第五章　一元积分学习题解答
(仅供教师参考)

5.1 不定积分

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1. (1)

$$\int(\frac{3}{x}+\frac{x}{3})^3dx=\int(\frac{27}{x^3}+\frac{x^3}{27}+\frac{9}{x}+x)dx=-\frac{27}{2x^2}+\frac{1}{108}x^4+9\ln|x|+\frac{1}{2}x^2+C.$$

(2)

$$\int(4\cos x+2-3x^2+\frac{1}{x}-\frac{7}{1+x^2})dx=4\sin x+2x-x^3+\ln|x|-7\arctan x+C.$$

(3)

$$\int 3^xe^xdx=\int(3e)^xdx=\frac{(3e)^x}{\ln(3e)}+C=\frac{3^xe^x}{1+\ln 3}+C.$$

(4)

$$\int\frac{\cos 2x}{\cos x-\sin x}dx=\int(\cos x+\sin x)dx=\sin x-\cos x+C.$$

(5)

$$\int\frac{1}{(x+3)(x+7)}dx=\frac{1}{4}\int(\frac{1}{x+3}-\frac{1}{x+7})dx=\frac{1}{4}\ln\left|\frac{x+3}{x+7}\right|+C.$$

(6)

$$\int\frac{x^4}{1+x^2}dx=\int\frac{x^4-1+1}{1+x^2}dx=\int(x^2-1+\frac{1}{1+x^2})dx=\frac{1}{3}x^3-x+\arctan x+C$$

(7)

$$\int\sqrt{x\sqrt{x\sqrt{x}}}dx=\int x^{\frac{7}{8}}dx=\frac{8}{15}x^{\frac{15}{8}}+C.$$

(8)

$$\int(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}})dx=\int\frac{2}{\sqrt{1-x^2}}dx=2\arcsin x+C.$$

(9)同(4)

(10)

$$\int e^x(2^x - \frac{e^{-x}}{\sqrt{1-x^2}})dx = \int ((2e)^x - \frac{1}{\sqrt{1-x^2}})dx = \frac{2^x e^x}{1+\ln 2} - \arcsin x + C.$$

(11)

$$\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \tan x - x + C.$$

(12)

$$\int \cos x \cos 2x dx = \frac{1}{2}\int (\cos x + \cos 3x)dx = \frac{1}{2}\sin x + \frac{1}{6}\sin 3x + C.$$

2. 解：因为$F'(x) = (\sin\frac{x}{2} - \cos\frac{x}{2})^2 = 1 - \sin x$, 所以$F(x) = x + \cos x + C$, 又因为$F(\frac{\pi}{2}) = \frac{\pi}{2} + C = 0$, 所以$C = -\frac{\pi}{2}$, 因此

$$F(x) = x + \cos x - \frac{\pi}{2}.$$

3. 解：由题意得, $f'(x) = 2x - 2$, 所以$f(x) = x^2 - 2x + C$, 又因为此曲线通过点$(1,0)$, 所以$f(1) = 0$, 得$C = 1$, 因此

$$f(x) = x^2 - 2x + 1.$$

4. 解：由题意得, $f'(x) = kx^3$(其中k为常数), 所以$f(x) = \frac{k}{4}x^4 + C$, 又因为此曲线通过点$(1,6),(2,-9)$, 所以

$$\begin{aligned} f(1) &= \frac{k}{4} + C = 6, \\ f(2) &= 4k + C = -9 \end{aligned}$$

解得, $k = -4, C = 7$, 因此

$$f(x) = -x^4 + 7.$$

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1. (1)

$$\int \frac{\ln x}{x}dx = \int \ln x d(\ln x) = \frac{\ln^2 x}{2} + C.$$

(2)

$$\int (1+x)^{2010}dx = \int (1+x)^{2010}d(1+x) = \frac{(1+x)^{2011}}{2011} + C.$$

(3)

$$\begin{aligned}
&\int (\frac{1}{\sqrt{3-x^2}}+\frac{1}{\sqrt{1-3x^2}})dx\\
=&\int \frac{1}{\sqrt{1-(x/\sqrt{3})^2}}d(\frac{x}{\sqrt{3}})+\frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-(\sqrt{3}x)^2}}d(\sqrt{3}x)\\
=&\arcsin\frac{x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\arcsin\sqrt{3}x+C.
\end{aligned}$$

(4)

$$\int \frac{1}{\cos^2 5x}dx=\frac{1}{5}\int \sec^2 5xd(5x)=\frac{1}{5}\tan 5x+C.$$

(5)

$$\begin{aligned}
\int \frac{1}{1+\cos x}dx &= \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx\\
&= \int \frac{1-\cos x}{\sin^2 x}dx\\
&= \int \frac{1}{\sin^2 x}dx-\int \frac{d\sin x}{\sin^2 x}\\
&= -\cot x+\frac{1}{\sin x}+C.
\end{aligned}$$

(6)

$$\int \frac{1}{x\ln^3 x}dx=\int \frac{1}{\ln^3 x}d(\ln x)=-\frac{1}{2\ln^2 x}+C.$$

(7)

$$\int \frac{\tan x}{\cos^2 x}dx=\int \tan xd\tan x=\frac{\tan^2 x}{2}+C$$

(8)

$$\int \frac{\sin 2x}{(1+\cos 2x)^2}dx=-\frac{1}{2}\int \frac{1}{(1+\cos 2x)^2}d(1+\cos 2x)=\frac{1}{2(1+\cos 2x)}+C.$$

(9)

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}dx=\int \arcsin xd\arcsin x=\frac{1}{2}(\arcsin x)^2+C.$$

(10)

$$\int \frac{1}{e^x+e^{-x}}dx=\int \frac{e^x}{1+e^{2x}}dx=\int \frac{1}{1+(e^x)^2}d(e^x)=\arctan e^x+C.$$

(11)令$x = a\tan t (\in (-\frac{\pi}{2}, \frac{\pi}{2})$, 则$dx = a\sec^2 t dt$, 且有$t = \arctan\frac{x}{a}$ 及$\cos t > 0$, 故$\sqrt{a^2+x^2} = a\sec t$, 所以

$$\begin{aligned}\text{原式} &= \int \frac{a\sec^2 t}{a\sec t}dt = \int \frac{1}{\cos t}dt = \ln|\sec t + \tan t| + C \\ &= \ln\frac{x+\sqrt{a^2+x^2}}{a} + C' = \ln(x+\sqrt{a^2+x^2}) + C.\end{aligned}$$

(12)

$$\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-(x^2)^2}}d(x^2) = \frac{1}{2}\arcsin x^2 + C.$$

(13)

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin\sqrt{x}d\sqrt{x} = -2\cos\sqrt{x} + C.$$

(14)

$$\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}dx = \int \frac{1}{\sqrt[3]{\sin x-\cos x}}d(\sin x-\cos x) = \frac{3}{2}(\sin x-\cos x)^{\frac{2}{3}} + C.$$

(15)令$x = t^6$, 则$t = \sqrt[6]{x}$, $dx = 6t^5dt$, 于是

$$\begin{aligned}\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}dx &= \int \frac{t^3}{1-t^2}6t^5dt = 6\int \frac{t^8}{1-t^2}dt \\ &= 6\int (-t^6-t^4-t^2-1+\frac{1}{2(1-t)}+\frac{1}{2(1+t)})dt \\ &= -\frac{6t^7}{7}-\frac{6t^5}{5}-2t^3-6t-3\ln|1-t|+3\ln|1+t|+C \\ &= -\frac{6x^{7/6}}{7}-\frac{6x^{5/6}}{5}-2\sqrt{x}-6\sqrt[6]{x}-3\ln|1-\sqrt[6]{x}|+3\ln|1+\sqrt[6]{x}|+C.\end{aligned}$$

(16)

$$\int \frac{1}{x\sqrt{1-\ln^2 x}}dx = \int \frac{1}{\sqrt{1-\ln^2 x}}d\ln x = \arcsin\ln x + C.$$

(17)

$$\int \frac{\sqrt{1+\ln x}}{x}dx = \int \sqrt{1+\ln x}d(1+\ln x) = \frac{2}{3}(1+\ln x)^{\frac{3}{2}} + C.$$

(18)

$$\int \frac{\sin x\cos x}{1+\sin^4 x}dx = \frac{1}{2}\int \frac{1}{1+(\sin^2 x)^2}d(\sin^2 x) = \frac{1}{2}\arctan(\sin^2 x) + C.$$

(19)

$$\int \frac{1}{(\arcsin x)^2\sqrt{1-x^2}}dx = \int \frac{1}{(\arcsin x)^2}d(\arcsin x) = -\frac{1}{\arcsin x} + C.$$

(20)令$\sqrt{1+x} = t$, 则$x = t^2-1$, $dx = 2tdt$,

$$\begin{aligned}\int \frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}dx &= \int \frac{2t(t-1)}{t+1}dt \\ &= 2\int (t-2+\frac{2}{t+1})dt \\ &= t^2-4t+4\ln|t+1|+C \\ &= 1+x-4\sqrt{1+x}+4\ln(\sqrt{1+x}+1)+C.\end{aligned}$$

(21)

$$\begin{aligned}&\int \frac{x}{\sqrt{1+x^2}}\tan\sqrt{1+x^2}dx = \int \tan\sqrt{1+x^2}d(\sqrt{1+x^2}) \\ &= -\int \frac{1}{\cos\sqrt{1+x^2}}d(\cos\sqrt{1+x^2}) = -\ln|\cos\sqrt{1+x^2}|+C.\end{aligned}$$

(22) 不妨设$\alpha < \beta$, 被积函数的存在域为$\alpha < x < \beta$, 因此可设$x-\alpha = (\beta-\alpha)\sin^2 t$, 并限制$0 < t < \frac{\pi}{2}$, 从而$\sqrt{(x-\alpha)(\beta-x)} = (\beta-\alpha)\sin t\cos t$, $dx = 2(\beta-\alpha)\sin t\cos t$, 代入得

$$\begin{aligned}\int \sqrt{(x-\alpha)(\beta-x)}dx &= 2(\beta-\alpha)^2\int \sin^2 t\cos^2 tdt \\ &= \frac{(\beta-\alpha)^2}{2}\int \sin^2 2tdt \\ &= \frac{(\beta-\alpha)^2}{4}\int (1-\cos 4t)dt \\ &= \frac{(\beta-\alpha)^2}{4}(t-\frac{1}{4}\sin t)+C \\ &= \frac{(\beta-\alpha)^2}{4}\arcsin\sqrt{\frac{x-\alpha}{\beta-\alpha}} + \frac{2x-(\alpha+\beta)}{4}\sqrt{(x-\alpha)(\beta-x)}+C.\end{aligned}$$

2. (1)

$$\begin{aligned}\int x^n\ln xdx &= \frac{1}{n+1}\int \ln xd(x^{n+1}) = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^{n+1}\cdot\frac{1}{x}dx \\ &= \frac{x^{n+1}}{n+1}(\ln x - \frac{1}{n+1})+C.\end{aligned}$$

(2)

$$\begin{aligned}\int e^x\cos x dx &= \int \cos x de^x = e^x\cos x + \int e^x \sin x dx\\ &= e^x\cos x + \int \sin x de^x = e^x\cos x + e^x\sin x - \int e^x\cos x dx\\ &= e^x(\cos x+\sin x) - \int e^x\cos x dx,\end{aligned}$$

故

$$\int e^x\cos x dx = \frac{1}{2}e^x(\cos x+\sin x)+C.$$

(3)

$$\int(\ln x)^2dx = x(\ln x)^2 - 2\int \ln x dx = x(\ln x)^2 - 2(x\ln x - x)+C.$$

(4)

$$\begin{aligned}\int x\arctan x dx &= \frac{1}{2}\int \arctan x d(x^2) = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx\\ &= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int(1-\frac{1}{1+x^2})dx = \frac{1+x^2}{2}\arctan x - \frac{x}{2}+C.\end{aligned}$$

(5)

$$\begin{aligned}\int\sqrt{x}\ln^2 x dx &= \frac{2}{3}\int \ln^2 x d(x^{\frac{3}{2}}) = \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{4}{3}\int x^{\frac{3}{2}}\ln x\frac{1}{x}dx\\ &= \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{8}{9}\int \ln x d(x^{\frac{3}{2}}) = \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{8}{9}\int x^{\frac{3}{2}}\frac{1}{x}dx\\ &= \frac{2}{3}x^{\frac{3}{2}}(\ln^2 x - \frac{4}{3}\ln x + \frac{8}{9})+C.\end{aligned}$$

(6)

$$\begin{aligned}\int\ln(x+\sqrt{1+x^2})dx &= x\ln(x+\sqrt{1+x^2}) - \int\frac{x}{\sqrt{1+x^2}}dx\\ &= x\ln(x+\sqrt{1+x^2}) - \frac{1}{2}\int\frac{1}{\sqrt{1+x^2}}d(1+x^2)\\ &= x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2}+C.\end{aligned}$$

(7)

$$\begin{aligned}\int\frac{\arcsin\sqrt{x}}{\sqrt{x}}dx &= 2\int\arcsin\sqrt{x}d(\sqrt{x}) = 2\int\arcsin t dt\\ &= 2t\arcsin t - 2\int\frac{t}{\sqrt{1-t^2}}dt = 2t\arcsin t + 2\sqrt{1-t^2}+C\\ &= 2\sqrt{x}\arcsin\sqrt{x} + 2\sqrt{1-x}+C.\end{aligned}$$

(8)令$x=4\sin t,\ dx=4\cos tdt$,

$$\begin{aligned}\int\frac{x^2}{\sqrt{16-x^2}}dx &= \int 16\sin^2 tdt = 8\int(1-\cos 2t)dt\\ &= 8t-4\sin 2t+C = 8\arcsin(x/4)-x\sqrt{16-x^2}/2+C.\end{aligned}$$

(9)

$$\begin{aligned}\int(\ln(\ln x)+\frac{1}{\ln x})dx &= \ln(\ln x)x-\int\frac{1}{\ln x}\frac{1}{x}xdx+\int\frac{1}{\ln x}dx\\ &= \ln(\ln x)x+C.\end{aligned}$$

(10)

$$\begin{aligned}\int xe^x\sin xdx &= \int x\sin xd(e^x)\\ &= xe^x\sin x-\int e^x(\sin x+\cos x)dx\\ &= xe^x\sin x-\int(\sin x+\cos x)d(e^x)\\ &= e^x(x\sin x-\sin x-x\cos x)+\int e^x(2\cos x-x\sin x)dx\\ &= e^x(x\sin x-\sin x-x\cos x)+2\int e^x\cos xdx-\int xe^x\sin xdx,\end{aligned}$$

于是,

$$\begin{aligned}\int xe^x\sin xdx &= \frac{e^x}{2}(x\sin x-\sin x-x\cos x)+\int e^x\cos xdx\\ &= \frac{e^x}{2}(x\sin x-\sin x-x\cos x)+\frac{e^x}{2}(\sin x+\cos x)+C\\ &= \frac{e^x}{2}[x(\sin x-\cos x)+\cos x]+C.\end{aligned}$$

注：由类似方法可得

$$\int x^2e^x\sin xdx = -\frac{1}{2}(x-1)^2e^x\cos x+\frac{1}{2}(x^2-1)e^x\sin x+C.$$

(11)

$$
\begin{aligned}
\text{原式} &= \int x\sqrt{1+x^2}\ln\sqrt{x^2-1}dx = \frac{1}{2}\int\sqrt{1+x^2}\ln\sqrt{x^2-1}d(x^2) \\
&= \frac{1}{3}\int\ln(x^2-1)d(1+x^2)^{3/2} \\
&= \frac{1}{3}(1+x^2)^{3/2}\ln(x^2-1) - \frac{1}{3}\int\frac{(1+x^2)^{3/2}}{x^2-1}d(x^2) = \cdots \\
&= \frac{1}{3}(1+x^2)^{3/2}\ln(x^2-1) - \frac{2}{9}(1+x^2)^{3/2} - \frac{4}{3}\sqrt{1+x^2} \\
&\quad + \frac{2\sqrt{2}}{3}\ln\Big|\frac{\sqrt{1+x^2}-\sqrt{2}}{\sqrt{1+x^2}+\sqrt{2}}\Big| + C.
\end{aligned}
$$

(12)

$$
\begin{aligned}
\int(\frac{\ln x}{x})^3 dx &= -\frac{1}{2}\int\ln^3 x d(\frac{1}{x^2}) = -\frac{1}{2x^2}\ln^3 x + \frac{3}{2}\int\frac{\ln^2 x}{x^3}dx \\
&= -\frac{1}{2x^2}\ln^3 x - \frac{3}{4}\int\ln^2 x d(\frac{1}{x^2}) \\
&= -\frac{1}{2x^2}\ln^3 x - \frac{3}{4x^2}\ln^2 x + \frac{3}{2}\int\frac{\ln x}{x^3}dx \\
&= -\frac{1}{2x^2}\ln^3 x - \frac{3}{4x^2}\ln^2 x - \frac{3}{4}\int\ln x d(\frac{1}{x^2}) \\
&= -\frac{1}{2x^2}\ln^3 x - \frac{3}{4x^2}\ln^2 x - \frac{3}{4x^2}\ln x + \frac{3}{4}\int\frac{dx}{x^3} \\
&= -\frac{1}{2x^2}(\ln^3 x + \frac{3}{2}\ln^2 x + +\frac{3}{2}\ln x + \frac{3}{4}) + C.
\end{aligned}
$$

(13)

$$
\begin{aligned}
\int\frac{\ln(1+e^{-x})}{1+e^x}dx &= \int\ln(1+e^{-x})(\ln(1+e^{-x}))'dx \\
&= \int\ln(1+e^{-x})d(\ln(1+e^{-x})) = \frac{1}{2}\ln^2(1+e^{-x}) + C.
\end{aligned}
$$

(14)若$\alpha=\beta=0$, 则积分为$x+C$; 以下设$\alpha^2+\beta^2\neq 0$, 则有

$$
\begin{aligned}
\int e^{\alpha x}\sin\beta x dx &= \frac{1}{\alpha}\int\sin\beta x d(e^{\alpha x}) \\
&= \frac{1}{\alpha}e^{\alpha x}\sin\beta x - \frac{\beta}{\alpha}\int e^{\alpha x}\cos\beta x dx \\
&= \frac{1}{\alpha}e^{\alpha x}\sin\beta x - \frac{\beta}{\alpha^2}\int\cos\beta x d(e^{\alpha x}) \\
&= \frac{1}{\alpha}e^{\alpha x}\sin\beta x - \frac{\beta}{\alpha^2}e^{\alpha x}\cos\beta x - -\frac{\beta^2}{\alpha^2}\int e^{\alpha x}\sin\beta x dx,
\end{aligned}
$$

故
$$\int e^{\alpha x}\sin\beta x dx = \frac{e^{\alpha x}(\alpha\sin\beta x - \beta\cos\beta x)}{\alpha^2+\beta^2} + C.$$
类似地，有
$$\int e^{\alpha x}\cos\beta x dx = \frac{e^{\alpha x}(\beta\sin\beta x + \alpha\cos\beta x)}{\alpha^2+\beta^2} + C.$$

(15)
$$\int \frac{1-\ln x}{(x-\ln x)^2}dx \quad = \quad \int (\frac{x}{x-\ln x})' dx = \frac{x}{x-\ln x} + C.$$

(16)
$$\begin{aligned}\int \frac{\ln x}{\sqrt{(1+x^2)^3}}dx &= \int \ln x d(\frac{x}{\sqrt{1+x^2}}) = \frac{x\ln x}{\sqrt{1+x^2}} - \int \frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{x}dx \\ &= \frac{x\ln x}{\sqrt{1+x^2}} - \ln|\sqrt{1+x^2}+x| + C.\end{aligned}$$

3. (1)
$$\begin{aligned}\int \frac{dx}{\sqrt{e^x-1}} &= \int \frac{dx}{e^{\frac{x}{2}}\sqrt{1-(e^{-\frac{x}{2}})^2}} = -2\int \frac{d(e^{-\frac{x}{2}})}{\sqrt{1-(e^{-\frac{x}{2}})^2}} \\ &= -2\arcsin(e^{-\frac{x}{2}}) + C.\end{aligned}$$

或令$\sqrt{e^x-1}=t$, 可得$\displaystyle\int\frac{dx}{\sqrt{e^x-1}} = 2\arctan\sqrt{e^x-1} + C.$

(2)
$$\begin{aligned}\int \frac{dx}{\sqrt{e^x+1}} &= \int \frac{dx}{e^{\frac{x}{2}}\sqrt{1+(e^{-\frac{x}{2}})^2}} = -2\int \frac{d(e^{-\frac{x}{2}})}{\sqrt{1+(e^{-\frac{x}{2}})^2}} \\ &= -2\ln(e^{-x/2}+\sqrt{e^{-x}+1}) + C.\end{aligned}$$

或令$\sqrt{e^x+1}=t$, 可得$\displaystyle\int\frac{dx}{\sqrt{e^x+1}} = x - 2\ln(1+\sqrt{e^x+1}) + C.$

(3)
$$\int \frac{1}{x(x^4+1)}dx \quad = \quad \int (\frac{1}{x} - \frac{x^3}{x^4+1})dx = \ln|x| - \frac{1}{4}\ln(x^4+1) + C.$$

(4)
$$\begin{aligned}\int \frac{x^2\arctan x}{1+x^2}dx &= \int \arctan x dx - \int \frac{1}{1+x^2}\arctan x dx \\ &= x\arctan x - \int \frac{x}{1+x^2}dx - \frac{1}{2}\arctan^2 x \\ &= x\arctan x - \frac{1}{2}\arctan^2 x - \frac{1}{2}\ln(x^2+1) + C.\end{aligned}$$

(5)

$$\begin{aligned}\int \frac{1}{1-x^2}\ln\frac{1+x}{1-x}dx &= \frac{1}{2}\int \ln\frac{1+x}{1-x}(\ln\frac{1+x}{1-x})'dx \\ &= \frac{1}{2}\int \ln\frac{1+x}{1-x}d(\ln\frac{1+x}{1-x}) = \frac{1}{4}\ln^2\frac{1+x}{1-x}+C.\end{aligned}$$

(6)

$$\begin{aligned}\int \frac{1}{1+e^x}dx &= \int \frac{1+e^x-e^x}{1+e^x}dx \\ &= \int (1-\frac{e^x}{1+e^x})dx = x-\ln(1+e^x)+C.\end{aligned}$$

(7)

$$\begin{aligned}\int x\ln\frac{1+x}{1-x}dx &= \frac{1}{2}\int \ln\frac{1+x}{1-x}d(x^2) = \frac{x^2}{2}\ln\frac{1+x}{1-x}-\int\frac{x^2}{1-x^2}dx \\ &= \frac{x^2}{2}\ln\frac{1+x}{1-x}+\int(1-\frac{1}{1-x^2})dx \\ &= x-\frac{1-x^2}{2}\ln\frac{1+x}{1-x}+C.\end{aligned}$$

(8)

$$\begin{aligned}\int \frac{x\ln(1+\sqrt{1+x^2})}{\sqrt{1+x^2}} &= \int \ln(1+\sqrt{1+x^2})d(1+\sqrt{1+x^2}) \\ &= (1+\sqrt{1+x^2})\ln(1+\sqrt{1+x^2})-\int\sqrt{1+x^2}\cdot\frac{1}{\sqrt{1+x^2}}dx \\ &= (1+\sqrt{1+x^2})\ln(1+\sqrt{1+x^2})-x+C.\end{aligned}$$

(9) 令$\sqrt{\sin x}=t$,

$$\begin{aligned}\int \cos^5 x\sqrt{\sin x}dx &= \int(1-\sin^2 x)^2\sqrt{\sin x}d(\sin x) \\ &= \int 2(1-t^4)^2t^2dt = \int(2t^2-4t^6+2t^{10})dt \\ &= \frac{2}{3}t^3-\frac{4}{7}t^7+\frac{2}{11}t^{11}+C \\ &= \frac{2}{3}\sin x^{\frac{3}{2}}-\frac{4}{7}\sin x^{\frac{7}{2}}+\frac{2}{11}\sin x^{\frac{11}{2}}+C.\end{aligned}$$

(10) 令$\sqrt[3]{1-x}=t$, 则$x=1-t^3$, $dx=-3t^2dt$, 故

$$\begin{aligned}\int x^2\sqrt[3]{1-x}dx &= -3\int(1-t^3)^2t^3dt = \int(-3t^3+6t^6-3t^9)dt \\ &= -\frac{3}{4}t^4+\frac{6}{7}t^7-\frac{3}{10}t^{10}+C \\ &= -\frac{3}{4}(1-x)^{\frac{4}{3}}+\frac{6}{7}(1-x)^{\frac{7}{3}}-\frac{3}{10}(1-x)^{\frac{10}{3}}+C.\end{aligned}$$

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1. (1)

$$
\begin{aligned}
& \int \frac{2x^2+2x+13}{(x-2)(x^2+1)^2}dx = \int (\frac{1}{x-2} - \frac{x+2}{x^2+1} - \frac{3x+4}{(x^2+1)^2})dx \\
= \;& \ln|x-2| - \frac{1}{2}\ln(x^2+1) - 2\arctan x + \frac{3}{2(x^2+1)} - 4\int \frac{1}{(x^2+1)^2}dx \\
= \;& \ln|x-2| - \frac{1}{2}\ln(x^2+1) - 2\arctan x + \frac{3}{2(x^2+1)} - 4(\frac{1}{2}\arctan x + \frac{x}{2(x^2+1)}) + C \\
= \;& \ln|x-2| - \frac{1}{2}\ln(x^2+1) - 4\arctan x + \frac{3}{2(x^2+1)} + \frac{2x}{x^2+1} + C.
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int \frac{3x-7}{x^3+x^2+4x+4} &= \int \frac{d(x^2+4)}{x^2+4} + \int \frac{dx}{x^2+4} - 2\int \frac{d(x+1)}{x+1} \\
&= \ln|\frac{x^2+4}{(x+1)^2}| + \frac{1}{2}\int \frac{1}{1+(\frac{x}{2})^2}d\frac{x}{2} \\
&= \ln|\frac{x^2+4}{(x+1)^2}| + \frac{1}{2}\arctan\frac{x}{2} + C.
\end{aligned}
$$

(3)

$$
\begin{aligned}
& \int \frac{1}{(x+1)(x^2+x+1)^2}dx \\
= \;& \frac{1}{3}\int (\frac{1}{x+1} - \frac{x}{x^2+x+1} - \frac{3(x-1)}{(x^2+x+1)^2})dx \\
= \;& \frac{1}{3}(\ln|x+1| - \frac{1}{2}(\int \frac{d(x^2+x+1)}{x^2+x+1} - \int \frac{1}{x^2+x+1}dx) \\
& -\frac{3}{2}(\int \frac{d(x^2+x+1)}{(x^2+x+1)^2} - \int \frac{3}{(x^2+x+1)^2})) \\
= \;& \frac{1}{3}\ln|x+1| - \frac{1}{6}(\ln|x^2+x+1| - \int \frac{1}{x^2+x+1}dx) \\
& -\frac{1}{2}(\int \frac{d(x^2+x+1)}{x^2+x+1} - \int \frac{3}{(x^2+x+1)^2}dx) \\
= \;& \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} \\
& -\ln(x^2+x+1) + \frac{3}{2}\int \frac{dx}{(x^2+x+1)^2} \\
= \;& \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} \\
& -\ln(x^2+x+1) + \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + \frac{x+\frac{1}{2}}{x^2+x+1} + C.
\end{aligned}
$$

(4)

$$\begin{aligned}
\int \frac{1}{x^4+1}dx &= \frac{1}{2}\int \frac{((x^2+1)-(x^2-1))}{x^4+1}dx \\
&= \frac{1}{2}[\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx] \\
&= \frac{1}{2}[\int \frac{d(x-\frac{1}{x})}{(x+\frac{1}{x})^2-2}]dx \\
&= \frac{1}{2\sqrt{2}}(\arctan\frac{x^2-1}{\sqrt{2}x} - \frac{1}{2}\ln\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}) + C.
\end{aligned}$$

(5)

$$\begin{aligned}
\int \frac{1}{x(1+x^2)^2}dx &= \int (\frac{1}{x} - \frac{x}{1+x^2} - \frac{x}{(1+x^2)^2})dx \\
&= \ln|x| - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}\int \frac{1}{(1+x^2)^2}d(1+x^2) \\
&= \ln|x| - \frac{1}{2}\ln(1+x^2) + \frac{1}{2(1+x^2)} + C.
\end{aligned}$$

(6)

$$\begin{aligned}
& \int \frac{2x^4-x^3+4x^2+9x-10}{x^5+x^4-5x^3-2x^2+4x-8}dx \\
= {} & \int \frac{dx}{x-2} + 2\int \frac{dx}{x+2} - \int \frac{dx}{(x+2)^2} + \int \frac{-x+1}{x^2-x+1}dx \\
= {} & \ln|(x-2)(x+2)^2| + \frac{1}{x+2} - \frac{1}{2}\int \frac{d(x^2-x+1)}{x^2-x+1} + \frac{1}{2}\int \frac{dx}{x^2-x+1} \\
= {} & \ln\frac{|(x-2)(x+2)^2|}{\sqrt{|x^2-x+1|}} + \frac{1}{x+2} + \frac{\sqrt{3}}{3}\arctan[\frac{2}{\sqrt{3}}(x-\frac{1}{2})] + C.
\end{aligned}$$

2. (1)

$$\begin{aligned}
\int \sin^4 x dx &= \int [\frac{1}{2}(1-\cos 2x)]^2 dx = \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)dx \\
&= \frac{1}{4}[\int dx - 2\int \cos 2x dx + \int \frac{1}{2}(1+\cos 4x)dx] \\
&= \frac{1}{4}(x - \sin 2x + \frac{x}{2} + \frac{1}{8}\sin 4x) + C \\
&= \frac{1}{4}(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x) + C.
\end{aligned}$$

(2) 令$\cos x = t$, 则$dt = -\sin x dx$, 故

$$\begin{aligned}\int \frac{\sin^3 x}{\sqrt[3]{\cos^4 x}}dx &= -\int \frac{(1-t^2)dt}{t^{\frac{4}{3}}} = -\int (t^{-\frac{4}{3}} - t^{\frac{2}{3}})dt \\ &= 3t^{-\frac{1}{3}} + \frac{3}{5}t^{\frac{5}{3}} + C = \frac{3}{\sqrt[3]{\cos x}} + \frac{3}{5}\sqrt[3]{\cos^5 x} + C.\end{aligned}$$

(3) 令$t = \tan\frac{x}{2}$, 则$\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$, 于是,

$$\begin{aligned}\int \frac{1}{4-\sin x}dx &= \int \frac{dt}{2t^2 - t + 2} = \frac{1}{2}\int \frac{1}{t^2 - \frac{1}{2}t + 1}dt \\ &= \frac{1}{2}\int \frac{1}{(t-\frac{1}{4})^2 + \frac{15}{16}}dt = \frac{2}{\sqrt{15}}\arctan\frac{4t-1}{\sqrt{15}} + C \\ &= \frac{2}{\sqrt{15}}\arctan\frac{4\tan\frac{x}{2} - 1}{\sqrt{15}} + C.\end{aligned}$$

(4)

$$\int \frac{\cos x}{1+\cos x}dx = \int \frac{\cos x + 1 - 1}{1+\cos x}dx = \int (1 - \frac{1}{1+\cos x})dx = x - \tan\frac{x}{2} + C.$$

(5)

$$\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}dx = \int (-1 + \frac{2}{1+\sin x - \cos x})dx,$$

令$t = \tan\frac{x}{2}$, 则$\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $x = 2\arctan t$, $dx = \frac{2dt}{1+t^2}$, 于是,

$$\int \frac{2}{1+\sin x-\cos x}dx = \int \frac{2}{t(1+t)}dt = 2\int (\frac{1}{t} - \frac{1}{1+t})dt = 2\ln|\frac{t}{1+t}| + C,$$

故

$$\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}dx = \int (-1 + \frac{2}{1+\sin x-\cos x})dx = -x + 2\ln|\frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}| + C.$$

(6) $\int \frac{1}{\sqrt{\sin x\cos^7 x}}dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}\sec^2 x dx$, 令$\tan x = t$, 则$dt = \sec^2 x dx$, 故,

$$\begin{aligned}\int \frac{1}{\sqrt{\sin x\cos^7 x}}dx &= \int \frac{\sec^2 x}{\sqrt{\tan x}}\sec^2 x dx = \int \frac{1+t^2}{\sqrt{t}}dt = \int (t^{-\frac{1}{2}} - t^{\frac{3}{2}})dt \\ &= 2t^{\frac{1}{2}} + \frac{2}{5}t^{\frac{5}{2}} + C = 2\sqrt{\tan x} + \frac{2}{5}\tan x^{\frac{5}{2}} + C.\end{aligned}$$

(7)令$x=t^6$, 则$t=\sqrt[6]{x}$, $dx=6t^5dt$, 于是

$$\begin{aligned}
\int\frac{\sqrt{x}-1}{\sqrt[3]{x}+1}dx &= \int\frac{t^3-1}{t^2+1}6t^5dt=6\int\frac{t^8-t^5}{t^2+1}dt\\
&= 6\int(t^6-t^4-t^3+t^2+t-1+\frac{-t+1}{t^2+1})dt\\
&= \frac{6t^7}{7}-\frac{6t^5}{5}-\frac{3t^4}{2}+2t^3+3t^2-t-\frac{1}{2}\ln(1+t^2)+\arctan t+C\\
&= \frac{6x^{7/6}}{7}-\frac{6x^{5/6}}{5}-\frac{3x^{2/3}}{2}+2\sqrt{x}+3\sqrt[3]{x}-\sqrt[6]{x}-\frac{1}{2}\ln(1+\sqrt[3]{x})\\
&\quad+\arctan\sqrt[6]{x}+C.
\end{aligned}$$

(8) 令 $\sqrt[12]{x}=t$, 则$x=t^{12}$, $dx=12t^{11}dt$,

$$\begin{aligned}
\int\frac{\sqrt[4]{x}}{\sqrt[3]{x}+\sqrt{x}}dx &= \int\frac{12t^{10}}{t^2+1}dt\\
&= 12\int(t^8-t^6+t^4-t^2+1-\frac{1}{t^2+1})dt\\
&= \frac{4}{3}t^9-\frac{12}{7}t^7+\frac{12}{5}t^5-4t^3+12t-12\arctan t+C\\
&= \frac{4}{3}x^{\frac{3}{4}}-\frac{12}{7}x^{\frac{7}{12}}+\frac{12}{5}x^{\frac{5}{12}}-4\sqrt[4]{x}+12\sqrt[12]{x}-12\arctan\sqrt[12]{x}+C.
\end{aligned}$$

(9) 令$\sqrt{x-2}=t$, 则$x=t^2+2$, $dx=2tdt$

$$\begin{aligned}
\int\frac{x+1}{x\sqrt{x-2}}dx &= \int\frac{t^2+3}{t(t^2+2)}2tdt=2\int\frac{t^2+3}{t^2+2}dt\\
&= 2\int(1+\frac{1}{t^2+2})dt=2t+\sqrt{2}\arctan\frac{t}{\sqrt{2}}+C\\
&= 2\sqrt{x-2}+\sqrt{2}\arctan\sqrt{\frac{x-2}{2}}+C.
\end{aligned}$$

(10) 令$\sqrt{\frac{1-x}{1+x}}=t$, 则$x=\frac{1-t^2}{1+t^2}$, $dx=\frac{-4t}{(1+t^2)^2}dt$,

$$\begin{aligned}
\int\frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx &= \int\frac{-4t^2}{(1-t^2)^2}dt=-2\int(\frac{1}{(1-t^2)^2}-\frac{1}{(1+t^2)^2})dt\\
&= -\int(\frac{1}{(1+t)^2}+\frac{1}{(1-t)^2}+\frac{1}{t-1}-\frac{1}{t+1})dt\\
&= \frac{1}{t+1}+\frac{1}{t-1}+\ln|\frac{t+1}{t-1}|+C\\
&= -\frac{\sqrt{1-x^2}}{x}+\ln|\frac{1+\sqrt{1-x^2}}{x}|+C.
\end{aligned}$$

(11) 令$\sqrt{\dfrac{2-x}{2+x}}=t$, 则$x=\dfrac{2(1-t^2)}{1+t^2}$, $dx=\dfrac{-8t}{(1+t^2)^2}dt$, $2-x=\dfrac{4t^2}{1+t^2}$

$$\int\sqrt{\frac{2-x}{2+x}}\cdot\frac{1}{(2-x)^2}dx = -\int\frac{1}{2t^2}dt=\frac{1}{2t}+C=\frac{1}{2}\sqrt{\frac{2+x}{2-x}}+C.$$

(12)

$$\begin{aligned}
\int\frac{x-2}{\sqrt{2x^2+4x+5}}dx &= \frac{1}{4}\int\frac{4x+4-12}{\sqrt{2x^2+4x+5}}dx\\
&= \frac{1}{4}(\int\frac{d(2x^2+4x)}{\sqrt{2x^2+4x+5}}-\int\frac{12}{\sqrt{2x^2+4x+5}})dx\\
&= \frac{\sqrt{2x^2+4x+5}}{2}-3\int\frac{1}{\sqrt{2x^2+4x+5}}dx\\
&= \frac{\sqrt{2x^2+4x+5}}{2}-3\int\frac{1}{\sqrt{2}\sqrt{(x+1)^2+\frac{3}{2}}}dx\\
&= \frac{\sqrt{2x^2+4x+5}}{2}-\frac{3}{\sqrt{2}}\ln(x+1+\sqrt{(x+1)^2+\frac{3}{2}})+C.
\end{aligned}$$

(13) 令$\sqrt{x^2-x+1}=t-x$, $x=\dfrac{t^2-1}{2t-1}$, $dx=\dfrac{2(t^2-t+1)}{(2t-1)^2}dt$, 则

$$\begin{aligned}
\int\frac{1}{x+\sqrt{x^2-x+1}}dx &= 2\int\frac{t^2-t+1}{t(2t-1)^2}dt\\
&= \int[\frac{2}{t}-\frac{3}{2t-1}+\frac{3}{(2t-1)^2}]dt\\
&= 2\ln|t|-\frac{3}{2}\ln|2t-1|-\frac{3}{2(2t-1)}+C\\
&= 2\ln|x+\sqrt{x^2-x+1}|-\frac{3}{2}\ln|2x+2\sqrt{x^2-x+1}-1|\\
&\quad -\frac{3}{2(2x+2\sqrt{x^2-x+1}-1)}+C.
\end{aligned}$$

(14)

$$\begin{aligned}
\text{原式} &= \int(\frac{x^2+x+2}{x^2\sqrt{x^2+x+1}}-\frac{2}{x^2})dx\\
&= \frac{2-2\sqrt{x^2+x+1}}{x}+\ln(2x+1+2\sqrt{x^2+x+1})+C.
\end{aligned}$$

(15)

$$\begin{aligned}
\int \frac{x+3}{\sqrt{1+4x-5x^2}}dx &= -\frac{1}{10}\int \frac{-10x+4-34}{\sqrt{1+4x-5x^2}}dx \\
&= -\frac{1}{10}(\int \frac{d(-10x+4)}{\sqrt{1+4x-5x^2}} - \frac{34}{\sqrt{1+4x-5x^2}})dx \\
&= -\frac{1}{5}\sqrt{1+4x-5x^2} + \frac{17}{5}\int \frac{1}{\sqrt{1+4x-5x^2}}dx \\
&= -\frac{1}{5}\sqrt{1+4x-5x^2} + \frac{17}{5\sqrt{5}}\arcsin(\frac{5}{3}x-\frac{2}{3}) + C.
\end{aligned}$$

(16)令$\sqrt[4]{x}=t$, 则$x=t^4$, $dx=4t^3dt$,

$$\begin{aligned}
\int \frac{1}{\sqrt{x}(1+\sqrt[4]{x})^3}dx &= \int \frac{4t^3dt}{t^2(1+t)^3}dt \\
&= \int \frac{4tdt}{(1+t)^3}dt \\
&= \int (\frac{4}{(1+t)^2}) - \frac{4}{(1+t)^3})dt \\
&= -\frac{4}{1+t} + \frac{2}{(1+t)^2} + C \\
&= -\frac{4}{1+\sqrt[4]{x}} + \frac{2}{(1+\sqrt[4]{x})^2} + C.
\end{aligned}$$

5.2 定积分

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1.由题意, $f(x)$在$[0,1]$上单调增加, 由推论5.2.3得$f(x)$在$[0,1]$上可积.

2.证明: 用推论5.2.1证. $\forall\varepsilon>0$, 取充分大的q, 使$\dfrac{1}{q}<\dfrac{\varepsilon}{2}$, 则在$[0,1]$上使$R(x)=\dfrac{1}{q}\geqslant\dfrac{\varepsilon}{2}$的有理点$x=\frac{p}{q}$只有有限个, 设它们为$r_1,r_2,...,r_k$. 现对$[0,1]$做分割$T=\{\Delta_1,\Delta_2\cdots\Delta_n\}$, 使细度$\parallel T\parallel<\dfrac{\varepsilon}{2k}$. 将$T$的小区间分为两类，其中$\triangle_{i'}$为含有$\{r_i\}$中点的小区间(其个数$\leqslant 2k$), $\triangle_{i''}$为不含$\{r_i\}$中点的小区间，则在$\triangle_{i'}$上$f(x)$的振幅$\omega_{i'}$满足$\omega_{i'}\leqslant\frac{1}{2}$, 从而

$$\sum_{i'}\omega_{i'}\triangle x_{i'}\leqslant\frac{1}{2}\sum_{i'}\triangle x_{i'}\leqslant\frac{1}{2}\cdot 2k||T||<\frac{\varepsilon}{2}.$$

在$\triangle_{i''}$上$f(x)$的振幅$\omega_{i''}\leqslant\dfrac{\varepsilon}{2}$,从而

$$\sum_{i''}\omega_{i''}\triangle x_{i''}\leqslant\frac{\varepsilon}{2}\sum_{i''}\triangle x_{i''}<\frac{\varepsilon}{2}.$$

所以

$$\sum_{i}\omega_{i}\triangle x_{i}=\sum_{i'}\omega_{i'}\triangle x_{i'}+\sum_{i''}\omega_{i''}\triangle x_{i''}<\varepsilon.$$

故$f(x)$在$[0,1]$上黎曼可积，且$\int_0^1 f(x)dx=0$.

3. (1)由$f(x)$的定义可知$0\leqslant f(x)<1$, 且$f(x)$的不连续点为$x=0$和$x=\frac{1}{n}(n=1,2\cdots)$. 因此, 对$\forall\varepsilon>0$, 在区间$[\varepsilon,1]$上$f(x)$只有有限个不连续点, 从而$f(x)$在$[\varepsilon,1]$上可积. 因此存在划分$T$, 使$\displaystyle\sum_{i=1}^{n}\omega_i\Delta x_i<\varepsilon$. 现在划分$T$中增加分点$O$, 构成$[0,1]$上的一个划分, 且在小区间$[0,\varepsilon]$上有$\omega_0\Delta x_0<\varepsilon$, 从而

$$\sum_{i=0}^{n}\omega_i\Delta x_i=\sum_{i=1}^{n}\omega_i\Delta x_i+\omega_0\Delta x_0<\varepsilon+\varepsilon=2\varepsilon.$$

因此, 由推论5.5.1知$f(x)$在$[0,1]$上可积.

(2)由$g(x)$的定义可知$-1\leqslant g(x)\leqslant 1$, 且$g(x)$的不连续点为$x=0$和$x=\frac{1}{n}(n=1,2\cdots)$. 因此, 对$\forall\varepsilon>0$, 在区间$[\varepsilon,1]$上, $g(x)$只有有限个不连续点, 从而$g(x)$在$[\varepsilon,1]$上可积. 因此存在划分T, 使$\displaystyle\sum_{i=1}^{n}\omega_i\Delta x_i<\varepsilon$. 现在划分$T$中增加分点$O$, 构成$[0,1]$上的一

个划分, 且在小区间$[0,\varepsilon]$上有$\omega_0\Delta x_0<\varepsilon$, 从而

$$\sum_{i=0}^{n}\omega_i\Delta x_i=\sum_{i=1}^{n}\omega_i\Delta x_i+\omega_0\Delta x_0<\varepsilon+\varepsilon=2\varepsilon.$$

因此$g(x)$在$[0,1]$上可积.

4. 设T是$[0,1]$上的任一划分, 由实数稠密性可知, 在任一小区间$[x_{i-1},x_i]$上, 有$\omega_i=2$. 从而$\sum\limits_{i=1}^{n}\omega_i\Delta x_i=2$, 故$f(x)$不可积. $|f(x)|\equiv 1$显然可积.

5. 反证法. 假设对$[0,1]$的任意闭子区间$[\alpha,\beta]$, 都存在$\eta\in[\alpha,\beta]$, 使得$f(\eta)\leqslant 0$. 对$[0,1]$的任一分割:

$$0=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=1,$$

取$\eta_k\in[x_k,x_{k+1}]$, 使$f(x_k)\leqslant 0$, $1\leqslant k\leqslant n$, 则Riemann和满足

$$\sum_{k=1}^{n}f(\eta_k)\Delta x_k\leqslant 0.$$

由f可积, 得到

$$\lim_{\|T\|\to 0}\sum_{k=1}^{n}f(\eta_k)\Delta x_k=\int_0^1 f(x)dx\leqslant 0,$$

与已知相矛盾. 故存在某个闭区间$[\alpha,\beta]$, $\forall x\in[\alpha,\beta]$, 有$f(x)>0$.

6. 因为$f(x)$在$[a,b]$上可积, 所以对$\forall\varepsilon>0$,存在一种划分, 使$\sum\limits_{i=1}^{n}\omega_i\Delta x_i<\varepsilon$, 其中$\omega_i=M_i-m_i$是$f(x)$在第$i$个区间上的振幅. 于是, $\dfrac{1}{f(x)}$在该区间上的振幅为

$$\eta_i=\frac{1}{m_i}-\frac{1}{M_i}=\frac{M_i-m_i}{m_iM_i}\leqslant\frac{1}{\Lambda^2}\omega_i,$$

因此

$$\sum_{i=1}^{n}\eta_i\Delta x_i\leqslant\sum_{i=1}^{n}\frac{1}{\Lambda^2}\omega_i\Delta x_i<\frac{\varepsilon}{\Lambda^2},$$

即$\dfrac{1}{f(x)}$在$[a,b]$上也可积.

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1. 证明: 取$\varphi(x)=f(x)$, 则$\int_{\alpha}^{\beta} f^2(x)dx=0$. 假设存在$x_0\in[\alpha,\beta]$, 使$f(x_0)\neq 0$, 则由连续函数的局部保号性可知, 存在含$x_0$的区间$[a,b]\subset[\alpha,\beta]$, 使对任意的$x\in[a,b]$, 有$f^2(x)>f^2(x_0)/2>0$. 于是,

$$0=\int_{\alpha}^{\beta} f^2(x)dx \geqslant \int_a^b f^2(x)dx > \frac{f^2(x_0)}{2}(b-a)>0,$$

矛盾. 故在$[\alpha,\beta]$上, $f(x)\equiv 0$.

2. 由积分中值公式, 有

$$\begin{aligned}\lim_{n\to\infty}\int_{n^2}^{n^2+n}\frac{1}{\sqrt{x}}\,e^{-\frac{1}{x}}dx &= \lim_{n\to\infty} e^{-\frac{1}{\xi}}\int_{n^2}^{n^2+n}\frac{dx}{\sqrt{x}}\\ &= \lim_{n\to\infty} e^{-\frac{1}{\xi}}\frac{2n}{\sqrt{n^2+n}+n}=1,\end{aligned}$$

其中$n^2<\xi<n^2+n$, 从而$1/\xi\to 0\ (n\to\infty)$.

3.(1) 对$\forall x\in[0,\frac{\pi}{2}]$, $0\leqslant \sin x\leqslant 1$, 但对$0<x<\frac{\pi}{2}$, 有$\sin^{n+1}x<\sin^n x$, 故

$$\int_0^{\frac{\pi}{2}}\sin^{n+1}x dx<\int_0^{\frac{\pi}{2}}\sin^n x dx.$$

(2) 当$x\in(0,\frac{\pi}{2})$时, $\frac{\pi}{2}<\frac{\sin x}{x}<1$; 又$\lim\limits_{x\to 0^+}\frac{\sin x}{x}=1$, 所以

$$1=\int_0^{\frac{\pi}{2}}\frac{2}{\pi}dx<\int_0^{\frac{\pi}{2}}\frac{\sin}{x}dx<\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}.$$

(3) 当$x\in(0,\frac{\pi}{2})$时, $1>\sqrt{1-\frac{1}{2}\sin^2 x}>\sqrt{\frac{1}{2}}$, 于是

$$\frac{\pi}{2}<\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-\frac{1}{2}\sin^2 x}}dx<\frac{\pi}{\sqrt{2}}.$$

(4) $3\sqrt{e}=\int_e^{4e}\frac{1}{\sqrt{x}}dx<\int_e^{4e}\frac{\ln x}{\sqrt{x}}dx$. 又

$$\max_{x\in[e,4e]}\frac{\ln x}{\sqrt{x}}=\frac{\ln e^2}{\sqrt{e^2}}=\frac{2}{e},$$

所以

$$\int_e^{4e} \frac{\ln x}{\sqrt{x}} dx < 3e \cdot \frac{2}{e} = 6.$$

故,

$$3\sqrt{e} < \int_e^{4e} \frac{\ln x}{\sqrt{x}} dx < 6.$$

4. 由于$\alpha \int_0^1 f(x)dx = \alpha \int_0^{\alpha} f(x)dx + \alpha \int_{\alpha}^1 f(x)dx$, 所以要证

$$\int_0^{\alpha} f(x)dx \geqslant \alpha \int_0^1 f(x)dx,$$

只需证

$$(1-\alpha) \int_0^{\alpha} f(x)dx \geqslant \alpha \int_{\alpha}^1 f(x)dx.$$

因为$f(x)$是递减函数, 所以有

$$\int_0^{\alpha} f(x)dx \geqslant \int_0^{\alpha} f(\alpha)dx = \alpha f(\alpha), \quad \int_{\alpha}^1 f(x)dx \leqslant \int_{\alpha}^1 f(\alpha)dx = (1-\alpha)f(\alpha).$$

从而

$$(1-\alpha) \int_0^{\alpha} f(x)dx \geqslant (1-\alpha)\alpha f(\alpha) \geqslant \alpha \int_{\alpha}^1 f(x)dx.$$

5. $f(x)$在$[\alpha, \beta]$上连续, 则由积分中值公式可知, 存在$\eta \in (\alpha, (\alpha+\beta)/2)$, 使得

$$\int_{\alpha}^{\frac{\alpha+\beta}{2}} f(x)dx = f(\eta) \cdot \frac{\beta - \alpha}{2}.$$

于是$f(\eta) = f(\beta)$. 在区间$[\eta, \beta]$上用Rolle中值定理即得结论.

注: 请指出下列证明中的错误.

$f(x)$在(α, β)内可导, 则由Lagrange中值定理, 对$\forall x \in (\alpha, \beta)$ 存在$\xi \in (x, \beta)$, 使

$$\frac{f(x) - f(\beta)}{x - \beta} = f'(\xi),$$

即

$$f(x) = f(\beta) + f'(\xi)(x - \beta).$$

从而

$$\begin{aligned}\int_\alpha^{\frac{\alpha+\beta}{2}} f(x)dx &= \int_\alpha^{\frac{\alpha+\beta}{2}} f(\beta)dx + \int_\alpha^{\frac{\alpha+\beta}{2}} f'(\xi)(x-\beta)dx \\ &= f(\beta)(\frac{\alpha+\beta}{2}-\alpha) + f'(\xi)\frac{(x-\beta)^2}{2}|_\alpha^{\frac{\alpha+\beta}{2}} \\ &= f(\beta)\frac{\beta-\alpha}{2} - \frac{3}{8}f'(\xi)(\alpha-\beta)^2.\end{aligned}$$

又

$$\int_\alpha^{\frac{\alpha+\beta}{2}} f(x)dx = f(\beta)\frac{\beta-\alpha}{2},$$

代入上式得

$$\frac{3}{8}f'(\xi)(\alpha-\beta)^2 = 0.$$

所以

$$f'(\xi) = 0, \xi \in (\alpha,\beta).$$

6. 令$F(x) = \int_0^x f(\theta)\sin(\theta)d\theta, x \in [0,\pi]$, 则$F(x) \in C[0,\pi]$, 在$(0,\pi)$内可导，且$F(0) = F(\pi) = 0$, 由Rolle定理可知，$\exists\alpha \in (0,\pi)$, 使得

$$F'(\alpha) = 0, \Rightarrow f(\alpha)\sin(\alpha) = 0.$$

因$\alpha \in (0,\pi)$, 故$\sin(\alpha) \neq 0$, 因此必有$f(\alpha) = 0$.

往证$\exists\beta \in (0,\pi)(\beta \neq \alpha)$, 使得$f(\beta) = 0$, 用反证法.

假设$f(x)$在$(0,\pi)$内只有唯一零点$x = \alpha$, 则$f(x)$在$(0,\alpha)$和(α,π)内必反号，否则不可能有$\int_0^\pi f(\theta)\sin(\theta)d\theta = 0$. 而$\sin(\theta-\alpha)$在$(0,\alpha)$和$(\alpha,\pi)$内符号也相反，故$f(\theta)\sin(\theta-\alpha)$这两个区间内必同号. 于是有

$$\int_0^\pi f(\theta)\sin(\theta-\alpha)d\theta > 0.$$

另一方面，由题设条件又有

$$\begin{aligned}&\int_0^\pi f(\theta)\sin(\theta-\alpha)d\theta = \int_0^\pi f(\theta)(\sin(\theta)\cos(\alpha) - \cos(\theta)\sin(\alpha))d\theta \\ &= \cos(\alpha)\int_0^\pi f(\theta)\sin(\theta) - \sin(\alpha)\int_0^\pi f(\theta)\cos(\theta)d\theta = 0.\end{aligned}$$

从而推出矛盾. 由此可得$f(x)$在$(0,\pi)$内至少有两个零点.

7.(1) 设t是任一实数, 则$[tf(x) - g(x)]^2 \geqslant 0$, 即

$$t^2f^2(x) - 2tf(x)g(x) + g^2(x) \geqslant 0.$$

两边积分得

$$t^2\int_\alpha^\beta f^2(x)dx-2t\int_\alpha^\beta f(x)g(x)dx+\int_\alpha^\beta g^2(x)dx\geqslant 0.$$

从而关于t的二次三项式的判别式非正, 即

$$[2\int_\alpha^\beta f(x)g(x)dx]^2-4\int_\alpha^\beta f^2(x)dx\int_\alpha^\beta g^2(x)dx\leqslant 0.$$

整理可得

$$[\int_\alpha^\beta f(x)g(x)dx]^2\leqslant\int_\alpha^\beta f^2(x)dx\int_\alpha^\beta g^2(x)dx.$$

注: 可令$h(x)=\int_\alpha^x f^2(t)dt\int_\alpha^x g^2(t)dt-[\int_\alpha^x f(t)g(t)dt]^2$.

(2) 证法一: 利用刚证明的Schwarz不等式, 得到

$$\begin{aligned}\int_\alpha^\beta [f(x)+g(x)]^2dx &= \int_\alpha^\beta f(x)[f(x)+g(x)]dx+\int_\alpha^\beta g(x)[f(x)+g(x)]dx\\ &\leqslant \Big(\int_\alpha^\beta f^2(x)dx\Big)^{1/2}\cdot\Big(\int_\alpha^\beta [f(x)+g(x)]^2dx\Big)^{1/2}\\ &\quad +\Big(\int_\alpha^\beta g^2(x)dx\Big)^{1/2}\cdot\Big(\int_\alpha^\beta [f(x)+g(x)]^2dx\Big)^{1/2}.\end{aligned}$$

两端同除以$\Big(\int_\alpha^\beta [f(x)+g(x)]^2dx\Big)^{1/2}$即得Minkowski不等式.

证法二: 因为

$$\begin{aligned}&\Big\{[\int_\alpha^\beta f^2(x)dx]^{\frac12}+[\int_\alpha^\beta g^2(x)dx]^{\frac12}\Big\}^2\\ &=\int_\alpha^\beta f^2(x)dx+\int_\alpha^\beta g^2(x)dx+2[\int_\alpha^\beta f^2(x)dx\int_\alpha^\beta g^2(x)dx]^{\frac12},\end{aligned}$$

而由(1)知

$$\int_\alpha^\beta f^2(x)dx\int_\alpha^\beta g^2(x)dx\geqslant[\int_\alpha^\beta f(x)g(x)dx]^2,$$

从而

$$\begin{aligned}&\Big\{[\int_\alpha^\beta f^2(x)dx]^{\frac12}+[\int_\alpha^\beta g^2(x)dx]^{\frac12}\Big\}^2\\ &\geqslant\int_\alpha^\beta f^2(x)dx+\int_\alpha^\beta g^2(x)dx+2\int_\alpha^\beta f(x)g(x)dx\\ &=\int_\alpha^\beta (f^2(x)+2f(x)g(x)+g^2(x))dx\\ &=\int_\alpha^\beta [f(x)+g(x)]^2dx.\end{aligned}$$

两边开方得

$$\Big\{\int_\alpha^\beta [f(x)+g(x)]^2dx\Big\}^{\frac{1}{2}} \leqslant [\int_\alpha^\beta f^2(x)dx]^{\frac{1}{2}} + [\int_\alpha^\beta g^2(x)dx]^{\frac{1}{2}}.$$

8.

$$\Big|\int_0^1 f(x)dx - \frac{1}{n}\sum_{i=1}^n f(\frac{i}{n})\Big| = \Big|\sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x)dx - \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(\frac{i}{n})dx\Big|$$

$$= \Big|\sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} [f(x)-f(\frac{i}{n})]dx\Big| \leqslant \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big|f(x)-f(\frac{i}{n})\Big|dx \leqslant \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} L\Big|x-\frac{i}{n}\Big|dx$$

$$\leqslant \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \frac{L}{n}dx \leqslant \frac{L}{n}.$$

9. 将$[0,1]$区间n等分，分点为$0,\frac{1}{n},\cdots,\frac{n}{n}$. 因$f$在$[0,1]$上单调减, 所以对$\forall x \in [\frac{i-1}{n},\frac{i}{n}](i=1,2,\cdots,n)$, $f(\frac{i-1}{n}) \geqslant f(x) \geqslant f(\frac{i}{n})$, 如上题，有

$$\begin{aligned}
&\int_0^1 f(x)dx - \frac{1}{n}\sum_{i=1}^n f(\frac{i}{n}) = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x)dx - \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(\frac{i}{n})dx \\
&= \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} [f(x)-f(\frac{i}{n})]dx \leqslant \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big[f(\frac{i-1}{n})-f(\frac{i}{n})\Big]dx \\
&= \sum_{i=1}^n \frac{1}{n}[f(\frac{i-1}{n})-f(\frac{i}{n})] = \frac{f(0)-f(1)}{n}.
\end{aligned}$$

10. 由$1 \leqslant f(x) \leqslant 2 \Rightarrow [f(x)-1][f(x-2)] \leqslant 0 \Rightarrow f^2(x)-3f(x)+2 \leqslant 0$

$$\Rightarrow f(x)-3+\frac{2}{f(x)} \leqslant 0 \Rightarrow \int_0^1 f(x)dx - 3 + 2\int_0^1 \frac{1}{f(x)}dx \leqslant 0$$

$$\Rightarrow 3 \geqslant \int_0^1 f(x)dx + 2\int_0^1 \frac{1}{f(x)}dx \geqslant 2\Big[2\int_0^1 f(x)dx \int_0^1 \frac{1}{f(x)}dx\Big]^{\frac{1}{2}}$$

$$\Rightarrow \int_0^1 f(x)dx \int_0^1 \frac{1}{f(x)}dx] \leqslant \frac{9}{8}.$$

11. 因$g(x)$在$[\alpha,\beta]$上连续, 所以$\exists M, m(M \geqslant m > 0)$, 使对$\forall x \in [\alpha,\beta]$, 有$m \leqslant g(x) \leqslant M$. 设$f(x_0) = \max\limits_{\alpha\leqslant x\leqslant\beta} f(x)$, $x_0 \in (\alpha,\beta)$, 则对$\forall \varepsilon > 0, \exists \delta > 0$, 使得当$x \in [x_0-\delta, x_0+\delta] \subset [\alpha,\beta]$时, 有

$$f(x_0) - \frac{\varepsilon}{2} < f(x) \leqslant f(x_0).$$

记$I_n = [\int_\alpha^\beta f^n(x)g(x)dx]^{\frac{1}{n}}$, 则

$$[2\delta(f(x_0)-\frac{\varepsilon}{2})^n m]^{\frac{1}{n}} \leqslant [\int_{x_0-\delta}^{x_0+\delta} f^n(x)g(x)dx]^{\frac{1}{n}} \leqslant I_n \leqslant [f^n(x_0)M(\alpha-\beta)]^{\frac{1}{n}}.$$

令$n\to\infty$, 得

$$f(x_0)-\frac{\varepsilon}{2} \leqslant \lim_{n\to\infty} I_n \leqslant f(x_0).$$

由ε的任意性, 得

$$\lim_{n\to\infty}[\int_\alpha^\beta f^n(x)g(x)dx]^{\frac{1}{n}}) = f(x_0) = \max_{\alpha\leqslant x\leqslant\beta} f(x).$$

当$x_0=a$, 或$x_0=b$时, 类似可证.

12. $\{\alpha_n\}$和$\{\alpha_n\}$如下：

$$\{\alpha_n\} = \int_0^1 \max\{x,\beta_{n-1}\}dx, \quad \{\beta_n\} = \int_0^1 \min\{x,\alpha_{n-1}\}dx, (n=2,3,\cdots) \tag{1}$$

由上式，有

$$\alpha_n \geqslant \int_0^1 xdx = \frac{1}{2}, \quad \beta_n \leqslant \int_0^1 xdx = \frac{1}{2}, \quad (n=2,3,\cdots) \tag{2}$$

将（2）代人（1），有

$$\alpha_n \leqslant \int_0^{\frac{1}{2}} \frac{1}{2}dx + \int_{\frac{1}{2}}^1 xdx = \frac{5}{8}, \quad \beta_n \geqslant \int_0^{\frac{1}{2}} xdx + \int_{\frac{1}{2}}^1 \frac{1}{2}dx = \frac{3}{8}, \quad (n=2,3,\cdots) \tag{3}$$

$$\Rightarrow \frac{1}{2} \leqslant \alpha_n \leqslant \frac{5}{8}, \quad \frac{3}{8} \leqslant \beta_n \leqslant \frac{1}{2}, \quad (n=2,3,\cdots) \qquad (4)$$

由(4)及（1）可得

$$\left\{\begin{aligned} &2\alpha_{n+1} = 2(\int_0^{\beta_n}\beta_n dx + \int_{\beta_n}^1 xdx) = 1+\beta_n^2, \quad (5)\\ &2\beta_{n+1} = 2(\int_0^{\alpha_n} xdx + \int_{\alpha_n}^1 \alpha_n dx = 2\alpha_n - \alpha_n^2, \quad (6)\end{aligned}\right. \qquad (n=2,3,\cdots)$$

$\Rightarrow \quad \alpha_{n+2}-\alpha_{n+1} = \dfrac{\beta_{n+1}-\beta_n}{2}(\beta_{n+1}-\beta_n),$

$\beta_{n+1}-\beta_n = \dfrac{2-\alpha_n-\alpha_{n-1}}{2}(\alpha_n-\alpha_{n-1}), \quad (n=2,3,\cdots)$

$\Rightarrow \quad \alpha_{n+2}-\alpha_{n+1} = \dfrac{\beta_{n+1}+\beta_n}{2}\cdot\dfrac{2-\alpha_n-\alpha_{n-1}}{2}(\alpha_n-\alpha_{n-1}),$

由（4）$\Rightarrow |\alpha_{n+2}-\alpha_{n+1}| \leqslant |\alpha_n-\alpha_{n-1}|, \quad n=2,3,\cdots$

反复用上式，得

$$|\alpha_{2m+2}-\alpha_{2m+1}| \leqslant \frac{1}{4^m}|\alpha_2-\alpha_1|, \quad |\alpha_{2m+3}-\alpha_{2m+2}| \leqslant \frac{1}{4^m}|\alpha_3-\alpha_2|, \quad m=1,2,\cdots$$

令$m\to\infty$, 得$\alpha_{2m+2}-\alpha_{2m+1}\to 0, \quad \alpha_{2m+3}-\alpha_{2m+2}\to 0,$, 从而可得$\lim\limits_{n\to\infty}\alpha_n$存在，同理可知，$\lim\limits_{n\to\infty}\beta_n$存在. 设$\lim\limits_{n\to\infty}\alpha_n=\alpha$，$\lim\limits_{n\to\infty}\beta_n=\beta$, 由（5）（6）可得

$$2\alpha = 1+\beta^2, \quad 2\beta = 2\alpha-\alpha^2 \qquad (7)$$

解得: $\alpha = 2-\sqrt{2}, \quad \beta=\sqrt{2}-1.$

13. 证：$\Lambda_{n+1}=\int_\alpha^\beta \varphi(x)f^{n+1}(x)dx \leqslant M\int_\alpha^\beta \varphi(x)f^n(x)dx = M\Lambda_n$,

其中$M=\max\limits_{\alpha\leqslant x\leqslant\beta} f(x)$.

$\Rightarrow \dfrac{\Lambda_{n+1}}{\Lambda_n}\leqslant M$, 即数列$\{\dfrac{\Lambda_{n+1}}{\Lambda_n}\}$有上界. 再由Cauchy-Schwartz不等式，有

$$\begin{aligned}\Lambda_{n+1}^2 &= (\int_\alpha^\beta \varphi(x)f^{n+1}(x)dx)^2 = (\int_\alpha^\beta [\varphi(x)]^{\frac12}[f(x)]^{\frac{n+2}{2}}[\varphi(x)]^{\frac12}[f(x)]^{\frac{n}{2}}dx)^2 \\ &\leqslant \int_\alpha^\beta [\varphi(x)][f(x)]^{n+2}dx\int_\alpha^\beta [\varphi(x)][f(x)]^n dx = \Lambda_{n+2}\cdot\Lambda_n,\end{aligned}$$

于是，$\dfrac{\Lambda_{n+1}}{\Lambda_n}\leqslant\dfrac{\Lambda_{n+2}}{\Lambda_{n+1}}$, 即数列$\{\dfrac{\Lambda_{n+1}}{\Lambda_n}\}$是单调增加的，故其极限存在. 再由命题（*）及第11题知，

$$\lim_{n\to\infty}\frac{\Lambda_{n+1}}{\Lambda_n} = \lim_{n\to\infty}\sqrt[n]{\Lambda_n} = \lim_{n\to\infty}(\int_\alpha^\beta [\varphi(x)][f(x)]^n dx)^{\frac1n} = \max_{\alpha\leqslant x\leqslant\beta} f(x).$$

注：命题（*）：若$x_n>0(n=1,2,\cdots,)$且$\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n}$存在, 则$\lim\limits_{n\to\infty}\sqrt[n]{x_n}=\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n}$.

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1.(1) $\lim\limits_{n\to\infty}\int_0^{\frac23}\dfrac{x^n}{1+x}\,dx = \lim\limits_{n\to\infty}\dfrac{1}{1+\xi}\int_0^{\frac23}x^n\,dx = \lim\limits_{n\to\infty}\dfrac{2^{n+1}}{3^{n+1}(1+\xi)(1+n)}=0$, 其中$\xi\in[0,2/3]$, 所以$1/(\xi+1)$有界.

(2) 由积分第一中值定理, 存在$\xi\in[n,n+1]$, 使得

$$\lim_{n\to\infty}\int_n^{n+1}\frac{\sin x}{x}\,dx = \lim_{n\to\infty}\sin\xi\cdot\int_n^{n+1}\frac1x\,dx = \lim_{n\to\infty}\sin\xi\cdot\ln\frac{n+1}{n}=0$$

(3) 由罗比达法则得, 原式$=\lim\limits_{x\to\infty}\dfrac{e^{x^2}}{e^{2x^2}}=\lim\limits_{x\to\infty}\dfrac{1}{e^{x^2}}=0.$

(4) 令$r=t^2$, 则

$$\int_x^{x+1}\sin t^2\ dt=\frac{1}{2}\int_{x^2}^{(x+1)^2}\frac{\sin r}{\sqrt{r}}\ dr=-\frac{\cos r}{2\sqrt{r}}\Big|_{x^2}^{(x+1)^2}-\frac{1}{4}\int_{x^2}^{(x+1)^2}\frac{\cos r}{r\sqrt{r}}\ dr.$$

因为当$x>0$时, 有

$$\Big|\frac{\cos x^2}{x}-\frac{\cos(x^2+1)}{x+1}\Big|\leqslant\frac{1}{x}+\frac{1}{x+1}\to 0\quad(x\to+\infty),$$

$$\Big|\frac{1}{2}\int_{x^2}^{(x+1)^2}\frac{\cos r}{r\sqrt{r}}\ dr\Big|\leqslant\frac{1}{2}\int_{x^2}^{(x+1)^2}\frac{1}{r\sqrt{r}}\ dr=\frac{1}{x}-\frac{1}{x+1}\to 0\quad(x\to+\infty),$$

所以$\lim\limits_{x\to+\infty}\int_x^{(x+1)}\sin t^2\ dt=0.$

(5)令$S(x)=\int_0^x|\cos t|dt$. 由于$|\cos t|$是以π为周期的周期函数, 故在任一周期长的区间上定积分值相同. 设$n\pi\leqslant x<(n+1)\pi$(n为正整数), 则

$$\int_0^{n\pi}|\cos t|dt\leqslant S(x)<\int_0^{(n+1)\pi}|\cos t|dt.$$

又

$$\int_0^{\pi}|\cos t|dt=\int_0^{\frac{\pi}{2}}\cos xdx-\int_{\frac{\pi}{2}}^{\pi}\cos xdx=2,$$

故$2n\leqslant S(x)<2(n+1)$. 因此

$$\frac{2n}{(n+1)\pi}<\frac{S(x)}{x}<\frac{2(n+1)}{n\pi}.$$

因为当$x\to+\infty$时, $n\to\infty$且

$$\lim_{n\to\infty}\frac{2n}{(n+1)\pi}=\lim_{n\to\infty}\frac{2(n+1)}{n\pi}=\frac{2}{\pi},$$

故$\lim\limits_{n\to\infty}\frac{S(x)}{x}=\frac{2}{\pi}.$

2.(1) 原式$=\lim\limits_{n\to\infty}(\frac{1}{n}\sum\limits_{i=1}^{n}\frac{i}{n}-\frac{1}{n})=\int_0^1 xdx=\frac{1}{2}.$

(2)原式$=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\sin\frac{i}{n}\pi=\int_0^1\sin\pi xdx=-\frac{1}{\pi}\cos\pi x\Big|_0^1=\frac{2}{\pi}.$

(3)原式$=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}(\frac{i}{n})^3=\int_0^1 x^3dx=\frac{x^3}{3+1}\Big|_0^1=\frac{1}{4}.$

3. (1) 由page125的3（1）题知

$$\int_{\ln 2}^{1}\frac{dx}{\sqrt{e^x-1}}=-2\arcsin(e^{-\frac{x}{2}})\big|_{\ln 2}^{1}=-2\arcsin(e^{-\frac{1}{2}})+2\arcsin(e^{-\frac{\ln 2}{2}})\approx 0.267.$$

(2) 由page152的4（4）知

$$\begin{aligned}\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx &= \int_0^{\pi/2}\frac{\cos^2 x}{\sin x+\cos x}dx=\frac{1}{2}\int_0^{\pi/2}\frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}dx\\ &= \frac{1}{2}\int_0^{\pi/2}\frac{1}{\sin x+\cos x}dx=\frac{1}{2}\int_0^{\pi/2}\frac{1}{\sqrt{2}\sin(x+\pi/4)}dx\\ &= \frac{1}{2\sqrt{2}}\ln|\csc(x+\pi/4)-\cot(x+\pi/4)|_0^{\pi/2}=\frac{\sqrt{2}}{2}\ln(\sqrt{2}+1).\end{aligned}$$

(3) 原式$=x\arcsin x\Big|_0^1-\int_0^1\frac{x}{\sqrt{1-x^2}}\ dx=\frac{\pi}{2}-\int_0^1\frac{x}{\sqrt{1-x^2}}\ dx.$
令$t=1-x^2$, 则$dt=-2xdx$. 当$x=0$时, $t=1$; 当$x=1$时, $t=0$. 于是

$$\int_0^1\arcsin x dx=\frac{\pi}{2}+\frac{1}{2}\int_1^0\frac{1}{\sqrt{t}}\ dt=\frac{\pi}{2}+\sqrt{t}\Big|_1^0=\frac{\pi}{2}-1.$$

(4) 令$\ln x=t$, 则$dx=e^t dt$. 当$x=1$时, $t=0$; 当$x=e$时, $t=1$. 于是

$$\int_1^e\sin(\ln x)dx=\int_0^1 e^t\sin t dt=e^t\sin t\Big|_0^1-\int_0^1 e^t\cos t dt,$$

其中

$$\begin{aligned}\int_0^1 e^t\cos t dt &= e^t\sin t\Big|_0^1-\int_0^1 e^t\sin t dt\\ &= e\sin 1+e^t\cos t\Big|_0^1-\int_0^1 e^t\cos t dt\\ &= e\sin 1+e\cos 1-1-\int_0^1 e^t\cos t dt,\end{aligned}$$

于是

$$\int_0^1 e^t\cos t dt=\frac{1}{2}(e\sin 1+e\cos 1)-\frac{1}{2}.$$

故

$$\int_1^e\sin(\ln x)dx=e\sin 1-\frac{1}{2}e(\sin 1+\cos 1)+\frac{1}{2}=\frac{1}{2}e(\sin 1-\cos 1)+\frac{1}{2}.$$

(5) (题有问题, 下限不对. 原积分发散) 令$x=\tan t$, 则$dx=\sec^2 t dt$. 当$x=0$时, $t=0$; 当$x=1$时, $t=\frac{\pi}{4}$. 于是

$$原式=\int_0^{\frac{\pi}{4}}\frac{\sec t}{\tan t}dt=\int_0^{\frac{\pi}{4}}\csc t dt=\ln|\csc t-\cot t|\Big|_0^{\frac{\pi}{4}}=+\infty.$$

注：将原题积分区间改为$[1,2]$, 则

$$\int_1^2 \frac{1}{x\sqrt{1+x^2}}dx = [\ln x - \ln(\sqrt{x^2+1}+1)]_1^2 = \ln 2 - \ln(\sqrt{5}+1) + \ln(\sqrt{3}+1).$$

(6) 原式$=\int_0^1 \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\ dx = \int_0^1 \frac{1}{(x-\frac{1}{x})^2+2}d(x-\frac{1}{x})$. 令$t=x-\frac{1}{x}$, 则当$x\to 0^+$时, $t\to-\infty$; 当$x=1$时, $t=0$. 于是

$$\text{原式} = \int_{-\infty}^0 \frac{1}{t^2+2}\ dt = \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\Big|_{-\infty}^0 = \frac{\sqrt{2}}{4}\pi.$$

注：上述方法需要用到反常积分, 可用常规的有理函数积分法求原函数.

(7) 记原积分为I_n, 则

$$I_n = \frac{1}{2}x^2\ln^n x\Big|_1^e - \frac{1}{2}\int_1^e x^2\cdot n\ln^{n-1}x\cdot\frac{1}{x}dx = \frac{1}{2}e^2 - \frac{n}{2}\int_1^e x\ln^{n-1}x dx,$$

即

$$I_n = \frac{1}{2}e^2 - \frac{n}{2}I_{n-1}.$$

注意到

$$I_1 = \int_1^e x\ln x dx = \frac{1}{2}e^2 - \frac{1}{4}(e^2-1),$$

得到

$$\begin{aligned}
I_n &= \frac{1}{2}e^2 - \frac{n}{2}I_{n-1} \\
&= \frac{1}{2}e^2 - \frac{n}{4}e^2 + \frac{n(n-1)}{4}I_{n-2} \\
&= \cdots \\
&= \frac{1}{2}e^2[1-\frac{n}{2}+\frac{n(n-1)}{2^2}+\cdots+(-1)^{n-1}\frac{n!}{2^{n-1}}] + (-1)^n\frac{n!(e^2-1)}{2^{n+1}} \\
&= \frac{1}{2}e^2[1-\frac{n}{2}+\frac{n(n-1)}{2^2}+\cdots+(-1)^{n-1}\frac{n!}{2^n}] + (-1)^{n+1}\frac{n!}{2^{n+1}}.
\end{aligned}$$

(8) 原式$=\int_0^2(4^x+2\cdot 6^x+9^x)dx = \Big(\frac{4^x}{2\ln 2}+\frac{2\cdot 6^x}{\ln 6}+\frac{9^x}{2\ln 3}\Big)\Big|_0^2$

$$= \frac{15}{2\ln 2}+\frac{70}{\ln 6}+\frac{40}{\ln 3}.$$

(9) 令$t=\sqrt{x}$,则$dx=2tdt$. 当$x=0$时, $t=0$; 当$x=1$时, $t=1$. 于是
原式$=\int_0^1 2te^t dt = 2(te^t-e^t)\Big|_0^1 = 2(e-e+1) = 2.$

(11)当$\alpha \leqslant 0$时, 原式$=\int_0^1 x(x-\alpha)dx = (\frac{x^3}{3} - \frac{ax^2}{2})\Big|_0^1 = \frac{1}{3} - \frac{\alpha}{2}$;
当$0 < \alpha < 1$时,
原式$=\int_0^{\alpha} x(\alpha-x)dx + \int_{\alpha}^1 x(x-\alpha)dx = (-\frac{x^3}{3} + \frac{ax^2}{2})\Big|_0^{\alpha} + (\frac{x^3}{3} - \frac{ax^2}{2})\Big|_{\alpha}^1 = \frac{\alpha^3}{3} - \frac{\alpha}{2} + \frac{1}{3}$;
当$\alpha \geqslant 1$时, 原式$=\int_0^1 x(\alpha-x)dx = (-\frac{x^3}{3} + \frac{ax^2}{2})\Big|_0^1 = \frac{\alpha}{2} - \frac{1}{3}$.

(12) 原式$=\int_0^{\ln 2} 1dx + \int_{\ln 2}^{\ln 3} 2dx + \int_{\ln 3}^{\ln 4} 3dx + \int_{\ln 4}^{\ln 5} 4dx + \int_{\ln 5}^{\ln 6} 5dx + \int_{\ln 6}^{\ln 7} 6dx + \int_{\ln 7}^{2} 7dx$
$= \ln 2 + 2(\ln 3 - \ln 2) + 3(\ln 4 - \ln 3) + 4(\ln 5 - \ln 4) + 5(\ln 6 - \ln 5) + 6(\ln 7 - \ln 6) + 7(2 - \ln 7)$
$= \ln[2 \cdot (\frac{3}{2})^2 \cdot (\frac{4}{3})^3 \cdot (\frac{5}{4})^4 \cdot (\frac{6}{5})^5 \cdot (\frac{7}{6})^6 \cdot (\frac{1}{7})^7] + 14$
$= 14 - \ln(7!)$.

4. (1)

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = -\int_a^0 f(-x)dx + \int_0^a f(x)dx \\ &= \int_0^a [f(-x) + f(x)]dx = 2\int_0^a f(x)dx.\end{aligned}$$

(2)

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = \int_a^0 f(-x)d(-x) + \int_0^a f(x)dx \\ &= \int_0^a f(-x)dx + \int_0^a f(x)dx = \int_0^a [f(-x) + f(x)]dx = 0.\end{aligned}$$

(3)

$$\begin{aligned}\int_a^{a+T} f(x)dx - \int_0^T f(x)dx &= \int_a^{a+T} f(x)dx + \int_T^a f(x)dx - \int_0^T f(x)dx - \int_T^a f(x)dx \\ &= \int_T^{a+T} f(x)dx - \int_0^a f(x)dx \\ &= \int_T^{a+T} f(x)dx - \int_0^a f(x+T)d(x+T) \\ &= \int_T^{a+T} f(x)dx - \int_T^{a+T} f(x)dx = 0.\end{aligned}$$

(4)令$t = \frac{\pi}{2} - x$,则

$$\int_0^{\frac{\pi}{2}} f(\cos x)dx = \int_0^{\frac{\pi}{2}} f[\cos(\frac{\pi}{2} - t)]dt = \int_0^{\frac{\pi}{2}} f(\sin t)dt = \int_0^{\frac{\pi}{2}} f(\sin x)dx$$

(5)令$t=\pi-x$, 则

$$\begin{aligned}\int_0^\pi xf(\sin x)dx &= -\int_\pi^0(\pi-t)f[\sin(\pi-t)]dt=\int_0^\pi(\pi-t)f(\sin t)dt\\ &= \pi\int_0^\pi f(\sin t)dt-\int_0^\pi tf(\sin t)dt\\ &= \pi\int_0^\pi f(\sin x)dx-\int_0^\pi xf(\sin x)dx,\end{aligned}$$

于是

$$\int_0^\pi xf(\sin x)dx=\frac{\pi}{2}\int_0^\pi f(\sin x)dx.$$

原式$=\frac{\pi}{2}\int_0^\pi\frac{\sin x}{1+\cos^2 x}\,dx=-\frac{\pi}{2}\int_0^\pi\frac{1}{1+\cos^2 x}\,d(\cos x)=-\frac{\pi}{2}\arctan(\cos x)\Big|_0^\pi$
$=-\frac{\pi}{2}[\arctan(-1)-\arctan 1]=-\frac{\pi}{2}(-\frac{\pi}{4}-\frac{\pi}{4})=\frac{\pi^2}{4}.$

5. 对于左端的不等式，注意到当$k-1<x<k$时，有$\sqrt{k}>\sqrt{x}$, 故有$\sqrt{k}>\int_{k-1}^k\sqrt{x}dx$, 从而得

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}>\int_0^n\sqrt{x}dx=\frac{2}{3}n^{3/2}.$$

对于右端不等式，因曲线$y=\sqrt{x}$在$(0,+\infty)$上是凸的，所以有

$$\frac{\sqrt{k-1}+\sqrt{k}}{2}<\int_{k-1}^k\sqrt{x}dx.$$

由此可得

$$\begin{aligned}\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} &= \frac{\sqrt{0}+\sqrt{1}}{2}+\frac{\sqrt{1}+\sqrt{2}}{2}+\cdots+\frac{\sqrt{n-1}+\sqrt{n}}{2}+\frac{\sqrt{n}}{2}\\ &< \int_0^n\sqrt{x}dx+\frac{\sqrt{n}}{2}=\frac{4n+3}{6}\sqrt{n}.\end{aligned}$$

6.

$$\begin{aligned}\int_0^1 x^n f(x)dx &= \frac{1}{n+1}\int_0^1 f(x)dx^{n+1}=\frac{1}{n+1}x^{n+1}f(x)\Big|_0^1-\frac{1}{n+1}\int_0^1 x^{n+1}f'(x)dx\\ &= \frac{f(1)}{n+1}-\frac{1}{(n+1)(n+2)}\int_0^1 f'(x)dx^{n+2}\\ &= \frac{f(1)}{n+1}-\frac{f'(1)}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)}\int_0^1 x^{n+2}f''(x)dx\end{aligned}$$

$$\int_0^1 x^n f(x)dx - [\frac{f(1)}{n} - \frac{f(1)+f'(1)}{n^2}] = [\frac{1}{n^2} - \frac{1}{n(n+1)}]f(1)$$
$$+ \quad [\frac{1}{n^2} - \frac{1}{(n+1)(n+2)}]f'(1) + \frac{1}{(n+1)(n+2)}\int_0^1 x^{n+2}f''(x)dx \qquad (1)$$

因$f''(x)$在$[0,1]$上连续，所以$\exists M>0, \forall x\in[0,1], |f''(x)|\leqslant M$.

$$\Rightarrow |\int_0^1 x^{n+2}f''(x)dx| \leqslant M\int_0^1 x^{n+2} = \frac{M}{(n+3)} \to 0 (n\to\infty).$$

由(1)可得

$$\lim_{n\to\infty} n^2\{\int_0^1 x^n f(x)dx - [\frac{f(1)}{n} - \frac{f(1)+f'(1)}{n^2}]\} = 0$$
$$\Rightarrow \quad \int_0^1 x^n f(x)dx = \frac{f(1)}{n} - \frac{f(1)+f'(1)}{n^2} + o(\frac{1}{n^2})\,(n\to\infty).$$

7. 令$F(x) = 2\int_a^x tf(t)dt - x\int_0^x f(t)dt + a\int_0^a f(x)dx, x\in[a,b]$, 则$F(a)=0$, 因$f(x)$单调减，所以有

$$F'(x) = xf(x) - \int_0^x f(t)dt = \int_0^x [f(x)-f(t)]dt \leqslant 0,$$

故$F(x)$在$[a,b]$上单调减，$\Rightarrow F(b)\leqslant F(a)=0$,

$$\Rightarrow 2\int_a^b xf(x)dx \leqslant b\int_0^b f(x)dx + a\int_0^a f(x)dx.$$

8. 由定积分第一中值定理知存在$\xi\in(0,a)$, 满足$|f(\xi)| = \frac{1}{a}\int_0^a |f(x)|dx$. 于是由于$f(x)$在$[0,2\pi]$上连续可导, 可得

$$\begin{aligned} |f(0)| - \frac{1}{a}\int_0^a |f(x)|dx &= |f(0)| - |f(\xi)| \leqslant |f(0)-f(\xi)| = \Big|\int_0^\xi f'(x)dx\Big| \\ &\leqslant \int_0^\xi |f'(x)|dx \leqslant \int_0^a |f'(x)|dx. \end{aligned}$$

因此

$$|f(0)| \leqslant \frac{1}{a}\int_0^a |f(x)|dx + \int_0^a |f'(x)|dx.$$

9. 因$f(x)$在$[0,1]$上连续, 则可设$f(\eta) = \max\limits_{0\leqslant x\leqslant 1}|f(x)|, \eta\in[0,1]$. 由积分中值定理可知, 存在$\xi\in[0,1]$, 使得$|\int_0^1 f(x)dx| = |f(\xi)|$.
若$\xi=\eta$, 则不等式显然成立. 下设$\xi\neq\eta$, 则

$$|f(\eta)-f(\xi)| = \Big|\int_\eta^\xi f'(x)dx\Big| \leqslant \int_0^1 |f'(x)|dx,$$

即

$$|f(\xi)| \geqslant |f(\eta)| - \int_0^1 |f'(x)|dx.$$

因此

$$\Big|\int_0^1 f(x)dx\Big| \geqslant |f(\eta)| - \int_0^1 |f'(x)|dx.$$

即

$$\max_{0\leqslant x\leqslant 1} |f(x)| \leqslant \Big|\int_0^1 f(x)dx\Big| + \int_0^1 |f'(x)|dx.$$

因为对$\forall x \in [0,1]$, $|f(x)| \leqslant \max\limits_{0\leqslant x\leqslant 1} |f(x)|$, 所以

$$|f(x)| \leqslant \Big|\int_0^1 f(x)dx\Big| + \int_0^1 |f'(x)|dx \leqslant \int_0^1 [|f(x)| + |f'(x)|]dx.$$

10. 由题设$\exists x_1 \in [0, 1/2]$, 使得$f(1) - 2x_1 f(x_1)(1-1/2)$, 即$f(1) = x_1 f(x_1)$. 再对$F(x) = xf(x)$在区间$[x_1, 1]$上用Rolle定理即可.

11.设$G(u) = \int_0^u f(t)dt$, 则$f(x)$的连续性知，$G(u)$可导，且$G'(u) = f(u)$. 由复合函数求导法则，有

$$\Big(\int_0^{v(x)} f(t)dt\Big)' = G[v(x)]' = G'[v(x)]v'(x) = f[v(x)]v'(x).$$

于是，

$$F'(x) = \Big(\int_{u(x)}^{v(x)} f(t)dt\Big)' = \Big(\int_0^{v(x)} f(t)dt - \int_0^{v(x)} f(t)dt\Big)' = f[v(x)]v'(x) - f[u(x)]u'(x).$$

12. 证明: 若函数$f(x)$在$(-\infty, +\infty)$的任意有界闭区间$[\alpha, \beta]$上可积，且对$\forall x, y \in [\alpha, \beta]$, 有$f(x+y) = f(x) + f(y)$, 则$f(x) = cx, c = f(1)$.

证：$\forall x \in \mathbb{R}, x \neq 0, f(t+y) = f(t) + f(y)$, 两边对$t$从0到$x$积分，得

$$\int_0^x f(t+y)dt = \int_0^x f(t)dt + \int_0^x f(y)dt = \int_0^x f(t)dt + xf(y),$$

或

$$xf(y) = \int_0^x f(t+y)dt - \int_0^x f(t)dt.$$

令$t + y = u$, 有

$$\int_0^x f(t+y)dt = \int_y^{x+y} f(u)du = \int_0^{x+y} f(u)du - \int_0^y f(u)du,$$

$$\Rightarrow xf(y)=\int_0^{x+y}f(u)du-\int_0^y f(u)du-\int_0^x f(u)du,$$

交换x与y的位置，右端积分的代数和不变，即

$$xf(y)=yf(x)\quad \text{或}\quad \frac{f(x)}{x}=\frac{f(y)}{y}.$$

于是$\dfrac{f(x)}{x}=c$, 即$f(x)=cx$. 当$x=y=0$时，$f(0)=2f(0),\Rightarrow f(0)=0$, 上式也成立. 令$x=1,\Rightarrow c=f(1)$.

13. 作变换$t=nx$, 由定积分第一中值定理知存在$\varepsilon_k\in(2(k-1)\pi,2k\pi)$, 使

$$\begin{aligned}\int_0^{2\pi}f(x)|\sin nx|dx &= \frac{1}{n}\int_0^{2n\pi}f(\frac{x}{n})|\sin x|dx=\frac{1}{n}\sum_{k=1}^n\int_{2(k-1)\pi}^{2k\pi}f(\frac{x}{n})|\sin x|dx\\ &= \frac{1}{n}\sum_{k=1}^n f(\frac{\xi_k}{n})\int_{2(k-1)\pi}^{2k\pi}|\sin x|dx=\frac{4}{n}\sum_{k=1}^n f(\frac{\xi_k}{n})\\ &= \frac{2}{\pi}\sum_{k=1}^n f(\frac{\xi_k}{n})\frac{2\pi}{n},\end{aligned}$$

而$\displaystyle\sum_{k=1}^n f(\frac{\xi_k}{n})\frac{2\pi}{n}$是$f(x)$将$[0,2\pi]$区间$n$等分的积分和, 由于$f(x)$在$[0,2\pi]$上连续, 故

$$\lim_{n\to\infty}\sum_{k=1}^n f(\frac{\xi_k}{n})\frac{2\pi}{n}=\int_0^{2\pi}f(x)dx,$$

从而

$$\lim_{n\to\infty}\int_0^{2\pi}f(x)|\sin nx|dx=\frac{2}{\pi}\int_0^{2\pi}f(x)dx.$$

14. 证：不妨设$0<h<1$($-1<h<0$时，同法可证). 因

$$\begin{aligned}\int_{-1}^1\frac{h}{h^2+x^2}f(x)dx &= \int_{-1}^{-\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx+\int_{-\sqrt{h}}^{\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx\\ &+ \int_{\sqrt{h}}^1\frac{h}{h^2+x^2}f(x)dx \qquad (1)\end{aligned}$$

对（1）式右端第一个积分，由于$h\to 0^+$时，有

$$\Big|\int_{-1}^{-\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx\Big|\leqslant M\int_{-1}^{-\sqrt{h}}\frac{h}{h^2+x^2}dx=M(-\arctan\frac{1}{\sqrt{h}}+\arctan\frac{1}{h})\to 0,$$

故$\lim\limits_{h\to 0^+}\int_{-1}^{-\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx = 0$. 同理可得$\lim\limits_{h\to 0^+}\int_{\sqrt{h}}^{1}\frac{h}{h^2+x^2}f(x)dx = 0$. 对(1)式右端第二个积分，由积分中值定理，$\exists\xi_h\in(-\sqrt{h},\sqrt{h})$, 使得

$$\begin{aligned}\int_{-\sqrt{h}}^{\sqrt{h}}\frac{h}{h^2+x^2}f(x)dx &= f(\xi_h)\int_{-\sqrt{h}}^{\sqrt{h}}\frac{h}{h^2+x^2}dx = f(\xi_h)\arctan\frac{x}{h}\Big|_{-\sqrt{h}}^{\sqrt{h}}\\ &= f(\xi_h)\cdot 2\arctan\frac{1}{\sqrt{h}}\to\pi f(0)\quad(h\to 0^+),\end{aligned}$$

所以$\lim\limits_{h\to 0+}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)dx = \pi f(0)$.

类似可证，$\lim\limits_{h\to 0-}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)dx = \pi f(0)$. 故，原式成立.

15. 证：

$$\int_{\pi}^{n\pi}\frac{|\sin(x)|}{x}dx = \sum_{k=1}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(x)|}{x}dx,$$

$$\int_{k\pi}^{(k+1)\pi}\frac{|\sin(x)|}{x}dx \overset{x=k\pi+t}{=} \int_0^{\pi}\frac{|\sin(t)|}{k\pi+t}dt > \int_0^{\pi}\frac{\sin(t)}{(k+1)\pi}dt = \frac{2}{(k+1)\pi}.$$

又$\int_n^{n+1}\frac{dx}{x} < \int_n^{n+1}\frac{dx}{n} = \frac{1}{n}$, 于是

$$\begin{aligned}\int_{\pi}^{n\pi}\frac{|\sin(x)|}{x}dx &= \sum_{k=1}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(x)|}{x}dx > \sum_{k=1}^{n-1}\frac{2}{(k+1)\pi} = \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{k+1}\\ &> \frac{2}{\pi}\sum_{k=1}^{n-1}\int_{k+1}^{k+2}\frac{1}{x}dx = \frac{2}{\pi}\int_2^{n+1}\frac{1}{x}dx = \frac{2}{\pi}\ln\frac{n+1}{2}.\end{aligned}$$

16.提示：当$n\neq m$时，不防设$n<m$, 并记$a_n=\frac{1}{2^n n!}$. 连续应用m次分部积分公式：

$$\int_{-1}^{1}P_m(x)P_n(x)dx = a_n\int_{-1}^{1}P_m(x)d\Big(\frac{d^n-1}{dx^{n-1}}(x^2-1)^n\Big) = \cdots$$

注意，当$k\leqslant m$时，有$P_m^{(k)}(x)\frac{d^{k-1}}{dx^{k-1}}(x^2-1)^n\Big|_{-1}^{1} = 0$.

当$n=m$时，连续应用n次分部积分：

$$\int_{-1}^{1}P_n(x)P_n(x)dx = \frac{2(2n)!}{2^{2n}(n!)^2}\int_0^1(1-x^2)^n dx,$$

令$x=\cos(t)$, 有

$$\int_{-1}^{1}[P_n(x)]^2dx=\frac{2(2n)!}{2^{2n}(n!)^2}\int_{0}^{\pi/2}\sin^{2n+1}(t)dt,$$

再应用Page150，例5.2.11.

习题 5.3

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1. (1)

$$A=\int_0^2[(2x-x^2)-(2x^2-4x)]dx=3\int_0^2(2x-x^2)dx=3(x^2-\frac{x^3}{3})\Big|_0^2=4.$$

(2)

$$\begin{aligned}
A &= \int_0^{2n\pi}\Big|e^{-x}\sin x\Big|dx=\int_0^{2n\pi}e^{-x}\Big|\sin x\Big|dx \\
&= \sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}e^{-x}\sin xdx-\sum_{k=0}^{n-1}\int_{(2k+1)\pi}^{(2k+2)\pi}e^{-x}\sin xdx \\
&= \sum_{k=0}^{n-1}[-\frac{1}{2}e^{-x}(\sin x+\cos x)\Big|_{2k\pi}^{(2k+1)\pi}]+\sum_{k=0}^{n-1}[\frac{1}{2}e^{-x}(\sin x+\cos x)]\Big|_{(2k+1)\pi}^{(2k+2)\pi} \\
&= \sum_{k=0}^{n-1}\frac{1}{2}e^{-2k\pi}(1+e^{-\pi})^2=\frac{e^{\pi}+1}{2(e^{\pi}-1)}(1-e^{-2n\pi}).
\end{aligned}$$

(3) $A=\int_0^1(1-\sqrt{x})^2dx=\frac{1}{6}.$

(4) $A=\int_0^2(\sin x-\cos x)dx=\sin 1+\cos 1-\sin 2-\cos 2.$

(5) 当t由0变到2时，动点(x,y)在第一象限描绘的闭曲线围城一区域，如图1. 所求面积为

$$A=\int_0^2|y(t)x'(t)|dt=\int_0^2|(2t^2-t^3)(2-2t)|dt=\frac{8}{15}.$$

(6) 所求面积为

$$A=\frac{1}{2}\int_0^{2\pi}(xy'_t-x'_ty)dt=3a^2\int_0^{2\pi}(1-\cos t)dt=6\pi a^2.$$

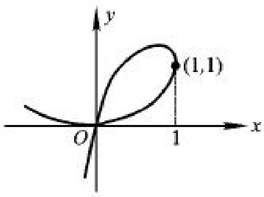

图 1

(7) 曲线为半径为$\dfrac{\alpha}{2}$的圆，面积为$A=\pi(\dfrac{\alpha}{2})^2=\dfrac{\pi\alpha^2}{4}$.

(8) 所求面积为

$$A=\frac{1}{2}\int_0^{2\pi}\frac{p^2}{(1+\varepsilon\cos\theta)^2}d\theta=\int_0^{\pi}\frac{p^2}{(1+\varepsilon\cos\theta)^2}d\theta=\frac{\pi p^2}{(1-\varepsilon^2)^{3/2}}.$$

2. 取焦点为极坐标原点，抛物线的轴为极轴. 则抛物线$y^2=4ax$的极坐标方程为

$$r=\frac{2a}{1-\cos\theta}\quad(0<\theta<2\pi).$$

设过焦点的动弦为$\theta=t$, 由对称性，不妨限定$0<t<\pi$, 则动弦与抛物线所围的面积为

$$A(t)=\int_t^{t+\pi}\frac{r^2}{2}d\theta=\frac{1}{2}\int_t^{t+\pi}\big(\frac{2a}{1-\cos\theta}\big)^2d\theta.$$

$$A'(t)=\frac{1}{2}\big[\big(\frac{2a}{1-\cos(\pi+t)}\big)^2-\big(\frac{2a}{1-\cos t}\big)^2\big]=-\frac{8a^2\cos t}{(\sin t)^4}.$$

令$A'(t)=0$, 解得唯一驻点$t=\dfrac{\pi}{2}$，即当弦与极轴垂直时，所围面积最小，最小面积为

$$A(\frac{\pi}{2})=\frac{1}{2}\int_{\pi/2}^{3\pi/2}\big(\frac{2a}{1-\cos\theta}\big)^2d\theta=\frac{8a^2}{3}.$$

3. 面积之比为：$3\pi+2:9\pi-2$.

4. 两个圆柱面围成的立体关于三个坐标面都对称，它的体积是第一卦限那部分体积的8倍，如图2.

$\forall x\in[0,a]$, 过x且垂直于x轴的平面截第一卦限那部分立体的截口是正方形，其边长是$\sqrt{a^2-x^2}$, 总体积为

$$V=8\int_0^a(\sqrt{a^2-x^2})^2dx=\frac{16a^2}{3}.$$

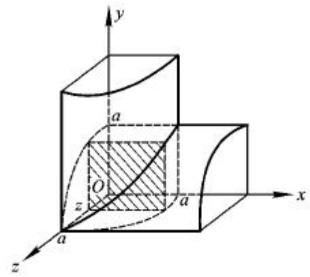

图 2

5. (1) $V=\int_0^{\pi}\pi\sin^2 xdx=\pi^2/2.$

(2) $V=\int_{-b}^{b}\pi x^2(y)dy=\pi\int_{-b}^{b}(a\sqrt{1-y^2/b^2})^2dy=\frac{4}{3}\pi a^2 b.$

(3) $V=\int_0^{\pi/2}\pi|\sin x-\cos x|^2dx=\frac{\pi^2}{2}-\pi.$

(4)$V=\frac{2\pi}{3}\int_0^{\pi}\rho^3(\theta)\sin\theta d\theta=\frac{2\pi}{3}\int_0^{\pi}\alpha^3(1+\cos\theta)^3\sin\theta d\theta=\frac{8}{3}\pi\alpha^3.$

6. 椭圆与两条切线围成的区域关于x轴对称, 因此只讨论位于第一象限那部分区域绕y轴旋转的体积再2倍即可, 如图3. 设位于第一象限内椭圆上的切点坐标

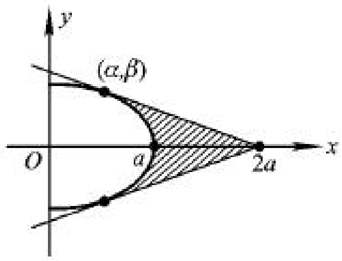

图 3

是(a,b) 由解析几何知，切线方程是

$$\frac{ax}{\alpha^2}+\frac{by}{\beta^2}=1.$$

已知切线通过点$(2\alpha,0)$, 有$\frac{2a}{\alpha}=1$与$\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}=1$, 解得切点坐标为$(\frac{\alpha}{2},\frac{\sqrt{3}\beta}{2})$. 切线

方程是

$$x = 2\alpha(1 - \frac{\sqrt{3}}{2\beta}y).$$

于是，该区域绕y轴旋转所得旋转体的体积

$$\begin{aligned}
V &= 2\pi\int_0^b 4\alpha^2\Big(1-\frac{\sqrt{3}}{2\beta}y\Big)dy - \int_0^b \alpha^2\Big(1-\frac{y^2}{\beta^2}\Big)dy \\
&= 2\alpha^2\pi\int_0^b \Big(3-\frac{4\sqrt{3}}{2\beta}y+\frac{4}{\beta^2}y^2\Big)dy \\
&= 2\alpha^2\pi\Big(3y-\frac{2\sqrt{3}}{\beta}y^2+\frac{4}{3\beta^2}y^3\Big)\Big|_0^b \quad (b=\frac{\sqrt{3}}{2}\beta) \\
&= \sqrt{3}\alpha^2\beta\pi.
\end{aligned}$$

7.(1) 曲线关于x轴对称，在x轴上方$y = x^{3/2}\ (x\in[0,1])$, 弧长为

$$l = 2\int_0^1 \sqrt{1+y'^2(x)}dx = 2\int_0^1 \sqrt{1+\frac{9}{4}x}dx = \frac{26\sqrt{13}-16}{27}.$$

(2) 原题曲线改为$y = \ln\frac{e^x+1}{e^x-1}, x\in[0,2]$, 则$y' = \frac{-2e^x}{e^{2x}-1}$, 弧长为

$$\begin{aligned}
l &= \int_1^2 \sqrt{1+y'^2(x)}dx = \int_1^2 \sqrt{1+\big(\frac{-2e^x}{e^{2x}-1}\big)^2}dx \\
&= \ln(e^2+1)-1.
\end{aligned}$$

(3) $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$, $x' = at\cos t, y' = a\sin t$, 弧长为

$$\begin{aligned}
l &= \int_0^{2\pi} \sqrt{x'^2(t)+y'^2(t)}dt = \int_0^{2\pi} \sqrt{a^2t^2\cos^2 t + a^2t^2\sin t}dt \\
&= a\int_0^{2\pi} tdt = 2\pi^2 a.
\end{aligned}$$

(4) $l = \int_0^3 \sqrt{x'^2(t)+y'^2(t)}dt = \int_0^3 \sqrt{(6t)^2+(3-3t)^2}dt = 36.$

(5) $r' = a\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}$. 曲线的弧长

$$\begin{aligned}
l &= \int_0^{3\pi} \sqrt{r^2+r'^2}d\theta = \int_0^{3\pi} \sqrt{a^2\sin^6\frac{\theta}{3}+a^2\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}}d\theta \\
&= a\int_0^{3\pi} \sin^3\frac{\theta}{3}d\theta = \frac{3\pi a}{2}.
\end{aligned}$$

(6) 由$\theta=\frac{1}{2}(\rho+\frac{1}{\rho})$ 得：$\rho^2-2\theta\rho+1=0$, 两边对θ求导,得: $2\rho\rho'-2\theta\rho'-2\rho=0$, 解得: $\rho'=\frac{\rho}{\rho-\theta}$. 从而$\sqrt{\rho^2+\rho'^2}=\frac{\rho\theta}{\rho-\theta}=\frac{\rho^3+\rho}{\rho^2-1}$, 又$d\theta=\frac{1}{2}(1-\frac{1}{\rho^2})d\rho$, 所以曲线的弧长为

$$l=\int_{\theta(1)}^{\theta(3)}\sqrt{\rho^2+\rho'^2}d\theta\overset{\theta=\theta(\rho)}{=}\int_1^3\frac{\rho^3+\rho}{\rho^2-1}\cdot\frac{1}{2}(1-\frac{1}{\rho^2})d\rho=\frac{1}{2}\int_1^3(\rho+\frac{1}{\rho})d\rho=2+\frac{\sqrt{3}}{2}.$$

8.（1）

$$\begin{aligned}S&=2\pi\int_0^{\pi/4}\tan x\sqrt{1+(\sec^2x)^2}dx=\pi\int_0^{\pi/4}\sqrt{1+\cos^4x}d(\frac{1}{\cos^2x})\\&=\pi[\frac{\sqrt{1+\cos^4x}}{\cos^2x}-\ln(\cos^2x+\sqrt{\cos^4x+1}]_0^{\pi/4}\\&=\pi[\sqrt{5}-\sqrt{2}+\ln\frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2}].\end{aligned}$$

（2）将原题改为：曲线$a^2y=x^3,x\in[0,a]$绕x轴. 则$y=x^3/a^2,y'=3x^2/a^2$, 所求面积为

$$S=2\pi\int_0^a\frac{x^3}{a^2}\sqrt{1+(\frac{3x^2}{a^2})^2}dx=\frac{\pi a^2}{27}\Big(1+\frac{9x^4}{a^4}\Big)^{3/2}\Big|_0^a=\frac{\pi a^2}{27}(10^{3/2}-1).$$

（3）曲线为星形线，将其化为直角坐标方程为：$x^{2/3}+y^{2/3}=a^{2/3}$, 则有

$$y'=-\sqrt{\frac{y}{x}},\quad\sqrt{1+y'^2}=(a/x)^{1/3},$$

曲线关于两个坐标轴都对称，由对称性，所求面积为

$$S=2\cdot2\pi\int_0^a y(x)\sqrt{1+y'^2}dx=2\cdot2\pi\int_0^a(a^{2/3}-x^{2/3})^{3/2}(a/x)^{1/3}dx=\frac{12\pi a^2}{5}.$$

（4）曲线是心形线，其参数方程为：

$$x(\theta)=\alpha(1+\cos\theta)\cos\theta,\ y(\theta)=\alpha(1+\cos\theta)\sin\theta\,(\theta\in[0,2\pi])$$

所求面积为

$$\begin{aligned}S&=2\pi\int_0^\pi y(\theta)\sqrt{x'^2(\theta)+y'^2(\theta)}d\theta\\&=2\pi\int_0^\pi\alpha(1+\cos\theta)\sin\theta\cdot2\alpha\cos\frac{\theta}{2}d\theta\\&=2\pi\int_0^\pi8\alpha^2\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}d\theta=\frac{32\pi a^2}{5}.\end{aligned}$$

习题 5.4

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1. 计算下列无穷积分:

(1) $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\ \mathrm{d}x$;

(2) $\displaystyle\int_0^{+\infty} \frac{x}{1+x^4}\ \mathrm{d}x$;

(3) $\displaystyle\int_0^{+\infty} e^{-\alpha x}\sin\beta x\ \mathrm{d}x$;

(4) $\displaystyle\int_0^{+\infty} e^{-x}x^n\ \mathrm{d}x$.

解: (1) $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\ \mathrm{d}x = \lim_{A\to+\infty}\int_1^{A} \frac{1}{x^2}\ \mathrm{d}x = 1$.

(2) $\displaystyle\int_0^{+\infty} \frac{x}{1+x^4}\ \mathrm{d}x = \lim_{A\to+\infty}\int_0^{A} \frac{x}{1+x^4}\ \mathrm{d}x = \lim_{A\to+\infty}\int_0^{A} \frac{1}{2(1+x^4)}\ \mathrm{d}(x^2) = \lim_{A\to+\infty}\frac{1}{2}\int_0^{A^2}\frac{1}{1+t^2}dt = \lim_{A\to+\infty}\frac{1}{2}\arctan t\Big|_0^{A^2} = \frac{\pi}{4}$.

(3) $\displaystyle\int_0^{+\infty} e^{-\alpha x}\sin\beta x\ \mathrm{d}x = \frac{e^{-\alpha x}}{\alpha^2+\beta^2}(-\alpha\sin\beta x-\beta\cos\beta x)\Big|_0^{+\infty} = \frac{\beta}{\alpha^2+\beta^2}$.

(4) $\displaystyle\int_0^{+\infty} e^{-x}x^n\ \mathrm{d}x = -x^ne^{-x}\Big|_0^{+\infty} + n\int_0^{+\infty} x^{n-1}e^{-x}\ \mathrm{d}x = 0 - nx^{n-1}e^{-x}\Big|_0^{+\infty} + n(n-1)\int_0^{+\infty} x^{n-2}e^{-x}\ \mathrm{d}x = n(n-1)\int_0^{+\infty} x^{n-2}e^{-x}\ \mathrm{d}x = \ldots = n!\int_0^{+\infty} e^{-x}\ \mathrm{d}x = n!$.

2. 判别下列无穷积分的敛散性：

(1) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt[3]{1+x^4}}\ \mathrm{d}x$;

(2) $\displaystyle\int_1^{+\infty} \frac{x\arctan x}{1+x^3}\ \mathrm{d}x$;

(3) $\displaystyle\int_1^{+\infty} \sin\frac{1}{x^2}\ \mathrm{d}x$;

(4) $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x$.

解：(1) $\displaystyle\lim_{x\to+\infty} x^{\frac{4}{3}}\frac{1}{\sqrt[3]{1+x^4}} = 1, p=\frac{4}{3}, \lambda=1$，故$\displaystyle\int_0^{+\infty} \frac{1}{\sqrt[3]{1+x^4}}\ \mathrm{d}x$ 收敛.

(2) 在$[1,+\infty)$ 上，$\displaystyle\frac{x\arctan x}{1+x^3} \leqslant \frac{\frac{\pi}{2}x}{1+x^3} \leqslant \frac{\frac{\pi}{2}}{x^2}$, 又$\displaystyle\int_1^{+\infty}\frac{\pi}{2}\frac{1}{x^2}\ \mathrm{d}x = \frac{\pi}{2}\int_1^{+\infty}\frac{1}{x^2}\ \mathrm{d}x = \frac{\pi}{2}$. 由比较法则知，$\displaystyle\int_1^{+\infty}\frac{\arctan x}{1+x^3}\ \mathrm{d}x$ 收敛.

(3) $\displaystyle\lim_{x\to+\infty}\frac{\sin\frac{1}{x^2}}{\frac{1}{x^2}} = 1$, 且$\displaystyle\int_1^{+\infty}\frac{1}{x^2}$ 收敛，所以$\displaystyle\int_1^{+\infty}\sin\frac{1}{x^2}\ \mathrm{d}x$ 收敛.

(4)

$$\int_{-\infty}^{+\infty} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x = \int_{-\infty}^{-1} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x \int_{-1}^{1} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x + \int_{1}^{+\infty} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x,$$

由于$\lim\limits_{x\to-\infty} x^2\dfrac{x}{e^x+e^{-x}} = 0$, 且$\displaystyle\int_{-\infty}^{-1} \frac{1}{x^2}$ 收敛, 所以$\displaystyle\int_{-\infty}^{-1} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x$ 收敛，定积分$\displaystyle\int_{-1}^{1} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x = 0$; 又因为$\lim\limits_{x\to+\infty} \dfrac{x}{e^x+e^{-x}}x^2 = 0$, 且$\displaystyle\int_{1}^{+\infty} \frac{1}{x^2}\ \mathrm{d}x$ 收敛，所以$\displaystyle\int_{1}^{+\infty} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x$ 收敛，故$\displaystyle\int_{-\infty}^{+\infty} \frac{x}{e^x+e^{-x}}\ \mathrm{d}x$ 收敛.

3. 证明：无穷积分$\displaystyle\int_{1}^{+\infty} \frac{\sin x}{x^p}\ \mathrm{d}x$ 与$\displaystyle\int_{1}^{+\infty} \frac{\cos x}{x^p}\ \mathrm{d}x$在$p\in(0,1)$ 时是条件收敛的.

证：先证$\displaystyle\int_{1}^{+\infty} \frac{\sin x}{x^p}\ \mathrm{d}x$ 和$\displaystyle\int_{1}^{+\infty} \frac{\cos x}{x^p}\ \mathrm{d}x$ 收敛. 由于对任意$A>1$, 有$|F(A)| = \left|\displaystyle\int_{1}^{A} \sin x\,\mathrm{d}x\right| \leqslant 2$, 且$\dfrac{1}{x^p}$ 当$x\to+\infty$ 时, 单调趋于0. 由狄利克雷判别法知, $\displaystyle\int_{1}^{+\infty} \frac{\sin x}{x^p}\ \mathrm{d}x$ 收敛. 同理可得, $\displaystyle\int_{1}^{+\infty} \frac{\cos x}{x^p}\ \mathrm{d}x$收敛. 再证$\displaystyle\int_{1}^{+\infty} \left|\frac{\sin x}{x^p}\right|\ \mathrm{d}x$和$\displaystyle\int_{1}^{+\infty} \left|\frac{\cos x}{x^p}\right|\ \mathrm{d}x$ 发散. 由于对任意$x\in[1,+\infty)$,

$$\left|\frac{\sin x}{x^p}\right| \geqslant \frac{\sin^2 x}{x^p} = \frac{1}{2x^p} - \frac{\cos 2x}{2x^p}, \quad \left|\frac{\cos x}{x^p}\right| \geqslant \frac{\cos^2 x}{x^p} = \frac{1}{2x^p} + \frac{\cos 2x}{2x^p};$$

又$p\in(0,1)$, $\displaystyle\int_{1}^{+\infty} \frac{1}{2x^p}\ \mathrm{d}x$ 发散，由前面的证明可知, $\displaystyle\int_{1}^{+\infty} \frac{\cos 2x}{2x^p}\ \mathrm{d}x$ 收敛，则由比较判别法知, $\displaystyle\int_{1}^{+\infty} \left|\frac{\sin x}{x^p}\right|\ \mathrm{d}x$和$\displaystyle\int_{1}^{+\infty} \left|\frac{\cos x}{x^p}\right|\ \mathrm{d}x$均发散. 综上所述，$\displaystyle\int_{1}^{+\infty} \frac{\sin x}{x^p}\ \mathrm{d}x$ 与$\displaystyle\int_{1}^{+\infty} \frac{\cos x}{x^p}\ \mathrm{d}x$ 在$p\in(0,1)$ 时是条件收敛的.

4. 证明：若无穷积分$\displaystyle\int_{a}^{+\infty} f(x)\mathrm{d}x$ 绝对收敛，且$\lim\limits_{x\to+\infty} f(x) = 0$，则$\displaystyle\int_{a}^{+\infty} f^2(x)\mathrm{d}x$ 收敛.

证：由$\lim\limits_{x\to+\infty} |f(x)| = 0$可知, 存在$A>a$, 对任意$x\in[A,+\infty), 0\leqslant|f(x)|\leqslant 1$, 此时有$0\leqslant f^2(x)\leqslant|f(x)|$. 由$\displaystyle\int_{a}^{+\infty} f(x)\mathrm{d}x$ 绝对收敛，得到$\displaystyle\int_{A}^{+\infty} f^2(x)\mathrm{d}x$ 收敛. 又由于$\displaystyle\int_{a}^{+\infty} f^2(x)\mathrm{d}x = \int_{a}^{A} f^2(x)\mathrm{d}x + \int_{A}^{+\infty} f^2(x)\mathrm{d}x$, 而$\int_a^A f^2(x)\ \mathrm{d}x$ 为定积分，故$\displaystyle\int_{a}^{+\infty} f^2(x)\mathrm{d}x$ 收敛.

5. 证明：若函数$f(x)$在$[0,+\infty)$ 上一致连续，且无穷积分$\int_0^{+\infty} f(x)\ \mathrm{d}x$ 收敛，则

$$\lim_{x\to+\infty} f(x)=0$$

.

证：用反证法. 若在所给条件下$\lim\limits_{x\to+\infty} f(x)\neq 0$, 则存在$\varepsilon_0>0$, 对于任意给定的$X>0$, 存在$x_0>X$, 使得$|f(x_0)|\geqslant\varepsilon_0$. 又因为$f(x)$在$[0,+\infty)$上一致连续，所以存在$\delta_0\in(0,1)$, 使得对任意的$x_1,x_2\in(0,+\infty)$, 当$|x_1-x_2|<\delta_0$时，有$|f(x_1)-f(x_2)|<\dfrac{\varepsilon}{2}$. 对任意给定的$A_0\geqslant 0$, 取$X=A_0+1$, 并设$x_0>X$满足$|f(x_0)|\geqslant\varepsilon_0$. 不妨设$f(x_0)>0$, 则对满足$|x-x_0|\leqslant\delta_0$的任意$x$, 有

$$f(x)>f(x_0)-\frac{\varepsilon_0}{2}\geqslant\frac{\varepsilon_0}{2}>0.$$

取$A_1=x_0-\frac{\varepsilon_0}{2}$, $A_2=x_0+\frac{\varepsilon_0}{2}$, 则$A_2>A_1>A_0$, 且有

$$\Big|\int_{A_1}^{A_2} f(x)dx\Big|>\frac{\varepsilon_0}{2}\delta_0>0.$$

由Cauchy收敛准则, $\int_0^\infty f(x)dx$不收敛, 与已知矛盾. 故$\lim\limits_{x\to+\infty} f(x)=0$.

6. 证明：若函数$f(x)$ 在$[a,+\infty)$ 上连续可微，且无穷积分$\int_a^{+\infty} f(x)\,\mathrm{d}x$ 与$\int_a^{+\infty} f'(x)\,\mathrm{d}x$ 都收敛，则$\lim\limits_{x\to+\infty} f(x)=0$.

证：因为$\int_a^{+\infty} f'(x)\ \mathrm{d}x$ 收敛，所以对任意$\varepsilon>0$，存在$M>0$，当$x_1,x_2>M$ 时，有$|\int_{x_1}^{x_2} f'(x)\ \mathrm{d}x|<\varepsilon$, 即$|f(x_1)-f(x_2)|<\varepsilon$，所以$\lim\limits_{x\to+\infty} f(x)$ 存在. 若$\lim\limits_{x\to+\infty} f(x)=A\neq 0$ ，不妨设$A>0$ 则对$\varepsilon=\frac{A}{2}$ ，有M 存在，当$x>M$ 时，有$f(x)\geqslant\frac{A}{2}$ ，所以$\int_M^{+\infty} f(x)\ \mathrm{d}x\geqslant\int_M^{+\infty}\dfrac{A}{2}\ \mathrm{d}x=+\infty$, 于是$\int_M^{+\infty} f(x)\ \mathrm{d}x$ 发散, 从而$\int_a^{+\infty} f(x)\ \mathrm{d}x$ 发散. 与题设矛盾. 故$\lim\limits_{x\to+\infty} f(x)=0$.

注: 由$\lim\limits_{x\to+\infty} f(x)$存在可知, $f(x)$在$[a,+\infty)$上一致连续, 再由上题也可得证结论.

7. 证：若无穷积分$\int_a^{+\infty} f(x)\ \mathrm{d}x$ 收敛，且$xf(x)$ 在$[a,+\infty)$ 上单调减少，则

$$\lim_{x\to+\infty} xf(x)\ln x=0$$

.

证: 若$a<1$，则$\int_a^{+\infty} f(x)\,\mathrm{d}x=\int_a^1 f(x)\,\mathrm{d}x+\int_1^{+\infty} f(x)\,\mathrm{d}x$ 则$\int_a^1 f(x)\,\mathrm{d}x$ 为定积分，只需考虑$\int_1^{+\infty} f(x)\,\mathrm{d}x$ ，故不妨设$a\geqslant 1$. 于是, 对任意$x\geqslant a, xf(x)\geqslant 0$. 否则, 存在$x_0\geqslant a$, 使得$x_0 f(x_0)=c<0$. 由$xf(x)$ 单调递减，对任意$x\geqslant x_0$, 有$xf(x)\leqslant c<0$,可得$f(x)\leqslant \frac{c}{x}$, 所以$\int_{x_0}^{+\infty} f(x)\,\mathrm{d}x\leqslant c\int_{x_0}^{+\infty}\frac{1}{x}\,\mathrm{d}x=-\infty$, 与题设矛盾. 由广义积分收敛柯西准则，对任意$\varepsilon>0$，存在$A>a\geqslant 1$, 对任意$x>\sqrt{x}>A$，有$|\int_{\sqrt{x}}^{x} f(t)dt|<\varepsilon$, 而$\left|\int_{\sqrt{x}}^{x} f(t)dt\right|\geqslant xf(x)\left|\int_{\sqrt{x}}^{x}\frac{1}{t}dt\right|=\frac{1}{2}xf(x)\ln x$, 所以$\lim\limits_{x\to+\infty} xf(x)\ln x=0$.

8. 证明: 若无穷积分$\int_a^{+\infty} f(x)\,\mathrm{d}x$ 收敛，且f 在$[a,+\infty)$ 上单调，则$\lim\limits_{x\to+\infty} xf(x)=0$.
证：不妨设$f(x)$单减的, 则$f(x)$非负. 否则, 存在x_0, 使得$f(x_0)<0$, 于是对任意$x>x_0$, 有$f(x)\leqslant f(x_0)<0$, 从而$\int_{x_0}^{+\infty} f(x)\,\mathrm{d}x\leqslant\int_{x_0}^{+\infty} f(x_0)\,\mathrm{d}x$，这与$\int_a^{+\infty} f(x)\,\mathrm{d}x$收敛矛盾. 因为$\int_a^{+\infty} f(x)\,\mathrm{d}x$收敛，所以对任意$\varepsilon>0$, 存在$M>a$, 对任意$A_1,A_2>M$，有$|\int_{A_1}^{A_2} f(x)\,\mathrm{d}x|<\varepsilon$. 取$A_1=x/2, A_2=x$，则当$x>2M$ 时，有$0\leqslant xf(x)/2\leqslant\int_{x/2}^{x} f(t)dt<\varepsilon$, 所以$\lim\limits_{x\to+\infty} xf(x)=0$.

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1. 计算下列积分：

(1) $\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$;

(2) $\int_0^1\frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x$;

(3) $\int_\alpha^\beta\frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,\mathrm{d}x$;

解: (1) $\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\int_{-1}^{0}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x+\int_0^1\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\lim\limits_{a\to-1^+}\int_a^0\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x+\lim\limits_{b\to1^-}\int_0^b\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\lim\limits_{a\to-1^+}\arcsin x|_a^0+\lim\limits_{b\to1^-}\arcsin x|_0^b=0-(-\frac{\pi}{2})+\frac{\pi}{2}-0=\pi$.

(2) $\int_0^1\frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x=\lim\limits_{a\to1^-}\int_0^a\frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x=\lim\limits_{a\to1^-}\int_0^a(-\sqrt{1-x^2}+\frac{1}{\sqrt{1-x^2}})\,\mathrm{d}x=-(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta|_0^{\frac{\pi}{2}})+\frac{\pi}{2}=-\frac{\pi}{4}+\frac{\pi}{2}=\frac{\pi}{4}$.

(3) 对任意$x\in(\alpha,\beta)$, 有$0<\frac{x-\alpha}{\beta-\alpha}<1, 0<\frac{\beta-x}{\beta-\alpha}<1$. 令$\frac{x-\alpha}{\beta-\alpha}=\sin^2 t$,

则$x=\alpha+(\beta-\alpha)sin^2 t=\cos^2 t+\beta\sin^2 t$, $\mathrm{d}x=2(\beta-\alpha)\sin t\cos t dt$, 所以

$$\int_\alpha^\beta \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\ \mathrm{d}x=2\int_0^{\frac{\pi}{2}} dt=\pi.$$

2.判别下列瑕积分的敛散性:

(1) $\int_0^1 \frac{1}{\sqrt{x}\ln x}\ \mathrm{d}x$;

(2) $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin\theta}}\ \mathrm{d}\theta$;

(3) $\int_0^1 \frac{1}{\sqrt[3]{x^2(1-x)}}\ \mathrm{d}x$;

(4) $\int_0^1 \frac{\ln x}{x^2-1}\ \mathrm{d}x$.

解: (1) 瑕点有$x=0, x=1$, 又$\int_0^1 \frac{1}{\sqrt{x}\ln x}\ \mathrm{d}x=\int_0^A \frac{1}{\sqrt{x}\ln x}\ \mathrm{d}x+\int_A^1 \frac{1}{\sqrt{x}\ln x}\ \mathrm{d}x(A\in(0,1))$. 由于$\lim\limits_{x\to 0^+} x^{\frac{1}{2}}\frac{1}{\sqrt{x}\ln x}=0$, 即$\int_0^A \frac{1}{\sqrt{x}\ln x}\ \mathrm{d}x$ 收敛, 而$\lim\limits_{x\to 1^-}(x-1)\frac{1}{\sqrt{x}\ln x}=1$, 即$\int_A^1 \frac{1}{\sqrt{x}\ln x}\ \mathrm{d}x$ 发散, 所以$\int_0^1 \frac{1}{\sqrt{x}\ln x}\ \mathrm{d}x$ 发散.

(2) 令$t=\sin\theta$, $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin\theta}}\ \mathrm{d}\theta=\int_0^1 \frac{1}{(1-t)\sqrt{1+t}}\ \mathrm{d}t$, $t=1$为其瑕点，由于$\lim\limits_{t\to 1^-}(1-t)\frac{1}{(1-t)\sqrt{1+t}}=\frac{\sqrt{2}}{2}$, 即$\int_0^1 \frac{1}{(1-t)\sqrt{1+t}}\ \mathrm{d}t$ 发散，所以$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin\theta}}\ \mathrm{d}x$ 发散.

(3) $x=0, x=1$ 都是瑕点, 对$A\in(0,1)$, 考虑

$$\int_0^1 \frac{1}{\sqrt[3]{x^2(1-x)}}\ \mathrm{d}x=\int_0^A \frac{1}{\sqrt[3]{x^2(1-x)}}\ \mathrm{d}x+\int_A^1 \frac{1}{\sqrt[3]{x^2(1-x)}}\ \mathrm{d}x.$$

由于$\lim\limits_{x\to 0^+} x^{\frac{2}{3}}\frac{1}{\sqrt[3]{x^2(1-x)}}=1$, 即$\int_0^A \frac{1}{\sqrt[3]{x^2(1-x)}}\ \mathrm{d}x$ 收敛; $\lim\limits_{x\to 1^-}(1-x)^{\frac{2}{3}}\frac{1}{\sqrt[3]{x^2(1-x)}}=1$, 即$\int_A^1 \frac{1}{\sqrt[3]{x^2(1-x)}}\ \mathrm{d}x$ 发散, 所以$\int_0^1 \frac{1}{\sqrt[3]{x^2(1-x)}}\ \mathrm{d}x$ 发散.

(4) $x=0$ 为其瑕点, $x=1$ 不是其瑕点. 由于$\lim\limits_{x\to 0^+}\sqrt{x}\cdot\frac{\ln x}{x^2-1}=0$, 所以$\int_0^1 \frac{\ln x}{x^2-1}\ \mathrm{d}x$收敛.

3. 求下列函数:

$\Gamma(\alpha)=\int_0^{+\infty} x^{\alpha-1}e^{-x}\ \mathrm{d}x$ 与$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\ \mathrm{d}x$

的定义域.

解：(1) 反常积分$\int_0^{\infty} x^{\alpha-1}e^{-x}\mathrm{d}x$是带有奇点$0(\alpha<1)$的无穷积分. 为了研究它的收敛性，需要把它分成两个积分，即

$$\int_0^{+\infty} x^{\alpha-1}e^{-x}\mathrm{d}x=\int_0^{1} x^{\alpha-1}e^{-x}\mathrm{d}x+\int_1^{+\infty} x^{\alpha-1}e^{-x}\mathrm{d}x$$

其中右端第一个是奇异积分$(\alpha<1)$，而第二个是无穷积分. 对于右端第一个积分，因为

$$0\leqslant x^{\alpha-1}e^{-x}=\frac{1}{x^{1-\alpha}}\ (0<x\leqslant 1)$$

根据柯西判别法，当$\alpha>0$时，积分$\int_0^1 x^{\alpha-1}e^{-x}\mathrm{d}x$收敛；对于右端第二个积分，根据不等式

$$e^x\geqslant 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\geqslant\ \frac{x^n}{n!}\quad(x\geqslant 0)$$

只要取正整数$n>\alpha$，则有

$$0\leqslant x^{\alpha-1}e^{-x}=\frac{x^{\alpha-1}}{e^x}\leqslant\frac{n!}{x^{1+n-\alpha}}\ (1\leqslant x<+\infty)$$

根据柯西判别法，积分$\int_1^{+\infty} x^{\alpha-1}e^{-x}\mathrm{d}x$收敛. 因此，积分$\int_0^{+\infty} x^{\alpha-1}e^{-x}\mathrm{d}x$对任意$\alpha>0$都收敛. 当$\alpha\leqslant 0$时，在积分$\int_0^1 x^{\alpha-1}e^{-x}\mathrm{d}x$中，由于

$$x^{\alpha-1}e^{-x}=\frac{e^{-x}}{x^{1-\alpha}}\geqslant\frac{e^{-1}}{x^{1-\alpha}}(\text{注意}:1-\alpha\geqslant 1)$$

根据柯西判别法，奇异积分$\int_0^1 x^{\alpha-1}e^{-x}\mathrm{d}x$发散，因此，积分$\int_0^{+\infty} x^{\alpha-1}e^{-x}\mathrm{d}x$也发散. 综上所述，定义域为$\alpha>0$.

(2)在积分$\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x$中有两个参数$p$和$q$. 因为点0和点1都有可能是奇点，所以要把它分成两个积分来讨论它的收敛性，即

$$\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x=\int_0^{1/2} x^{p-1}(1-x)^{q-1}\mathrm{d}x+\int_{1/2}^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x\quad(*)$$

当$p>0$且$q>0$时，在右端第一个积分中（0可能是奇点），因为（注意，$0<x\leqslant 1/2$）

$$0\leqslant x^{p-1}(1-x)^{q-1}=\frac{(1-x)^{q-1}}{x^{1-p}}\leqslant\begin{cases}\dfrac{1}{x^{1-p}}(q\geqslant 1)\\ \dfrac{2^{1-q}}{x^{1-p}}(0<q<1)\end{cases}$$

根据柯西判别法，所以右端第一个积分收敛；在右端第二个积分中（1可能是奇点），因为（注意，$1/2\leqslant x<1$）

$$0\leqslant x^{p-1}(1-x)^{q-1}=\frac{x^{p-1}}{(1-x)^{1-q}}\leqslant\begin{cases}\dfrac{1}{(1-x)^{1-q}}(p\geqslant 1)\\ \dfrac{2^{1-p}}{(1-x)^{1-q}}(0<p<1)\end{cases}$$

根据柯西判别法，所以右端第二个积分也收敛.

当$p \leqslant 0$时，在式(*)右端第一个积分中（0是奇点），因为

$$x^{p-1}(1-x)^{q-1}=\frac{(1-x)^{q-1}}{x^{1-p}} \geqslant \begin{cases} \dfrac{2^{1-q}}{x^{1-p}}(q \geqslant 1) \\ \dfrac{1}{x^{1-p}}(q < 1) \end{cases}$$

所以右端第一个积分发散；同理，当$q \leqslant 0$时，右端第二个积分也发散.

综合以上结果: 当$p > 0$且$q > 0$时，积分

$$\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x$$

收敛；而当$p \leqslant 0$或$q \leqslant 0$时，积分

$$\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x$$

发散. 故, 定义域为$p > 0, q > 0$.

4.证明：

(1)设函数$f(x)$在$[0,+\infty)$上连续，且$\lim\limits_{x \to +\infty} f(x)=k$，则

$$\int_0^{+\infty} \frac{f(\alpha x)-f(\beta x)}{x}\ \mathrm{d}x=[f(0)-k]\ln\frac{\beta}{\alpha}(\beta > \alpha > 0).$$

(2)若上述条件$\lim\limits_{x \to +\infty} f(x)=k$ 改为$\displaystyle\int_0^{+\infty} \frac{f(x)}{x}\ \mathrm{d}x$存在，则

$$\int_0^{+\infty} \frac{f(\alpha x)-f(\beta x)}{x}\ \mathrm{d}x=f(0)\ln\frac{\beta}{\alpha}(\beta > \alpha > 0).$$

证明：(1)

$$
\begin{aligned}
\text{左边} &= \lim_{m\to0^+,M\to+\infty}\int_m^M \frac{f(\alpha x)-f(\beta x)}{x}\,\mathrm{d}x\\
&= \lim_{m\to0^+,M\to+\infty}\int_m^M \frac{f(\alpha x)}{x}\,\mathrm{d}x-\int_m^M \frac{f(\beta x)}{x}\,\mathrm{d}x\\
&= \lim_{m\to0^+,M\to+\infty}\int_{m\alpha}^{M\alpha} \frac{f(x)}{x}\,\mathrm{d}x-\int_{m\beta}^{M\beta} \frac{f(x)}{x}\,\mathrm{d}x\\
&= \lim_{m\to0}\int_{m\alpha}^{M\alpha} \frac{f(x)}{x}\,\mathrm{d}x-\lim_{M\to+\infty}\int_{m\beta}^{M\beta} \frac{f(x)}{x}\\
&= \lim_{m\to0} f(m\alpha+\theta_1(m\beta-m\alpha))\ln\frac{\beta}{\alpha}\\
&\quad -\lim_{M\to+\infty} f(M\alpha+\theta_2(M\beta-M\alpha))\ln\frac{\beta}{\alpha}\quad (0<\theta_1<1,0<\theta_2<1)\\
&= [f(0)-k]\ln\frac{\beta}{\alpha}.
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_0^{+\infty}\frac{f(\alpha x)-f(\beta x)}{x}\,\mathrm{d}x &= \lim_{m\to0}\int_m^{+\infty}\frac{f(\alpha x)-f(\beta x)}{x}\,\mathrm{d}x\\
&= \lim_{m\to0}\int_m^{+\infty}\frac{f(\alpha x)}{x}\,\mathrm{d}x-\int_m^{+\infty}\frac{f(\beta x)}{x}\,\mathrm{d}x\\
&= \lim_{m\to0}\int_{m\alpha}^{+\infty}\frac{f(x)}{x}\,\mathrm{d}x-\int_{m\beta}^{+\infty}\frac{f(x)}{x}\,\mathrm{d}x=\lim_{m\to0}\int_{m\alpha}^{m\beta}\frac{f(x)}{x}\,\mathrm{d}x\\
&= \lim_{m\to0} f(\xi)\ln\frac{\beta}{\alpha}\,(m\alpha<\xi<m\beta)\\
&= f(0)\ln\frac{\beta}{\alpha}.
\end{aligned}
$$