qquad i = 1, 2, 3, \dots,$$

$$f_i(x_n) = \sum_{j=1}^{n} a_{ij}, \qquad i, n = 1, 2, 3, \dots,$$

$$g(x) = \sum_{i=1}^{\infty} f_i(x), \qquad x \in E.$$

For each fixed i, since $\sum\limits_{j=1}^{\infty} |a_{ij}|$ converges, we know that

$$\lim f_i(x_n) = f_i(x_0)$$

It is clear that for each i, f_i is continuous on x_0. Since x_0 is the only limit point of E, f_i is continuous on E. By the hypotheses, we see that $|f_i(x)| \le b_i$ for $x \in E$, so that $\sum f_i(x)$ converges uniformly on E. By Theorem 7.5, we have

$$\lim_{k\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x_k) = \lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=1}^{n} f_i(x_k).$$

The left hand side is

$$\begin{aligned}
\lim_{k\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x_k) &= \lim_{k\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \sum_{j=1}^{k} \sum_{i=1}^{n} a_{ij} \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} \left(\lim_{n\to\infty} \sum_{i=1}^{n} a_{ij} \right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.
\end{aligned}$$

Similarly, the right hand side is

$$\lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=1}^{n} f_i(x_k) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.$$

We obtain the desired identity. ◀

Justification: uniqueness of power series

For the sake of simplicity, we prove the result in the case $z_0 = 0$ and $R = 1$. Put $c_n = a_n - b_n$. Then, by the hypotheses, there is a function f such that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad z \in S,$$

and $f(z) = 0$ on E.

Let A be the set of all limit points of E in S, and let B consist of all other points of S. Then, for any point in B, there is a neighborhood that contains no point in A, otherwise it would be a limit point of A. Hence B is open. Suppose we can prove that A is also open. Then A and B are disjoint open sets. It is clear that both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. Thus, by the definition, A and B are separated. Since $S = A \cup B$ and S is connected, one of A and B must be empty. By hypothesis, A is not empty. Hence B is empty, and $A = S$. Since f is continuous in S, $A \subset E$, This gives $E = S$. This concludes that $c_n = 0$ for $n = 0, 1, 2, \ldots$.

It remains to prove that A is open. Let $a \in A$. Then $|a| < R$. Theorem 8.3 states that f can be expressed as a power series about $z = a$:

$$f(z) = \sum_{n=0}^{\infty} d_n (z-a)^n, \qquad |z-a| < R - |a|.$$

We claim that $d_n = 0$ for all n. Otherwise, let k be the smallest nonnegative integer such that $d_k \neq 0$. Then

$$f(z) = (z-a)^k g(z), \qquad |z-a| < R - |a|,$$

where

$$g(z) = \sum_{m=0}^{\infty} d_{k+m} (z-a)^m.$$

Since g is continuous at a and $g(a) = d_k \neq 0$, there exists $\delta > 0$ such that $g(z) \neq 0$ if $|z-a| < \delta$. Thus, $f(z) \neq 0$ in $|z-a| < \delta$ except at $z = a$. This contradicts to the fact that a is a limit point of E.

Thus $d_n = 0$ for all n, so that $f(z) = \sum_{n=0}^{\infty} d_n (z-a)^n = 0$ in a neighborhood of a. This shows that A is open, and completes the proof. ◀

Justification: properties of the exponential function

The Ratio Test shows that this series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for every complex z, so that the complex function E is well-defined on $\mathbb{C}$.

Lemma

1. For complex numbers z, w, the equality $E(z+w) = E(z)E(w)$ holds.
2. On $\mathbb{R}$, the function E is strictly increasing, and $E(x) > 0$ for all real x;
3. The function E is continuous and differentiable for all real x with $E'(x) = E(x)$;
4. $\lim_{x\to\infty} x^k E(x) = \infty$ and $\lim_{x\to-\infty} x^k E(x) = 0$ for any integer k. ▶

By repeatedly applying item 1 of the lemma above, we have

$$E(z_1 + \cdots + z_n) = E(z_1)\cdots E(z_n).$$

Putting $z_1 = \cdots = z_n = 1$, since $E(1) = e$ by Definition 3.19, we get

$$E(n) = e^n.$$

If $p = n/m$, where n, m are positive integers, then, by the lemma above,

$$[E(p)]^m = E(mp) = E(n) = e^n = e^{mp} = (e^p)^m.$$

Thus, we have

$$E(p) = e^p, \qquad p > 0,\ p \text{ rational},$$

by Proposition 1.11. Hence, $E(-p) = 1/E(p) = e^{-p}$. So, $E(p) = e^p$ for all rational p.

It is known that $e > 1$. For each real x, we define

$$e^x = \sup e^p,$$

where the sup is taken over all rational p such that $p \le x$. Thus, by the continuity and monotonicity properties of E as stated in the lemma, the equality $E(p) = e^p$ gives the desired relation:

$$E(x) = e^x$$

for all real x.

The properties stated in items 1 are the laws of exponents, while the properties stated in items 2 – 4 are given by the lemma. ◀

Justification: properties of the exponential function - lemma

Item 1 Applying Proposition 3.25 on multiplication of absolutely convergent series, we obtain

$$\begin{aligned} E(z)E(w) &= \left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{w^m}{m!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{z^k w^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}z^k w^{n-k} = \sum_{n=0}^{\infty}\frac{(z+w)^n}{n!} = E(z+w). \end{aligned}$$

Item 2 By the definition of E, it is clear that $E(x) = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!} > 0$ if $x > 0$. It follows from item 1 that, for any complex z, we have

$$E(z)E(-z) = E(z-z) = E(0) = 1.$$

Thus, for $x < 0$, we have $E(-x) = 1/E(x) > 0$. Hence, we know that $E(x) > 0$ for all real x.

For $x > 0$, it is clear that $E(x) = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$ is strictly increasing. For $x < 0$, it is also strictly increasing because of the equality $E(-x) = 1/E(x)$.

Item 3 For any real x, by item 1, we have

$$\begin{aligned} \frac{E(x+h)-E(x)}{h} &= E(x)\times\frac{E(h)-1}{h} \\ &= E(x)\times\frac{\sum\limits_{n=0}^{\infty}\frac{h^n}{n!}-1}{h} = E(x)\times\sum_{n=1}^{\infty}\frac{h^{n-1}}{n!}. \end{aligned}$$

Letting $h \to 0$, we get

$$[E(x)]' = E(x),$$

so that E is differentiable for all real x. The result of item 3 follows.

Item 4 It is clear that $E(x) = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!} \to \infty$ as $x \to \infty$. Thus, for any nonnegative integer k, $\lim\limits_{x\to\infty} x^k E(x) = \infty$. For any negative integer k, there is a sufficient large n such that $n+k > 0$. Since $x^k E(x) > \dfrac{x^{n+k}}{n!} \to \infty$ as $x \to \infty$, we also have $\lim\limits_{x\to\infty} x^k E(x) = \infty$.

By the equality

$$x^k E(x) = \frac{x^k}{E(-x)} = \frac{(-1)^k}{(-x)^{-k}E(-k)},$$

the limit $\lim\limits_{x\to-\infty} x^k E(x) = 0$ follows from the above limit $\lim\limits_{x\to\infty} x^k E(x) = \infty$.

◀

Justification: properties of the logarithmic function

As stated in Exercise **??**, a strictly monotonic continuous function on an interval has a strictly monotonic continuous inverse function. Hence, the function $L(x) = \ln x$, as the inverse of E, is well-defined and continuous on $E(\mathbb{R}) = (0, \infty)$. In fact, the function L is the logarithmic function with base e. According to Exercise **??**, the laws of logarithms (Item 1 of the theorem) hold.

By the equality $E(L(x)) = x$ for $x > 0$, the addition formula of E gives

$$x^n = E(L(x)) \cdots E(L(x)) = E(nL(x)) = e^{n \ln x}, \qquad \text{x} > 0 \text{ and } n \text{ positive integer.}$$

By Proposition 1.11 we have

$$x^p = E(pL(x)) = e^{p \ln x}, \qquad p > 0, p \text{ rational.}$$

Since $E(-x) = 1/E(x)$, we have $x^p = E(pL(x)) = e^{p \ln x}$ all rational p.

Let $x > 1$, as suggested in Exercise **??**, for α real, we define

$$x^\alpha = \sup x^p,$$

where the sup is taken over all rational p such that $p \le \alpha$. Thus, by the continuity and monotonicity properties of E as stated in the lemma, we have the relation:

$$x^\alpha = e^{\alpha \ln x} = E(\alpha L(x)), \qquad x > 1.$$

For $0 < x < 1$, define $x^\alpha = (x^{-1})^{-\alpha}$. Then

$$x^\alpha = (x^{-1})^{-\alpha} = E(-\alpha L(x^{-1}) = e^{-\alpha \ln(x^{-1})} = e^{\alpha \ln x} = E(\alpha L(x)).$$

Hence, for $x > 0$, we alway have $x^\alpha = E(\alpha L(x))$.

For the remaining properties, we prove them as follows.

Item 1 The properties stated in items 1 are the laws of logarithms.

Item 2 The monotonicity follows from Exercise **??**.

For $x > 1$, since $e > 1$, the equality $x = e^{\ln x}$ implies that $\ln x > 0$. For the same reason, for $0 < x < 1$, we have $\ln x < 0$.

Item 3 The differentiability of $\ln x$ follows from Proposition 5.5. To find the derivative, applying the chain rule to the equality $L(E(x)) = x$ (where $x \in \mathbb{R}$), we get $L'(E(x)) \cdot E'(x) = 1$, or $L'(E(x)) = 1/E'(x) = 1/e^x$. Replacing e^x by x, we have

$$(\ln x)' = \frac{1}{x}.$$

Item 4 Since $e > 1$ and $e^{\ln x} = x \to \infty$ as $x \to \infty$, we know that $\lim_{x\to\infty} \ln x = \infty$. Similarly, Since $e^{\ln x} = x \to 0$ as $x \to 0^+$, we have that $\lim_{x\to 0^+} \ln x = -\infty$.

To prove $\lim_{x\to\infty} x^{-k} \ln x = 0$ for any positive k, we take any ε satisfying $0 < \varepsilon < k$. Then, when $x > 1$, we have

$$x^{-k} \ln x = x^{-k} \int_1^x t^{-1}\,\mathrm{d}t < x^{-k} \int_1^x t^{\varepsilon-1}\,\mathrm{d}t = x^{-k} \cdot \frac{x^\varepsilon - 1}{\varepsilon} < \frac{x^{\varepsilon-k}}{\varepsilon},$$

which implies $\lim_{x\to\infty} x^{-k} \ln x = 0$.

To prove $\lim_{x\to 0^+} x^k \ln x = 0$ for any positive k, by making change of variable $x = y^{-1}$, we have

$$\lim_{x\to 0^+} x^k \ln x = \lim_{y\to\infty} (y^{-1})^k \ln(y^{-1}) = (-1) \cdot \lim_{y\to\infty} y^{-k} \ln y = (-1) \cdot 0 = 0.$$

◀

Justification: properties of the trigonometric functions

We establish the following lemmas:

Lemma 1

1. For real x, the functions $C(x)$ and $S(x)$ are real, and
$$E(ix) = C(x) + iS(x), \qquad [C(x)]^2 + [S(x)]^2 = 1.$$
2. $C(0) = 1$, $S(0) = 0$, and $C'(x) = -S(x)$, $S'(x) = C(x)$.

Lemma 2

Let x_0 be the smallest number such that $C(x_0) = 0$. Then the following properties hold:

1. Both C and S are periodic functions, with the smallest positive period $4x_0$.
2. The complex function $C(t) + iS(t)$ is 1-to-1 and onto from the set $[0, 4x_0)$ to the set $\{z \in \mathbb{C} \,|\, |z| = 1\}$.

From Lemma 1 and 2, what remain to prove are the following

1. For $\tilde{\pi} = 2x_0$, we have $\tilde{\pi} = \pi$.
2. For real x, $C(x) = \cos x$, $S(x) = \sin x$.

Part 1 By item 2 of Lemma 2 and item 1 of Lemma 1, the curve γ defined by
$$\gamma(t) = E(it), \qquad 0 \le t \le 2\tilde{\pi},$$
is a simple closed curve whose range is the unit circle in the plane.

Let γ be a given curve on $[a, b]$. For each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, we define
$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|.$$
which is the length of a polygonal path with vertices at $\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_n)$, in this order. Define the ***length*** (長度) of γ as
$$\Lambda(\gamma) = \sup_P \Lambda(P, \gamma).$$

length formula

Let γ be a curve defined on $[a, b]$. If γ' is continuous on $[a, b]$, then
$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \,\mathrm{d}t.$$

Since $\gamma'(t) = iE(it)$, the length of γ is
$$\int_0^{2\tilde{\pi}} |\gamma'(t)| \,\mathrm{d}t = \int_0^{2\tilde{\pi}} |E(it)| \,\mathrm{d}t = \int_0^{2\tilde{\pi}} 1 \,\mathrm{d}t = 2\tilde{\pi}.$$
On the other hand, since the circumference of the unit circle is 2π. Hence, we have $\tilde{\pi} = \pi$.

Part 2 More generally we notice that $E(it)$ parametrizes the unit circle by arc length. That is, t measures the arc length in radians. Hence the definitions of S and C we have used above agree with the standard geometric definitions of the sine and cosine functions:
$$C(t) = \cos t, \qquad S(t) = \sin t.$$

Justification: properties of the trigonometric functions - lemma 1

Item 1 By the definition of the complex function E, we see that

$$\overline{E(z)} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \sum_{n=0}^{\infty} \frac{(\overline{z})^n}{n!} = E(\overline{z}).$$

Thus, for real x,

$$\overline{C(x)} = \frac{1}{2}[\overline{E(ix)} + \overline{E(-ix)}] = \frac{1}{2}[E(-ix) + E(ix)] = C(x),$$

so C is real. Similarly, $S(x)$ is real for real x. It is easy to have

$$C(x) + iS(x) = \frac{1}{2}[E(ix) + E(-ix)] + i \cdot \frac{1}{2i}[E(ix) - E(-ix)] = E(ix),$$

so that $C(x) = \operatorname{Re} E(ix)$ and $S(x) = \operatorname{Im} E(ix)$. Since

$$|E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = E(ix - ix) = E(0) = 1,$$

we have $[C(x)]^2 + [S(x)]^2 = 1$.

Item 2 From the definition of the function C, we have

$$C(0) = \frac{1}{2}[E(0) + E(0)] = 1.$$

Similarly, we have $S(0) = 0$.

Direct computation gives, for real x,

$$C(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!}\right] = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!},$$

$$S(x) = \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!}\right] = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

By applying Theorem 8.5, we get $C'(x) = -S(x)$ and $S'(x) = C(x)$. ◀

Justification: properties of the trigonometric functions - lemma 2

Item 1 We assert that there exist positive numbers x such that $C(x) = 0$. If it is not the case, since $C(0) = 1$, it then follows that $C(x) > 0$ for all $x > 0$, hence $S'(x) > 0$ by item 2 of Lemma 2, so that is strictly increasing. Thus, $S(x) > S(0) = 0$ for $x > 0$. Hence, if $0 < x < y$, we have

$$S(x)(y-x) < \int_x^y S(t)\,\mathrm{d}t = C(x) - C(y) \le 2.$$

Since $S(x) > 0$, the last inequality cannot be true for large y, and we have a contradiction.

Let x_0 be the smallest positive number such that $C(x_0) = 0$. This exists, since the set of zeros of a continuous function is closed (Exercise 4.5), and $C(0) \neq 0$. We define the number $\tilde{\pi}$ by $\boxed{\tilde{\pi} = 2x_0}$, so that $C(\tilde{\pi}/2) = 0$. By item 1 of Lemma 1, $S(\tilde{\pi}/2) = \pm 1$. Since S is increasing in $(0, \tilde{\pi}/2)$, we have $S(\tilde{\pi}/2) = 1$. Thus, $E(\tilde{\pi}i/2) = i$.

$$E(\tilde{\pi}i) = E(\tilde{\pi}i/2)E(\tilde{\pi}i/2) = -1, \qquad E(2\tilde{\pi}i) = E(\tilde{\pi}i)E(\tilde{\pi}i) = 1.$$

Hence, for any complex z,

$$E(z + 2\tilde{\pi}i) = E(z)E(2\cdot\tilde{\pi}i) = E(z).$$

It follows that the function C is of $2\tilde{\pi}$ periodic, since

$$\begin{aligned} C(x+2\tilde{\pi}) &= \frac{1}{2}\left[E(i(x+2\tilde{\pi})) + E(-i(x+2\tilde{\pi}))\right] \\ &= \frac{1}{2}\left[E(ix+2\tilde{\pi}i) + E(-ix-2\tilde{\pi}i)\right] = \frac{1}{2}\left[E(ix) + E(-ix)\right] = C(x). \end{aligned}$$

Similarly, the function S is also of $2\tilde{\pi}$ periodic.

We assert that C and S are not periodic with a smaller period. It would be enough to show that if $E(ix) = 1$ for the smallest positive x, then $x = 2\tilde{\pi}$. Obviously, $x \le 2\tilde{\pi}$. By the addition formula of E, we have $[E(ix/4)]^4 = 1$. If $E(ix/4) = a + ib$, then

$$(a+ib)^4 = a^4 - 6a^2b^2 + b^4 + i\cdot 4ab(a^2 - b^2) = 1.$$

As $x/4 < \tilde{\pi}/2$, then $a = C(x/4) \ge 0$ and $b = S(x/4) > 0$. Thus, either $a^2 = b^2$ so that $1 = a^4 - 6a^2b^2 + b^4 = -4a^4$ which is impossible, or $a = 0$, in which case $x/4 = x_0 = \tilde{\pi}/2$ so that $x = 2\tilde{\pi}$.

Item 2 First, $E(it)$ is 1-to-1 from $[0, 2x_0)$ to $\{z \in \mathbb{C} \,|\, |z| = 1\}$: If $E(ix) = E(iy)$ with $x > y$, then $E(i(x-y)) = 1$, so that $x - y$ is a multiple of $2\tilde{\pi}$. This leads to $x = y$.

Next, $E(it)$ is onto from $[0, 2x_0)$ to $\{z \in \mathbb{C} \,|\, |z| = 1\}$: Pick real numbers a and b such that $a^2 + b^2 = 1$.

(1) Suppose $a \ge 0$ and $b \ge 0$. By the Intermediate Value Theorem, there must exist $t \in [0, \tilde{\pi}/2]$ such that $a = C(t)$. Thus, $b^2 = 1 - [C(t)]^2 = [S(t)]^2$ by item 1 of Lemma 1. Since both b and $S(t)$ are nonnegative, we get $b = S(t)$.

(2) Suppose $a < 0$ and $b \ge 0$. Then $-i(a+ib) = b - ia = E(it)$ for some $t \in [0, \tilde{\pi}/2]$ by the preceding case. Since $i = E(\tilde{\pi}i/2)$, we have $a + ib = i(b - ia) = E(\tilde{\pi}i/2)E(it) = E(i(t + \tilde{\pi}/2))$, with $t + \tilde{\pi}/2 \in [\tilde{\pi}/2, \tilde{\pi}]$.

(3) Suppose $b < 0$. $-(a+ib) = -a + i(-b)$. By the preceding two cases, there exists some $t \in (0, \tilde{\pi})$ such that $-(a+ib) = E(it)$. Hence, $a + ib = -E(it) = E(\tilde{\pi}i)E(it) = E(i(t+\tilde{\pi}))$, with $t + \tilde{\pi} \in (\tilde{\pi}, 2\tilde{\pi})$. ◀

Justification: length formula

For any given partition P of $[a,b]$,

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\, \mathrm{d}t \right| \le \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, \mathrm{d}t.$$

Hence,

$$\Lambda(P,\gamma) = \sum |\gamma(x_i) - \gamma(x_{i-1})| \le \sum \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, \mathrm{d}t = \int_a^b |\gamma'(t)| \, \mathrm{d}t,$$

which implies

$$\Lambda(\gamma) \le \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Since γ' is uniformly continuous on $[a,b]$, there exists $\delta > 0$ such that if $|s-t| < \delta$, $s,t \in [a,b]$, then

$$|\gamma'(s) - \gamma'(t)| < \varepsilon.$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ such that $\Delta x_i < \delta$ for all i. If $x_{i-1} \le t \le x_i$, then

$$|\gamma'(t)| \le |\gamma'(x_i)| + |\gamma'(t) - \gamma'(x_i)| < |\gamma'(x_i)| + \varepsilon.$$

Hence,

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\gamma'(t)| \, \mathrm{d}t &\le |\gamma'(x_i)| \, \Delta x_i + \varepsilon \Delta x_i \\
&= \left| \int_{x_{i-1}}^{x_i} [\gamma'(t) + \gamma'(x_i) - \gamma'(t)] \, \mathrm{d}t \right| + \varepsilon \Delta x_i \\
&\le \left| \int_{x_{i-1}}^{x_i} \gamma'(t) \, \mathrm{d}t \right| + \left| \int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)] \, \mathrm{d}t \right| + \varepsilon \Delta x_i \\
&\le |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon \Delta x_i.
\end{aligned}$$

Adding these inequalities gives

$$\int_a^b |\gamma'(t)| \, \mathrm{d}t \le \Lambda(P,\gamma) + 2\varepsilon(b-a) \le \Lambda(\gamma) + 2\varepsilon(b-a).$$

Since ε is arbitrary, we obtain

$$\int_a^b |\gamma'(t)| \, \mathrm{d}t \le \Lambda(\gamma).$$

This completes the proof. ◀

Example: orthogonal systems

(1) The functions e^{-inx}, $n = 0, \pm 1, \pm 2, \ldots$, form an orthogonal system on $[-\pi, \pi]$.

In fact, if $m \neq n$, then

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-inx} \cdot \overline{e^{-imx}}\, \mathrm{d}x = \int_{-\pi}^{\pi} e^{i(m-n)x}\, \mathrm{d}x = \left.\frac{e^{i(m-n)x}}{i(m-n)}\right|_{-\pi}^{\pi} \\ = \frac{1}{i(m-n)}\left[e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right] \\ = \frac{1}{i(m-n)e^{i(m-n)\pi}}\left[e^{2i(m-n)\pi} - 1\right] = 0.\end{aligned}$$

(2) The functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

form an orthogonal system on $[-\pi, \pi]$.

To show that the given functions form an orthogonal system, we need to use the following product to sum formulas

$$\begin{aligned}\sin\alpha\, \cos\beta &= \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right], \\ \sin\alpha\, \sin\beta &= -\frac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right], \\ \cos\alpha\, \cos\beta &= \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right].\end{aligned}$$

These give the following integrals.

① For any integers $m \geq 1$ and $n \geq 0$,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx\, \cos nx\, \mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} \left[\sin(m+n)x + \sin(m-n)x\right]\, \mathrm{d}x \\ &= \begin{cases} \dfrac{1}{2}\left[-\dfrac{\cos(m+n)x}{m+n} - \dfrac{\cos(m-n)x}{m-n}\right]\Bigg|_{-\pi}^{\pi}, & \text{if } m \neq n, \\ \dfrac{1}{2}\left[-\dfrac{\cos 2mx}{2m}\right]\Bigg|_{-\pi}^{\pi}, & \text{if } m = n \end{cases} = 0.\end{aligned}$$

② For any integers $m \geq 1$ and $n \geq 1$, with $m \neq n$,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx\, \sin nx\, \mathrm{d}x &= -\frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m+n)x - \cos(m-n)x\right]\, \mathrm{d}x \\ &= -\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} - \frac{\sin(m-n)x}{m-n}\right]\Bigg|_{-\pi}^{\pi} = 0.\end{aligned}$$

③ For any integers $m \geq 0$ and $n \geq 0$, with $m \neq n$,

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx\, \cos nx\, \mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m+n)x + \cos(m-n)x\right]\, \mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]\Bigg|_{-\pi}^{\pi} = 0.\end{aligned}$$

The orthogonality of the given functions follows from these three equalities.

(3) The functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots,$$

form an orthonormal system on $[-\pi, \pi]$.

We notice the functions given in (2) are orthogonal. Furthermore, we have the following equalities. ◀

① For any integers $n \geq 1$,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2 nx \,\mathrm{d}x &= -\frac{1}{2}\int_{-\pi}^{\pi} (\cos 2nx - 1)\,\mathrm{d}x \\ &= -\frac{1}{2}\left(\frac{\sin 2nx}{2n} - x\right)\Big|_{-\pi}^{\pi} = \pi.\end{aligned}$$

② For any integers $n \geq 0$,

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2 nx \,\mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} (\cos 2nx + 1)\,\mathrm{d}x \\ &= \begin{cases} \frac{1}{2}\left(\frac{\sin 2nx}{2n} + x\right)\Big|_{-\pi}^{\pi}, & \text{if } n \neq 0, \\ \frac{1}{2}\,(2x)\big|_{-\pi}^{\pi}, & \text{if } n = 0 \end{cases} = \begin{cases} \pi, & \text{if } n \neq 0, \\ 2\pi, & \text{if } n = 0. \end{cases}\end{aligned}$$

Thus, use these values to rescale to the functions in (2), we get the functions, given at the beginning of (3), that form an orthonormal system. ◀

Justification: Bessel's inequality - lemma

Since $\{\phi_n\}$ is orthonormal, elementary calculations give

$$\begin{aligned}\int_a^b f\cdot\bar{t}_n\,\mathrm{d}x &= \int_a^b f\cdot\left(\sum_{m=1}^{n}\overline{\gamma}_m\overline{\phi}_m\right)\,\mathrm{d}x\\ &= \sum_{m=1}^{n}\overline{\gamma}_m\int_a^b f\cdot\overline{\phi}_m\,\mathrm{d}x = \sum_{m=1}^{n}c_m\overline{\gamma}_m,\end{aligned}$$

and

$$\begin{aligned}\int_a^b |t_n|^2\,\mathrm{d}x &= \int_a^b t_n\cdot\bar{t}_n\,\mathrm{d}x\\ &= \int_a^b\left(\sum_{m=1}^{n}\gamma_m\phi_m\right)\cdot\left(\sum_{m=1}^{n}\overline{\gamma}_m\overline{\phi}_m\right)\,\mathrm{d}x = \sum_{m=1}^{n}|\gamma_m|^2.\end{aligned}$$

Hence

$$\begin{aligned}\int_a^b |f-t_n|^2\,\mathrm{d}x &= \int_a^b (f-t_n)\cdot(\overline{f}-\bar{t}_n)\,\mathrm{d}x\\ &= \int_a^b |f|^2\,\mathrm{d}x - \int_a^b f\cdot\bar{t}_n\,\mathrm{d}x - \int_a^b \overline{f}\cdot t_n\,\mathrm{d}x + \int_a^b |t_n|^2\,\mathrm{d}x\\ &= \int_a^b |f|^2\,\mathrm{d}x - \sum_{m=1}^{n}c_m\overline{\gamma}_m - \sum_{m=1}^{n}\overline{c}_m\gamma_m + \sum_{m=1}^{n}|\gamma_m|^2\\ &= \int_a^b |f|^2\,\mathrm{d}x - \sum_{m=1}^{n}|c_m|^2 + \sum_{m=1}^{n}|\gamma_m-c_m|^2.\end{aligned}$$

In particular, if $\gamma_m = c_m$, we have

$$\int_a^b |f-s_n|^2\,\mathrm{d}x = \int_a^b |f|^2\,\mathrm{d}x - \sum_{m=1}^{n}|c_m|^2.$$

These give

$$\int_a^b |f-t_n|^2\,\mathrm{d}x = \int_a^b |f-s_n|^2\,\mathrm{d}x + \sum_{m=1}^{n}|\gamma_m-c_m|^2.$$

The results follow this inequality. ◀

Justification: Bessel's inequality

In the proof of the lemma, we have

$$0 \leq \int_a^b |f - s_n|^2 \, \mathrm{d}x = \int_a^b |f|^2 \, \mathrm{d}x - \sum_{m=1}^{n} |c_m|^2.$$

Let $n \to \infty$, we have Bessel's inequality. ◀

Justification: triangle inequality of L^2 norm

We shall prove the following equivalent inequality

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2.$$

For any $f, g \in \mathscr{R}[a,b]$, we define the mapping $\langle , \rangle \colon \mathscr{R}[a,b] \times \mathscr{R}[a,b] \to \mathbb{R}$ by

$$\langle f, g \rangle = \int_a^b f\, g \,\mathrm{d}x.$$

The mapping is well-defined, by Exercise 6.4. It is obvious that $\langle f, f \rangle = \|f\|_2^2 \geq 0$. For fixed f and g, consider the real function F on $\mathbb{R}$, defined by

$$F(t) = \langle f + tg, f + tg \rangle.$$

By the definition, we have

$$F(t) = \langle f, f \rangle + 2t\langle f, g \rangle + t^2 \langle g, g \rangle.$$

The function F is a real quadratic function in t. Since $F \geq 0$, we know that the discriminant satisfies

$$4\langle f, g \rangle^2 \leq 4\langle f, f \rangle \cdot \langle g, g \rangle,$$

which implies

$$\langle f, g \rangle \leq \|f\|_2 \, \|g\|_2.$$

Hence,

$$\begin{aligned}
\|f+g\|_2^2 &= \langle f+g, f+g \rangle \\
&= \langle f, g \rangle + 2\langle f, g \rangle + \langle g, g \rangle \\
&\leq \|f\|_2^2 + 2\|f\|_2 \, \|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2,
\end{aligned}$$

which gives the triangle inequality

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2.$$

◀

Justification: Dirichlet kernel

The Dirichlet kernel is

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = e^{-iNx}[1 + e^{ix} + \cdots + e^{2iNx}]$$
$$= e^{-iNx} \cdot \frac{1 - e^{2iNx+ix}}{1 - e^{ix}} = \frac{\sin(N + \frac{1}{2})x}{\sin(\frac{1}{2}x)}.$$

Thus,

$$s_N(f;x) = \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}te^{inx}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-N}^{N} e^{in(x-t)}\mathrm{d}t$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)D_N(x-t)\,\mathrm{d}t.$$

For f being 2π-periodic, by a change of variable, the last integral is the same as

$$s_N(f;x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_N(t)\,\mathrm{d}t.$$

◀

Justification: pointwise convergence of Fourier series

By a change of variable, since $D_N(-t) = D_N(t)$, it is easy to have

$$s_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)D_N(t)\,\mathrm{d}t.$$

Direct calculation gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-N}^{N} e^{inx}\,\mathrm{d}x = 1.$$

Thus,

$$\begin{aligned}
&s_N(f;x) - \frac{f(x+)+f(x-)}{2}\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(x+t)+f(x-t)}{2}\cdot D_N(t)\,\mathrm{d}t - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(x+)+f(x-)}{2}\cdot D_N(t)\,\mathrm{d}t\\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{f(x+t)+f(x-t)-f(x+)-f(x-)}{\sin(\frac{1}{2}t)}\cdot \sin(N+\tfrac{1}{2})t\,\mathrm{d}t
\end{aligned}$$

If we put

$$g(t) = \begin{cases} \dfrac{f(x+t)+f(x-t)-f(x+)-f(x-)}{\sin(\frac{1}{2}t)}, & \text{if } t \neq 0,\\[2ex] 0, & \text{if } t = 0,\end{cases}$$

by the hypothesis, then g is bounded, so that $g \in \mathscr{R}$ on $[-\pi, \pi]$. Re-writing the expression as

$$s_N(f;x) - \frac{f(x+)+f(x-)}{2} = \frac{1}{4\pi}\int_{-\pi}^{\pi}\big[g(t)\cos(\tfrac{1}{2}t)\big]\sin Nt\,\mathrm{d}t + \frac{1}{4\pi}\int_{-\pi}^{\pi}\big[g(t)\sin(\tfrac{1}{2}t)\big]\cos Nt\,\mathrm{d}t,$$

by Bessel's inequality, we know that both integrals tends to 0 as $N \to \infty$. ◀

Justification: uniform approximation by trigonometric polynomials

Consider

$$u_n(x) = \frac{1}{\alpha_n}\left(\frac{1+\cos x}{2}\right)^n, \qquad n = 1, 2, 3, \ldots,$$

where $\alpha_n = \int_{-\pi}^{\pi}\left(\frac{1+\cos x}{2}\right)^n \mathrm{d}x$. Then $u_n(x) \geq 0$, $u_n(x+2\pi) = u_n(x)$, and $\int_{-\pi}^{\pi} u_n(x)\,\mathrm{d}x = 1$. We claim that for any fixed $\delta \in (0,\pi)$, $\int_{\delta\leq|x|\leq\pi} u_n(x)\,\mathrm{d}x \to 0$ as $n \to \infty$. In fact, since $1+\cos x$ is decreasing on $[0,\pi]$, for any $x \in [\delta,\pi]$ and $y \in [0,\delta/2]$, we have $\frac{1+\cos x}{1+\cos y} \leq \frac{1+\cos\delta}{1+\cos(\delta/2)}$, so that

$$1+\cos x < r(1+\cos y),$$

where $r = \frac{1+\cos\delta}{1+\cos(\delta/2)} < 1$. Taking powers and dividing by α_n, we have $u_n(x) \leq r^n u_n(y)$. This implies

$$\frac{\delta}{2}u_n(x) = \int_0^{\delta/2} u_n(x)\,\mathrm{d}y \leq r^n \int_0^{\delta/2} u_n(y)\,\mathrm{d}y \leq r^n \to 0,$$

as $n \to \infty$. Thus $u_n(x) \to 0$ uniformly on $[\delta,\pi]$ as $n \to \infty$, so that $\int_{\delta\leq|x|\leq\pi} u_n(x)\,\mathrm{d}x \to 0$ by Theorem 7.8.

By Euler's formula $e^{ix} = \cos x + i\sin x$, we see that $u_n(x) = \sum_{m=-n}^{n} \beta_m e^{imx}$ for some constants β_m. It is known that f is continuous and 2π-periodic. If we put

$$P_n(x) = \int_{-\pi}^{\pi} u_n(y) f(x-y)\,\mathrm{d}y,$$

then P_n is a trigonometric polynomial for each n, since

$$\begin{aligned} P_n(x) &= (-1)\int_{x+\pi}^{x-\pi} u_n(x-z) f(z)\,\mathrm{d}z \\ &= \int_{x-\pi}^{x+\pi} \sum_{m=-n}^{n} \beta_m e^{im(x-z)} f(z)\,\mathrm{d}z = \sum_{m=-n}^{n} \beta_m e^{imx} \int_{-\pi}^{\pi} f(z) e^{-inz}\,\mathrm{d}z. \end{aligned}$$

Let $\varepsilon > 0$ be given. We shall prove that there is N such that $n \geq N$ implies

$$|P_n(x) - f(x)| < \varepsilon,$$

for all $x \in [-\pi,\pi]$. Then the assertion of the theorem follows.

Since f is uniformly continuous on $[-2\pi, 2\pi]$, there exists $\delta > 0$ such that for all $x \in [-\pi,\pi]$ and y with $|y| < \delta$,

$$|f(x-y) - f(x)| < \varepsilon/2.$$

Put $M = \sup_{-\infty<y<\infty} |f(y)| < \infty$. Then, from the discussion above, there is N such that $n \geq N$ implies

$$\int_{\delta\leq|y|\leq\pi} u_n(y)\,\mathrm{d}y < \varepsilon/(4M).$$

Thus, for $n \geq N$,

$$\begin{aligned} |P_n(x) - f(x)| &= \left|\int_{-\pi}^{\pi} u_n(y) f(x-y)\,\mathrm{d}y - \int_{-\pi}^{\pi} u_n(y) f(x)\,\mathrm{d}y\right| \\ &\leq \int_{-\pi}^{\pi} u_n(y)\,|f(x-y) - f(x)|\,\mathrm{d}y \\ &= \int_{|y|<\delta} u_n(y)\,|f(x-y) - f(x)|\,\mathrm{d}y + \int_{\delta\leq|y|\leq\pi} u_n(y)\,|f(x-y) - f(x)|\,\mathrm{d}y \\ &< \frac{\varepsilon}{2}\cdot\int_{|y|<\delta} u_n(y)\,\mathrm{d}y + 2M\int_{\delta\leq|y|\leq\pi} u_n(y)\,\mathrm{d}y \\ &< \frac{\varepsilon}{2}\cdot 1 + 2M\cdot\frac{\varepsilon}{4M} = \varepsilon. \end{aligned}$$

◀

Justification: uniform convergence of Fourier series

For simplicity, we assume that f' has only one discontinuity at $a \in [-\pi, \pi]$. The proof can be easily extended to cover functions with multiple discontinuities.

We expand f as a Fourier series, $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$, with $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$.

The proof is divided into two steps:

1. Prove that the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly.

2. Let $S(x)$ be the uniform limit of $\sum_{n=-\infty}^{\infty} c_n e^{inx}$. Prove $f(x) = S(x)$ for all x.

Step 1 By the Weierstrass M-Test, it is sufficient to show that $\sum |c_n|$ converges. In, fact, since f is continuously differentiable on $(-\pi, a)$ and (a, π), we integrate by parts on each of these intervals to have

$$\begin{aligned} 2\pi c_n &= \int_{-\pi}^{a} f(x)e^{-inx}\,dx + \int_{a}^{\pi} f(x)e^{-inx}\,dx \\ &= \left[f(x)\frac{e^{-inx}}{-in}\bigg|_{x=-\pi}^{a} - \int_{-\pi}^{a} f'(x)\frac{e^{-inx}}{-in}\,dx \right] + \left[f(x)\frac{e^{-inx}}{-in}\bigg|_{x=a}^{\pi} - \int_{a}^{\pi} f'(x)\frac{e^{-inx}}{-in}\,dx \right]. \end{aligned}$$

Put $\gamma_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx$. Since $f(-\pi) = f(\pi)$, the above calculation gives

$$c_n = \frac{\gamma_n}{in}, \qquad n = \pm 1, \pm 2, \pm 3, \ldots.$$

By Bessel's inequality for f', we have

$$\sum_{n=-N}^{N} |\gamma_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2\,dx < \infty.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{1\le|n|\le N} |c_n| = \sum_{1\le|n|\le N} \frac{1}{n}|\gamma_n| \le \left(\sum_{1\le|n|\le N} \frac{1}{n^2}\right)^{1/2} \left(\sum_{1\le|n|\le N} |\gamma_n|^2\right)^{1/2} < \infty,$$

so that $\sum |c_n|$ converges.

Step 2 By the corollary of Theorem 7.5, the uniform limit $S(x)$ of $\sum_{-\infty}^{\infty} c_n e^{inx}$ is continuous. For $s_N(f; x) = \sum_{n=-N}^{N} c_n e^{inx}$, we know that the limit of $\int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2\,dx$ exists, since the value of the integral decreases as N increases by the lemma of Theorem 8.10. Thus, for any N,

$$\int_{-\pi}^{\pi} |f(x) - S(x)|^2\,dx = \lim_{N\to\infty} \int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2\,dx \le \int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2\,dx.$$

It follows, again the lemma of Theorem 8.10, that for any trigonometric polynomial P,

$$\int_{-\pi}^{\pi} |f(x) - S(x)|^2\,dx \le \int_{-\pi}^{\pi} |f(x) - P(x)|^2\,dx.$$

Let $\varepsilon > 0$ be given. By Theorem 8.13, there is a trigonometric polynomial P such that $|P(x) - f(x)| < \varepsilon$ for all x. Thus,

$$\int_{-\pi}^{\pi} |f(x) - S(x)|^2\,dx < 2\pi\varepsilon^2,$$

which implies the value of the integral is zero. We conclude that $f(x) = S(x)$ for all x, since $f(x)$ and $S(x)$ are continuous. ◀

Justification: Parseval's identity

Let $\varepsilon > 0$ be given. We show that for 2π-periodic Riemann-integrable function f, there is a function h that is continuous, 2π-periodic and piecewise differentiable such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-h(x)|^2\,\mathrm{d}x<\varepsilon.$$

In fact, we only prove this assertion for real f. If f is complex, we just need to apply the conclusion for real and imaginary parts separately.

Since $f \in \mathscr{R}$ on $[-\pi,\pi]$, there is a partition $P=\{x_0,\ldots,x_n\}$ of $[-\pi,\pi]$, such that

$$U(P,f)-L(P,f)=\sum_{n=1}^{n}(M_i-m_i)\Delta x_i<\frac{\pi\varepsilon}{M},$$

where $M_i=\sup\limits_{x_{i-1}\le x\le x_i} f(x)$, $m_i=\inf\limits_{x_{i-1}\le x\le x_i} f(x)$, and $M=\sup\limits_{-\pi\le x\le\pi}|f(x)|$. Now we define a piecewise linear (continuous) function on $[x_{i-1},x_i]$ by

$$h(x)=\frac{x_i-x}{x_i-x_{i-1}}f(x_{i-1})+\frac{x-x_{i-1}}{x_i-x_{i-1}}f(x_i).$$

The facts that $h(x_{i-1})=f(x_{i-1})$, $h(x_i)=f(x_i)$, and f is 2π-periodic imply that h can be continuously extended to the whole real line. We still denote the extended function as h. Hence we have a continuous, 2π-periodic, and piecewise differentiable function h. To see the estimate, we notice that on $[x_{i-1},x_i]$, $m_i\le h(x)\le M_i$, $1\le i\le n$. Hence,

$$\int_{-\pi}^{\pi}|f(x)-h(x)|^2\,\mathrm{d}x\le\sum(M_i-m_i)^2\Delta x_i\le 2M\sum(M_i-m_i)\Delta x_i<2\pi\varepsilon.$$

To prove the limit in the theorem, for the function h, we know that, by Theorem 8.14, there exists N such that

$$|h(x)-s_N(h;x)|<\sqrt{\varepsilon},\qquad n\ge N,$$

for all $x\in[-\pi,\pi]$. By Bessel's inequality, we know that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|s_N(f;x)-s_N(h;x)|^2\,\mathrm{d}x=\frac{1}{2\pi}\int_{-\pi}^{\pi}|s_N(f-h;x)|^2\,\mathrm{d}x\le\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-h(x)|^2\,\mathrm{d}x<\varepsilon.$$

Thus, when $n\ge N$, by writing $f(x)-s_N(f;x)=f(x)-h(x)+h(x)-s_N(h;x)+s_N(h;x)-s_N(f;x)$, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-s_N(f;x)|^2\,\mathrm{d}x\\
&\le\frac{3}{2\pi}\left[\int_{-\pi}^{\pi}|f(x)-h(x)|^2\,\mathrm{d}x+\int_{-\pi}^{\pi}|h(x)-s_N(h;x)|^2\,\mathrm{d}x+\int_{-\pi}^{\pi}|s_N(h;x)-s_N(f;x)|^2\,\mathrm{d}x\right]\\
&<3(\varepsilon+\varepsilon+\varepsilon)=9\varepsilon.
\end{aligned}$$

This gives $\lim\limits_{N\to\infty}\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}|f(x)-s_N(f;x)|^2\,\mathrm{d}x=0$. Finally, Parseval's identity follows from the following:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-s_N(f;x)|^2\,\mathrm{d}x=\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-s_N(f;x)]\cdot\overline{[f(x)-s_N(f;x)]}\,\mathrm{d}x\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,\mathrm{d}x-\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\cdot\left(\sum_{n=-N}^{N}\overline{c}_ne^{-inx}\right)\mathrm{d}x\\
&\quad-\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-N}^{N}c_ne^{inx}\right)\cdot\overline{f(x)}\,\mathrm{d}x+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-N}^{N}c_ne^{inx}\right)\cdot\left(\sum_{n=-N}^{N}\overline{c}_ne^{-inx}\right)\mathrm{d}x\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,\mathrm{d}x-\sum_{n=-N}^{N}|c_n|^2.
\end{aligned}$$

◀

Appendix B: historical notes

Historical Note: Field

The concept of a **field** in mathematics was introduced by the German mathematician, Ernst Steinitz (1871–1928), in the early 20th century. Steinitz's work on fields was influenced by the earlier work of mathematicians such as Richard Dedekind (1831–1916) and David Hilbert (1862–1943), who had studied algebraic number theory and the theory of algebraic equations. ◀

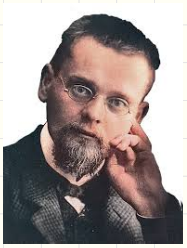
Steinitz
(1871–1928)

Dedekind
(1831–1916)

Hilbert
(1862–1943)

Historical Note: Ordered Field

The concept of an **ordered field** in mathematics was introduced by the German mathematician, Ernst Friedrich Ferdinand Zermelo (1871–1953), in 1904. Zermelo's work on ordered fields was motivated by his interest in the foundations of mathematics and set theory. He showed that any ordered field can be embedded in the real numbers, which he used to prove the well-ordering theorem for sets. ◄



Zermelo
(1871–1953)

Historical Note: the least-upper-bound property

The concepts of **supremum, infimum, and least-upper-bound** were introduced by several mathematicians in the 19th century. The German mathematician, Karl Weierstrass (1815–1897), introduced the concepts of sup and inf in his development of the theory of real analysis. The German mathematician, Richard Dedekind (1831–1916), introduced the concept of the least upper bound, also known as the supremum, as part of his work on the foundations of the real numbers. Dedekind used the least-upper-bound property to define the real numbers in terms of cuts, which are partitions of the rational numbers into two non-empty sets that satisfy certain conditions. ◀

Weierstrass (1815–1897)

Dedekind (1831–1916)

Historical Note: the least-upper-bound property of $\mathbb{R}$

The first proof that **the set of real numbers has the least-upper-bound property** is attributed to the German mathematician, Georg Cantor (1845–1918), in the late 19th century. Cantor's proof used the concept of a nested interval, which is a sequence of closed intervals that contain each other, and the completeness axiom of the real numbers. The proof is considered to be one of the most elegant and insightful proofs in mathematics and has had a profound impact on the development of analysis and topology. ◀

Cantor
(1845–1918)

Historical Note: the archimedean property

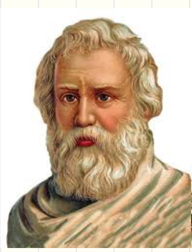
The **archimedean property** is named after the ancient Greek mathematician, Archimedes of Syracuse (c. 287–c. 212 BC). It was Otto Stolz (in the 1880s) who gave the axiom of Archimedes its name because it appears as Axiom V of Archimedes' *On the Sphere and Cylinder*.

In his Book V of *Elements*, Euclid of Alexandria (c. 325–c. 265 BC) included the Archimedean property as Definition 4:

Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.

The archimedean property is also referred to as the "Theorem of Eudoxus" or the *Eudoxus axiom*, as it was named after Eudoxus of Cnidus (c. 390–c. 340 BC), whom Archimedes credited for its discovery.

In the 19th century, the German mathematician, Karl Weierstrass (1815–1897), used the archimedean property to develop the theory of real analysis. Weierstrass showed that the archimedean property is a fundamental property of the real numbers and is equivalent to the completeness axiom of the real numbers.



Archimedes (c. 287– c. 212 BC)

Euclid (c. 325– c. 265 BC)

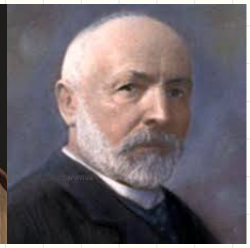
Eudoxus (c. 390– c. 340 BC)

Weierstrass (1815–1897)

Historical Note: $\mathbb{Q}$ is dense in $\mathbb{R}$

The ancient Greek philosopher and mathematician, Pythagoras (c. 570 BC–c. 495 BC), is credited with discovering that **the rational numbers are dense in the set of real numbers**. However, the first rigorous proof of this fact is attributed to the 19th-century German mathematician, Georg Cantor (1845–1918).

Pythagoras (c. 570– c. 495 BC)



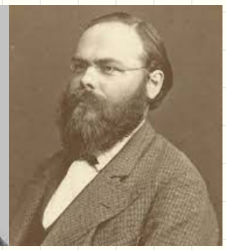
Cantor (1845–1918)

Historical Note: the Cauchy-Schwarz inequality

The **Cauchy-Schwarz inequality** is a fundamental result in mathematics that relates to the dot product or inner product of vectors. It was first published by Augustin-Louis Cauchy (1789–1857) in 1821 and then independently discovered by Viktor Yakovlevich Bunyakovsky (1804–1889) in 1859. However, it was Hermann Amandus Schwarz (1843–1921) who gave the inequality its modern form in 1885. Schwarz was a German mathematician who made significant contributions to the field of complex analysis, and the inequality is named after both Cauchy and Schwarz in recognition of their work. Today, the Cauchy-Schwarz inequality is widely used in various branches of mathematics, including linear algebra, calculus, and probability theory.

Cauchy
(1789–1857)

Bunyakovsky
(1804–1889)



Schwarz
(1843–1921)

Historical Note: the extended real number system $\overline{\mathbb{R}}$

The **symbol** ∞ for infinity is believed to have been introduced by the English mathematician John Wallis (1616–1703) in the 17th century. Wallis used the symbol in his writings on calculus and infinite series. However, the use of the symbol ∞ became more widespread in the 18th and 19th centuries, particularly in the works of mathematicians such as Leonhard Euler (1707–1783) and Augustin-Louis Cauchy (1789–1857).

Wallis
(1616–1703)

Euler
(1707–1783)

Cauchy
(1789–1857)

The concept of the **extended real number system** was introduced by the German mathematician, Friedrich Wilhelm Bessel (1784–1846), in the early 19th century. Bessel used the extended real number system to study functions that have infinite limits or singularities, such as the gamma function and the zeta function. The modern formulation of the extended real number system, which includes the symbols ∞ and $-\infty$ to represent positive and negative infinity, respectively, was developed in the 20th century by mathematicians such as Henri Lebesgue (1875–1941) and Stefan Banach (1892–1945). ◀

Bessel
(1784–1846)

Lebesgue
(1875–1941)

Banach
(1892–1945)

Historical Note: complex numbers

The history of **complex numbers** dates back to the 16th century when Italian mathematician Gerolamo Cardano (1501–1576) first introduced the idea of complex numbers as solutions to cubic equations. However, the concept of complex numbers was not widely accepted until the 18th century when mathematicians such as Leonhard Euler (1707–1783) and Carl Friedrich Gauss (1777–1855) began to study them more extensively. Euler used the symbol "i" to represent the square root of -1, and he also established the formula "$e^{i\pi} + 1 = 0$", which is known as Euler's identity. Gauss further developed the concept of complex numbers and established the *fundamental theorem of algebra*, which states that every non-constant polynomial equation with complex coefficients has at least one complex root.

The concept of complex conjugate was introduced by Irish mathematician William Rowan Hamilton (1805–1865) in the 19th century. Hamilton defined the complex conjugate of a complex number as the number obtained by changing the sign of the imaginary part of the number. ◀


Cardano (1501–1576)


Euler (1707–1783)


Gauss (1777–1855)


Hamilton (1805–1865)

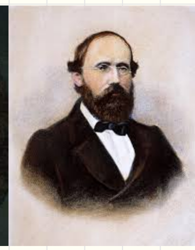
Historical Note: euclidean spaces

The concept of **euclidean spaces** dates back to ancient Greece, where the philosopher Euclid (c. 325–c. 265 BC) developed the principles of geometry in his work "*Elements*" around 300 BC. Euclid's work laid the foundation for the study of geometry and the concept of space. In the 17th century, French mathematician René Descartes (1596–1650) introduced the concept of analytic geometry, which allowed geometric problems to be solved algebraically using coordinates. This led to the development of the Cartesian coordinate system and the ability to represent geometric shapes and spaces algebraically.

In the 19th century, German mathematician Georg Friedrich Bernhard Riemann (1826–1866) introduced the concept of non-euclidean geometry, which challenged the traditional Euclidean view of space. This led to the development of new geometries, such as hyperbolic and elliptic geometry, and a deeper understanding of the nature of space. ◀

Euclid (c. 325– c. 265 BC)

Descartes (1596–1650)



Riemann (1826–1866)

Historical Note: laws of exponents

The **laws of exponents** are fundamental rules in mathematics that describe the behavior of exponential functions. The earliest known use of exponents dates back to ancient Babylonian and Egyptian mathematics, where they were used in computations involving large numbers. However, the laws of exponents as we know them today were first introduced by European mathematicians in the 16th and 17th centuries.

The product law of exponents, which states that $a^m \cdot a^n = a^{m+n}$, was introduced by Michael Stifel (1487–1567) in 1544 in his book *Arithmetica Integra*. The power law, which states that $(a^m)^n = a^{mn}$, was introduced by René Descartes (1596–1650) in 1637 in his work *La Geometrie*. The third law, which states that $(ab)^n = a^n \cdot b^n$, was also introduced by Descartes in the same work.

Later, Leonard Euler (1707–1783) in the 18th century introduced the negative exponent law, which states that $a^{-n} = 1/a^n$, and the zero exponent law, which states that $a^0 = 1$.

Today, the laws of exponents are widely used in various fields of mathematics, including calculus, algebra, and number theory. ◀

Stifel
(1487–1567)

Descartes
(1596–1650)

Euler
(1707–1783)

Historical Note: laws of logarithms

The **laws of logarithms** are fundamental rules that describe the behavior of logarithmic functions. The concept of logarithms was first introduced by John Napier of Merchiston (1550–1617) in the early 17th century as a way to simplify calculations involving large numbers. However, it was not until the 18th century that the laws of logarithms were developed.

The product law, which states that $\log(ab) = \log(a) + \log(b)$, was introduced by the Swiss mathematician Johann Bernoulli (1667–1748) in 1694. The quotient law, which states that $\log(a/b) = \log(a) - \log(b)$, was discovered by the English mathematician Henry Briggs (1561–1630) in 1617. The power law, which states that $\log(a^b) = b\log(a)$, was introduced by the Scottish mathematician Colin Maclaurin (1698–1746) in 1742.

Later, in the 19th century, the laws of logarithms were extended to include complex and imaginary numbers. The laws of logarithms are widely used in various fields of mathematics, including calculus, algebra, and number theory. They are also used in science and engineering, particularly in the fields of physics and chemistry, to model exponential growth and decay processes. ◀


Napier (1550–1617)


Johann Bernoulli (1667–1748)


Briggs (1561–1630)


Maclaurin (1698–1746)

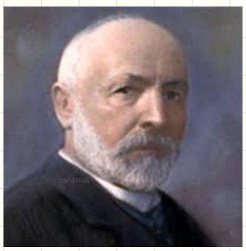
Historical Note: cardinality

The concept of **cardinality** in mathematics, which refers to the size or number of elements in a set, has a long and complex history. The ancient Greeks were the first to study the concept of infinity, but it was not until the 19th century that the concept of cardinality was developed.

One of the first mathematicians to study cardinality was Georg Cantor (1845–1918), a German mathematician who is considered the founder of set theory. In the late 19th century, Cantor developed the concept of one-to-one correspondence, which states that two sets have the same cardinality if there exists a one-to-one correspondence between their elements.

Using this concept, Cantor was able to prove that there are different sizes of infinity, and that some infinities are larger than others. He also developed the concept of cardinal numbers, which are used to represent the size or cardinality of sets.

Cantor's work on cardinality and set theory was controversial at the time, and he faced criticism and opposition from many mathematicians. However, his ideas eventually gained acceptance, and today they are considered fundamental to modern mathematics.



Cantor
(1845–1918)

Historical Note: equivalence relation

The concept of **equivalence relation** in mathematics has a long history, dating back to the ancient Greeks. However, it was not until the 19th century that the concept was formally defined and studied.

The first mathematician to study equivalence relations was Augustin-Louis Cauchy (1789–1857), a French mathematician who worked in the early 19th century. Cauchy introduced the concept of a "complete system of invariants," which is a set of properties that are preserved under a particular transformation. He used this concept to study the properties of algebraic equations, and he showed that certain transformations could be used to simplify and solve these equations.

Later in the 19th century, the German mathematician Felix Klein (1849–1925) developed the concept of a "group," which is a set of elements that can be combined using a binary operation. Klein showed that groups could be used to study symmetry and transformation, and he introduced the concept of a "transformation group," which is a group that acts on a set of objects.

Using the concept of a transformation group, Klein was able to define the concept of an equivalence relation, which is a relation that divides a set into disjoint subsets, or "equivalence classes," that are related by a particular transformation. He showed that equivalence relations could be used to study symmetry and invariance in geometry and other areas of mathematics.

Today, equivalence relations are an important concept in many areas of mathematics, including algebra, geometry, topology, and analysis. They are also used in computer science and other fields to study algorithms and data structures, and they have many practical applications in engineering, physics, and other sciences.

Cauchy
(1789–1857)

Klein
(1849–1925)

Historical Note: topology on $\mathbb{R}$

Topology on $\mathbb{R}$, also known as point-set topology, is a branch of mathematics that studies the properties of real numbers and their subsets. The history of topology on $\mathbb{R}$ can be traced back to the ancient Greeks, who were the first to study the properties of real numbers and their relationships to geometry.

In the 19th century, mathematicians began to formalize the study of real numbers and their properties. One of the first mathematicians to do so was Augustin-Louis Cauchy (1789–1857), a French mathematician who worked in the early 19th century. Cauchy introduced the concept of a "limit," which is a value that a function approaches as the input approaches a certain value.

Later in the 19th century, the German mathematician Georg Cantor (1845–1918) introduced the concept of a "set," which is a collection of objects that share a common property. Cantor used sets to study the properties of real numbers and their subsets, and he introduced the concept of a "cardinality," which is a measure of the size or number of elements in a set.

Some of the most fundamental concepts in topology on $\mathbb{R}$ include:

- **Open sets**: A set of real numbers is open if it does not include its boundary points. Open sets play a crucial role in topology on $\mathbb{R}$ because they allow mathematicians to define and study many important concepts, such as continuity, convergence, and compactness. The concept of open sets was first introduced by Cauchy in the early 19th century. However, it was not until the 20th century that the concept was fully developed and formalized by mathematicians such as Henri Lebesgue (1875–1941) and Felix Hausdorff (1868–1942).
- **Closed sets**: A set of real numbers is closed if it contains all of its boundary points. Closed sets are important in topology on $\mathbb{R}$ because they allow mathematicians to define and study many important concepts, such as completeness and connectedness. The concept of closed sets was also introduced by Cauchy in the early 19th century, but it was not until the later part of the century that mathematicians such as Karl Weierstrass (1815–1897) and Georg Cantor began to study the properties of closed sets in depth.
- **Continuity**: A function f is said to be continuous if the inverse image of an open set under f is also an open set. Continuity is a fundamental concept in topology on $\mathbb{R}$ because it allows mathematicians to study the behavior of functions and their limits. The concept of continuity can be traced back to the ancient Greeks, who were the first to study the properties of continuous functions and their relationships to geometry. However, it was not until the 19th century that the concept was formalized and studied in depth by mathematicians such as Cauchy, Weierstrass, and Bernhard Riemann (1826–1866).
- **Convergence**: A sequence of real numbers is said to converge to a limit if every open set containing the limit contains all but finitely many terms of the sequence. Convergence is a fundamental concept in topology on $\mathbb{R}$ because it allows mathematicians to study the behavior of sequences and their limits. The concept of convergence can also be traced back to the ancient Greeks, who were the first to study the properties of infinite sequences and their limits. However, it was not until the 19th century that the concept was formalized and studied in depth by mathematicians such as Cauchy, Weierstrass, and Karl Gustav Jacobi (1804–1851).
- **Compactness**: A set of real numbers is said to be compact if every open cover of the set has a finite subcover. Compactness is a fundamental concept in topology on $\mathbb{R}$ because it allows mathematicians to study the properties of sets that are bounded and closed. The concept of compactness was first introduced by the German mathematician Eduard Heine (1821–1881) in the mid-19th century, but it

was not until the 20th century that the concept was fully developed and formalized by mathematicians such as Hausdorff and Lebesgue.

- **Connectedness**: A subset of $\mathbb{R}$ is said to be connected if it cannot be divided into two disjoint nonempty open sets. Connectedness is a fundamental concept in topology on $\mathbb{R}$ because it allows mathematicians to study the properties of sets that are "unbroken" and cannot be separated into distinct parts. The concept of connectedness was first introduced by the French mathematician Henri Poincaré (1854–1912) in the late 19th century, but it was not until the 20th century that the concept was fully developed and formalized by mathematicians such as Hausdorff and Lebesgue.

These are just a few of the most fundamental concepts in topology on $\mathbb{R}$, and there are many more that are important in the study of point-set topology.


Cauchy (1789–1857)

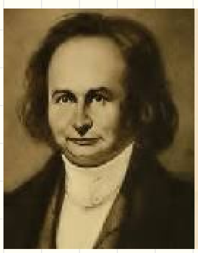

Cantor (1845–1918)


Lebesgue (1875–1941)


Hausdorff (1849–1925)


Weierstrass (1815–1897)


Riemann (1826–1866)


Jacobi (1804–1851)


Heine (1821–1881)


Poincaré (1854–1912)

Historical Note: charaterizations of compact sets in $\mathbb{R}$ and the Heine-Borel theorem

The study of **compact** sets in the real numbers dates back to the 19th century, with the work of mathematicians such as Bernard Bolzano (1781–1848), Karl Weierstrass (1815–1897), and Eduard Heine (1821–1881). However, it was the French mathematician Émile Borel (1871–1956) who first gave a rigorous definition of compactness in 1909.

Borel defined a set to be compact if every open cover of the set has a finite subcover. In other words, if a set can be covered by a collection of open intervals, then there exists a finite subcollection that still covers the set.

In 1872, Heine, a German mathematician, proved what is now known as the **Heine-Borel theorem**, which characterizes compact sets in the real numbers. Heine showed that a set in the real numbers is compact if and only if it is closed and bounded. ◀

Bolzano (1781–1848)

Weierstrass (1815–1897)

Heine (1821–1881)

Borel (1871–1956)

Historical Note: the Bolzano-Weierstrass theorem

The **Bolzano-Weierstrass theorem**, also known as the Bolzano theorem or Weierstrass theorem, is a fundamental result in real analysis that states that every bounded sequence in the real numbers has a convergent subsequence. The theorem is named after the mathematicians Bernhard Bolzano and Karl Weierstrass, who both contributed to its development.

Bolzano, a Czech mathematician, first stated the theorem in the early 19th century in his work on the foundations of analysis. However, his proof was incomplete and lacked rigor. It was Weierstrass, a German mathematician, who provided a rigorous proof of the theorem in the mid-19th century.

Weierstrass's proof used the concept of nested intervals, which is also used in the proof of Cantor's intersection theorem. He showed that any bounded sequence can be divided into two subsequences, one of which is bounded above and the other of which is bounded below. By repeating this process with the bounded subsequences, he constructed a nested sequence of closed intervals whose lengths approach zero, and he used Cantor's intersection theorem to show that the sequence has a limit. ◀

Bolzano
(1781–1848)

Weierstrass
(1815–1897)

Historical Note: convergence sequence

The concept of a **convergent sequence** is a fundamental concept in analysis and dates back to the 17th century, when mathematicians such as John Wallis (1616–1703) and James Gregory (1638–1675) studied infinite series and their convergence.

However, the modern definition of a convergent sequence is usually credited to the French mathematician Augustin-Louis Cauchy (1789−1857). Cauchy defined a sequence to be convergent if its terms approach a limit as the index of the terms approaches infinity.

Cauchy's definition of a convergent sequence was based on the concept of a Cauchy sequence, which he also introduced. A **Cauchy sequence** is a sequence in which the terms become arbitrarily close to each other as the index of the terms approaches infinity. Cauchy showed that every convergent sequence is a Cauchy sequence, and that every Cauchy sequence in the real numbers has a limit. ◀

Wallis
(1616–1703)

Gregory
(1638–1675)

Cauchy
(1789–1857)

Historical Note: upper and lower limits

The concept of **upper and lower limits** is a fundamental concept in analysis and dates back to the 19th century, when mathematicians such as Augustin-Louis Cauchy (1789 – 1857) and Karl Weierstrass (1815–1897) studied the convergence of sequences and series.

The idea of upper and lower limits is closely related to the concept of a limit of a sequence. A sequence is said to converge to a limit if its terms become arbitrarily close to the limit as the index of the terms approaches infinity. However, not all sequences converge to a limit, and some sequences may have multiple limits or no limits at all.

To address this issue, mathematicians introduced the concepts of upper and lower limits. The upper limit of a sequence is the supremum of the set of all limits of its subsequences, while the lower limit is the infimum of the set of all limits of its subsequences. If the upper and lower limits of a sequence are equal, then the sequence converges to that limit. ◀

Cauchy
(1789–1857)

Weierstrass
(1815–1897)

Historical Note: Euler's number e

The **number e** is a mathematical constant that is approximately equal to 2.71828. It is named after the Swiss mathematician Leonhard Euler (1707–1783), although the number itself was known to mathematicians before Euler's time.

The first appearance of the number e in mathematics was in the work of the Scottish mathematician John Napier (1550–1617). Napier discovered that the logarithms of numbers could be expressed as the sum of an infinite series, and he used this idea to develop logarithmic tables that were used for many years in scientific and engineering calculations.

In the 17th century, the Swiss mathematician Jacob Bernoulli (1655–1705) studied the same infinite series that Napier had used, and he discovered that the limit of the series was a new mathematical constant, which he denoted by the letter b. Bernoulli showed that b was equal to e, although he did not use this notation.

Euler was the first mathematician to extensively study the properties of the number e. He showed that e is a transcendental number, meaning that it is not the root of any polynomial equation with rational coefficients. Euler also discovered many important relationships between e and other mathematical constants, such as π and the imaginary unit i. ◄


Euler (1707–1783)


Napier (1550–1617)


Jacob Bernoulli (1655–1705)

Historical Note: timeline of the convergence tests

Here is a timeline of the discovery of some of the most important **convergence tests** for infinite series:

- 1687: Jacob Bernoulli (1655–1705) discovers the Divergence Test.
- 1748: Johann Bernoulli (1667–1748) and Leonhard Euler (1707–1783) develop the Alternating Series Test.
- 1748: Euler discovers Euler's formula, which relates the complex exponential function to the trigonometric functions.
- 1821: Augustin-Louis Cauchy (1789–1857) develops the Comparison Test.
- 1826: Cauchy develops the Root Test and the Ratio Test.
- 1826: Niels Henrik Abel (1802–1829) develops Abel's Test.
- 1846: Peter Gustav Lejeune Dirichlet (1805–1859) develops Dirichlet's test.
- 1867: Georg Friedrich Bernhard Riemann (1826–1866) develops the Riemann Rearrangement Theorem, which states that any conditionally convergent series can be rearranged in such a way that it converges to any desired limit, or it can be made to diverge.
- 1872: Karl Weierstrass (1815–1897) develops the Weierstrass M-Test, which is used to determine the convergence of a series whose terms involve a sequence of functions. ◀

Jacob Bernoulli (1655–1705)

Johann Bernoulli (1667–1748)

Euler (1707–1783)

Cauchy (1789–1857)

Abel (1802–1829)

Dirichlet (1805–1859)

Riemann (1826–1866)

Weierstrass (1815–1897)

Historical Note: absolute convergence and conditional convergence

The concepts of **absolute convergence** and **conditional convergence** are closely related to the development of convergence tests for infinite series.

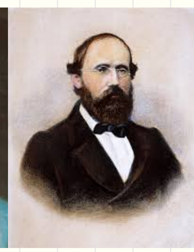
In the 18th century, mathematicians such as Johann Bernoulli (1667–1748) and Leonhard Euler (1707–1783) studied infinite series and their convergence properties. They discovered that some series converge absolutely, meaning that the series of absolute values of the terms converges, while others converge conditionally, meaning that the series itself converges but the series of absolute values of the terms diverges.

The study of absolute convergence and conditional convergence became more formalized in the 19th century, with the development of convergence tests such as the Ratio Test, the Root Test, and Dirichlet's Test. These tests allowed mathematicians to determine whether a given series converges absolutely or conditionally, and to analyze the behavior of series with alternating signs or complex terms.

One of the most important results in the theory of conditional convergence is the Riemann Rearrangement Theorem, which was proved by Georg Friedrich Bernhard Riemann (1826–1866) in 1854. This theorem states that for any conditionally convergent series, its terms can be rearranged in such a way that the series converges to any desired limit, or it can be made to diverge. This result has important implications for the study of infinite series and their convergence properties. ◀

Johann Bernoulli (1667–1748)

Euler (1707–1783)



Riemann (1826–1866)

Historical Note: limit of function

The concept of the **limit of a function** is a fundamental idea in calculus and analysis, and it has a long and rich history dating back to ancient Greek mathematics.

The ancient Greeks were interested in the concept of infinity and the behavior of quantities as they approached infinity. They developed the method of exhaustion, which involved approximating a quantity by successively smaller quantities until the difference between them was negligible. This method was used to calculate areas and volumes of geometric shapes.

In the 17th century, mathematicians such as Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) developed the calculus, which provided a more rigorous framework for studying the behavior of functions as they approach certain values. They introduced the concept of the limit of a function and used it to define derivatives and integrals.

The 18th and 19th centuries saw the development of more rigorous methods for analyzing limits and their properties. Mathematicians such as Augustin-Louis Cauchy (1789–1857) and Karl Weierstrass (1815–1897) developed the $\varepsilon - \delta$ definition of limits, which provided a precise way of defining and analyzing the behavior of functions as they approach certain values.

In the 20th century, the concept of limits was further refined and extended to more general settings, such as metric spaces and topological spaces. The development of topology and analysis allowed mathematicians to study the behavior of functions in more abstract and general settings, and to develop powerful tools for analyzing the properties of limits and their applications to other areas of mathematics and science. ◀

Newton
(1642–1727)

Leibniz
(1646–1716)

Cauchy
(1789–1857)

Weierstrass
(1815–1897)

Historical Note: derivative

The concept of the **derivative**, slope of tangent, and rate of change are all closely related and have a rich history in mathematics.

The derivative of a function is a measure of how quickly the function changes at a given point, and can be interpreted as the slope of the tangent line to the curve at that point. It can also be interpreted as the instantaneous rate of change of the function at that point. The concept of the derivative was first introduced by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) in the late 17th century, and has since become a fundamental concept in calculus and analysis.

The meaning of these concepts is that they allow us to study the behavior of functions and systems in a precise and quantitative way. By calculating the derivative, slope of tangent, and rate of change of a function, we can understand how it is changing at any given point, and use this information to make predictions and solve problems in a wide range of fields.

The concept of the derivative was further developed by mathematicians such as Leonhard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813), and Augustin-Louis Cauchy (1789–1857) in the 18th and 19th centuries. They developed the theory of differential equations, which used the derivative to study the behavior of functions and systems of equations.

Today, the derivative is a fundamental concept in many areas of mathematics, science, and engineering, and is used to study a wide range of phenomena, from the motion of objects to the behavior of financial markets. The concept of rate of change remains a central idea in many fields, and is used to describe the behavior of systems ranging from physical processes to biological systems to social networks. ◂

Newton
(1642–1727)

Leibniz
(1646–1716)

Euler
(1707–1783)

Lagrange
(1736–1813)

Cauchy
(1789–1857)

Historical Note: l'Hôpital's Rule

L'Hôpital's Rule is a mathematical theorem that provides a method for evaluating certain limits involving indeterminate forms. The rule is named after the French mathematician Guillaume de l'Hôpital (1661–1704), who published it in his book *Analyse des Infiniment Petits* in 1696. This was the first textbook on infinitesimal calculus. It presented the ideas of differential calculus and their applications to differential geometry of curves in a clear way with numerous figures, but did not cover integration.

The publication of the book was surrounded by controversy due to a proposal made by l'Hôpital to Johann Bernoulli (1667–1748) in 1694. In exchange for an annual payment of 300 Francs, Bernoulli would inform l'Hôpital of his latest mathematical discoveries, withholding them from correspondence with others. Bernoulli agreed and subsequently provided l'Hôpital with statements and portions of the text of *Analyse*. However, after l'Hôpital's death, Bernoulli publicly revealed their agreement and claimed credit for his contributions to the book.

For many years, Bernoulli's claims were not considered credible by some historians of mathematics, but in 1921 a manuscript of Bernoulli's lectures on differential calculus was discovered, showing remarkable similarities to l'Hôpital's writing.

Despite the controversy, *Analyse* was successful in popularizing the ideas of differential calculus stemming from Leibniz (1646 – 1716) and is recognized for its pedagogical brilliance. l'Hôpital's Rule remains an important tool in calculus and analysis, widely used in mathematics, science, and engineering to solve a variety of problems. ◀

l'Hôpital (1661–1704)

Johann Bernoulli (1667–1748)

Leibniz (1646–1716)

Historical Note: Taylor's Theorem

Taylor's theorem is a mathematical theorem that provides an approximation of a function as a sum of its derivatives at a single point. It was first discovered by James Gregory (1638–1675) in the 17th century, but it was Brook Taylor (1685–1731) who gave the theorem its modern form in 1712.

Taylor's theorem states that a function f can be represented as an infinite sum of its derivatives evaluated at a single point a, multiplied by powers of $(x-a)$. This is known as the Taylor series expansion of f about a. The theorem is important in calculus, analysis, and applied mathematics, as it allows for the approximation of functions and the calculation of integrals and derivatives.

The development of Taylor's theorem was a gradual process, with contributions from various mathematicians over several centuries. In the 14th century, Madhava of Sangamagrama (c. 1340–c. 1425), a mathematician from Kerala, India, discovered a series expansion for trigonometric functions using a method similar to Taylor's theorem.

In the 17th century, James Gregory derived a series expansion for the arctangent function, which was later extended to other functions by John Wallis (1616–1703) and Isaac Barrow (1630–1677). In 1671, Johann Bernoulli (1667–1748) discovered a series expansion for the exponential function, which was later generalized by his brother, Jacob Bernoulli (1655–1705).

Brook Taylor's contribution to the development of Taylor's theorem was significant, as he gave the theorem its modern form and applied it to a wide range of problems in mathematics and physics. His work on the theorem was published in his book *Methodus Incrementorum Directa et Inversa* in 1715.

Taylor's theorem has become a valuable tool in various fields such as mathematics, science, and engineering. It is utilized to estimate functions, find integrals and derivatives, and resolve differential equations. The widespread use of this theorem is a tribute to the innovative and imaginative efforts of the mathematicians who worked on its evolution throughout the years. ◀

Gregory (1638–1675)

Taylor (1685–1731)

Madhava (c. 1340–c. 1425)

Wallis (1616–1703)

Barrow (1630–1677)

Johann Bernoulli (1667–1748)

Jacob Bernoulli (1655–1705)

Historical Note: Riemann integral

Before the German mathematician Bernhard Riemann (1826–1866) introduced the concept of the **Riemann integral** in the mid-19th century, mathematicians had been using methods that were not completely rigorous to calculate areas under curves. They had been using approximations and limiting processes that were not well-defined, which led to some inconsistencies and inaccuracies in their calculations.

For example, the ancient Greeks had been using the method of exhaustion, which involved approximating the area under a curve by inscribing and circumscribing it with a series of polygons. This method was not always accurate, and it was difficult to determine the exact value of the area.

In the 17th century, mathematicians such as Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) developed the fundamental ideas of calculus, but their methods were still not completely rigorous. It wasn't until Riemann's introduction of the Riemann integral that a more rigorous and precise method for calculating areas under curves was developed.

Riemann's approach was to divide the area under the curve into a series of small rectangles and then add up the areas of these rectangles to get an approximation of the total area. As the width of the rectangles became smaller and smaller, the approximation became more and more accurate, and eventually converged to the exact value of the area under the curve.

Riemann's definition of the integral was based on the concept of a limit, which he used to formalize the idea of the sum of an infinite number of infinitely small quantities. This was a major breakthrough in calculus, and it paved the way for many important developments in mathematics and science.

The Riemann integral remains a crucial tool in various fields such as mathematics, physics, engineering, and other related disciplines. It is utilized as a fundamental method to compute areas, volumes, and other quantities that can be expressed as integrals. ◀

Riemann
(1826–1866)

Newton
(1642–1727)

Leibniz
(1646–1716)

Historical Note: the Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** is a central theorem in calculus that establishes a connection between differentiation and integration. It states that if a function is continuous on a closed interval, then the integral of the function over that interval can be evaluated by finding an antiderivative of the function at the endpoints of the interval and subtracting the values of the antiderivative.

The theorem was first discovered independently by two mathematicians, Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716), in the 17th century. Newton and Leibniz were both working on developing the fundamental ideas of calculus, and they realized that differentiation and integration were inverse operations of each other.

However, the theorem was not stated in its modern form until the 18th century, when mathematicians such as Johann Bernoulli (1667–1748) and Leonhard Euler (1707 – 1783) began to formalize the ideas of calculus. Euler was the first to use the term "Fundamental Theorem of Calculus" in his work.

The modern version of the theorem was developed by the French mathematician Augustin-Louis Cauchy (1789–1857) in the early 19th century. Cauchy's version of the theorem stated that if a function is continuous on a closed interval, then the integral of the function over that interval can be evaluated by finding an antiderivative of the function at any point in the interval.

The Fundamental Theorem of Calculus is a significant concept in calculus that has become central in various fields such as mathematics, physics, engineering, and other related disciplines. It is a fundamental tool for evaluating integrals and is essential for comprehending many crucial concepts in calculus, including optimization, differential equations, and Fourier analysis. ◀

Newton (1642–1727)

Leibniz (1646–1716)

Johann Bernoulli (1667–1748)

Euler (1707–1783)

Cauchy (1789–1857)

Historical Note: uniform convergence

The concept of **uniform convergence** was first introduced by Augustin-Louis Cauchy (1789–1857) in the early 19th century. He used the term “convergence uniforme” in his work on complex analysis and calculus. However, the idea of uniform convergence was not fully developed until the late 19th century, when mathematicians such as Karl Weierstrass (1815–1897) and Georg Cantor (1845–1918) began to study it in more detail.

Weierstrass was particularly interested in the concept of uniform convergence and used it to prove his famous approximation theorem, which states that any continuous function on a closed interval can be uniformly approximated by a polynomial function. He also introduced the notion of a uniform limit, which is a limit that holds uniformly over a given domain.

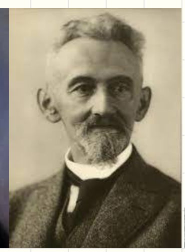
Cantor also made significant contributions to the study of uniform convergence, particularly in his work on the theory of sets. He used the concept of uniform convergence to define a new type of function, which he called a “uniformly continuous function.” This concept was later refined by other mathematicians, including Felix Hausdorff (1868–1942) and Henri Lebesgue (1875–1941).

Today, the concept of uniform convergence is an important tool in many areas of mathematics, including analysis, topology, and functional analysis. It is used to study the behavior of functions and sequences of functions, and plays a crucial role in the development of modern analysis. ◀


Cauchy (1789–1857)


Weierstrass (1815–1897)


Cantor (1845–1918)


Hausdorff (1849–1925)


Lebesgue (1875–1941)

Historical Note: the Weierstrass approximation theorem

Karl Weierstrass (1815–1897) is often referred to as the father of approximation theory because of his significant contributions to the field. He made several important discoveries related to the approximation of functions, including the **Weierstrass approximation theorem**, which states that any continuous function on a closed interval can be uniformly approximated by a polynomial function.

Weierstrass also developed a method for approximating functions using trigonometric series, which became known as the Weierstrass transform. He used this method to prove a number of important results in analysis, including the uniform convergence of Fourier series.

In addition to his specific contributions to approximation theory, Weierstrass was also known for his rigorous approach to mathematical analysis. He emphasized the importance of rigor and precision in mathematical proofs, and his work helped to establish the modern standards for mathematical rigor.

Overall, Weierstrass's work in approximation theory and his emphasis on rigor and precision in mathematical analysis have had a significant impact on the field of mathematics, and his contributions have earned him the title of father of approximation theory. ◂

Weierstrass
(1815–1897)

Appendix C: solutions to the exercises

Chapter 1 Quiz Answers

①D: The set of rational numbers is an example of an ordered field, by Proposition 1.7. An ordered field is a field.

②D: The set of real numbers is an ordered field, by Proposition 1.7.

③A: The set of odd numbers is not closed under addition, since the sum of two odd number is not an odd number.

④C: The set $\{x : x < 3\}$ is not a bounded set in $\mathbb{R}$ because it is not bounded below.

⑤D: Every nonempty bounded-above subset of $\mathbb{R}$ has a least upper bound (supremum). This is not necessarily true for minimum or maximum elements. For example, the set $(0, 1)$ has no minimum or maximum element, but it does have a supremum (which is 1). The set $[0, 1)$ has a minimum element (which is 0) but no maximum element, and its supremum is 1.

⑥C: The least upper bound of the set, $\{x : x \text{ is an irrational number and } x^2 \leq 2\}$, is $\sqrt{2}$, which is irrational and hence contained within the set.

⑦E: The inequality $|x + y| \geq |x| + |y|$ violates the triangle inequality because the sum of the absolute values of two real numbers should always be greater than or equal to the absolute value of the sum of these two real numbers.

⑧D: The Cauchy-Schwarz inequality is

$$|a_1b_1 + \cdots + a_nb_n|^2 \leq \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right).$$

By taking $a_k = x_k$, $b_k = x_{k+1}^{-1}$, $1 \leq k \leq n$ (denote $x_{n+1} = x_1$), we get

$$\begin{aligned}
&a_1b_1 + \cdots + a_nb_n = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1},\\
&a_1^2 + \cdots + a_n^2 = x_1^2 + \cdots + x_n^2,\\
&b_1^2 + \cdots + b_n^2 = x_2^{-2} + \cdots + x_{n+1}^{-2} = x_1^{-2} + \cdots + x_n^{-2}.
\end{aligned}$$

Thus, it follows from the Cauchy-Schwarz inequality that

$$\left|\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1}\right|^2 \leq \left(x_1^2 + \cdots + x_n^2\right)\left(x_1^{-2} + \cdots + x_n^{-2}\right).$$

The left-hand side of option D should be squared. ◀

Chapter 1 Exercise Solutions

Ex. 1.1 We prove by contradiction. Suppose that $\sqrt{6}$ is a rational number. Put $\sqrt{6} = m/n$, with m, n being co-prime. This gives $m^2 = 6n^2$, which implies that m is a multiple of 3. Put $m = 3k$. Then we have $2n^2 = 3k^2$. This implies that n is also a multiple of 3. This contradicts to the hypothesis that m and n are co-prime. Thus, the number $\sqrt{6}$ is irrational.

Ex. 1.2 We prove by contradiction. Suppose that the set $\{\sqrt{n}: n \in \mathbb{N}\}$ is bounded. Denote B an upper bound of the set. Thus, we have that $\sqrt{n} \le B$ for all $n \in \mathbb{N}$. It gives that

$$n \le B^2, \qquad \text{for all } n \in \mathbb{N}.$$

This contradicts to the archimedean property.

Ex. 1.3 Suppose E is bounded above, and suppose β_1 and β_2 are two distinct suprema of E.

For any $x \in E$, we have $x \le \beta_1$ and $x \le \beta_2$. If $\beta_1 < \beta_2$, then

$$x \le \beta_1 < \beta_2,$$

so that β_2 is not a supremum of E by definition. This contradicts to the hypothesis. Hence, $\beta_1 \ge \beta_2$.

In the similar manner, we have $\beta_2 \ge \beta_1$.

Therefore, we must have $\beta_2 = \beta_1$.

Ex. 1.4 Denote $A = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \dots\right\}$.

- Clearly, for any $x \in A$, we have $x \ge 1$, so that 1 is a lower bound of A. Furthermore, for any positive ε, since $1 < 1+\varepsilon$ and $1 \in A$, we know that $1+\varepsilon$ is not a lower bound of A. By the definition, we have $\inf A = 1$.
- Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. We prove by induction that $x_{2^n} \ge 1 + \frac{n}{2}$, so that the sequence $\{x_n\}$ is not bounded above. It follows that $\sup A = \infty$.

  In fact, we have $x_2 = 1 + \frac{1}{2}$ so that the desired inequality holds for $n = 1$.

  Assume that $x_{2^k} \ge 1 + \frac{k}{2}$ for an integer $n = k$. Then

$$x_{2^{k+1}} = x_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \ge x_{2^k} + \underbrace{\frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}}_{2^k \text{ terms}}$$

$$= x_{2^k} + \frac{2^k}{2^{k+1}} = x_{2^k} + \frac{1}{2} \ge 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2},$$

  so that the desired inequality holds for $n = k+1$.

Ex. 1.5 ($\Rightarrow$) Assume that $|x-a| < \delta$.

- If $x - a \ge 0$, then the inequality $|x-a| < \delta$ gives $x - a < \delta$, so that $x < a + \delta$. Thus, $a \le x < a + \delta$.
- If $x - a < 0$, then the inequality $|x-a| < \delta$ gives $-(x-a) < \delta$, so that $a - \delta < x$. Thus, $a - \delta < x < a$.

Combining above we conclude that we always have $a - \delta < x < a + \delta$.

($\Leftarrow$) Assume that $a - \delta < x < a + \delta$.

- If $x - a \ge 0$, then $a \le x < a + \delta$, or $0 \le x - a < \delta$. Thus, we have $|x-a| < \delta$.
- If $x - a < 0$, then $a - \delta < x < a$, or $-\delta < x - a < 0$. Thus, we also have $|x-a| < \delta$.

Combining above we conclude that we always have $|x-a| < \delta$. ◀

Ex. 1.6 Denote $S = \Big\{|a+b| : \; a^2 < 2, \, |b+1| < 3\Big\}$.

First, we prove that the set is bounded. In fact, by the triangle inequality, we have

$$\begin{aligned} 0 \le |a+b| &= |a+b+1-1| \le |a| + |b+1| + |-1| \\ &= \sqrt{a^2} + |b+1| + 1 < \sqrt{2} + 3 + 1 = 4 + \sqrt{2}. \end{aligned}$$

Next, we find the infimum and the supremum of S.

- Take $a = 0$ and $b = 0$. Then $a^2 < 2$ and $|b+1| < 3$. Clearly, $|a+b| = 0$. Hence, we have $\inf S = 0$.
- For any $0 < \varepsilon < 1$, let
$$a = -\sqrt{2} + \tfrac{1}{4}\varepsilon, \qquad b = -4 + \tfrac{1}{4}\varepsilon.$$
Then $-\sqrt{2} < a < 0 < \sqrt{2}$ and $-3 < b+1 < -2 < 3$. Thus, $a^2 < 2$ and $|b+1| < 3$. Since
$$a + b = -4 - \sqrt{2} + \tfrac{1}{2}\varepsilon < 0,$$
we have
$$|a+b| = 4 + \sqrt{2} - \tfrac{1}{2}\varepsilon > 4 + \sqrt{2} - \varepsilon.$$
This demonstrates that $4 + \sqrt{2} - \varepsilon$ is not an upper bound of S. Hence, by definition, we have $\sup S = 4 + \sqrt{2}$.

Ex. 1.7 By the triangle inequality,

$$|x_1| = |x_1 - x_2 + x_2| \le |x_1 - x_2| + |x_2|,$$

so that $|x_1| - |x_2| \le |x_1 - x_2|$. Similarly, we have $|x_2| - |x_1| \le |x_1 - x_2|$. Combining these two inequalities, we get

$$\big||x_1| - |x_2|\big| \le |x_1 - x_2|.$$

Ex. 1.8 Let $A = \sum a_j^2$, $B = \sum b_j^2$, $C = \sum a_j b_j$.

If $B = 0$, then $b_j = 0$ for $j = 1, \ldots, n$. For $\lambda = 0$ and any $\mu \neq 0$, these values λ and μ are not both zero. Obviously, we have $\lambda a_j = \mu b_j$, $j = 1, 2, \ldots, n$.

If $B \neq 0$, then

$$\begin{aligned} 0 \le \sum_{j=1}^{n} (Ba_j - Cb_j)^2 &= B^2 \sum_{j=1}^{n} a_j^2 - BC \sum_{j=1}^{n} a_j b_j - BC \sum_{j=1}^{n} a_j b_j + C^2 \sum_{j=1}^{n} b_j^2 \\ &= B^2 A - BC^2 - BC^2 + BC^2 \\ &= B(AB - C^2). \end{aligned}$$

Since $AB - C^2 = 0$, we have $Ba_j - Cb_j = 0$, $j = 1, 2, \ldots, n$. If we take $\lambda = B$ and $\mu = C$, then, λ and μ are not both zero such that $\lambda a_j = \mu b_j$, $j = 1, 2, \ldots, n$. ◀

Chapter 2 Quiz Answers

①E: The set of all infinite sequences of 0's and 1's is not countable, by Proposition 2.8.

②B: By Proposition 2.11, we know that $\bigcap_{\alpha} A_{\alpha}^{c} = \left(\bigcup_{\alpha} A_{\alpha}\right)^{c}$. Thus, the given set relation $\bigcup_{\alpha} A_{\alpha} = \bigcap_{\alpha} A_{\alpha}^{c}$ is equivalent to $\bigcup_{\alpha} A_{\alpha} = \left(\bigcup_{\alpha} A_{\alpha}\right)^{c}$. The latter is equivalent to $\bigcup_{\alpha} A_{\alpha} = \varnothing$.

③B: The interior of A is the union of all open sets contained in A. Hence, the interior of A is a subset of A.

④E: Every compact set is closed, by Proposition 2.18.

⑤D: Take $E = (0, 1) \subset \mathbb{R}$ and $K = [0, 1] \subset \mathbb{R}$. By Theorem 2.21. We know that K is compact in $\mathbb{R}$, but E is not. Obviously, E is a bounded subset of K.

⑥A: Let S be a compact subset of $\mathbb{R}$. By Theorem 2.21, S is bounded and closed. Since $\mathbb{R}$ possesses the least-upper-bound property, $y = \sup S$ is finite. By Proposition 2.15, $y \in \overline{S}$. Because S is closed, by Proposition 2.14, $\overline{S} = S$. Hence, $\sup S \in S$. This means that S has a maximum element.

⑦C: By the corollary of Proposition 2.18, the intersection of a compact set and a closed set is compact.

⑧D: A perfect set is a closed set with no isolated points. Option D is true by definition.

Option A is false: Consider the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. It is compact but not perfect.

Option B is false: $\mathbb{R}$ is perfect, but not compact.

Option C is false: $\mathbb{R}$ is closed, but not compact.

Option E is false: The set of all rational numbers of $\mathbb{R}$ is dense in $\mathbb{R}$, but not closed, so not compact. ◀

Chapter 2 Exercise Solutions

Ex. 2.1 Define a function $f\colon (0,1) \to [0,1]$ by

$$f(x) = \begin{cases} \dfrac{1}{n-2}, & \text{if } x = \dfrac{1}{n},\ n = 3, 4, \ldots, \\ 0, & \text{if } x = \frac{1}{2}, \\ x, & \text{otherwise.} \end{cases}$$

Then f is bijective from $(0,1)$ onto $[0,1]$.

Proof: f is injective. Denote $S = \left\{\frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Let $x_1, x_2 \in (0,1)$, with $x_1 \neq x_2$.

① If $x_1, x_2 \in S$, then, there are distinct $m, n \geq 3$, such that $x_1 = \dfrac{1}{m}$, $x_2 = \dfrac{1}{n}$. Thus, $f(x_1) = \dfrac{1}{m-2} \neq \dfrac{1}{n-2} = f(x_2)$.

② If $x_1 = \dfrac{1}{n} \in S$ $(n \geq 3)$ and $x_2 = \frac{1}{2}$, then $f(x_1) = \dfrac{1}{n-2} \neq 0 = f(x_2)$.

③ If $x_1 = \dfrac{1}{n} \in S$ $(n \geq 3)$ and $x_2 \in (0,1) \backslash \left(S \cup \left\{\frac{1}{2}\right\}\right)$, then $f(x_1) = \dfrac{1}{n-2} \neq x_2 = f(x_2)$.

④ If $x_1 = \frac{1}{2}$ and $x_2 \in (0,1) \backslash \left(S \cup \left\{\frac{1}{2}\right\}\right)$, then $f(x_1) = 0 \neq x_2 = f(x_2)$.

⑤ If $x_1, x_2 \in (0,1) \backslash \left(S \cup \left\{\frac{1}{2}\right\}\right)$, then $f(x_1) = x_1 \neq x_2 = f(x_2)$.

In summary, for all $x_1, x_2 \in (0,1)$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Proof: f is surjective. For any $y \in [0,1]$, we have

$$y = \begin{cases} f(\frac{1}{2}), & \text{if } y = 0, \\ f\left(\dfrac{1}{n+2}\right), & \text{if } y = \dfrac{1}{n},\ n \geq 1, \\ f(y), & \text{if } y \neq 0 \text{ or } y \neq \dfrac{1}{n},\ n \geq 1. \end{cases}$$

Ex. 2.2 ($\Rightarrow$) Assume that E is open.

If $p \in E$, then there exists $r > 0$ such that $N_r(p) \subset E$. This means that p is an interior point, so $p \in E^\circ$. Hence, $E \subset E^\circ$.

On the other hand, interior points in E are necessarily in E, since any neighborhood of a point contains that point. Hence $E^\circ \subset E$.

Therefore, we conclude that $E = E^\circ$.

($\Leftarrow$) Assume that $E = E^\circ$. To show E is open, we only need to prove that E° is open.

Suppose $p \in E^\circ$. Then there exists $r > 0$ such that $N_r(p) \subset E$. Since $N_r(p)$ is open, for any point $x \in N_r(p)$, there exists $\delta > 0$ such that $N_\delta(x) \subset N_r(p) \subset E$. This means that every point in $N_r(p)$ is an interior point of E. Hence $N_r(p) \subset E^\circ$. This implies that E° is open.

Ex. 2.3 We prove that the interior E° of E is the largest open set contained in E by completing the following steps:

1. E° is an open set contained in E.
2. Any open set U contained in E is a subset of E°.

Step 1 Suppose $p \in E^\circ$. Then there exists $r > 0$ such that $N_r(p) \subset E$. Clearly, $p \in N_r(p)$. Thus, $p \in E$. Hence, E° is a subset of E.

Furthermore, since $N_r(p)$ is open, for any point $x \in N_r(p)$, there exists $\delta > 0$ such that $N_\delta(x) \subset N_r(p) \subset E$. This means that every point in $N_r(p)$ is an interior point of E, that is, $N_r(p) \subset E^\circ$. Hence, E° is an open set contained in E. ◀

Step 2 Let U be an open set contained in E. For any point $p \in U$, there exists an open neighborhood $N_r(p) \subset U$. Since U is a subset of E, we have $N_r(p) \subset E$. This means that p is an interior point of E, and hence belongs to E°. Therefore, U is a subset of E°.

Ex. 2.4 We only prove the relation for two subsets A and B, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. By repeating applying the relation for two subsets, one can easily to have the desired relation for n subsets.

Since $A \subset A \cup B$ and $B \subset A \cup B$, we have $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$. Thus, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

On the other hand, since $A \subset \overline{A}$ and $B \subset \overline{B}$, we have $A \cup B \subset \overline{A} \cup \overline{B}$. Thus, $\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}}$. Since the closures $\overline{A}$ and $\overline{B}$ are closed and the union of two closed sets is closed, we know that $\overline{A} \cup \overline{B}$ is closed. By Proposition 2.14, we have $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Hence, $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

Therefore, we conclude that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Ex. 2.5

1. By the archimedean property, we can choose a positive integer N such $\varepsilon > \dfrac{1}{N}$. Then the interval $(-\varepsilon, \varepsilon)$ contains $0, \dfrac{1}{N}, \dfrac{1}{N+1}, \dfrac{1}{N+2}, \ldots$. Clearly, the finite collection
$$(-\varepsilon, \varepsilon), (1-\varepsilon, 1+\varepsilon), (\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon), \ldots, (\tfrac{1}{N-1}-\varepsilon, \tfrac{1}{N-1}+\varepsilon)$$
is a subcover of S.

2. To show that S is compact, by Theorem 2.21, we only need to prove that S is bounded and closed.

   Proof: S is bounded. For any $x \in S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \right\}$, we have $|x| \le 1$, so that S is bounded.

   Proof: S is closed. By the archimedean property, for any $\varepsilon > 0$, there is a positive integer n, such that $\varepsilon > \dfrac{1}{n}$. Thus, any neighborhood of 0 contains infinitely many points in S, so that 0 is a limit point of S.

   For any $x \in (0, 1]$, it is easy to see that there exists $r > 0$ such that the neighborhood $N_r(x)$ contains at most one point of S. Hence, any point in $(0, 1]$ is not a limit point of S.

   Therefore, 0 is the only limit point of S. Since $0 \in S$, we conclude that S is closed.

Ex. 2.6 For each n, since A_n is a nonempty bounded open subset, there is a bounded closed interval I_n such that $A_n \subset I_n$. Thus, $\overline{A}_n \subset \overline{I}_n = I_n$. Hence, $\overline{A}_n$ is a nonempty bounded closed subset of $\mathbb{R}$, so it is compact by Theorem 2.21. Therefore, $\bigcap_{n=1}^{\infty} \overline{A}_n \ne \varnothing$, by Theorem 2.20.

If we can prove $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A}_n$, then we can conclude that $\bigcap_{n=1}^{\infty} A_n \ne \varnothing$.

Clearly, $A_n \subset \overline{A}_n$ for all $n = 1, 2, \ldots$. Thus, $\bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} \overline{A}_n$.

On the other hand, to show that $\bigcap_{n=1}^{\infty} \overline{A}_n \subset \bigcap_{n=1}^{\infty} A_n$, we notice that
$$\bigcap_{n=2}^{\infty} \overline{A}_n \subset \overline{A}_2 \subset A_1 \subset \overline{A}_1$$
and have
$$\bigcap_{n=1}^{\infty} \overline{A}_n = \left(\bigcap_{n=2}^{\infty} \overline{A}_n\right) \cap \overline{A}_1 = \bigcap_{n=2}^{\infty} \overline{A}_n.$$
Because $\overline{A}_n \subset A_{n-1}$ for all $n = 2, 3, \ldots$, we have $\bigcap_{n=2}^{\infty} \overline{A}_n \subset \bigcap_{n=1}^{\infty} A_n$. Hence $\bigcap_{n=1}^{\infty} \overline{A}_n \subset \bigcap_{n=1}^{\infty} A_n$.

Therefore, we conclude that $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A}_n$.

Ex. 2.7

1. We only prove that for two perfect sets A and B, the union $A \cup B$ is perfect. By repeating applying the result for two perfect sets, one can easily to have the same result holds for any finite collection of perfect sets. ◀

By definition, a set is perfect if it is closed and if every point of the set is a limit point of the set. We know that, by Proposition 2.13, the union $A \cup B$ is closed since A and B are closed. If $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$, then x is a limit point of A, so it is a limit point of $A \cup B$. Similarly, if $x \in B$, then x is also a limit point of $A \cup B$. Therefore, the union $A \cup B$ is perfect.

2. To see that the union of a countable collection of perfect sets may not be perfect, consider the collection $\{A_n\}$, where

$$A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right], \quad n = 1, 2, \ldots.$$

We claim that

$$\bigcup_{n=1}^{\infty} A_n = (-1, 1).$$

In fact, it is clear that $A_n \subset (-1, 1)$, so that $\bigcup_{n=1}^{\infty} A_n \subset (-1, 1)$.

On the other hand, if $x \in (-1, 1)$, then, by the archimedean property, there is a positive integer N_1 such that $N_1 > \frac{1}{1-x}$. For the same reason, there is a positive integer N_2 such that $N_2 > \frac{1}{1+x}$. Thus, for any positive integer $n > \max\{N_1, N_2\}$, we have

$$n > \frac{1}{1-x}, \qquad n > \frac{1}{1+x}.$$

These two inequalities give $-1 + \frac{1}{n} < x < 1 - \frac{1}{n}$. Consequently, we have $x \in A_n \subset \bigcup_{n=1}^{\infty} A_n$. Hence, we have $(-1, 1) \subset \bigcup_{n=1}^{\infty} A_n$.

Therefore, we have $\bigcup_{n=1}^{\infty} A_n = (-1, 1)$.

Since any closed interval is perfect, and any finite open interval is not perfect (because it is not closed), the proved equality shows that the union of a countable collection of perfect sets may not be perfect.

Ex. 2.8 Let G be an open set in $\mathbb{R}$. For each $x \in G$, there are y and z, with $z < x < y$, such that $(z, y) \subset G$. Let $b = \sup\{y \colon (x, y) \subset G\}$ and $a = \inf\{z \colon (z, x) \subset G\}$. Then $-\infty \le a < x < b \le \infty$. Put $I_x = (a, b)$. It is clear that I_x is an open interval.

We claim that $b \notin G$. In fact, there is nothing to prove if $b = \infty$. If b is finite, and $b \in G$, then there is some $\delta > 0$ such that $(b - \delta, b + \delta) \subset G$ since G is open. This contradicts to the definition of b. Similarly, $a \notin G$.

We shall prove that $I_x \subset G$. Let $w \in I_x$, say $x < w < b$. By the definition of b, there is $y > x$ such that $(x, y) \subset G$. Hence $w \in G$. We can similarly discuss the case of $a < w < x$.

For each $x \in G$, the above construction yields a collection of open intervals $\{I_x\}$. We claim that $G = \bigcup I_x$. In fact, since each $I_x \subset G$, we have $\bigcup I_x \subset G$. On the other hand, for any $x \in G$, we know there is I_x such that $x \in I_x$. This implies $x \in \bigcup I_x$, so that $G \subset \bigcup I_x$.

It remains to show that the collection of open intervals $\{I_x\}$ is disjoint and at most countable.

To show that $\{I_x\}$ is disjoint, we let (a, b) and (c, d) be any two open intervals in the collection with both containing a common point x. Since $a < x < b$ and $c < x < d$, we have $c < b$ and $a < d$. Since $c \notin G$, it does not belong to (a, b), so that $c \le a$. The reversed inequality $a \le c$ holds by the same argument. Hence $a = c$. Similarly, $b = d$. Thus, any two different open intervals in the collection $\{I_x\}$ are disjoint.

To show that the collection $\{I_x\}$ is countable, we choose a rational number in each I_x as its representative. This can be done since $\mathbb{Q}$ is dense in $\mathbb{R}$. Since we have a disjoint collection, each segment contains a different rational number. Hence the collection can be put in one-to-one correspondence with a subset of the rational numbers. Thus it is an at most countable collection. ◀

Chapter 3 Quiz Answers

①D: The sequence $1, 2, 3, \ldots$ contains no convergent subsequence.

②C: By the remark in Definition 3.4, a sequence converges if and only if every subsequence converges to the same limit.

③A: By the definition, every sequence of K has a subsequence that converges to a point in K.

④C: The Cauchy criterion states that a series converges if and only if for every $\varepsilon > 0$, there is an integer N such that

$$\left|\sum_{k=n}^{m} a_k\right| < \varepsilon,$$

if $m \geq n \geq N$. Hence, it is a necessary and sufficient condition for convergence of a series.

⑤E: The Monotone Convergence Theorem states that a monotonic sequence converges if and only if it is bounded. Hence, we know that a bounded monotonic sequence must converge. This gives a sufficient condition for a sequence to converge.

⑥B: If $\underline{\lim}_{n\to\infty} x_n = -\infty$, then there exists a subsequence of $\{x_n\}$ whose limit is $-\infty$. If $\underline{\lim}_{n\to\infty} x_n = -\infty$ and $\overline{\lim}_{n\to\infty} x_n$ is finite, then $\{x_n\}$ must be bounded above but unbounded below.

⑦D: If a series converges as determined by the Comparison Test, the Root Test, or the Ratio Test, then the seres converges absolutely. The Divergence Test cannot be used for convergence.

Dirichlet's Test can be used to determine conditional convergence. A successful example is the convergence of the alternating harmonic series.

⑧E: If the absolute value of a series converges, then the series converges. ◀

Chapter 3 Exercise Solutions

Ex. 3.1 Since $\lim_{n\to\infty} x_n = \alpha$, for $\varepsilon = 1 > 0$, there exists an integer N_1 such that $n \geq N_1$ implies $|x_n - \alpha| < 1$. Thus, by triangle inequality, for $n \geq N_1$, we have

$$|x_n| = |x_n - \alpha + \alpha| \leq |x_n - \alpha| + |\alpha| < 1 + |\alpha|.$$

Hence, for $n \geq N_1$,

$$|x_n - \alpha| \leq |x_n| + |\alpha| < 1 + |\alpha| + |\alpha| = 1 + 2|\alpha|.$$

Again since $\lim_{n\to\infty} x_n = \alpha$, for $\varepsilon > 0$, there is an integer N_2 such that $n \geq N_2$ implies $|x_n - \alpha| < \dfrac{\varepsilon}{1+2|\alpha|}$. Let $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we have

$$\left|x_n^2 - \alpha^2\right| = |x_n - \alpha| \cdot |x_n + \alpha| < |x_n - \alpha| \cdot (1 + 2|\alpha|) < \frac{\varepsilon}{1+2|\alpha|} \cdot (1 + 2|\alpha|) = \varepsilon.$$

Hence, by the definition of limit, we conclude that $\lim_{n\to\infty} x_n^2 = \alpha^2$.

Ex. 3.2 It is clear that $0 < s_1 = \sqrt{2} < 2$. Suppose $0 < s_n < 2$. Then

$$0 < s_{n+1} = \sqrt{2 + \sqrt{s_n}} < \sqrt{2 + \sqrt{2}} < 2,$$

Hence, by induction, we have $0 < s_n < 2$ for all $n \geq 1$.

We shall show that $\{s_n\}$ is an increasing sequence by induction. In fact,

$$s_2 = \sqrt{2 + \sqrt{\sqrt{2}}} > \sqrt{2} = s_1.$$

Suppose $s_n > s_{n-1}$. Then

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

Thus, $\{s_n\}$ is a bounded increasing sequence. By Theorem 3.10, $\{s_n\}$ converges.

Ex. 3.3 We show that

$$(x_{2n-1}, x_{2n}) = \left(\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^{n-1}-1}{2^n}\right), \qquad n \geq 1.$$

In fact, by the definition,

$$(x_1, x_2) = (0, \tfrac{0}{2}) = (0, 0).$$

Suppose the formula holds for $n = k$. Then

$$\begin{aligned}
\left(x_{2(k+1)-1}, x_{2(k+1)}\right) &= \left(\frac{1}{2} + x_{2k}, \frac{x_{2k+1}}{2}\right) = \left(\frac{1}{2} + \frac{2^{k-1}-1}{2^k}, \frac{\frac{1}{2} + x_{2k}}{2}\right) \\
&= \left(\frac{2^{k+1}-2}{2^{k+1}}, \frac{1}{4} + \frac{1}{2} \cdot \frac{2^{k-1}-1}{2^k}\right) \\
&= \left(\frac{2^{(k+1)-1}-1}{2^{(k+1)-1}}, \frac{2^{(k+1)-1}-1}{2^{k+1}}\right).
\end{aligned}$$

The proved expression for $\{x_n\}$ gives:

$$\lim_{n\to\infty} x_{2n-1} = 1, \quad \lim_{n\to\infty} x_{2n} = \frac{1}{2}.$$

Hence, by the definitions of upper and lower limits, we have

$$\limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}}\, x_n = 1, \qquad \liminf_{n\to\infty} x_n = \underline{\lim_{n\to\infty}}\, x_n = \frac{1}{2}.$$

Ex. 3.4 Let $\{x_n\}$ be monotonically increasing with a convergent subsequence $\{x_{n_k}\}$, $\lim_{k\to\infty} x_{n_k} = \alpha$. Since $\{x_{n_k}\}$ is also monotonically increasing, by Theorem 3.10, $\alpha = \sup\{x_{n_k}\}$. We claim that $x_n \leq \alpha$ for all $n \geq 1$. In fact, if there is $x_N > \alpha$, then we have $x_{n_N} \geq x_N > \alpha$. This violates the fact that $\alpha = \sup\{x_{n_k}\}$.

For any $\varepsilon > 0$, there exists an integer K such that $k \geq K$ implies

$$|x_{n_k} - \alpha| < \varepsilon.$$

The last inequality implies that $\alpha - \varepsilon < x_{n_k} < \alpha + \varepsilon$. Since $\{x_n\}$ is monotonically increasing, we have that, for $n \geq n_K$,

$$\alpha - \varepsilon < x_{n_K} \leq x_n \leq \alpha < \alpha + \varepsilon.$$

By definition, this means that $\lim_{n\to\infty} x_n = \alpha$.

Ex. 3.5 By applying Proposition 3.14 and its corollary, we have

$$L = \lim_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n \leq \underline{\lim}_{n\to\infty} y_n \leq \overline{\lim}_{n\to\infty} y_n \leq \overline{\lim}_{n\to\infty} z_n = \lim_{n\to\infty} z_n = L.$$

Thus, $\underline{\lim}_{n\to\infty} y_n = \overline{\lim}_{n\to\infty} y_n = L$, so that $\lim_{n\to\infty} y_n = L$.

1. Let $x_n = \sqrt[n]{n} - 1$. Then $n = (1 + x_n)^n$. It is clear that $x_n > 0$ for all $n > 1$. By the binomial formula, we get

$$n = (1 + x_n)^n > 1 + nx_n + \frac{n(n-1)}{2}x_n^2 > \frac{n(n-1)}{2}x_n^2, \qquad n > 1.$$

Thus, we have

$$0 < x_n < \sqrt{\frac{2}{n-1}}, \qquad n > 1.$$

It is easy to see that $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$. Hence, the hypotheses of the Squeeze Theorem hold. Consequently, we have $\lim_{n\to\infty} x_n = 0$. Therefore, $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

2. If $n > 2024$, by the binomial formula, for $p > 0$, we have

$$(1+p)^n > 1 + \binom{n}{1}p + \binom{n}{2}p^2 + \cdots + \binom{n}{2024}p^{2024} > \frac{n(n-1)\cdots(n-2023)}{2024!}p^{2024}.$$

Thus, we get

$$0 < \frac{n^{2023}}{(1+p)^n} < \frac{n^{2023}}{\frac{n(n-1)\cdots(n-2023)}{2024!}p^{2024}}, \qquad n > 2024.$$

It is easy to see that $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{n^{2023}}{\frac{n(n-1)\cdots(n-2023)}{2024!}p^{2024}} = 0$. Hence, the hypotheses of the Squeeze Theorem hold. Consequently, we have $\lim_{n\to\infty} \frac{n^{2023}}{(1+p)^n} = 0$.

Ex. 3.6 **1.** For each $n \geq 1$, the partial sum

$$s_n = \sum_{k=1}^{n} a_k = \sqrt{n+1} - 1.$$

Since $\{s_n\}$ is unbounded, so it diverges. Hence, $\sum a_n$ diverges.

2. Since

$$0 < a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \leq \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{n^{3/2}}.$$

Since $\sum \frac{1}{n^{3/2}}$ is a convergent p-series (with $p = \frac{3}{2}$), by the Comparison Test, we know that $\sum a_n$ converges.

3. Based on the result given in part **1** in Exercise 3.5, we know that $\lim \sqrt[n]{n} = 1$. Hence, there exists N_0 such that for $n \geq N_0$,

$$0 < \sqrt[n]{n} - 1 < \frac{1}{2},$$

which implies that for $n \geq N_0$,

$$0 < a_n = (\sqrt[n]{n} - 1)^n < \left(\frac{1}{2}\right)^n.$$

The series $\sum \left(\frac{1}{2}\right)^n$ is a convergent geometric series, thus, by the Comparison Test, we know that $\sum a_n$ converges.

4. If $|z| \leq 1$, then $|1 + z^n| \leq 1 + |z|^n \leq 2$, which implies

$$|a_n| = \left|\frac{1}{1+z^n}\right| \not\to 0.$$

◀

By the Divergence Test, $\sum a_n$ diverges.

If $|z| > 1$, by the estimation

$$|a_n| = \left|\frac{1}{1+z^n}\right| \le \frac{1}{|z|^n - 1},$$

The series $\sum \frac{1}{|z|^n - 1}$ converges by the Ratio Test, since

$$\lim \frac{\dfrac{1}{|z|^n - 1}}{\dfrac{1}{|z|^{n-1} - 1}} = \lim \frac{1}{|z|} \cdot \frac{\dfrac{1}{1 - 1/|z|^n}}{\dfrac{1}{1 - 1/|z|^{n-1}}} = \frac{1}{|z|} < 1.$$

Hence, we know that $\sum a_n$ converges by the Comparison Test.

Ex. 3.7 It is clear that the following inequality holds

$$0 \le \frac{\sqrt{a_n}}{n} = \sqrt{\frac{a_n}{n^2}} \le \frac{1}{2}a_n + \frac{1}{2} \cdot \frac{1}{n^2}, \qquad n = 1, 2, 3, \ldots.$$

Since $\sum a_n$ is convergent and $\sum \frac{1}{n^2}$ is a convergent p-series (with $p = 2$), by Proposition 3.18, we know that the series $\sum b_n$ converges, where $b_n = \frac{1}{2}a_n + \frac{1}{2} \cdot \frac{1}{n^2}$. Hence, by the Comparison Test, we know that $\sum \frac{\sqrt{a_n}}{n}$ converges.

Ex. 3.8 For convergent series $\sum a_n$, the series $\sum a_n^2$ may not be convergent. This can be illustrated in the following example.

Consider the series $\sum a_n$, with

$$a_{2n-1} = \frac{1}{\sqrt{n}}, \quad a_{2n} = -\frac{1}{\sqrt{n}}, \qquad n \ge 1.$$

It is easy to see that the partial sums of $\sum a_n$ are

$$s_{2n-1} = \frac{1}{\sqrt{n}}, \quad s_{2n} = 0, \qquad n \ge 1.$$

Thus, we have

$$0 \le s_n \le \sqrt{\frac{2}{n+1}}, \qquad n \ge 1.$$

Applying the Squeeze Theorem gives $\lim s_n = 0$ so that $\sum a_n$ converges.

On the other hand, let $t_n = \sum_{k=1}^{n} \frac{1}{k}$. Then $s_{2n} = 2t_n$, where s_n is the n-th partial sum of the series

$$\sum a_n^2 = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \cdots.$$

Since the harmonic series diverges, we know that $\{t_n\}$, which is the sequence of the partial sums of the harmonic series, diverges. Hence, the sequence $\{s_{2n}\}$ diverges. Because the latter is a subsequence of $\{s_n\}$. Hence, $\{s_n\}$ diverges.

Ex. 3.9

1. If $L^* = \overline{\lim\limits_{n\to\infty}} \frac{a_n}{b_n} < \infty$, then there is an integer N_0 such that $n \ge N_0$ implies $\frac{a_n}{b_n} < L^* + 1$. Hence,

$$0 \le a_n < (L^* + 1)b_n, \qquad n \ge N_0.$$

If $\sum b_n$ converges, then $\sum (L^* + 1)b_n$ converges. In this case, by the Comparison Test, we know that $\sum a_n$ converges.

2. If $L_* = \underline{\lim\limits_{n\to\infty}} \frac{a_n}{b_n} > 0$, then there is an integer N_0 such that $n \ge N_0$ implies $\frac{a_n}{b_n} > \frac{1}{2}L_*$. Hence,

$$0 \le \tfrac{1}{2}L_* b_n < a_n, \qquad n \ge N_0.$$

If $\sum b_n$ diverges, then $\sum \frac{1}{2}L_* b_n$ diverges. In this case, by the Comparison Test, we know that $\sum a_n$ diverges. ◀

Ex. 3.10 Let $A_n = \sum_{k=0}^{n} |a_k|$ and $B_n = \sum_{k=0}^{n} |b_k|$. By the hypothesis, we know that $\{A_n\}$ and $\{B_n\}$ are convergent. For their Cauchy product $\sum c_n$, by applying the triangle inequality, we have

$$\sum_{k=0}^{n} |c_k| = \sum_{k=0}^{n} \left| \sum_{j=0}^{k} a_j b_{k-j} \right| \le \sum_{k=0}^{n} \sum_{j=0}^{k} |a_j| \, |b_{k-j}| = A_n B_n.$$

Hence, by the Comparison Test, the Cauchy product $\sum c_n$ is absolutely convergent. ◀

Chapter 4 Quiz Answers

①A: Item A is the continuity of f at x_0. Item C is the uniform continuity of f on E.

②C: Any interval $[a, b]$ is compact. By the Extreme Value Theorem, the continuous function f on the compact set $[a, b]$ attains its maximum value.

③D: The function f is continuous on $\mathbb{R}$. Since

$$\{x \in \mathbb{R} : f(x) \neq 0\} = \{x \in \mathbb{R} : f(x) < 0\} \cup \{x \in \mathbb{R} : f(x) > 0\}$$

Each set on the right-hand side is open. In fact, the complement of $\{x \in \mathbb{R} : f(x) < 0\}$ is $\{x \in \mathbb{R} : f(x) \geq 0\}$. The latter is closed: if $\{x_n\}$ is a consequence in $\{x \in \mathbb{R} : f(x) \geq 0\}$, and $x_n \to x^*$, then by the continuity,

$$f(x^*) = \lim f(x_n) \geq 0,$$

so that $x^* \in \{x \in \mathbb{R} : f(x) \geq 0\}$.

Thus, the set $\{x \in \mathbb{R} : f(x) \neq 0\}$ is a union of two open sets, so it is open.

④E: Since $\overline{E}$ is closed by Proposition 2.14, we know that $(\overline{E})^c$ is open. Thus, by Theorem 4.17, $f^{-1}((\overline{E})^c)$ is open.

⑤D: According to Theorem 4.23, a continuous mapping maps a connected subset to a connected subset. Because $E = (a, b)$ is connected, the set $f(E)$ is connected.

⑥A: The interval $[a, b]$ is compact, by the Extreme Value Theorem, the supremum $\sup\limits_{x\in[a,b]} g(x)$ and the infimum $\inf\limits_{x\in[a,b]} g(x)$ can be attained by the function g. In fact, the values of g lie between these two values. So, option A is false.

⑦B: The function $f(x) = 1/x$ is continuous on $(0, 1)$, but not uniformly continuous on $(0, 1)$.

⑧D: Since $\lim\limits_{x\to 0-} f(x) = -1$, $\lim\limits_{x\to 0+} f(x) = 1$, and neither $\lim\limits_{x\to 0+} g(x)$ nor $\lim\limits_{x\to 0-} g(x)$ exists. Hence, at $x = 0$, f has a jump discontinuity and g has an essential discontinuity. ◀

Chapter 4 Exercise Solutions

Ex. 4.1 The original definition of limit is

"for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - q| \le \varepsilon$ for all points $x \in E$ for which $0 < |x - p| < \delta$."

Thus, if $\lim_{x \to p} f(x) = q$, then for every $\varepsilon^* = \varepsilon^2 > 0$, there exists a $\delta > 0$ such that $|f(x) - q| < \varepsilon^* = \varepsilon^2$ for all points $x \in E$ for which $0 < |x - p| < \delta$. Hence, the modified statement for limit holds.

On the other hand, assume that the modified statement for limit holds. Then, for every $\varepsilon^* > 0$, there exists a $\delta > 0$ such that $|f(x) - q| \le \varepsilon^{*2}$ for all points $x \in E$ for which $0 < |x - p| < \delta$. In particular, for every $\varepsilon > 0$, if we take $\varepsilon^* = \sqrt{\frac{1}{2}\varepsilon}$, then there exists a $\delta > 0$ such that $|f(x) - q| \le \frac{1}{2}\varepsilon < \varepsilon$ for all points $x \in E$ for which $0 < |x - p| < \delta$. This means that $\lim_{x \to p} f(x) = q$.

Ex. 4.2 The function $f(x) = \sqrt{x}$ is defined on $[0, \infty)$. We need to prove that $\lim_{x \to p} f(x) = f(p)$ for every $p \in [0, \infty)$.

Let $\varepsilon > 0$ be given.

Case 1: $\lim_{x \to 0} \sqrt{x} = 0$. In this case, we understand that when $x \to 0$, we have $x \ge 0$.

Take $\delta = \varepsilon^2$. If $|x - 0| < \delta$, then $\left|\sqrt{x} - \sqrt{0}\right| = \sqrt{x} < \sqrt{\delta} = \varepsilon$. Thus, by the definition, we have $\lim_{x \to 0} \sqrt{x} = 0$.

Hence, the function f is continuous at $x = 0$.

Case 2: $\lim_{x \to p} \sqrt{x} = \sqrt{p}$ for all $p > 0$.

Take $\delta = \sqrt{p} \cdot \varepsilon$. If $|x - p| < \delta$, then

$$\left|\sqrt{x} - \sqrt{p}\right| = \frac{|x - p|}{\sqrt{x} + \sqrt{p}} \le \frac{|x - p|}{\sqrt{p}} < \frac{\sqrt{p} \cdot \varepsilon}{\sqrt{p}} = \varepsilon.$$

Thus, by the definition, we have $\lim_{x \to p} \sqrt{x} = \sqrt{p}$ for all $p > 0$.

Hence, the function f is continuous at $x = p$ for all $p > 0$.

Ex. 4.3 Let $\varepsilon > 0$ be given.

For any $p \in (0, \infty)$, since $p > 0$, we take $\delta_1 = \frac{1}{2}p > 0$. If $|x - p| < \delta_1$, then $\frac{1}{2}p = p - \delta_1 < x < p + \delta_1$. It is obvious we have the following

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{p}}\right| = \frac{\left|\sqrt{x} - \sqrt{p}\right|}{\sqrt{x} \cdot \sqrt{p}} = \frac{|x - p|}{\left|\sqrt{x} + \sqrt{p}\right| \cdot \sqrt{x} \cdot \sqrt{p}} < \frac{|x - p|}{\sqrt{p} \cdot \sqrt{x} \cdot \sqrt{p}} = \frac{|x - p|}{\sqrt{x} \cdot p}$$

Take $\delta_2 = \dfrac{p^{3/2} \cdot \varepsilon}{\sqrt{2}} > 0$ and let $\delta = \min\{\delta_1, \delta_2\}$. Then, when $|x - p| < \delta$, we have

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{p}}\right| < \frac{\frac{p^{3/2} \cdot \varepsilon}{\sqrt{2}}}{\sqrt{\frac{1}{2}p} \cdot p} = \varepsilon.$$

Thus, by the definition, we have $\lim_{x \to p} \dfrac{1}{\sqrt{x}} = \dfrac{1}{\sqrt{p}}$ for all $p > 0$.

Hence, the function $\dfrac{1}{\sqrt{x}}$ is continuous on $(0, \infty)$.

Ex. 4.4

1. Assume that f and g are monotonically increasing. Thus, for $a \le x < y \le b$, we have

$$f(x) \le f(y), \qquad g(x) \le g(y).$$

Hence, for $a \le x < y \le b$

$$H(x) = \max\{f(x), g(x)\} \le \max\{f(y), g(y)\} = H(y),$$

so that H is monotonically increasing.

In the similar manner, we can show that if f and g are monotonically decreasing, so it H.

The analogous results hold for the function h.

2. It is easy to verify that the following identity holds:

$$H(x) = \max\{f(x), g(x)\} = \tfrac{1}{2}\left[f(x) + g(x) + |f(x) - g(x)|\right].$$

According to Item 1 of Proposition 4.18, a linear combination of continuous functions is continuous. Additionally, we know that the absolute value function is continuous. By using Item 3 of Proposition 4.18, we can conclude that the composition of the absolute value function and a continuous function is also continuous. Therefore, we can deduce that the function H is continuous.

In the similar manner, using the identity

$$h(x) = \min\{f(x), g(x)\} = \tfrac{1}{2}\left[f(x) + g(x) - |f(x) - g(x)|\right],$$

we can prove that h is continuous if f and g are continuous.

Ex. 4.5 Since $Z(f) = f^{-1}(\{0\})$, and $\{0\}$ is a closed set in $\mathbb{R}$, by Theorem 4.17, $Z(f)$ is closed if f is continuous.

Another alternative approach is to prove the result directly. Let p be a limit point of $Z(f)$. Then there exists a sequence $\{p_n\}$ in $Z(f)$ such that $|p_n - p| \to 0$. Since f is continuous at p, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - p| < \delta$ implies $|f(x) - f(p)| < \varepsilon$. By $|p_n - p| \to 0$, we know that there exists N such that $n \geq N$ implies $|p_n - p| < \delta$. Hence, if $n \geq N$, then $|f(p_n) - f(p)| < \varepsilon$. Since $f(p_n) = 0$, we know that $|f(p)| < \varepsilon$ for any $\varepsilon > 0$. This implies $f(p) = 0$, or $p \in Z(f)$. Therefore $z(f)$ is closed, since it contains all its limit points.

Ex. 4.6 Put $g(x) = x - f(x)$ for $x \in I$. It is clear that g is continuous on I. If $g(0) = 0$ or $g(1) = 0$, the conclusion of the problem holds either for $x = 0$ or $x = 1$. Otherwise, we have $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$. By the Intermediate Value Theorem, there exists a $x \in (0, 1)$ such that $g(x) = 0$, since $g(0) < 0 < g(1)$. This gives $f(x) = x$ for this x.

Ex. 4.7 Without loss of generality, assume that $x_1 < \cdots < x_n$. By the hypothesis, the function f is continuous on $[x_1, x_n]$. Denote

$$m = \min_{x\in[x_1,x_n]} f(x), \qquad M = \max_{x\in[x_1,x_n]} f(x).$$

Show that

$$m = \frac{m + m + \cdots + m}{n} \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq \frac{M + M + \cdots + M}{n} \leq M.$$

By the Intermediate Value Theorem, there exists $\xi \in [x_1, x_n] \subset (a, b)$ such that

$$f(\xi) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Ex. 4.8 By the hypothesis $\lim\limits_{x\to\infty} f(x) = A$, we know that for $\varepsilon = 1$, there exists a number b, with $b > a$, such that $x \geq b$ implies $|f(x) - A| < 1$. Thus, when $x \geq b$, we have

$$|f(x)| = |f(x) - A + A| \leq |f(x) - A| + |A| < 1 + |A|.$$

On the finite interval $[a, b]$, which is compact, the function f is continuous. By Theorem 4.19, we know that $f([a, b])$ is bounded, so that there is a real number M such that for all $x \in [a, b]$,

$$|f(x)| \leq M.$$

Denote $B = \max\{1 + |A|, M\}$. Then, for all $x \in [a, \infty) = [a, b] \cup [b, \infty)$, we have

$$|f(x)| \leq B,$$

that is, the function f is bounded on $[a, \infty)$.

Ex. 4.9 If f is a uniformly continuous on E, then there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for any $x, y \in E$ satisfying $|x - y| < \delta$. Let $\{x_n\}$ be a Cauchy sequence in E. By definition, there exists an integer N such that $|x_n - x_m| < \delta$ if $n, m \geq N$. Thus, if $n, m \geq N$, we have

$$|f(x_n) - f(x_m)| < \varepsilon.$$

This means that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

Ex. 4.10 Since f is uniformly continuous on E, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ for any $x, y \in E$ satisfying $|x - y| < \delta$. ◀

Because the set E is bounded, it is contained in a bounded closed interval I, that is, $E \subset I$. For each $x \in I$, the collection $\{(x-\delta, x+\delta)\}$ of open intervals is an open cover of I. Since I is compact, there is a finite subcover of I. Obviously, this finite collection is also a cover of E. We keep only the open intervals in the collection which intersect with E, say $(y_1-\delta, y_1+\delta), \ldots, (y_K-\delta, y_K+\delta)$. Let $x_i \in (y_i-\delta, y_i+\delta)$, $i = 1, \ldots, K$, where $x_i \in E$. Denote $M = \max\limits_{1\le i\le K}\{|f(x_i)|\}$.

For any fixed $x \in E$, there is i_0, $1 \le i_0 \le K$, such that $x \in (x_{i_0}-\delta, x_{i_0}+\delta)$. Thus,

$$|f(x)| \le |f(x) - f(x_{i_0})| + |f(x_{i_0})| < 1 + M,$$

so that f is bounded on E. ◀

Chapter 5 Quiz Answers

①B: If $f'(x)$ exists for $x \in (a,b)$, then both the left-hand derivative and the right-hand derivative exist and equal to $f'(x)$. So, item B is correct.

②A: Since f is differentiable on $[a,b]$, by Preposition 5.2, f is continuous on $[a,b]$. By the Mean Value Theorem, there is a point $x \in (a,b)$ such that

$$f(b) - f(a) = (b-a)f'(x).$$

Since $f(a) = f(b)$, we have $f'(x) = 0$.

③C: By the definition, a critical point of function f is the point x at which either $f'(x) = 0$ or $f'(x)$ is undefined.

④D: The Mean Value Theorem claims that

$$\frac{f(b) - f(a)}{b-a} = f'(x).$$

This can be interpreted that the average rate of change of a function over an interval is equal to its instantaneous rate of change at some point within the interval.

⑤E: By the Mean Value Theorem, for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| = |f'(c)| \cdot |x-y| \le M|x-y|.$$

Thus, for every $\varepsilon > 0$, when $|x-y| < \dfrac{\varepsilon}{M} = \delta$, then $\delta > 0$, and

$$|f(x) - f(y)| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Hence, f is uniformly continuous on $\mathbb{R}$.

⑥D: According to l'Hôpital's rule, if $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

⑦B: Taylor's theorem is a method for approximating a function with a polynomial. It gives the remainder $\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, which measures the accuracy of the approximation.

⑧D: Taylor's theorem gives

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \qquad x_0 \in [a,b].$$

The remainder $R_n(x) = f(x) - \sum\limits_{k=0}^{n} \dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ is the error of the approximation between $f(x)$ and $\sum\limits_{k=0}^{n} \dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k$. For any $x \in [a,b]$, we have

$$|R_n(x)| \le \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \right|.$$

The maximum possible error of the right term is $M(b-a)^{n+1}/(n+1)!$. ◀

Chapter 5 Exercise Solutions

Ex. 5.1 The inequality implies that, for $x \neq y$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq C|x - y|.$$

Taking $x \to y$, we have $|f'(y)| \leq 0$, which implies that $f'(y) = 0$ for all $y \in \mathbb{R}$. We conclude that f is constant, by Item 2 of the Monotone Test.

Ex. 5.2 If there is $x \in (a, \infty)$ such that $f(x) = f(a)$, then, by the Mean Value Theorem, there exists ξ between x and a such that

$$f'(\xi) = \frac{f(x) - f(a)}{x - a} = \frac{f(a) - f(a)}{x - a} = 0.$$

If for all $x \in (a, \infty)$ such that $f(x) \neq f(a)$, then, without loss of generality, we assume that there is a number $c \in (a, \infty)$ such that $f(c) > f(a)$. Denote $\varepsilon = \frac{1}{2}[f(c) - f(a)] > 0$. Since $\lim\limits_{x \to \infty} f(x) = f(a)$, there is $X > c$ such that $|f(x) - f(a)| < \varepsilon$ for $x > X$, so that

$$f(X + 1) < f(a) + \varepsilon = \tfrac{1}{2}[f(a) + f(c)] < f(c).$$

Hence, the function f is continuous on $[a, X + 1]$, differentiable in $(a, X + 1)$, and for $c \in (a, X + 1)$,

$$f(c) > f(a), \qquad f(c) > f(X + 1).$$

By the Extreme Value Theorem, the function f attains its maximum value at some point $\xi \in (a, X + 1)$. By Proposition 5.7, we know that $f'(\xi) = 0$.

Ex. 5.3 Put $M = \sup\limits_{x \in \mathbb{R}} |g'(x)|$. Take any ε satisfying $0 < \varepsilon < \dfrac{1}{2M + 1}$. For $x < y$, by the Mean Value Theorem, we have

$$\begin{aligned} f(y) - f(x) = f'(c)(y - x) &= [1 + \varepsilon g'(c)](y - x) \\ &\geq (1 - \varepsilon M)(y - x) \\ &> \left(1 - \frac{1}{2M + 1} \cdot M\right)(y - x) > \frac{1}{2}(y - x), \end{aligned}$$

which implies that f is strictly increasing. Hence f is one-to-one.

Ex. 5.4 Put

$$P(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n + 1} x^{n+1}.$$

It is clear that the function P is a polynomial, so it is differentiable everywhere on $\mathbb{R}$. We know $P(0) = 0$, and $P(1) = 0$ by the hypothesis. By the Mean Value Theorem, there is $c \in (0, 1)$ such that

$$C_0 + C_1 c + \cdots + C_{n-1} c^{n-1} + C_n c^n = P'(c) = \frac{P(1) - P(0)}{1 - 0} = 0.$$

This completes the proof.

Ex. 5.5 Without loss of generality, we assume that $f'_+(a) > 0$. Then, there is a $\delta_1 > 0$ such that, if $t \in [a, b]$ and $t - a < \delta_1$.

$$\left| \frac{f(t) - f(a)}{t - a} - f'_+(a) \right| < \tfrac{1}{2} f'_+(a).$$

It follows that there is $x_1 \in (a, b)$ such that

$$f(x_1) = f(x_1) - f(a) > \tfrac{1}{2} f'_+(a)(x_1 - a) > 0.$$

Similarly, since $f'_-(b) > 0$, there is a $\delta_2 > 0$ such that, if $t \in [a, b]$ and $b - t < \delta_2$.

$$\left| \frac{f(t) - f(b)}{t - b} - f'_-(b) \right| < \tfrac{1}{2} f'_-(b).$$

It follows that there is $x_2 \in (a, b)$ such that

$$f(x_2) = f(x_2) - f(b) < \tfrac{1}{2} f'_-(b)(x_2 - b) < 0.$$

Clearly, we can choose x_1 and x_2 so that $a < x_1 < x_2 < b$. ◀

Since $f(x_1) \cdot f(x_2) < 0$, by the Intermediate Value Theorem or Bolzano's Theorem, there is $\xi \in (x_1, x_2) \subset (a, b)$ such that $f(\xi) = 0$.

Ex. 5.6 By the definition of derivative,

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x), \qquad \lim_{t \to x} \frac{g(t) - g(x)}{t - x} = g'(x) \neq 0.$$

Apply Proposition 4.13, we have

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{\dfrac{f(t) - f(x)}{t - x}}{\dfrac{g(t) - g(x)}{t - x}} = \frac{\lim\limits_{t \to x} \dfrac{f(t) - f(x)}{t - x}}{\lim\limits_{t \to x} \dfrac{g(t) - g(x)}{t - x}} = \frac{f'(x)}{g'(x)}.$$

Ex. 5.7 By Taylor's Theorem, for any $y \in \mathbb{R}$, we have

$$f(y) = f(x) + f'(x)(y - x) + \frac{f''(\xi)}{2}(y - x)^2,$$

where ξ is a number between x and y. Putting $y = x + h, x - h$ into the formula, respectively, we have

$$f(x + h) = f(x) + f'(x)h + \frac{f''(\xi_1)}{2}h^2,$$
$$f(x - h) = f(x) - f'(x)h + \frac{f''(\xi_2)}{2}h^2,$$

where ξ_1 is between x and $x + h$ and ξ_2 is between x and $x - h$. The sum of these two equations yields

$$f(x + h) + f(x - h) = 2f(x) + \frac{f''(\xi_1) + f''(\xi_2)}{2} \cdot h^2.$$

Since $f''(x) > 0$ for all $x \in \mathbb{R}$, we see that $\dfrac{f''(\xi_1) + f''(\xi_2)}{2} \cdot h^2 > 0$ for $h \neq 0$. Thus,

$$f(x + h) + f(x - h) > 2f(x)$$

holds for all $x, h \in \mathbb{R}$, with $h \neq 0$. Finally, if we take $x = \frac{1}{2}(x_1 + x_2)$ and $h = \frac{1}{2}(x_1 - x_2)$, then the desired equality follows.

Ex. 5.8 Let $x_1, x_2 \in \mathbb{R}$, with $x_1 < x_2$. Denote $\overline{x} = \frac{1}{2}(x_1 + x_2)$. By Taylor's Theorem, for any $x \in \mathbb{R}$, we have

$$f(x) = f(\overline{x}) + f'(\overline{x})\,(x - \overline{x}) + \tfrac{1}{2}f''(\overline{x})\,(x - \overline{x})^2 + \tfrac{1}{6}f'''(\xi)(x - \overline{x})^3,$$

where ξ is a number between x and $\overline{x}$. Putting $x = x_1, x_2$ into the formula, respectively, we have

$$f(x_1) = f(\overline{x}) + f'(\overline{x})\,(x_1 - \overline{x}) + \tfrac{1}{2}f''(\overline{x})\,(x_1 - \overline{x})^2 + \tfrac{1}{6}f'''(\xi_1)(x_1 - \overline{x})^3,$$
$$f(x_2) = f(\overline{x}) + f'(\overline{x})\,(x_2 - \overline{x}) + \tfrac{1}{2}f''(\overline{x})\,(x_2 - \overline{x})^2 + \tfrac{1}{6}f'''(\xi_2)(x_2 - \overline{x})^3,$$

where ξ_1 is between x_1 and $\overline{x}$ and ξ_2 is between $\overline{x}$ and x_2. The difference of these two equations yields

$$f(x_2) - f(x_1) = f'(\overline{x})\,(x_2 - x_1) + \tfrac{1}{6}[f'''(\xi_2) + f'''(\xi_1)] \cdot 2 \cdot \frac{(x_2 - x_1)^3}{8}.$$

Since $f'''(x) > 0$ for all $x \in \mathbb{R}$, we see that $\frac{1}{6}[f'''(\xi_2) + f'''(\xi_1)] \cdot 2 \cdot \dfrac{(x_2 - x_1)^3}{8} > 0$. Thus,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > f'\big(\tfrac{1}{2}(x_1 + x_2)\big)$$

holds for all $x_1, x_2 \in \mathbb{R}$ with $x_2 > x_1$.

Ex. 5.9 When $M_0 = 0$, then $f(x) \equiv 0$, the inequality is trivial.

When $M_2 = 0$, then $f'(x)$ is constant and $f(x)$ is a linear function, by the Mean Value Theorem. In this case, if $f'(x) \equiv c \neq 0$, then M_0 is infinite, a contradiction to the hypothesis. If $f'(x) \equiv 0$, then $M_1 = 0$, again we have a trivial inequality.

When $M_0 > 0$ and $M_2 > 0$, by Taylor's Theorem, for any $h > 0$, there is a $\xi \in (x, x + 2h)$ such that

$$f(x + 2h) = f(x) + \frac{1}{1!}f'(x)(2h) + \frac{1}{2!}f''(\xi)(2h)^2,$$

which gives

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi).$$

By the triangle inequality, we have, for any $h > 0$,

$$|f'(x)| \le \frac{1}{2h}[|f(x+2h)| + |f(x)|] + h|f''(\xi)| \le \frac{1}{h}M_0 + hM_2.$$

In particular, if we take $h = \sqrt{M_0/M_2}$ in the last inequality, then

$$|f'(x)| \le 2\sqrt{M_0\, M_2}$$

for any $x \in (a, \infty)$. Since x is arbitrary, we have

$$M_1 \le 2\sqrt{M_0\, M_2},$$

which implies

$$M_1^2 \le 4M_0\, M_2.$$

Ex. 5.10

1. If x_1 and x_2 are two fixed points of f, and if $x_1 \ne x_2$, then by the Mean Value Theorem, there is a point ξ between x_1 and x_2 such that

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1.$$

This contradicts to the hypothesis.

2. For the sequence $\{x_n\}$ generated by $x_{n+1} = f(x_n)$, if $x_{n_0+1} = x_{n_0}$ for some n_0, then $x_{n_0+2} = f(x_{n_0+1}) = f(x_{n_0}) = x_{n_0+1}$. In this manner, we see that $x_k = x_{n_0}$ for all $k \ge n$. Hence, in this case, we have $\lim x_n = x_{n_0}$, and $x_{n_0} = x_{n_0+1} = f(x_{n_0})$.

If $x_{n+1} \ne x_n$ for all n, by the Mean Value Theorem, for every n, there is ξ_n between x_n and x_{n+1} such that

$$x_{n+2} - x_{n+1} = f(x_{n+1}) - f(x_n) = f'(\xi_n)(x_{n+1} - x_n).$$

Since $|f'(t)| \le A$ for all t, we have

$$|x_{n+2} - x_{n+1}| \le |f'(\xi_n)| \cdot |x_{n+1} - x_n| \le A \cdot |x_{n+1} - x_n|.$$

Hence,

$$|x_{n+1} - x_n| \le A^{n-1} \cdot |x_2 - x_1|, \qquad n = 1, 2, 3, \cdots.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Indeed, for $0 \le A < 1$, we have $\lim A^n = 0$ (see part 2 in Ex. 3.5). Let $\varepsilon > 0$ be given. Since $\dfrac{1-A}{|x_2 - x_1|} > 0$, there exists N such that $n \ge N$ implies $A^n < \varepsilon \cdot \dfrac{1-A}{|x_2 - x_1|}$. For $n > m \ge N$,

$$\begin{aligned}
|x_m - x_n| &\le |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + \cdots + |x_{n-1} - x_n| \\
&\le A^m \cdot |x_2 - x_1| + A^{m+1} \cdot |x_2 - x_1| + \cdots + A^{n-1} \cdot |x_2 - x_1| \\
&= \frac{A^m(1 - A^{n-m})}{1 - A} \cdot |x_2 - x_1| \\
&\le A^m \cdot \frac{|x_2 - x_1|}{1 - A} < \varepsilon.
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Put $\lim x_n = x$. Then, by the continuity of f,

$$x = \lim x_n = \lim x_{n+1} = \lim f(x_n) = f(x),$$

that is, x is a fixed point of f. ◀

Chapter 6 Quiz Answers

①C: Any Riemann integrable function must be bounded, as stated in the remark of Theorem 6.3.

②A: By item 1 of Theorem 6.4, if f is continuous on $[a,b]$, then $f \in \mathscr{R}[a,b]$.

③B: It is known that $\underline{\int_a^b} f \,\mathrm{d}x \le \overline{\int_a^b} f \,\mathrm{d}x$. Hence, option B implies $\underline{\int_a^b} f \,\mathrm{d}x = \overline{\int_a^b} f \,\mathrm{d}x$, so that $f \in \mathscr{R}[a,b]$.

Option A is not true, since $f \in \mathscr{R}[a,b]$ is equivalent to $\sup_P L(P,f) = \inf_P U(P,f)$.

Option C is not true, since the hypothesis requires only for some positive integer n. This is not sufficient for f being integrable.

Option D is not true for the same reason as option C.

④D: By the hypothesis, we have $g_2 - g_1 \ge 0$, so that $f_1(g_2 - g_1) \le f_2(g_2 - g_1)$. Applying the monotonicity of the Riemann integral, we have

$$\int_a^b f_1(g_2 - g_1)\,\mathrm{d}x \le \int_a^b f_2(g_2 - g_1)\,\mathrm{d}x,$$

so that

$$\int_a^b (f_1 g_2 + f_2 g_1)\,\mathrm{d}x \le \int_a^b (f_1 g_1 + f_2 g_2)\,\mathrm{d}x.$$

⑤E: The functions $\dfrac{1}{1+x^2}, x^2 + x^3, |x|, e^x$ are all continuous functions, so Theorem 6.5 applies.

⑥C: Let $F(x) = \int_0^x f(t)\,\mathrm{d}t$. Then $F\big(\alpha(x)\big) = \int_0^{\alpha(x)} f(t)\,\mathrm{d}t$. Thus, by the chain rule,

$$\big[F\big(\alpha(x)\big)\big]' = F'\big(\alpha(x)\big) \cdot \alpha'(x).$$

By Part 1 of the Fundamental Theorem of Calculus, we know $F'(x) = f(x)$, sp that

$$F'\big(\alpha(x)\big) = f\big(\alpha(x)\big).$$

Hence, option C is true.

⑦B: Since F_α is an antiderivative of $f\alpha'$, by Part 2 of the Fundamental Theorem of Calculus, we know that option B is true.

The function G and $F \cdot \alpha$ are not antiderivatives of $f\alpha'$. Thus, options A and C are not true.

⑧A: If $f =$ on $[a,b]$, then its antiderivative is a constant, so that $\int_a^b f\,\mathrm{d}x = 0$ by the Fundamental Theorem of Calculus.

The other options are not consequences of the Fundamental Theorem of Calculus. ◀

Chapter 6 Exercise Solutions

Ex. 6.1 Suppose $f(x^*) > 0$ for some $x^* \in [a,b]$. Since f is continuous on $[a,b]$, for $\varepsilon = \frac{1}{2}f(x^*) > 0$, there exist $\delta > 0$ such that $|x - x^*| < \delta$ and $x \in [a,b]$ imply

$$|f(x) - f(x^*)| < \tfrac{1}{2}f(x^*).$$

Thus, we know that there is an interval whose length is at least δ, say $[\gamma, \gamma+\delta] \subset [a,b]$, on which

$$f(x) > f(x^*) - \tfrac{1}{2}f(x^*) = \tfrac{1}{2}f(x^*).$$

By the monotonicity of the Riemann integral, we have

$$\int_a^b f\,\mathrm{d}x = \int_a^\gamma f\,\mathrm{d}x + \int_\gamma^{\gamma+\delta} f\,\mathrm{d}x + \int_{\gamma+\delta}^b f\,\mathrm{d}x \geq 0 + \tfrac{1}{2}f(x^*)\delta + 0 > 0,$$

which contradicts to the hypothesis $\int_a^b f\,\mathrm{d}x = 0$. Therefore, for every $x \in [a,b]$, we have $f(x) = 0$.

Ex. 6.2 By the fact that the rational and the irrational numbers are both sense in every $[a,b]$ for any $a < b$, we know that for every partition of $[a,b]$,

$$U(P,f) = \sum_{i=1}^n \sup_{[x_{i-1},x_i]} f(x)\,\Delta x_i = b - a, \qquad L(P,f) = \sum_{i=1}^n \inf_{[x_{i-1},x_i]} f(x)\,\Delta x_i = 0.$$

Hence $\underline{\int_a^b} f\,\mathrm{d}x = 0 < b - a = \overline{\int_a^b} f\,\mathrm{d}x$. By the definition, $f \notin \mathscr{R}[a,b]$.

Ex. 6.3 Claim 1: The condition $f^2 \in \mathscr{R}[a,b]$ does not imply $f \in \mathscr{R}[a,b]$.

Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ -1, & \text{if } x \text{ is irrational.} \end{cases}$$

It is clear that $f^2 \equiv 1$ is a constant function and is integrable on any finite interval $[a,b]$ $(a < b)$. However, the function f is not integrable on $[a,b]$. In fact, since the rational and the irrational numbers are both sense in $[a,b]$, we have

$$U(P,f) = \sum_{i=1}^n \sup_{[x_{i-1},x_i]} f(x)\,\Delta x_i = \sum_{i=1}^n \sup_{[x_{i-1},x_i]} (1)\,\Delta x_i = b - a,$$

$$L(P,f) = \sum_{i=1}^n \inf_{[x_{i-1},x_i]} f(x)\,\Delta x_i = \sum_{i=1}^n \inf_{[x_{i-1},x_i]} (-1)\,\Delta x_i = -(b-a).$$

Hence $\underline{\int_a^b} f\,\mathrm{d}x = -(b-a) < b - a = \overline{\int_a^b} f\,\mathrm{d}x$. By the definition, $f \notin \mathscr{R}[a,b]$.

Claim 2: The condition $f^3 \in \mathscr{R}[a,b]$ implies $f \in \mathscr{R}[a,b]$.

The function $g(x) = x^3 \colon [a,b] \to [a^3, b^3]$ is continuous and bijective, shown as follows.

- Continuous: The function g is an elementary function. All elementary functions are continuous in their domains.

- Injective: If $x_1^3 = x_2^3$, then, from the equality

$$\begin{aligned} 0 &= (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) \\ &= (x_1 - x_2)\left[(x_1 + \tfrac{1}{2}x_2)^2 + \tfrac{3}{4}x_2^2\right], \end{aligned}$$

  we have $x_1 = x_2$ (because $(x_1 + \frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 = 0$ also implies $x_1 = x_2$).
- Surjective: For ant $y \in [a^3, b^3]$, let $x = \operatorname{sgn} y \cdot \sqrt[3]{|y|}$. Then $x \in [a,b]$ and $x^3 = y$.

Thus, we apply item 3 in Proposition 4.18 to know that the function $\phi(x) = \sqrt[3]{x}$ is continuous. It is clear that $f = \phi \circ f^3$. Since $f^3 \in \mathscr{R}[a,b]$, we conclude that $f \in \mathscr{R}[a,b]$, by Theorem 6.5.

Ex. 6.4 **1.** Since $f, g \in \mathscr{R}[a,b]$, by the linearity of the Riemann integral, we know $f+g, f-g \in \mathscr{R}[a,b]$. Since the square of integrable function is integrable, we further know that $(f+g)^2, (f-g)^2 \in \mathscr{R}[a,b]$. Applying the linearity of the Riemann integral once more, we know that

$$fg = \tfrac{1}{2}[(f+g)^2 - (f-g)^2] \in \mathscr{R}[a,b].$$

2. By part **1**, it suffices to show that $1/g$ is integrable when g is integrable and $|g| \geq c > 0$. We note the equality

$$\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| = \frac{|g(x)-g(y)|}{|g(x)g(y)|} \leq \frac{1}{c^2}\left|g(x)-g(y)\right|.$$

Thus, for any partition P of $[a,b]$,

$$P\colon \quad a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

if we denote

$$M_i = \sup_{x_{i-1}\leq x\leq x_i} g(x), \qquad m_i = \inf_{x_{i-1}\leq x\leq x_i} g(x),$$

then we have

$$\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| \leq \frac{1}{c^2}(M_i - m_i), \qquad x, y \in [x_{i-1}, x_i],\ i = 1, 2, \ldots, n.$$

Hence,

$$U(P, 1/g) - L(P, 1/g) \leq \frac{1}{c^2}[U(P,g) - L(P,g)].$$

The integrability of $1/g$ follows from the integrability of g and the integrability criterion.

3. As in part **1**, we know $f+g, f-g \in \mathscr{R}[a,b]$. Since the absolute value function $\phi(x) = |x|$ is continuous, we know $|f-g| \in \mathscr{R}[a,b]$. Applying the linearity of the Riemann integral, we know that

$$\max\{f,g\} = \tfrac{1}{2}\left[f+g+|f-g|\right] \in \mathscr{R}[a,b],$$
$$\min\{f,g\} = \tfrac{1}{2}\left[f+g-|f-g|\right] \in \mathscr{R}[a,b].$$

Ex. 6.5 Let $h = g - f$. Then h is continuous on $[a,b]$ except possibly at x^*, so that $h \in \mathscr{R}[a,b]$ by item 2 of Theorem 6.4. Thus, the function $g = h + f$ is integrable, by the linearity of the Riemann integral. Furthermore, the desired equality follows from the fact

$$\int_a^b h\,\mathrm{d}x = 0.$$

To show the last equality, we note that $h \equiv 0$ except possibly at x^*. For any partition P of $[a,b]$, it is easy to have the following:

$$L(P,h) = 0, \quad \text{if } h(x^*) \geq 0;$$
$$U(P,h) = 0, \quad \text{if } h(x^*) \leq 0,$$

Thus, either $\underline{\int_a^b} h\,\mathrm{d}x = 0$ if $h(x^*) \geq 0$ or $\overline{\int_a^b} h\,\mathrm{d}x = 0$ if $h(x^*) \leq 0$. Since $h \in \mathscr{R}[a,b]$, we have

$$\int_a^b h\,\mathrm{d}x = \underline{\int_a^b} h\,\mathrm{d}x = \overline{\int_a^b} h\,\mathrm{d}x.$$

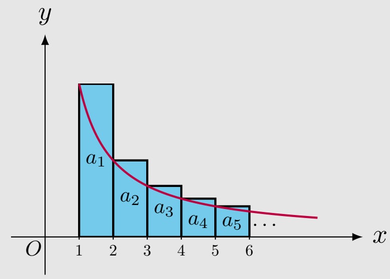
Hence, we conclude that $\int_a^b h\,\mathrm{d}x = 0$.

Ex. 6.6 ($\Rightarrow$) Suppose $\sum_{k=1}^{\infty} a_k$ converges. We prove that the limit $\lim_{n\to\infty}\int_1^n f(x)\,\mathrm{d}x$ exists and is finite. ◀

For each integer $k \geq 1$, put $g(x) = f(k)$ for $x \in [k, k+1)$. Then g is a function defined on $[1, \infty)$. Since f is monotonically decreasing, it is clear that $g \geq f$. Thus

$$\int_1^n f(x)\,\mathrm{d}x \leq \int_1^n g(x)\,\mathrm{d}x = \sum_{k=1}^{n-1} f(k) \leq \sum_{k=1}^{\infty} a_k,$$

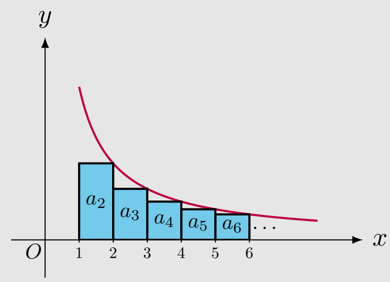
so that the increasing sequence $\left\{\int_1^n f(x)\,\mathrm{d}x\right\}$ is bounded above. Hence, the limit $\lim_{n\to\infty} \int_1^n f(x)\,\mathrm{d}x$ exists and is finite.



($\Leftarrow$) Suppose that the limit $\lim_{n\to\infty} \int_1^n f(x)\,\mathrm{d}x$ exists and is finite. We prove that $\sum_{k=1}^{\infty} a_k$ converges.

For each integer $k \geq 1$, put $h(x) = f(k+1)$ for $x \in [k, k+1)$. Then h is a function defined on $[1, \infty)$. Since f is monotonically decreasing, we know that $h \leq f$. Hence

$$\sum_{k=1}^{n} a_k = f(1) + \sum_{k=2}^{n} f(k)$$
$$\leq f(1) + \int_1^n h(x)\,\mathrm{d}x \leq f(1) + \int_1^n f(x)\,\mathrm{d}x.$$



Since

$$\int_1^n f(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \int_1^n f(x)\,\mathrm{d}x,$$

we know that the partial sums of the nonnegative series $\sum a_k$ are bounded above, so that $\sum_{k=1}^{\infty} a_k$ converges.

Ex. 6.7 Since f is continuous on $[a, b]$, by item 1 of Theorem 6.4, we know that $f \in \mathscr{R}[a, b]$. By Ex. 6.4, the product fg is a product of two integrable functions, so it is integrable.

Denote

$$m = \min_{x\in[a,b]} f(x), \qquad M = \max_{x\in[a,b]} f(x).$$

Without loss of generality, we assume that $g(x) \geq 0$ for all $x \in [a, b]$. By the monotonicity of the Riemann integral, we have

$$m \int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\mathrm{d}x \leq M \int_a^b g(x)\mathrm{d}x.$$

If $\int_a^b g(x)\,\mathrm{d}x = 0$, then the above inequalities yield $\int_a^b f(x)g(x)\,\mathrm{d}x = 0$, so that the desired equality holds for any fixed $\xi \in [a, b]$.

If $\int_a^b g(x)\,\mathrm{d}x > 0$, then we have $m \leq \lambda \leq M$, where

$$\lambda = \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}$$

By the Intermediate Value Theorem, there is $x \in [a, b]$, such that $f(\xi) = \lambda$. The desired equality follows.

In particular, when $g \equiv 1$, since $\int_a^b 1\,\mathrm{d}x = b - a$, we obtain

$$\frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x = f(\xi).$$

Ex. 6.8 Since $f \in \mathscr{R}[1, b]$, there are m and M such that

$$m \leq f(x) \leq M, \qquad x \in [a, b].$$

If $M = m$, then f is constant on $[a, b]$. The result holds trivially for $g = f$. ◀

Suppose $M \neq m$. By the integrability criterion, for any $\varepsilon > 0$, there exists a partition P of $[a,b]$:

$$P\colon a = x_0 < x_1 < \cdots < x_n = b,$$

such that

$$U(P,f) - L(P,f) < \varepsilon.$$

Define a function g_P on $[a,b]$ by

$$g_P(x) = \frac{x_i - x}{x_i - x_{i-1}} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i), \qquad x \in [x_{i-1}, x_i], \quad 1 \le i \le n.$$

It is a piecewise linear continuous function on $[a,b]$, so that $|f - g_P| \in \mathscr{R}[a,b]$. What remains to be proved is the claimed inequality.

In fact, by the construction of g_P, we know that

$$m_i \le g_P(x) \le M_i, \qquad x \in [x_{i-1}, x_i],$$

where

$$m_i = \inf_{x_{i-1} \le x \le x_i} f(x), \qquad M_i = \sup_{x_{i-1} \le x \le x_i} f(x).$$

It follows that

$$\begin{aligned}
\int_a^b |f - g_P| \,\mathrm{d}x &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f - g_P| \,\mathrm{d}x \\
&\le \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i - m_i) \,\mathrm{d}x \\
&= \sum_{i=1}^n (M_i - m_i) \Delta x_i = U(P,f) - L(P,f) < \varepsilon.
\end{aligned}$$

Ex. 6.9 We prove the remainder formula by induction.

For $n = 0$, the formula gives

$$\frac{1}{0!} \int_a^x (x-t)^0 f'(t) \,\mathrm{d}t = f(x) - f(a) = f(x) - T_0(x) = R_0(x).$$

So, the formula holds for $n = 0$.

Suppose the formula holds for $n = k$:

$$R_k(x) = \frac{1}{k!} \int_a^x (x-t)^k f^{(k+1)}(t) \,\mathrm{d}t, \qquad x \in I.$$

Then, by integration by parts, we get

$$\begin{aligned}
&\frac{1}{(k+1)!} \int_a^x (x-t)^{k+1} f^{(k+2)}(t) \,\mathrm{d}t \\
&= \left[\frac{1}{(k+1)!} (x-t)^{k+1} f^{(k+1)}(t) \right]_{t=a}^x + \frac{1}{k!} \int_a^x (x-t)^k f^{(k+1)}(t) \,\mathrm{d}t \\
&= R_k(x) - \frac{f^{(k+1)}(a)}{(k+1)} (x-a)^{k+1} \\
&= f(x) - T_k(x) - \frac{f^{(k+1)}(a)}{(k+1)} (x-a)^{k+1} \\
&= f(x) - T_{k+1}(x) = R_{k+1}(x).
\end{aligned}$$

So, the formula holds for $n = k+1$.

Hence, the remainder formula holds for all $n \ge 0$.

Ex. 6.10 By the Newton–Leibniz formula, for any $x \in [a,b]$,

$$\begin{aligned}
f(x) - f(a) &= \int_a^x f'(t) \,\mathrm{d}t, \\
f(x) - f(b) &= -\int_x^b f'(t) \,\mathrm{d}t.
\end{aligned}$$

The sum of these equalities gives ◀

$$2f(x) = 2f(x) - [f(a) + f(b)] = \int_a^x f'(t)\,\mathrm{d}t - \int_x^b f'(t)\,\mathrm{d}t.$$

Integrating both sides of the last equality gives

$$\int_a^b f(x)\,\mathrm{d}x = \frac{1}{2}\int_a^b \left[\int_a^x f'(t)\,\mathrm{d}t - \int_x^b f'(t)\,\mathrm{d}t\right]\mathrm{d}x.$$

Hence,

$$\begin{aligned}
\left|\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{1}{2}\int_a^b \left|\int_a^x f'(t)\,\mathrm{d}t - \int_x^b f'(t)\,\mathrm{d}t\right|\mathrm{d}x \\
&\le \frac{1}{2}\int_a^b \left[\left|\int_a^x f'(t)\,\mathrm{d}t\right| + \left|\int_x^b f'(t)\,\mathrm{d}t\right|\right]\mathrm{d}x \\
&\le \frac{1}{2}\int_a^b \left[\int_a^x \left|f'(t)\right|\,\mathrm{d}t + \int_x^b \left|f'(t)\right|\,\mathrm{d}t\right]\mathrm{d}x \\
&= \frac{1}{2}\int_a^b \left[\int_a^b \left|f'(t)\right|\,\mathrm{d}t\right]\mathrm{d}x \\
&= \frac{1}{2}\int_a^b \left|f'(t)\right|\,\mathrm{d}t \cdot \int_a^b 1\,\mathrm{d}x = \frac{b-a}{2}\int_a^b \left|f'(t)\right|\,\mathrm{d}t.
\end{aligned}$$

◀

Chapter 7 Quiz Answer

①C: By Theorem 7.3, a sequence of functions converges uniformly if and only if it satisfies the Cauchy Criterion for uniform convergence.

Option A is not true since uniform convergence implies pointwise convergence, but not the other way around, as shown by the examples given in the remarks for Definition 7.1.

Option B is not true because the sequence must be continuous. Consider the sequence $\{f_n\}$ and f, where

$$f_n(x) = f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ 0, & \text{if } x = 1. \end{cases}$$

Then the sequence $\{f_n\}$ converges uniformly on $[0,1]$ to the function f, which is clearly not continuous.

Option D is not true since the uniform convergence of $\{f_n\}$ does not imply the uniform convergence of $\{f_n'\}$. In fact, for the sequence $\{f_n\}$ on $[0, 2\pi]$,

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad n = 1, 2, \dots,$$

it is easy to see that

$$|f_n(x)| \leq \frac{1}{\sqrt{n}}, \quad x \in [0, 2\pi].$$

Thus, we know that $f_n \to 0$ uniformly on $[0, 2\pi]$. However, since $f_n'(x) = \sqrt{n} \sin nx$, we have

$$\lim_{n\to\infty} f_n'(x) \neq 0.$$

This implies that $\{f_n'\}$ does not converge uniformly on $[0, 2\pi]$, by Theorem 7.6.

②E: By Theorem 7.3, uniform convergence is equivalent to the Cauchy criterion. The latter requires the quantity $|f_n(x) - f_m(x)|$ to be small for all $x \in E$, as long as the indices m, n being sufficiently large.

The statements in Options A – D only require the mentioned quantity to be small for each prescribed point $x \in E$. This is Cauchy criterion for pointwise convergence, not uniform convergence.

③E: Since the series $\sum f_n(x)$ of functions passes the Weierstrass M-Test on E, the series converges uniformly on E, so that it also converges pointwise on E. Thus, options A and B are true.

It is clear that

$$f_n(x) = \sum_{j=1}^{n} f_j(x) - \sum_{j=1}^{n-1} f_j(x).$$

Since $\sum f_n(x)$ converges uniformly on E, the Cauchy Criterion implies that, for every $\varepsilon > 0$ and all $x \in E$, there exists an integer N such that if $n > N$, then

$$\left| \sum_{j=1}^{n} f_j(x) - \sum_{j=1}^{n-1} f_j(x) \right| < \varepsilon,$$

so that $|f_n(x)| < \varepsilon$. By the definition, the sequence $\{f_n\}$ converges uniformly to the zero function on E. Hence, options C and D are also true.

④D: For a sequence of continuous functions $\{f_n\}$ that pointwise converges on a closed and bounded interval, if $\{f_n(x)\}$ is decreasing for each point x, then, by Dini's theorem (Theorem 7.7), the sequence $\{f_n\}$ converges uniformly. Hence, option D is true.

⑤A: Consider the sequence of functions $\{f_n\}$,

$$f_n(x) = n, \quad n \geq 1.$$

It is obvious that $f_n \in \mathscr{C}(X)$. However, there exists no function M defined on X such that $|f_n(x)| < M(x)$ for all n.

By Theorem 7.3, we know that a sequence in $\mathscr{C}(X)$ is convergent (so is uniformly convergent) if and only if it satisfies the Cauchy Criterion, so option C and D are true.

The space $\mathscr{C}(X)$ is complete, so that the limit of convergent sequence in the space is continuous. Thus, option B is also true.

⑥B: According to the Weierstrass approximation theorem, every continuous function on a bounded closed interval can be approximated uniformly by polynomial functions. ◀

Chapter 7 Exercise Solutions

Ex. 7.1 It is known that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0, \qquad x \in (0,1),$$

that is, the sequence $\{f_n\}$ converges pointwise to the zero function on $(0,1)$.

To show that $\{f_n\}$ does not converge uniformly to the zero function on $(0,1)$, we take $x_n = 1 - \frac{1}{n+1} \in (0,1)$, $n = 1,2,3,\ldots$. Since

$$\begin{aligned}\lim_{n\to\infty} f_n(x_n) &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n \\ &= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e},\end{aligned}$$

we know that no matter how large n is, there exists a number $x \in (0,1)$ such that

$$|f_n(x)| > \frac{1}{2e}.$$

Consequently, the sequence $\{f_n\}$ does not converge uniformly to the zero function on $(0,1)$

Ex. 7.2 Suppose $|f_n(x)| \le M_n$ for each n and any $x \in E$, and $f_n \to f$ uniformly on E. Let N be such that

$$|f_n(x) - f(x)| < 1,$$

whenever $n \ge N$ and $x \in E$. Then for $n \ge N$ and $x \in E$,

$$|f_n(x)| \le |f_n(x) - f(x)| + |f(x) - f_N(x)| + |f_N(x)| < 2 + M_N.$$

Put

$$M = \max\{M_1, \cdots, M_{N-1}, 2 + M_N\}.$$

Then, for any $x \in E$,

$$|f_n(x)| \le M, \qquad n = 1,2,3,\ldots.$$

This means that $\{f_n\}$ is uniformly bounded.

Ex. 7.3 Denote $f_n(x) = c_n\, I(x - x_n)$. Since

$$|f_n(x)| \le c_n, \qquad n = 1,2,3,\ldots,$$

by the Weierstrass M-Test, the series $\sum c_n\, I(x - x_n)$ converges uniformly to $f(x) = \sum_{n=1}^{\infty} c_n\, I(x - x_n)$ on $[a,b]$.

Put $g_m(x) = \sum_{n=1}^{m} c_n\, I(x - x_n)$. To prove that f is continuous at every $x \in [a,b]\backslash\{x_n\}$, by Theorem 7.6, we only need to show that for every m, g_m is continuous at such x. In fact, if we put

$$\delta = \min\{|x - x_1|, \ldots, |x - x_m|\} > 0,$$

and if $|t - x| < \delta$, then

$$|t - x_n| \ge |x - x_n| - |x - t| > \delta - \delta = 0, \qquad n = 1,\ldots,m,$$

so that

$$I(t - x_n) = 1 = I(x - x_n), \qquad n = 1,\ldots,m.$$

Hence, for any $\varepsilon > 0$, if $|t - x| < \delta$, we have

$$\begin{aligned}|g_m(t) - g_m(x)| &= \left|\sum_{n=1}^{m} c_n\, I(t - x_n) - \sum_{n=1}^{m} c_n\, I(x - x_n)\right| \\ &= \left|\sum_{n=1}^{m} c_n - \sum_{n=1}^{m} c_n\right| = 0 < \varepsilon.\end{aligned}$$

It follows that for each m, g_m is continuous at every $x \in [a,b]\backslash\{x_n\}$.

Ex. 7.4 **1.** For convenience, we denote $f_n(x) = \frac{1}{1+n^2x}$ and the partial sums:

$$s_n(x) = \sum_{k=1}^{n} \frac{1}{1+k^2x}.$$

◀

We show that $s_n(x)$ converges absolutely on $(0,\infty)$. In fact, for any fixed $x>0$,

$$|f_n(x)| = \left|x^{-1}\right| \cdot \frac{1}{|x^{-1}+n^2|} \le \left|x^{-1}\right| \cdot \frac{1}{n^2}.$$

Since $\sum \dfrac{1}{n^2}$ converges, by Theorem 3.20 (the Comparison Test), $\sum |f_n(x)|$ converges, so that $\sum f_n(x)$ converges on $(0,\infty)$.

2. For $x\in[a,\infty)$ with $a>0$, since

$$0 < f_n(x) = \frac{1}{1+n^2x} \le \frac{1}{1+n^2a} < \frac{1}{n^2a},$$

by Theorem 7.4 (the Weierstrass M-Test), $\{s_n\}$ converges uniformly on $[a,\infty)$.

To show that $\{s_n\}$ does not converge uniformly on $(0,b)$, where $b>0$ is finite or $b=\infty$, by Ex. 7.2, we only need to show that $\{s_n\}$ is not uniformly bounded on $(0,b)$. In fact, it is easy to see that

$$\begin{aligned} s_{2n}(n^{-2}) &= \sum_{k=1}^{n-1} \frac{1}{1+k^2\cdot n^{-2}} + \frac{1}{1+n^2\cdot n^{-2}} + \cdots + \frac{1}{1+(2n)^2\cdot n^{-2}} \\ &> \frac{1}{1+n^2\cdot n^{-2}} + \cdots + \frac{1}{1+(2n)^2\cdot n^{-2}} \\ &\ge \frac{1}{1+4} + \cdots + \frac{1}{1+4} = \frac{n}{5}, \end{aligned}$$

so that there exists no number M such that $|s_n(x)|\le M$ for all $x\in(0,b)$.

3. For any $x_0>0$, by part **2**, the series is uniformly convergent on $[\frac{1}{2}x_0,\infty)$. Since the partial sums $\{s_n\}$ are clearly continuous on $[\frac{1}{2}x_0,\infty)$, by Theorem 7.6, the function f is continuous on $[\frac{1}{2}x_0,\infty)$. Because x_0 is arbitrary, we conclude that f is continuous on $(0,\infty)$.

4. For $x\in(0,\infty)$, by part **1**, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ converges. It is easy to see that

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{1+n^2x} \ge \sum_{k=1}^{n}\frac{1}{1+k^2x} \ge \frac{n}{1+n^2x}.$$

If we take $x_n = n^{-2}$, we have

$$f(x_n) \ge \frac{n}{2},$$

so that f is unbounded on $(0,\infty)$.

For $x\in[a,\infty)$ with $a>0$, since

$$0 < f(x) = \sum_{n=1}^{\infty}\frac{1}{1+n^2x} < \sum_{n=1}^{\infty}\frac{1}{n^2x} \le \sum_{n=1}^{\infty}\frac{1}{n^2a} = a^{-1}\cdot\sum_{n=1}^{\infty}\frac{1}{n^2},$$

we know that f is bounded on $[a,\infty)$ $(a>0)$.

Ex. 7.5 **1.** For any $x>0$, if n sufficiently large, the Archimedean property implies $1/n<x$. If $x\le 0$, it is clear that $x<1/(n+1)$. Hence, for any $x\in(-\infty,\infty)$, we know $f_n(x)=0$ for n sufficiently large. Thus, $f_n(x)\to 0$ at every point x. Apparently, $f=0$ is a continuous function everywhere.

Since $\{f_n\}$ converges pointwise to $f=0$, hence, if $\{f_n\}$ converges uniformly, by the definition, the uniform limit must be $f=0$. It follows that, for any $\varepsilon>0$, there exist N such that $n\ge N$ implies

$$|f_n(x)-0|<\varepsilon, \qquad x\in\mathbb{R}.$$

However, if we take $x_n = \dfrac{1}{n+\frac{1}{2}}$, then $\dfrac{1}{n+1} < x_n < \dfrac{1}{n}$ and $\sin^2\dfrac{\pi}{x_n}=1$, so that

$$|f_n(x_n)-0| = 1, \qquad n=1,2,3,\ldots.$$

Apparently, this implies that $\{f_n\}$ does not converge uniformly to $f=0$.

2. If $x\notin(0,1)$, then $f_n(x)=0$ for every n by definition. If $x\in(0,1)$, then there is a positive integer n such that $\dfrac{1}{n+1}<x\le\dfrac{1}{n}$. Hence, the series $\sum f_n$ actually contains only one nonzero term, so that the series converges absolutely. ◀

However, the series does not converge uniformly, since if again we take $x_n = \dfrac{1}{n+\frac{1}{2}}$, we have

$$|s_n(x_n) - s_{n-1}(x_n)| = |f(x_n)| = 1,$$

which implies that $\{s_n\}$ fails the Cauchy Criterion for uniform convergence.

Ex. 7.6 Let $\varepsilon > 0$ be given and $[a, b]$ be any bounded interval. Take a number M such that $x \in [a, b]$ implies $|x| \le M$. Since both $\sum(-1)^n \dfrac{1}{n}$ and $\sum(-1)^n \dfrac{x^2}{n^2}$ converge, we know that $\sum(-1)^n \dfrac{x^2+n}{n^2}$. Hence, there exist N_1, N_2, respectively, such that $m, n \ge N_1$ implies

$$\left| \sum_{k=m+1}^{n} (-1)^n \frac{1}{n} \right| < \tfrac{1}{2}\varepsilon,$$

and $m, n \ge N_2$ implies

$$\left| \sum_{k=m+1}^{n} \frac{M^2}{n^2} \right| < \tfrac{1}{2}\varepsilon.$$

Hence, $m, n \ge \max\{N_1, N_2\}$ implies that for any $x \in [a, b]$,

$$\begin{aligned} \left| \sum_{k=m+1}^{n} (-1)^n \frac{x^2+n}{n^2} \right| &\le \left| \sum_{k=m+1}^{n} (-1)^n \frac{x^2}{n^2} \right| + \left| \sum_{k=m+1}^{n} (-1)^n \frac{1}{n} \right| \\ &< \sum_{k=m+1}^{n} \frac{M^2}{n^2} + \tfrac{1}{2}\varepsilon < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

By the Cauchy Criterion for uniform convergence, we know that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly on $[a, b]$.

The series does not converge absolutely for any value of x, as confirmed by Theorem 3.20 (the Comparison Test) and the fact that $\sum \dfrac{1}{n}$ diverges, since

$$\left| (-1)^n \frac{x^2+n}{n^2} \right| = \frac{x^2+n}{n^2} \ge \frac{1}{n}.$$

Ex. 7.7 Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges to f uniformly on E, there exists N_1 such that $n \ge N_1$ implies

$$|f_n(t) - f(t)| < \tfrac{1}{2}\varepsilon,$$

whenever $t \in E$. In particular, if $n \ge N_1$, we have

$$|f_n(x_n) - f(x_n)| < \tfrac{1}{2}\varepsilon.$$

From the hypotheses, by Theorem 7.6, f is continuous. Hence, there exists $\delta > 0$ such that $|t - x| < \delta$ implies

$$|f(t) - f(x)| < \tfrac{1}{2}\varepsilon.$$

For $x_n \to x$, there exists N_2 such that $n \ge N_2$ implies

$$|x_n - x| < \delta.$$

Thus, if $n \ge N_2$, we have

$$|f(x_n) - f(x)| < \tfrac{1}{2}\varepsilon.$$

Put $N = \max\{N_1, N_2\}$. Then, if $n \ge N$, we have

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This implies that $\lim\limits_{n\to\infty} f_n(x_n) = f(x)$.

The converse statement is as follows: Let $\{f_n\}$, and f be continuous functions on a set E. If

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$, then $\{f_n\}$ converges to f uniformly on E.

We show that the above statement is false by an example. Take $E = (0, 1)$, and ◀

$$f_n(x) = \frac{1}{nx}, \qquad n = 1, 2, 3, \ldots.$$

It is clear that $\{f_n\}$ is a sequence of continuous functions on E, and $f_n \to f = 0$ pointwise on E.

For any sequence $x_n \to x \in (0,1)$, we have

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} \frac{1}{nx_n} = 0 \cdot \frac{1}{x} = 0.$$

Hence the conditions of the hypotheses are met.

However, $\{f_n\}$ does not converge to f on E uniformly. In fact,

$$|f_{2n}(1/n) - f_n(1/n)| = \left|\frac{1}{2} - \frac{1}{1}\right| = \frac{1}{2},$$

which implies that the Cauchy Criterion for uniform convergence fails.

Ex. 7.8 Let $\varepsilon > 0$ be given. Put $A_n(x) = \sum_{k=1}^{n} f_k(x)$.

By hypothesis (1), there is $M > 0$ such that $|A_n(x)| \le M$ for all $x \in E$, $n = 1, 2, 3, \ldots$. By hypothesis (2), there is N such that $g_N(x) < \frac{\varepsilon}{2M}$. Thus, if $p, q \ge N$, applying the summation by parts formula gives

$$\begin{aligned}
\left|\sum_{n=p}^{q} f_n(x) g_n(x)\right| &= \left|\sum_{n=p}^{q} [A_n(x) - A_{n-1}(x)] g_n(x)\right| \\
&= \left|\sum_{n=p}^{q-1} A_n(x)[g_n(x) - g_{n+1}(x)] + A_q(x) g_q(x) - A_{p-1}(x) g_p(x)\right| \\
&\le \sum_{n=p}^{q-1} |A_n(x)|\,|g_n(x) - g_{n+1}(x)| + |A_q(x)|\,|g_q(x)| + |A_{p-1}(x)|\,|g_p(x)| \\
&\le M\left(\sum_{n=p}^{q-1} |g_n(x) - g_{n+1}(x)| + |g_q(x)| + |g_p(x)|\right) \\
&= 2M g_p(x) \le 2M g_N(x) < \varepsilon.
\end{aligned}$$

Hence, the partial sums of the sequence $\sum f_n g_n$ satisfy the Cauchy Criterion for uniform convergence so that the series $\sum f_n g_n$ converges uniformly on E.

Ex. 7.9 Since f is continuous on $[0,1]$, by Theorem 4.19, there is $M > 0$ such that $|f(x)| \le M$ for all $x \in [0,1]$. By the Weierstrass approximation theorem, there is a sequence of polynomials $\{P_n\}$ such that $P_n \to f$ uniformly on $[0,1]$. Let $\varepsilon > 0$ be given. We know that there is N such that $n \ge N$ implies

$$|P_n(x) - f(x)| < \frac{\varepsilon}{M}, \qquad x \in [0,1].$$

Hence, if $n \ge N$, we have

$$\left|f(x)P_n(x) - [f(x)]^2\right| = |f(x)| \cdot |P_n(x) - f(x)| < M \cdot \frac{\varepsilon}{M} = \varepsilon, \qquad x \in [0,1].$$

that is, $fP_n \to f^2$ uniformly on $[0,1]$. By Theorem 7.8, we have

$$\int_0^1 f^2\,\mathrm{d}x = \lim_{n\to\infty} \int_0^1 fP_n\,\mathrm{d}x = 0.$$

Since $f^2 \ge 0$, we conclude $f^2 = 0$ on $[0,1]$ by Exercise 6.1, so that $f = 0$ on $[0,1]$.

Ex. 7.10 Let $\varepsilon > 0$ be given. By Exercise 6.8, there is a continuous function g on $[a,b]$ such that

$$\int_a^b |f - g|\,\mathrm{d}x < \tfrac{1}{2}\varepsilon.$$

By the Weierstrass approximation theorem, there is a sequence of polynomials $\{P_n\}$ such that $P_n \to g$ uniformly on $[a,b]$. Hence, there exists N such that $n \ge N$ implies

$$|P_n(x) - g(x)| < \frac{\varepsilon}{2(b-a)+1}, \qquad x \in [a,b].$$

Thus, if $n \ge N$, then ◀

$$\begin{aligned}\int_a^b |f - P_n| \, \mathrm{d}x &\le \int_a^b \left(|f - g| + |g - P_n|\right) \, \mathrm{d}x \\ &< \tfrac{1}{2}\varepsilon + \frac{\varepsilon}{2(b-a)+1} \cdot (b-a) < \varepsilon,\end{aligned}$$

so that $\lim_{n\to\infty} \int_a^b |f - P_n| \, \mathrm{d}x = 0$. ◀

Chapter 8 Quiz Answer

①D: The circle of convergence is given by the equation $|z - z_0| = R$, which is symmetric about its center z_0.

Option A is not true because $\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$ always exists, so the radius of convergence of the power series always exists.

Option B is not true since the radius of convergence can be zero.

Option C is not true since the power series $\sum_{k=0}^{\infty} c_n(z - z_0)^n$ always converges at $z = z_0$, even if the radius of convergence is zero.

②E: None of options A – D is true.

Options A and D are not true, for example, the series $\sum \frac{z^n}{n^2}$ converges at every point on the boundary of convergence $|z| = 1$.

Options B and C are not true, for example, the series $\sum z^n$ diverges at every point on the boundary of convergence $|z| = 1$.

③C: By Theorem 8.5, the function f is differentiable in $|x - x_0| < R$.

④B: Since

$$\begin{aligned}\int_{-1}^{1} x \cdot \sin \pi x \, dx &= \left(-\frac{1}{\pi} x \cos \pi x\right)\Big|_{x=-1}^{1} - \int_{-1}^{1} \left(\frac{1}{\pi} \cos \pi x\right) dx \\ &= \left(\frac{1}{\pi} + \frac{1}{\pi}\right) - \left(\frac{1}{\pi^2} \sin \pi x\right)\Big|_{x=-1}^{1} = \frac{2}{\pi} \neq 0,\end{aligned}$$

we know that the sequence in option B is not an orthogonal system.

⑤E: Option A is not true since the series $\sum_{n=1}^{\infty} |c_n|^2$ is convergent if $\{\phi_n(x)\}$ is orthonormal, by Theorem 8.10. If $\{\phi_n(x)\}$ is orthonormal, then we may consider the orthogonal system $\{|c_n|^{-1}\phi_n(x)\}$. It is easy to see that the corresponding Fourier coefficients are $\{|c_n|^{-1}c_n\}$ and

$$\sum \left| |c_n|^{-1} c_n \right|^2$$

diverges.

Options B – D are not true since, once again, the trigonometric series are under the orthonormal system $\{(2\pi)^{-1/2} e^{inx}\}$.

⑥C: The Fourier series may converge uniformly or pointwise, depending on the function. The convergence of the Fourier series depends on the properties of the function being approximated, see Theorems 8.12 and 8.14.