# Narrow-Band Random Processes

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1. The Definition of Band
2. An introduction of Hilbert Transform
3. Representation of Narrow-Band Signals
4. Narrow Band Random Processes
5. Gaussian Narrow-Band Random Processes
6. Sine Wave Plus Narrow-Band Noise

# 1. The Definition of Band

- In most cases, the signal we deal with is always band limited
- Band limited means that the band of a signal is finite.
- How can we define the bandwidth of a signal?
- Here, we just list some criterions for defining bandwidth

# 1.1 For Ideal Band Pass Signals

- Nonzero in a finite interval
  
  bandwidth=f2-f1
- Absolute bandwidth (cutoff bandwidth)
  
  bandwidth=f2-fc, fc is the carrier frequency

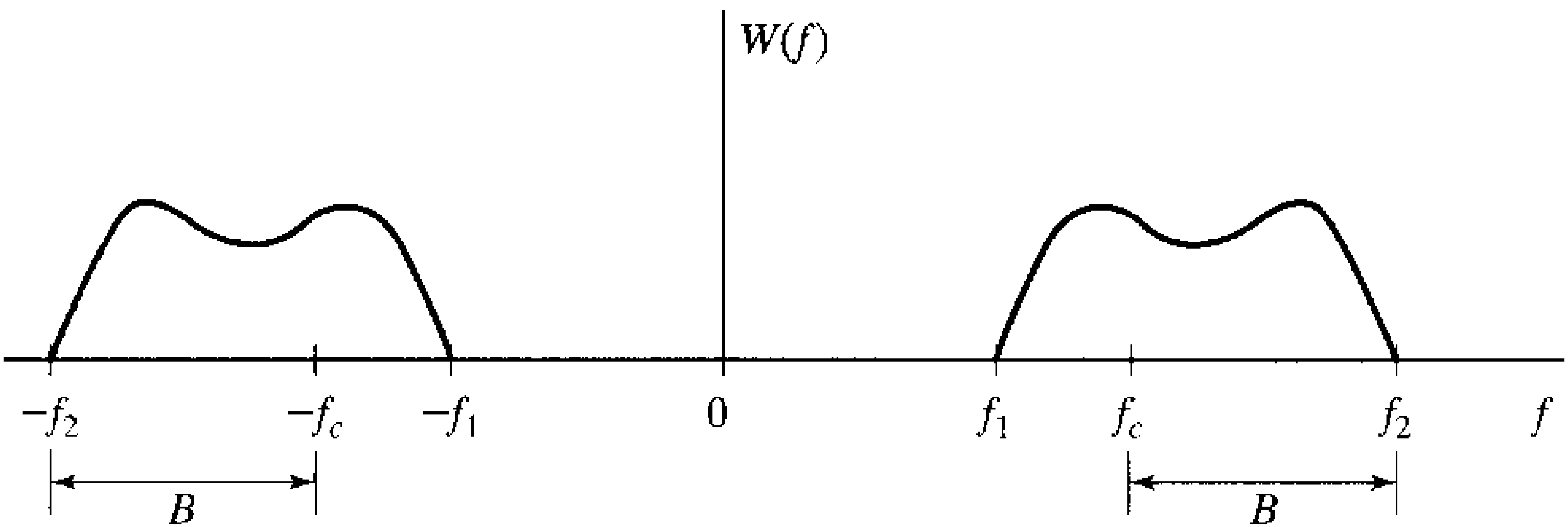


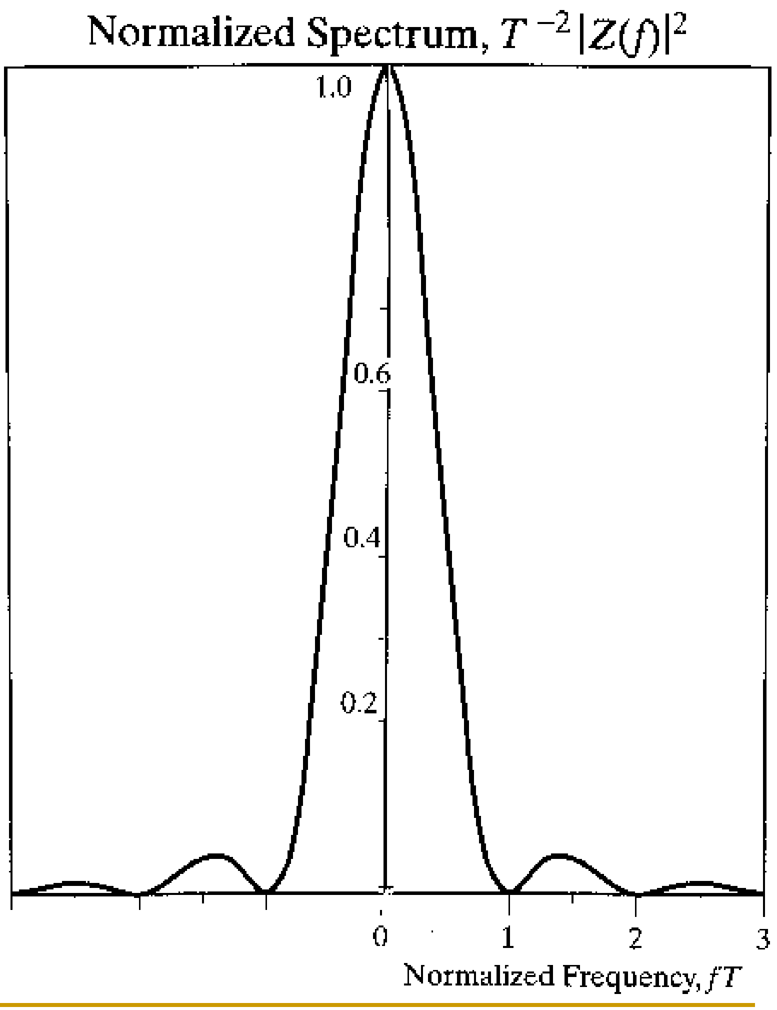

**FIGURE 4–3** An ideal band-pass frequency function.

# 1.2 For Nonideal Band Pass Signals

- Nonzero in infinite interval
- But the energy concentrates in a small interval
- Null-to-null bandwidth

  Normalized bandwidth

  =1-(-1)=2



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# 2 An introduction to Hilbert Transform(H.T.)

- Def. of Hilbert Transform

$$\begin{cases} \hat{g}(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty} \dfrac{g(\tau)}{t-\tau} d\tau = \dfrac{1}{\pi t} * g(t) \\ g(t) = -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty} \dfrac{\hat{g}(\tau)}{t-\tau} d\tau = -\dfrac{1}{\pi t} * \hat{g}(t) \end{cases}$$

- The functions above are Hilbert Transform pairs
- Hilbert Transform is an operation denoted by H[*]

# 2.1 The use of Hilbert Transform

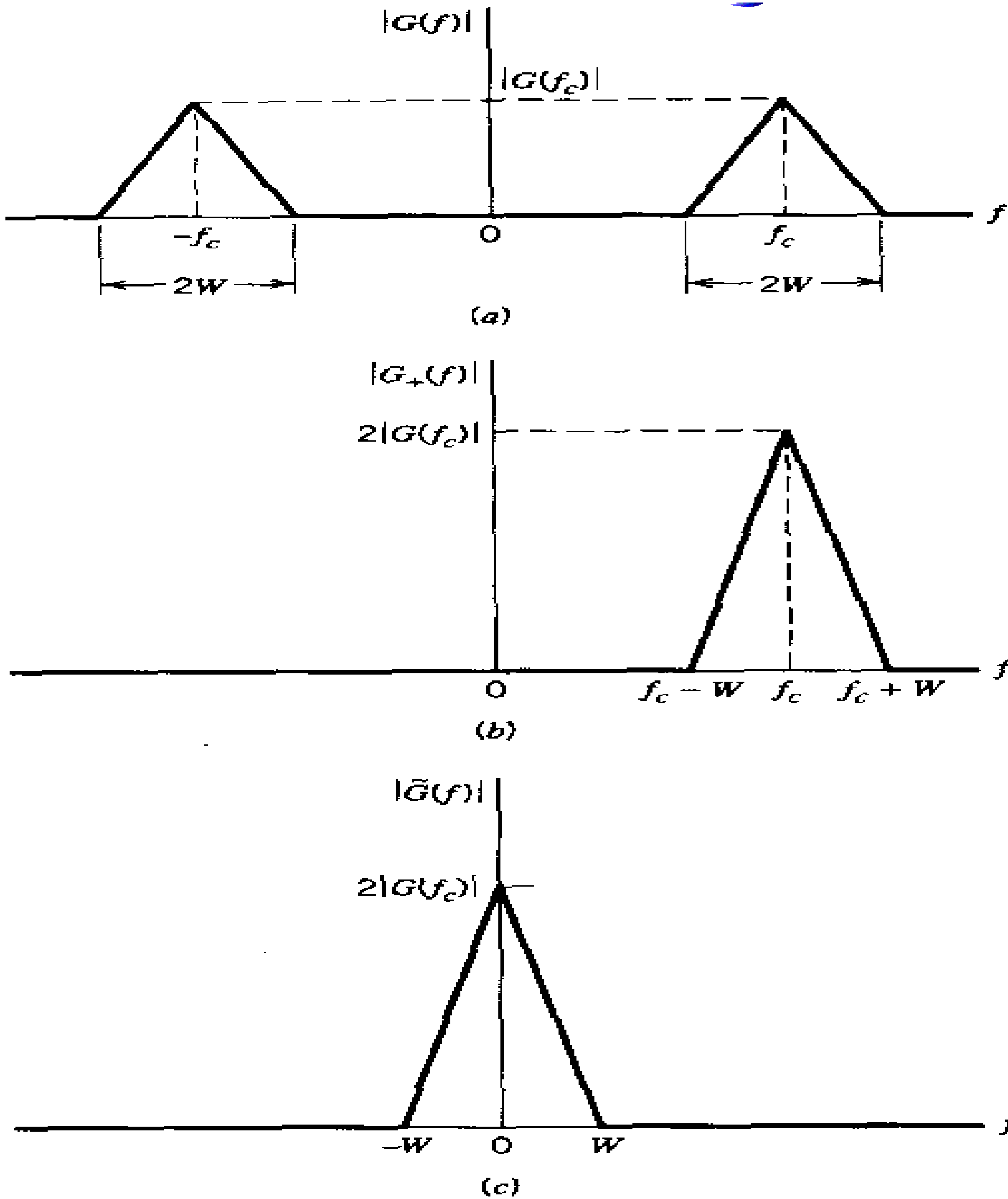
- Get the one-sided spectral: Let

$$\tilde{g}(t) = g(t) + j\hat{g}(t)$$

- Then:

$$\begin{aligned}\tilde{\mathrm{G}}(\mathrm{j}\omega) &= G(\mathrm{j}\omega) + \mathrm{j}\hat{\mathrm{G}}(\mathrm{j}\omega) \\ &= \mathrm{G}(\mathrm{j}\omega) + \mathrm{j}(\text{-}\mathrm{j}\mathrm{G}(\mathrm{j}\omega)\mathrm{sgn}\,\omega) \\ &= \mathrm{G}(\mathrm{j}\omega)[1 + \mathrm{sgn}\,\omega]\end{aligned}$$

- $\tilde{g}(t)$ is called the analytical signal or preenvelope of g(t)



|G(f)|
|G(f_c)|
-f_c
0
f_c
f
2W
2W
(a)
|G_+(f)|
2|G(f_c)|
0
f_c - W
f_c
f_c + W
f
(b)
|G̃(f)|
2|G(f_c)|
-W
0
W
f
(c)

# 2.2 Properties of Hilbert transform

- Linearity: $H\{ax(t)+by(t)\}=a\hat{x}(t)+b\hat{y}(t)$
- Modulation: $H[x(t)\cos 2\pi f_c t]=x(t)\sin 2\pi f_c t$

  $H[x(t)\sin 2\pi f_c t]=-x(t)\cos 2\pi f_c t$
- Successive Hilbert transforms: $H\{\hat{x}(t)\}=-x(t)$
- Orthogonality: $\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt=0$
- Convolution: $H\{x(t)*y(t)\}=\hat{x}(t)*y(t)=x(t)*\hat{y}(t)$
- Fourier Transform:

$$F\{\hat{x}(t)\}=\begin{cases}-iX(w), & w<0\\ 0, & w=0\\ iX(w), & w>0\end{cases}$$

# 2.2 Properties of Hilbert Transform

- Hilbert Transform is like a ideal phase shifter of 90 degree
- For a narrow-band signal x (t)

$$H[x(t)\cos 2\pi f_c t] = x(t)\sin 2\pi f_c t$$

$$H[x(t)\sin 2\pi f_c t] = -x(t)\cos 2\pi f_c t$$

- If A (t) and $\varphi(t)$ are low-frequency signal, then:

$$H[A(t)\cos(\omega_0 t + \varphi(t))] = A(t)\sin[\omega_0 t + \varphi(t)]$$

$$H[A(t)\sin(\omega_0 t + \varphi(t))] = -A(t)\cos[\omega_0 t + \varphi(t)]$$

# Example for Hilbert Transform

- Find the H.T. for $f(x) = \cos \omega t$ and its analytical signal.
- Solution:

$$\hat{f}(t) = \frac{1}{\pi t} * f(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\cos \omega\tau}{t-\tau} d\tau = \sin \omega t$$

$$\tilde{f}(t) = f(t) + j\hat{f}(t) = \cos \omega t + j \sin \omega t = e^{j\omega t}$$

# 2.3 Hilbert Transform For Random Processes

- If $X(t)$ is a WSS (wide-sense stationary) R.P. ,then $\hat{X}(t)$ is also a WSS R.P. .And they are jointly WSS.
- $R_{\hat{X}}(\tau) = R_X(\tau)$ , $S_{\hat{X}}(\omega) = S_X(\omega)$
- $R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau)$ , $R_{\hat{X}X}(\tau) = \hat{R}_X(\tau)$

  then: $R_{X\hat{X}}(\tau) = -R_{\hat{X}X}(\tau)$
- $R_{\hat{X}X}(-\tau) = -R_{\hat{X}X}(\tau)$ $\quad R_{\hat{X}X}(0) = 0$

# 2.3 H.T. for Random Process (cont.)

- Let

$$Z(t) = X(t) + j\hat{X}(t)$$

Then:

$$R_Z(\tau) = 2R_X(\tau) + j2\hat{R}_X(\tau)$$

$$S_{X\hat{X}}(\omega) = \begin{cases} -jS_X(\omega), & \omega \geq 0 \\ jS_X(\omega), & \omega < 0 \end{cases}$$

$$S_Z(\omega) = \begin{cases} 4S_X(\omega) & \omega \geq 0 \\ 0 & \omega < 0 \end{cases}$$

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# 3 Representation of Narrow-Band Signal (Canonic form)

- The narrow-band signal with following form:

$$X(t) = A(t)\cos[2\pi f_c t + \phi(t)]$$

- Then:

$$X(t) = A(t)\cos 2\pi f_c t \cos\phi(t) - A(t)\sin 2\pi f_c t \sin\phi(t)$$

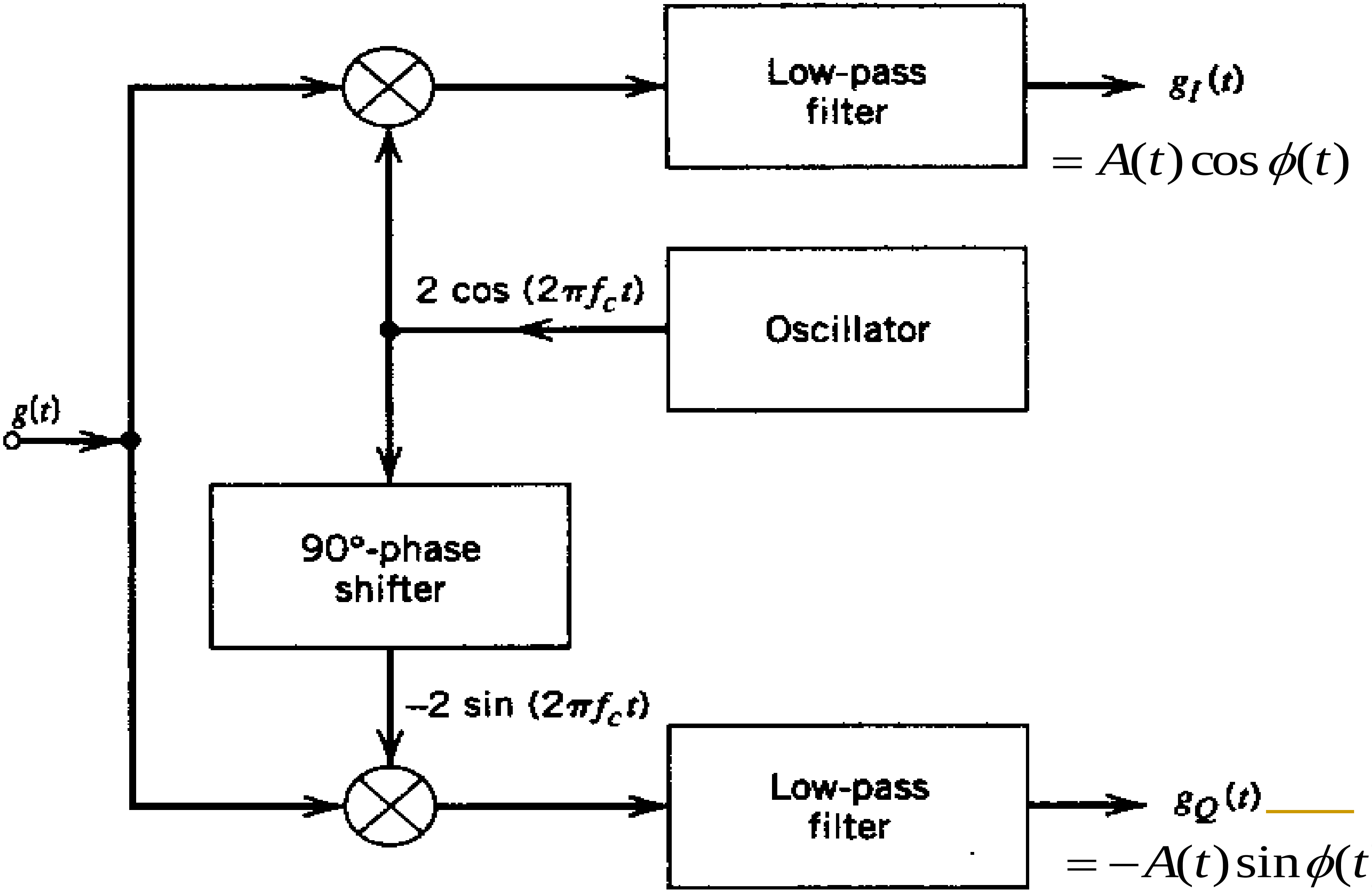
- Let:

$$X_I(t) = A(t)\cos\phi(t) \quad , \quad X_Q(t) = -A(t)\sin\phi(t)$$

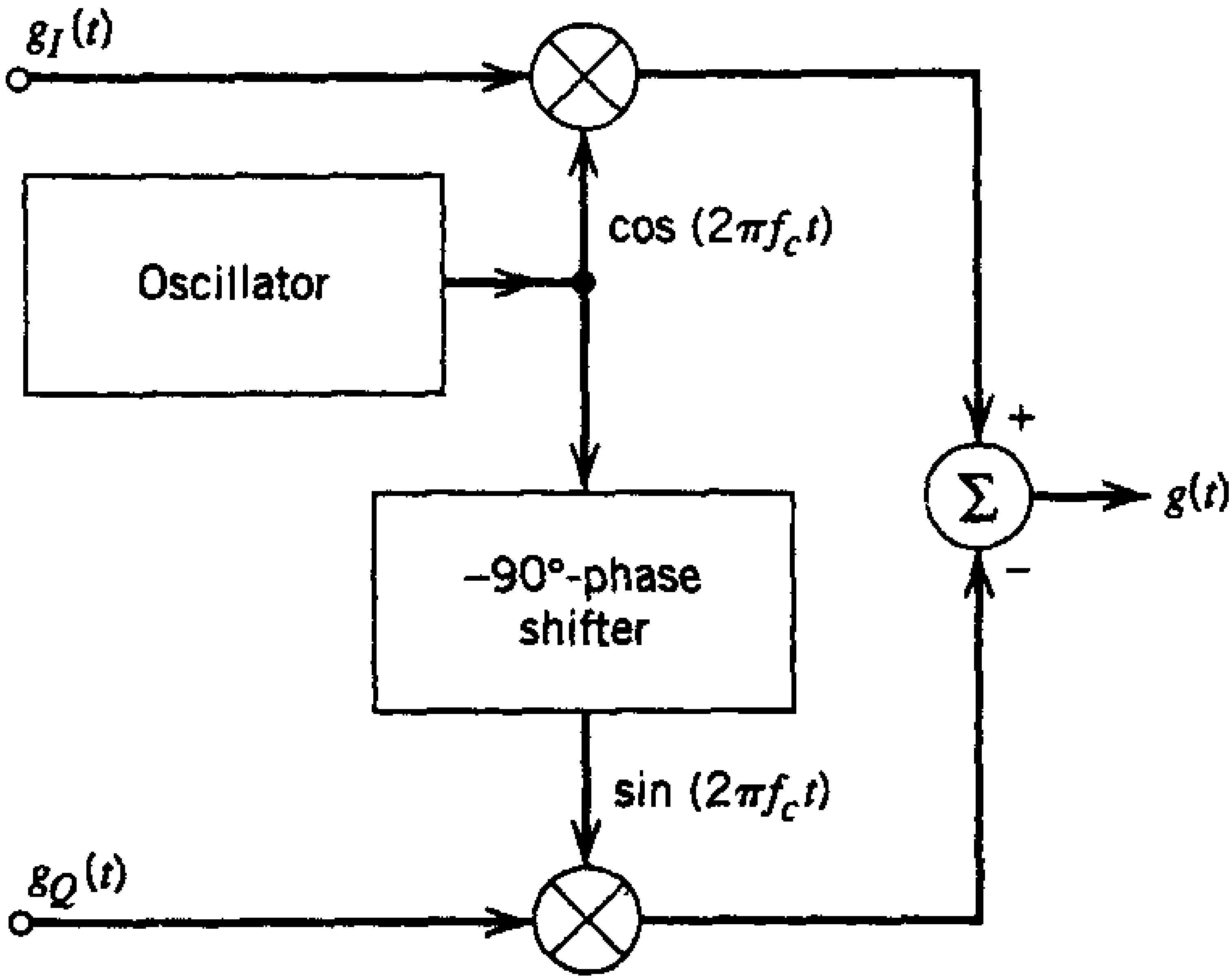
- Then

$$X(t) = X_I(t)\cos 2\pi f_c t + X_Q(t)\sin 2\pi f_c t$$

# Block Diagraph for decompose a narrow-band signal



# Block Diagram for recover the signal



# Getting the In-phase and Quadrature Component

- Make Hilbert Transform on both side of the canonic form, we get:

$$\hat{X}(t) = X_I(t)\sin 2\pi f_c t - X_Q(t)\cos 2\pi f_c t$$

- Then:

$$X_I(t) = X(t)\cos 2\pi f_c t + \hat{X}(t)\sin 2\pi f_c t$$

$$X_Q(t) = X(t)\sin 2\pi f_c t - \hat{X}(t)\cos 2\pi f_c t$$

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# 4 Narrow Band Random Process
## —— Facts about Narrow-band R.P.

- If X(t) is a WSS R.P. with zero mean ,then $X_I(t)$ and $X_Q(t)$ are also WSS with zero mean.

- $X_I(t)$ , $X_Q(t)$ has the same autocorrelation functions, and $\sigma_{X_I}^2 = \sigma_{X_Q}^2 = \sigma_X^2$

- $S_{X_I}(\omega), S_{X_Q}(\omega)$ concentrates in $|\omega| < \Delta\omega/2$ , so they are low-frequency R.P. ,then:

$$S_{X_I}(\omega) = S_{X_Q}(\omega) = S_X(\omega + \omega_c) + S_X(\omega - \omega_c) \quad |\omega| < \Delta\omega/2$$

# Facts About Narrow-band R.P.(cont.)

- If the S.D.F of X(t) is symmetry about $w_c$ ,then:

$$R_X(\tau) = R_{X_I}(\tau)\cos\omega_c\tau = R_{X_Q}(\tau)\cos\omega_c\tau$$

- And:

$$R_{X_I X_Q}(\tau) = R_{X_Q X_I}(\tau) = 0$$

$$S_{X_I X_Q}(\tau) = S_{X_Q X_I}(\tau) = 0$$

- As $X_I(t)$ and $X_Q(t)$ are WSS with zero mean, $X_I(t)$ and $X_Q(t)$ are independent at same times t.

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# 5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- Recall that :

$$X(t) = A(t)\cos[\omega_0 t + \phi(t)]$$
$$= X_I(t)\cos\omega_0 t + X_Q(t)\sin\omega_0 t$$

- And:

$$X_I(t) = X(t)\cos\omega_0 t + \hat{X}(t)\sin\omega_0 t$$
$$X_Q(t) = X(t)\sin\omega_0 t - \hat{X}(t)\cos\omega_0 t$$

- $X_I(t)$ and $X_Q(t)$ are all linear combination of X(t), so, if X(t) is a Gaussian random variable, then , $X_I(t)$ and $X_Q(t)$ are also Gaussian random variable.

# 5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

Suppose:

(1) X(t) is a narrow-band Gaussian Random Process with zero mean;

(2) The variance of X(t) is $\sigma^2$ ;

(3) The P.D.F of X(t) is symmetry about $f_c$

Then:

(1) Since $\boldsymbol{R}_{X_I X_Q}(\tau) = \boldsymbol{R}_{X_Q X_I}(\tau) = 0$, $X_I$(t) and $X_Q$(t) are independent;

(2) $$f_{X_I X_Q}(x_i, x_q) = f_{X_I}(x_i) f_{X_Q}(x_q) = \frac{1}{2\pi\sigma^2} \exp[-\frac{x_{it}^2 + x_{qt}^2}{2\sigma^2}]$$

# 5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- Let $A_t, \phi_t$ denote $A(t)$、$\phi(t)$ respectively, then:

$$f_{A\phi}(a_t, \varphi_t) = |J| f_{X_I X_Q}(x_{it}, x_{qt})$$

- And: $X_{it} = A_t \cos\phi_t, \qquad X_{qt} = A_t \sin\phi$

$$0 \le A_t < \infty,\ 0 \le \phi_t < 2\pi_t$$

- So: $f_{A\phi}(a_t, \varphi_t) = |J| f_{X_I X_Q}(x_{it}, x_{qt})$

$$= a_t f_{X_I X_Q}(x_{it}, x_{qt})$$

$$= \frac{a_t}{2\pi\sigma^2} \exp[-\frac{a_t^2}{2\sigma^2}] \quad a_t \ge 0, 0 \le \varphi_t < 2\pi$$

# 5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- Thus:

$$f_A(a_t) = \int_0^{2\pi} f_{A\phi}(a_t, \varphi_t) d\varphi_t = \frac{a_t}{\sigma^2} \exp(-\frac{a_t^2}{2\sigma^2}) \quad a_t \ge 0$$

$$f_\phi(\varphi_t) = \int_0^{\infty} f_{A\phi}(a_t, \varphi_t) da_t = \frac{1}{2\pi} \quad 0 \le \varphi_t < 2\pi$$

- That is: The envelope follows a Reyleigh distribution; the phase follows a uniform distribution.
- And: $f_{A\phi}(a_t, \varphi_t) = f_A(a_t) \cdot f_\phi(\varphi_t)$ illustrates that the envelope and phase are independent.

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# 6 Distribution of a Sinusoidal Signal Plus Noise

Given: r(t) is a sinusoidal signal with random phase plus a zero-mean, stationary Gaussian random process with a narrow-band spectrum,

$$r(t) = s(t) + n(t)$$

$$s(t) = A\cos[\omega_0 t + \theta]$$

$$n(t) = x(t)\cos(\omega_0 t) - y(t)\sin(\omega_0 t)$$

Obtain: Probability density function of amplitude and phase of r(t).

# 6 Distribution of a Sinusoidal Signal Plus Noise

Sln: $r(t) = A\cos[\omega_0 t + \theta] + x(t)\cos(\omega_0 t) - y(t)\sin(\omega_0 t)$

$$= [A\cos\theta + x(t)]\cos(\omega_0 t) - [A\sin\theta + y(t)]\sin(\omega_0 t)$$

$$= Z_c(t)\cos(\omega_0 t) - Z_s(t)\sin(\omega_0 t)$$

$$= Z(t)\cos[\omega_0 t + \varphi(t)]$$

Where,

$$Z_c(t) = Z(t)\cos\varphi(t)$$

$$Z_s(t) = Z(t)\sin\varphi(t)$$

$$Z(t) = \sqrt{Z_c^2(t) + Z_s^2(t)}$$

$$\varphi(t) = ac\tan\frac{Z_s(t)}{Z_c(t)}$$

Step 1: find the joint distribution of $Z_c(t)$ and $Z_s(t)$

Step 2: Using Jacobian transformation to find the distribution of $Z(t)$ and $\varphi(t)$

# 6 Distribution of a Sinusoidal Signal Plus Noise

- Modified Bessel function of order zero

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp[x\cos\theta]d\theta$$

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}} \qquad x >> 1$$

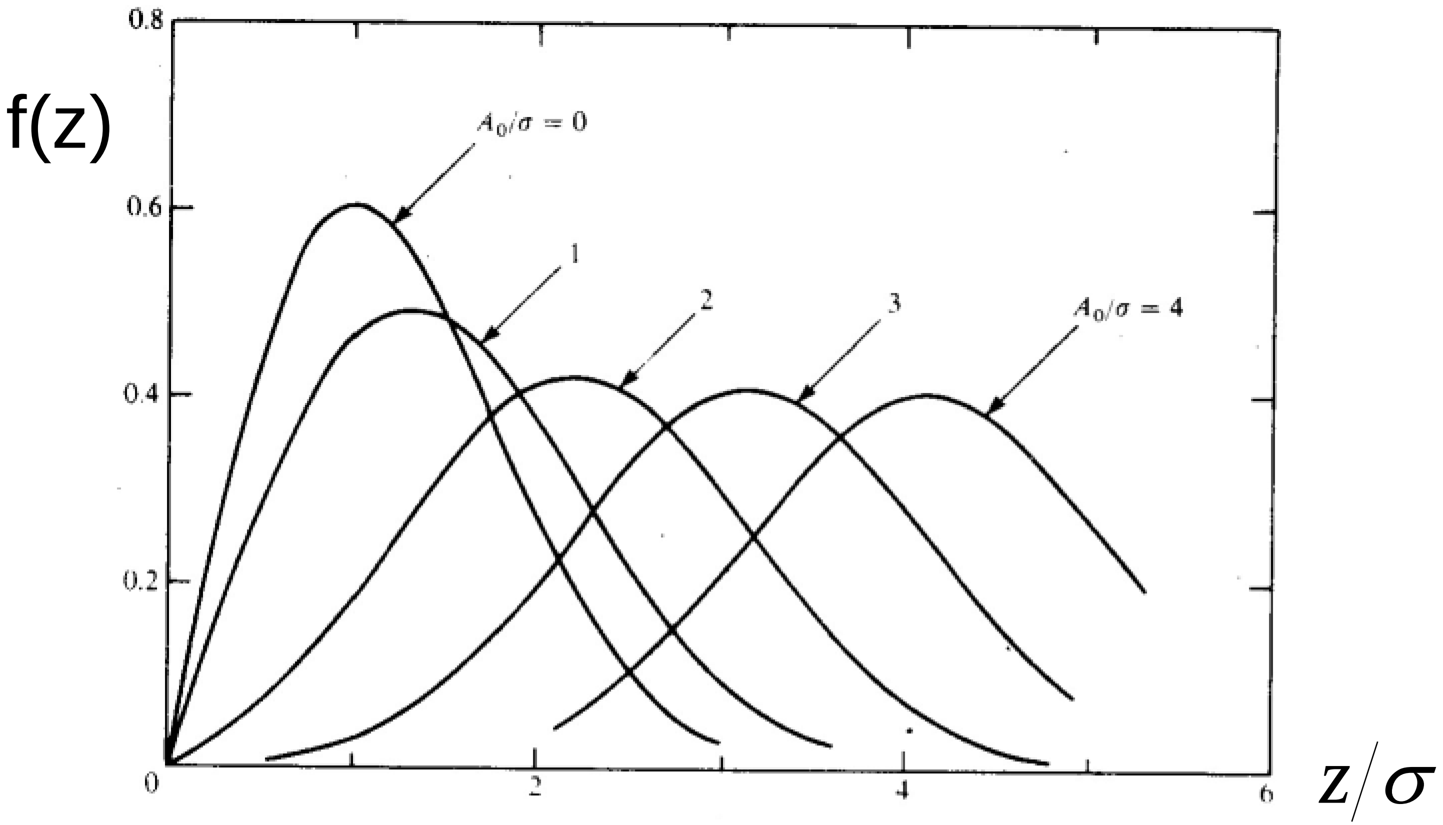
# 6 Distribution of a Sinusoidal Signal Plus Noise

- Probability densities of the envelope

$$f(z) = \frac{z}{\sigma^2} \exp[-\frac{1}{2\sigma^2}(z^2 + A^2)] I_0(\frac{Az}{\sigma^2}) \qquad z \geq 0$$

# 6 Distribution of a Sinusoidal Signal Plus Noise

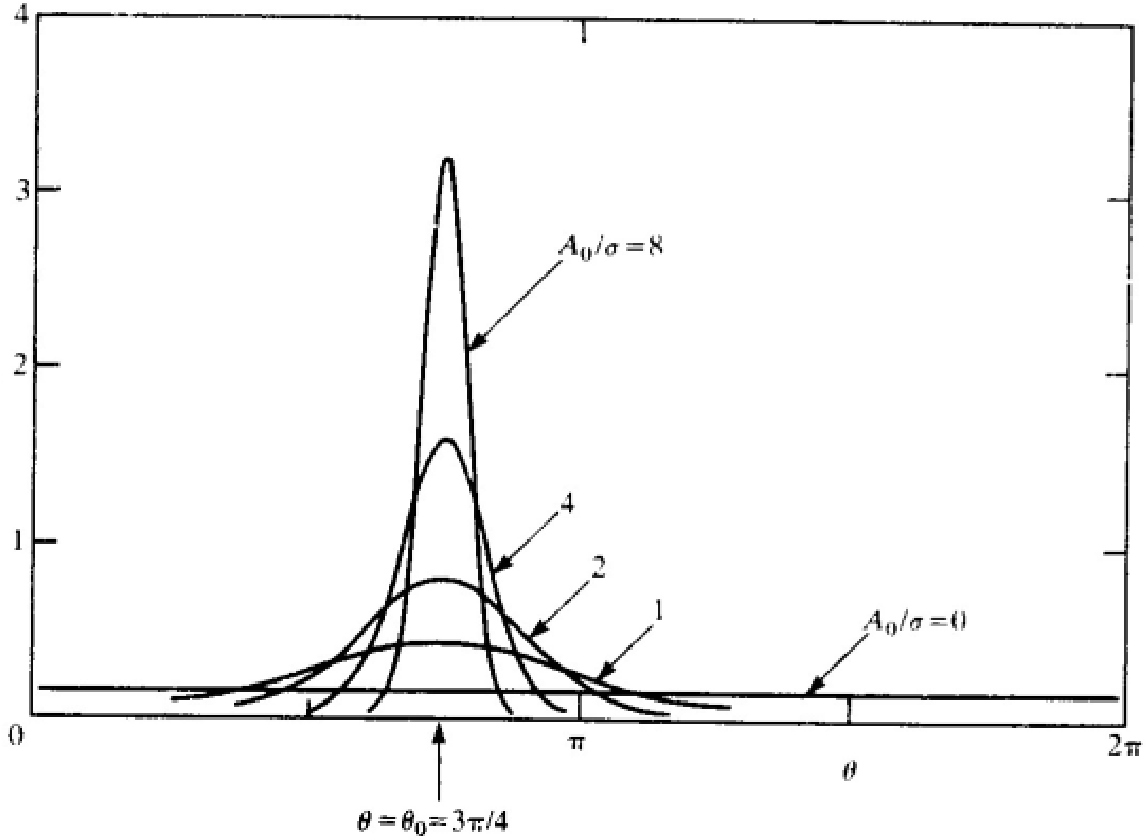
Probability densities of the envelope for various ratios $\frac{A}{\sigma}$



# 6 Distribution of a Sinusoidal Signal Plus Noise

Probability densities of the phase for various ratios $\frac{A}{\sigma}$.

$f(\varphi/\theta)$



# Homework

4.14,

4.16