Chap 4　Spectral Analysis of Stochastic Processes

Dong Yan
EIC. HUST.
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Chapter 4: Spectral Analysis

Content:

4.1 Spectral Density Functions

4.2 Spectral Analysis of Linear Systems

4.3 Spectrum of Amplitude-modulated Signals

4.4 Narrow-band Gaussian Processes

4.3 Spectrum of Amplitude-modulated Signals

Random Amplitude Processes

Consider a signal of the form

$$Y(t) = \sqrt{2}A(t)\cos(w_c t + \theta), \qquad t > 0$$

where A(t) is a random process representing the amplitude and θ is a random variable uniformly distributed between 0 and 2π , and w_c is constant. A(t) and θ are *independent*.

Obtain: power spectrum of Y(t)

- ✓ $\sqrt{2}\cos(w_c t + \theta)$ is seen as the *carrier signal*.
- ✓ A(t) is the *modulation signal or baseband signal*.
- ✓ For analog communications, A(t) may represent a speech signal. In digital communications, A(t) is a continuous-time wave form that represents a sequence of data pulses.

Random Amplitude Processes

$$Y(t) = \sqrt{2}A(t)\cos(w_c t + \theta), \qquad t > 0$$

$$\begin{aligned} E[Y(t)] &= \sqrt{2}E[A(t)\cos(w_c t + \theta)] \\ &= \sqrt{2}E[A(t)]E[\cos(w_c t + \theta)] = 0 \end{aligned}$$

$$\begin{aligned} &R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)] \\ &= 2E[A(t)\cos(w_c t + \theta)A(t+\tau)\cos(w_c t + w_c\tau + \theta)] \\ &= 2E[A(t)A(t+\tau)]E[\cos(w_c t + \theta)\cos(w_c t + w_c\tau + \theta)] \\ &= E[A(t)A(t+\tau)]\cos(w_c\tau) \ = R_{AA}(t, t+\tau)\cos(w_c\tau) \end{aligned}$$

If A(t) is a stationary process, then Y(t) is also stationary with the correlation function

$$R_{YY}(\tau) = R_{AA}(\tau)\cos(w_c\tau)$$

$How\ about\ Y(t) = \sqrt{2}A(t)\sin(w_c t + \theta), \qquad t > 0\ \ ?$

Random Amplitude Processes

Modulation theorem of Fourier transforms

- If x(t) has Fourier transform X(f), then the Fourier transform of the signal $y(t) = x(t)\cos(w_c t)$ is

$$Y(\omega) = \pi[X(\omega + \omega_c) + X(\omega - \omega_c)] \qquad w_c = 2\pi f_c$$

$$Y(f) = \frac{1}{2}[X(f + f_c) + X(f - f_c)]$$

The spectrum of $Y(t) = \sqrt{2}A(t)\cos(w_c t + \theta), \qquad t > 0$

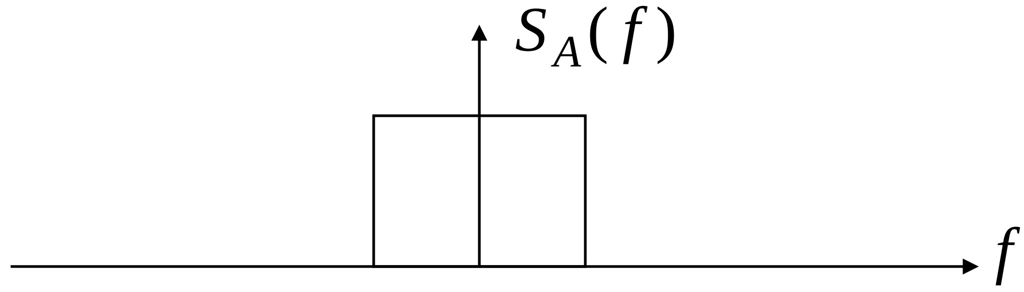
$$R_{YY}(\tau) = R_{AA}(\tau)\cos(w_c \tau)$$

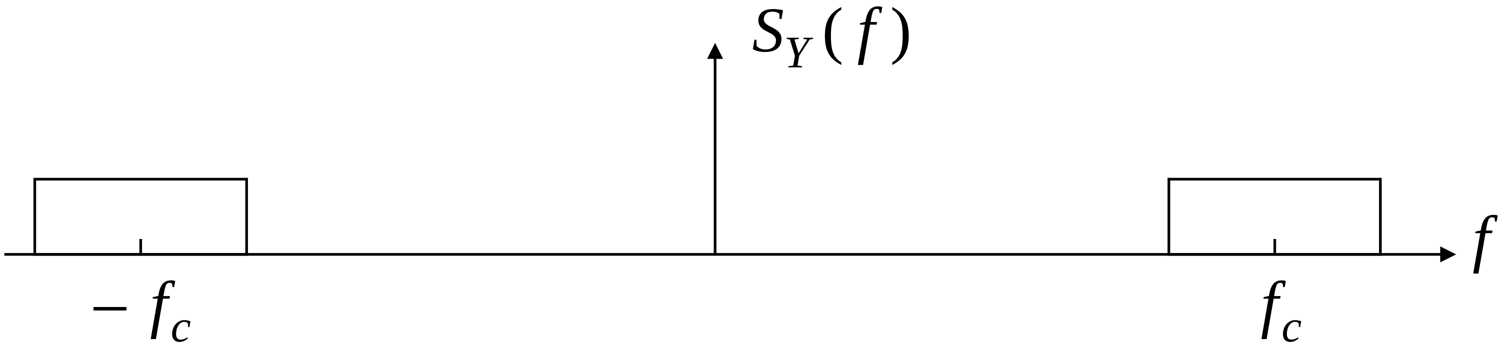
$$S_Y(\omega) = \pi[S_A(\omega + \omega_c) + S_A(\omega - \omega_c)]$$

$$S_Y(f) = \frac{1}{2}[S_A(f + f_c) + S_A(f - f_c)]$$

Random Amplitude Processes

- Spectrum of amplitude-modulated signal Y(t)





$$S_Y(f) = \frac{1}{2}[S_A(f - f_c) + S_A(f + f_c)]$$

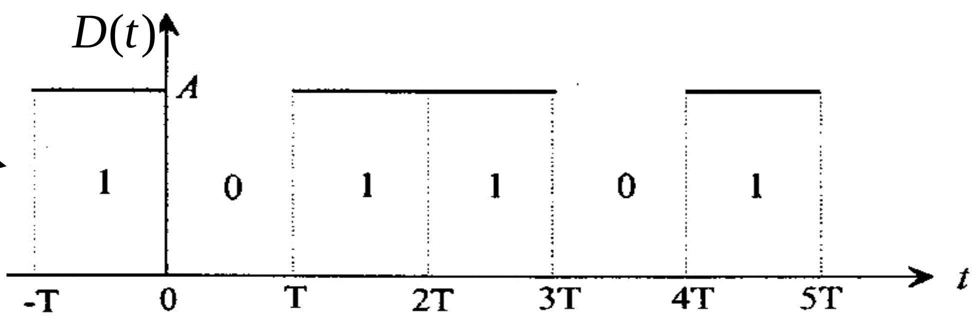
A random data signal with rectangular pulses

- Rectangular pulse: $$p_T(t) = \begin{cases} 1 & \text{for } 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$
- Define a random process as

$$D(t) = \sum_{n=-\infty}^{\infty} A_n p_T(t - nT), \;\; nT \le t < (n+1)T$$

- where *An* is a random variable representing the amplitude of the *n*th pulse. *An*, $-\infty < n < \infty$, are independent and identically distributed random variables.

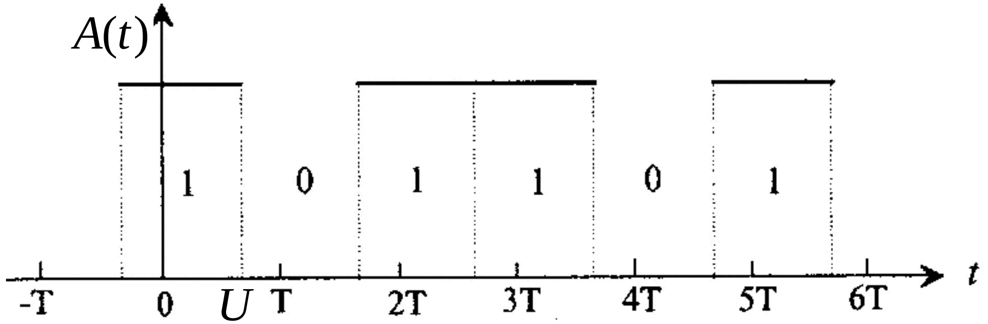
A sample wave of D(t)
Pulse Code Modulation



- D(t) is *not wide-sense stationary* because it's correlation function depends on the absolute value of time.

A random data signal with rectangular pulses

- Random delay of D(t): $A(t) = D(t-U), \quad -\infty < t < \infty$
- where U is uniformly distributed on the interval [0,T].



- A(t) is *wide-sense stationary*.
- Prove:
- For simple， let $p_T(t) = \begin{cases} 1 & \text{for } 0 \le t < T \\ -1 & \text{otherwise} \end{cases}$
- Let $\boxed{E[A_n]=0}$, $\boxed{E[A_n^2]=\alpha^2}$, then

$E[A(t)] = E[D(t)] = 0$, and $E[A_n A_k] = 0$ for $n \neq k$

A random data signal with rectangular pulses

- Correlation function of A(t)

$$E[A(t)A(t+\tau)] = E\left\{ \sum_{n=-\infty}^{\infty} A_n p_T(t-nT-U) \sum_{k=-\infty}^{\infty} A_k p_T(t+\tau-kT-U) \right\}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} E[A_n A_k] E[p_T(t-nT-U) p_T(t+\tau-kT-U)]$$

$$= \sum_{n=-\infty}^{\infty} \alpha^2 E[p_T(t-nT-U) p_T(t+\tau-nT-U)]$$

$$E[p_T(t-nT-U) p_T(t+\tau-nT-U)]$$

$$= \frac{1}{T}\int_0^T p_T(t-nT-u) p_T(t+\tau-nT-u) du \quad , \qquad v = u + nT$$

$$= \frac{1}{T}\int_{nT}^{(n+1)T} p_T(t-v) p_T(t+\tau-v) dv$$

A random data signal with rectangular pulses

$$E[A(t)A(t+\tau)] = \sum_{n=-\infty}^{\infty} \alpha^2 \frac{1}{T} \int_{nT}^{(n+1)T} p_T(t-v)\, p_T(t+\tau-v)\,dv$$

$$= \alpha^2 \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} p_T(t-v)\, p_T(t+\tau-v)\,dv$$

$$= \alpha^2 \frac{1}{T} \int_{-\infty}^{\infty} p_T(t-v)\, p_T(t+\tau-v)\,dv$$

Let $u = t+\tau-v$

$$E[A(t)A(t+\tau)] = \alpha^2 \frac{1}{T} \int_{-\infty}^{\infty} p_T(u-\tau)\, p_T(u)\,du$$

$$R_A(\tau) = \alpha^2 \frac{1}{T} p_T(\tau) * p_T(-\tau)$$

A random data signal with rectangular pulses

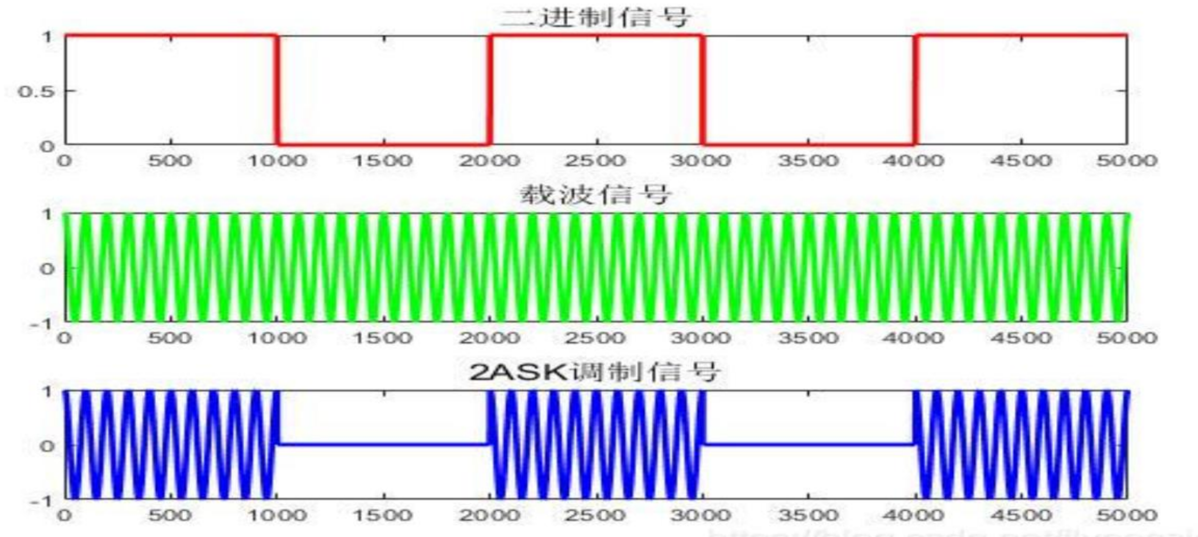
- Spectrum of A(t)

$$R_A(\tau) = \alpha^2 \frac{1}{T} p_T(\tau) * p_T(-\tau) \qquad S_A(f) = \alpha^2 \frac{1}{T} |P_T(f)|^2$$

- If $Y(t) = \sqrt{2}A(t)\cos(w_c t + \theta), \quad t > 0$

  then $$S_Y(f) = \frac{\alpha^2}{2T}[|P_T(f - f_c)|^2 + |P_T(f + f_c)|^2]$$

Sample wave of Y(t):



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4.4 Narrow-band Gaussian Processes

4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of BandWidth (BW)

4.4.2 Hilbert Transform and analytical signal

4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

4.4.1. The Definition of Band

- In most cases, the signal we deal with is always band limited
- Band limited means that the band of a signal is finite.
- How can we define the bandwidth of a signal?
- Here, we just list some criterions for defining bandwidth

Case1: Ideal Band Pass Signals

I) Nonzero in a finite interval: BW $= f_2 - f_1 \leq 2B$

(Absolute bandwidth, cutoff bandwidth)

f_c is the carrier frequency

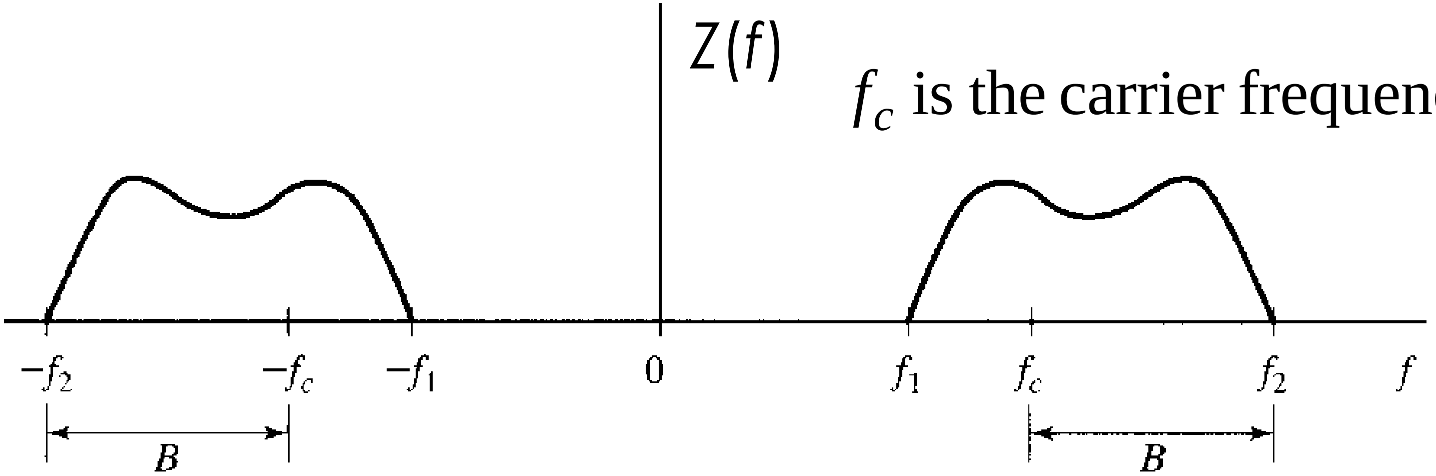


FIGURE 4–3 An ideal band-pass frequency function.

II) Half-power bandwidth (3dB bandwidth)

Let Zm be maximum value of $|Z$(f)|,

$$|Z(f_3)| = |Z(f_4)| = Z_m/\sqrt{2}, \quad |Z(f)| > Z_m/\sqrt{2} \text{ for } f_3 < f < f_4$$

$$\text{and} \quad |Z(f)| < Z_m/\sqrt{2} \text{ for } 0 < f < f_3 \text{ and } f_4 < f < \infty$$

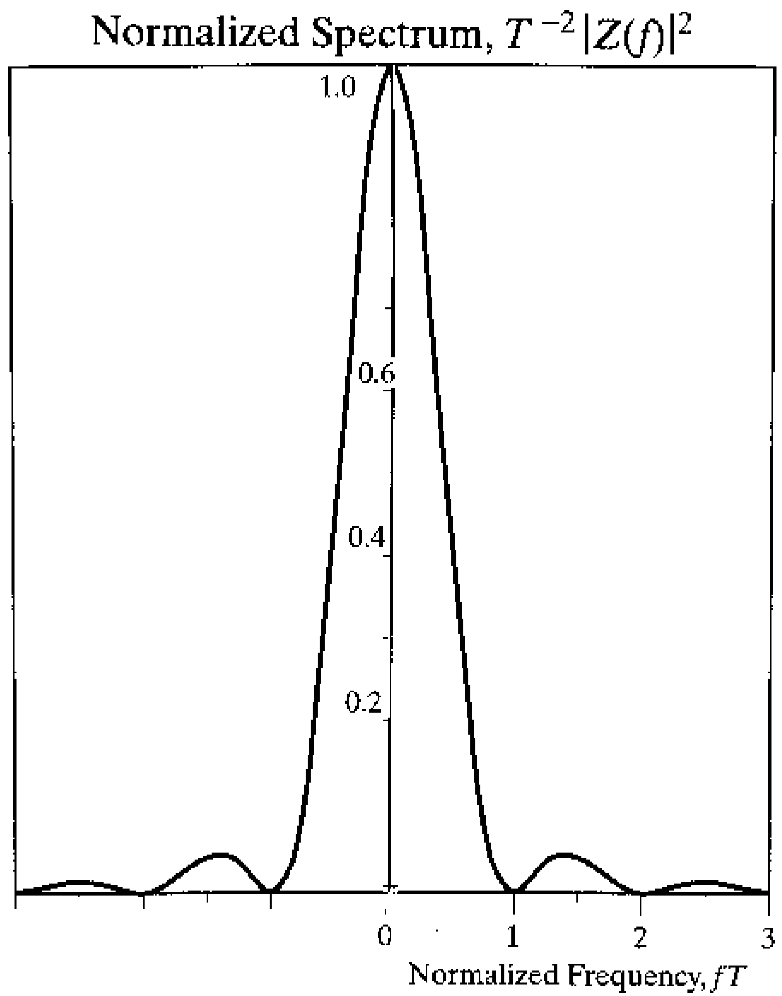
Case 2: Nonideal Band Pass Signals

III. Null-to-null bandwidth

- ✓ Nonzero in infinite interval
- ✓ But the energy concentrates in a small interval

IV. Normalized bandwidth

1-(-1)=2



4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of Band

4.4.2 Hilbert Transform and analytical signal

4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

通信系统中的数字信号正交变换理论

- 现实中产生的物理可实现的信号是实信号
- 但通信系统却提出复信号的表示方式

- 设有一个实信号 x(t)，z(t) 称为 x(t) 的解析表示

$$z(t)=x(t)+j\hat{x}(t)=a(t)e^{j\varphi(t)}$$

从这个表达式中，很容易得到信号的：

瞬时包络 $a(t)$

瞬时相位 $\varphi(t)$

瞬时频率 $\dfrac{d(\varphi(t))}{dt}$

这三个参数，恰好是信号分析、参数测量和识别调制的基础

- 这就是对实信号进行解析表示的意义所在。

通信系统中的数字信号正交变换理论

- 怎样对信号进行解析表示呢?
- 一个实信号的频谱具有共轭对称性。
- 所以，对于一个实信号，只要取其正频域部分或者负频域部分就能完全加以描述，而不会丢失任何信息！并且，所得的新信号是一个复信号！

通信系统中的数字信号正交变换理论

- 假设有一个信号 x(t)，取其正频域部分的频谱分量，这部分频谱可以用一个复函数 z(t)来表示。则：

$$Z(w)=\begin{cases}2X(w), & w>0\\ X(w), & w=0\\ 0, & w<0\end{cases}$$

w>0 的分量加倍是为了使 z(t) 与原信号能量相等。

- 再引入一个阶跃滤波器

$$\boxed{H(w)=\begin{cases}-j, & w>0\\ 0, & w=0\\ j, & w<0\end{cases}=-j\mathrm{sgn}(w)}$$

- 可以得到：

$$Z(w)=X(w)\,[1+jH(w)]$$

$$z(t)=x(t)+jx(t)*h(t)$$

$z(t)$称为$x(t)$的解析表示

通信系统中的数字信号正交变换理论

- 易于求出 $h(t)=\dfrac{1}{\pi t}$
- $\hat{x}(t)=x(t)*h(t)$ 称为 $x(t)$ 的 Hilbert 变换、正交分解。
- 一个实数的 Hilbert 变换同原信号正交。

$$\boxed{\boldsymbol{x(t)}\text{与 }\boldsymbol{\hat{x}(t)}\text{相正交，}\int_{-\infty}^{\infty}\boldsymbol{x(t)\hat{x}(t)dt=0}}$$

$$x(t) \longrightarrow \boxed{h(t)=\frac{1}{\pi t}} \longrightarrow \hat{x}(t)$$

Hilbert变换 是时域到时域的线性变换，LTI

4.4.2 Hilbert Transform

- Def. of Hilbert Transform

$$h(t) = \frac{1}{\pi t}$$

$$\begin{cases} \hat{x}(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau} d\tau = \frac{1}{\pi t} * x(t) \\ x(t) = -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty} \frac{\hat{x}(\tau)}{t-\tau} d\tau = -\frac{1}{\pi t} * \hat{x}(t) \end{cases}$$

- The functions above are Hilbert Transform pairs
- Hilbert Transform is an operation denoted by H[*]
- Relationship in frequency domain:

$$\boxed{H(w) = \begin{cases} -j & w \geq 0 \\ j & w < 0 \end{cases}} \qquad \boxed{\Rightarrow \quad |H(w)| = 1, \quad \varphi(w) = \begin{cases} -\pi/2 & w \geq 0 \\ \pi/2 & w < 0 \end{cases}}$$

$$\boxed{\hat{X}(w) = H(w)X(\mathrm{w}) = -j\,\mathrm{sgn}(w)X(\mathrm{w})}$$

Properties of Hilbert transform (1)

$$\boxed{\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x(\tau)}{t-\tau}d\tau = \frac{1}{\pi t} * x(t)}$$

- Fourier Transform:

$$\hat{X}(w) = F\{\hat{x}(t)\} = -j\,\mathrm{sgn}(w)X(w) = \begin{cases} -jX(w), & w \geq 0 \\ 0, & w = 0 \\ jX(w), & w < 0 \end{cases}$$

- Orthogonality: $\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$
- Linearity: $H\{ax(t)+by(t)\} = a\hat{x}(t)+b\hat{y}(t)$
- Successive Hilbert transforms: $H\{\hat{x}(t)\} = \frac{1}{\pi t} * \frac{1}{\pi t} * x(t) = -x(t)$
- Convolution: $H\{x(t)*y(t)\} = \hat{x}(t)*y(t) = x(t)*\hat{y}(t)$

The use of Hilbert Transform

- Get the one-sided spectral:

  Let $\boxed{\tilde{x}(t) = x(t) + j\hat{x}(t)}$

- Then:

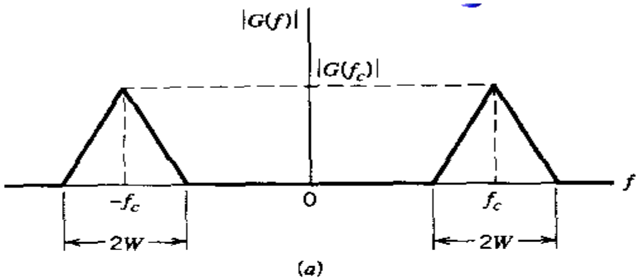
$$\tilde{X}(\omega) = X(\omega) + j\hat{X}(\omega)$$
$$= X(\omega) + j\,[-j\,\mathrm{sgn}\omega\; X(\omega)]$$
$$= X(\omega)[1 + \mathrm{sgn}\omega]$$
$$= \begin{cases} 2X(\omega), & \omega \geq 0 \\ 0, & \omega < 0 \end{cases}$$

$$\boxed{= 2X_+(\omega)}$$

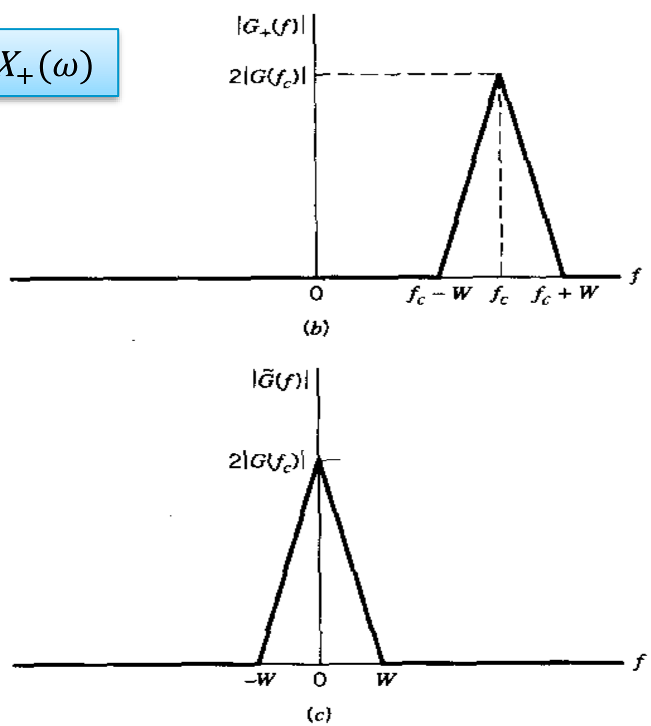
- $\tilde{x}(t)$ is called *the analytical signal* of $x(t)$

$$x(t) = A(t)\cos(w_c t)$$

$$X(\omega) = \pi[A(\omega + \omega_c) + A(\omega - \omega_c)]$$



$$\tilde{X}(\omega) = 2X_+(\omega)$$



Example for Hilbert Transform

e.g. Find the H.T. for $x(t) = \cos \omega\, t$ and its analytical signal.

Solution:

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\cos \omega\, \tau}{t - \tau} d\tau = \sin \omega\, t$$

- Hilbert Transform is like an ideal phase shifter of 90 degree
- $\tilde{x}(t) = x(t) + j\hat{x}(t)$ is *the analytical signal* of $x(t)$

$$\boxed{\tilde{x}(t) = \cos \omega\, t + j \sin \omega\, t = e^{j\omega t}}$$

- The compare between the frequency spectral of $x(t)$ and $\tilde{x}(t)$
- The spectral of $\tilde{x}(t)$ is the positive parts of the spectral of $x(t)$

$$x(t) = \cos \omega_c\, t \qquad X(\omega) = \pi(\delta(\omega + \omega_c) + \delta(\omega - \omega_c))$$

$$\hat{x}(t) = \sin \omega_c\, t \qquad \hat{X}(\omega) = j\pi(\delta(\omega + \omega_c) - \delta(\omega - \omega_c))$$

$$\boxed{\tilde{X}(\omega) = 2X_{+}(\omega)}$$

4.4 Narrow-band Gaussian Processes

- References

[1] Analysis of stochastic signals, 朱华，北京理工大学, *Chapter 5*

[2] Random signal analysis, 李晓风，电子工业出版社, *Chapter 6*

HT for Random Processes (1)

- If $X(t)$ is a WSS (wide-sense stationary) R.P., then $\hat{X}(t)$ is also a WSS R.P. . And they are jointly WSS.

$$\hat{X}(t) = X(t) * \frac{1}{\pi t}, \qquad H(\omega) = -j\,\mathrm{sgn}\omega$$

$$\boxed{R_{\hat{X}}(\tau) = R_X(\tau)\,,\ S_{\hat{X}}(\omega) = S_X(\omega)}$$

$$\boxed{R_{\hat{X}X}(\tau) = \hat{R}_X(\tau)\,, \qquad R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau), \qquad \tau = t_1 - t_2}$$

$$R_{\hat{X}X}(\tau) = E\left[\hat{X}(t_1)X(t_2)\right] = E\left[X(t_1) * \frac{1}{\pi t}X(t_2)\right]$$

$$= E\left[\int_{-\infty}^{\infty} X(t_1 - u)\frac{1}{\pi u}du\, X(t_2)\right]$$

$$= \int_{-\infty}^{\infty} E[X(t_1 - u)\, X(t_2)]\frac{1}{\pi u}du$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau - u)\frac{1}{\pi u}du = \hat{R}_X(\tau)$$

HT for Random Processes (1)

- If $X(t)$ is a WSS (wide-sense stationary) R.P., then $\hat{X}(t)$ is also a WSS R.P. . And they are jointly WSS.

$$\boxed{R_{\hat{X}}(\tau) = R_X(\tau)\,,\, S_{\hat{X}}(\omega) = S_X(\omega)}$$

$$\boxed{R_{\hat{X}X}(\tau) = \hat{R}_X(\tau)\,, \qquad R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau), \qquad \tau = t_1 - t_2}$$

Then, $R_{X\hat{X}}(\tau) = -R_{\hat{X}X}(\tau)$ (1)

- From the property of cross-correlation function:

$$R_{X\hat{X}}(\tau) = R_{\hat{X}X}(-\tau) \qquad (2)$$

$R_{X\hat{X}}(\tau)$ and $R_{\hat{X}X}(\tau)$ are odd functions: $\boxed{R_{\hat{X}X}(-\tau) = -R_{\hat{X}X}(\tau)}$

$R_{\hat{X}X}(0) = 0$

$\hat{X}(t)\ and\ X(t)$ are orthogonal at the same time *t*

H.T. for Random Process (2)

The spectrum of the *analytical signal* of R.P. *X*(*t*)

- Let $Z(t) = X(t) + j\hat{X}(t)$
- Then,

$$\boxed{\begin{aligned} R_Z(\tau) &= E[Z(t)Z^*(t+\tau)] \\ &= E[(X(t)+j\hat{X}(t))(X(t+\tau)-j\hat{X}(t+\tau))] \\ &= R_X(\tau) + R_{\hat{X}}(\tau) + jR_{X\hat{X}}(\tau) - jR_{\hat{X}X}(\tau) \\ &= 2R_X(\tau) + j2\hat{R}_X(\tau) \end{aligned}}$$

$$R_{\hat{X}X}(\tau) = \hat{R}_X(\tau) \quad \Rightarrow \quad S_{\hat{X}X}(\omega) = \begin{cases} -jS_X(\omega), & \omega \geq 0 \\ jS_X(\omega), & \omega < 0 \end{cases}$$

$$R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau) \quad \Rightarrow \quad S_{X\hat{X}}(\omega) = \begin{cases} jS_X(\omega), & \omega \geq 0 \\ -jS_X(\omega), & \omega < 0 \end{cases}$$

$$\boxed{S_Z(\omega) = \begin{cases} 4S_X(\omega) & \omega \geq 0 \\ 0 & \omega < 0 \end{cases}}$$

4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of Band

4.4.2 Hilbert Transform and analytical signal

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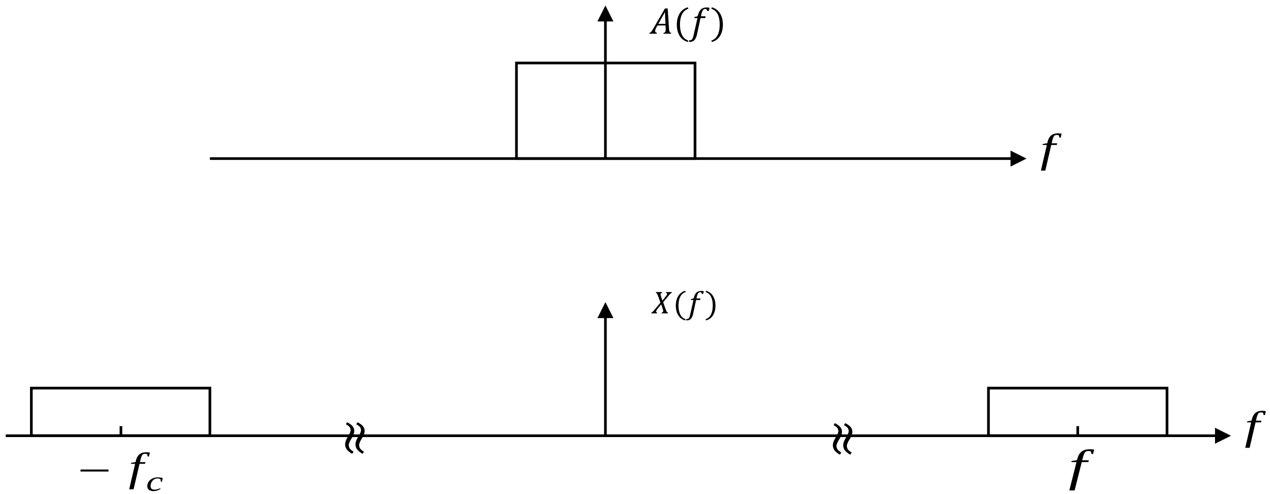
4.4.6 Sine Wave Plus Narrow-Band Noise

4.4.3 Representation of Narrow-Band Signal (Canonic form)

- The narrow-band signal with following form:

$$X(t) = A(t)\cos[\,2\pi f_c t + \phi(t)]$$

- $A(t)$ is the *envelop* and f_c is the *carrier frequency*.
- $A(t)$ and $\phi(t)$ are low-frequency bandlimited signals, their max frequencies are much lower than f_c
- $A(t)$ and $\phi(t)$ can be seen as the *baseband signals*.



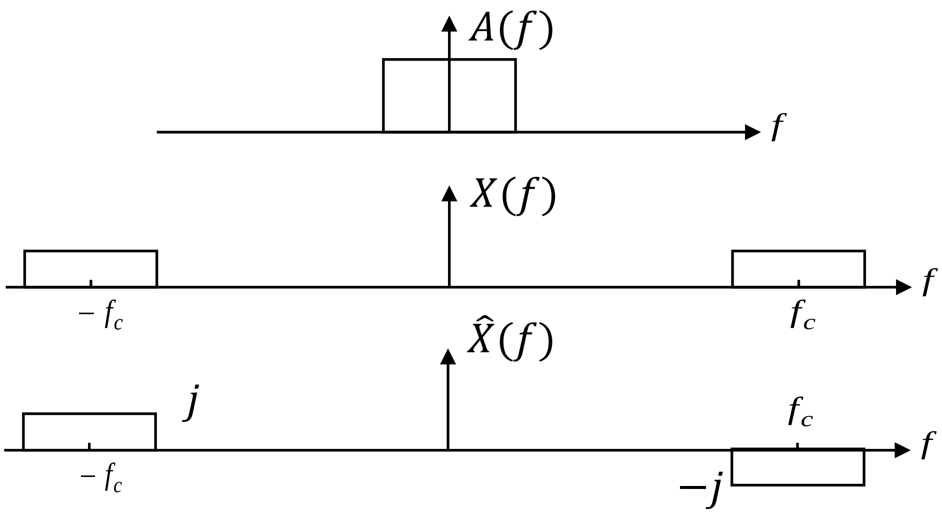
Hilbert Transform of a narrow-band signal

- For a narrow-band signal $X(t) = A(t)\cos 2\pi f_c t$

$$\boxed{\begin{aligned} H[A(t)\cos 2\pi f_c t] &= A(t)\sin 2\pi f_c t \\ H[A(t)\sin 2\pi f_c t] &= -A(t)\cos 2\pi f_c t \end{aligned}}$$

$$\boxed{\begin{aligned} H[A(t)\cos(\omega_c t + \phi(t))] &= A(t)\sin[\omega_c t + \phi(t)] \\ H[A(t)\sin(\omega_c t + \phi(t))] &= -A(t)\cos[\omega_c t + \phi(t)] \end{aligned}}$$

- Prove can be done easily in frequency domain



$$\hat{x}(t) = \frac{1}{\pi t} * x(t)$$

$$\begin{aligned} \hat{X}(w) = F\{\hat{x}(t)\} &= -j\,\mathrm{sgn}(w)X(w) \\ &= \begin{cases} -jX(w), & w \geq 0 \\ 0, & w = 0 \\ jX(w), & w < 0 \end{cases} \end{aligned}$$

4.4.3 Representation of Narrow-Band Signal (Canonic form)

- The narrow-band signal with following form:

$$X(t) = A(t)\cos[2\pi f_c t + \phi(t)]$$

- Then: $\boxed{X(t) = A(t)\cos\phi(t)\cos 2\pi f_c t - A(t)\sin\phi(t)\sin 2\pi f_c t}$
- Let: $\boxed{X_I(t) = A(t)\cos\phi(t)\ \ ,\ \ X_Q(t) = -A(t)\sin\phi(t)}$
- Then
- $\boxed{X(t) = X_I(t)\cos 2\pi f_c t + X_Q(t)\sin 2\pi f_c t}$

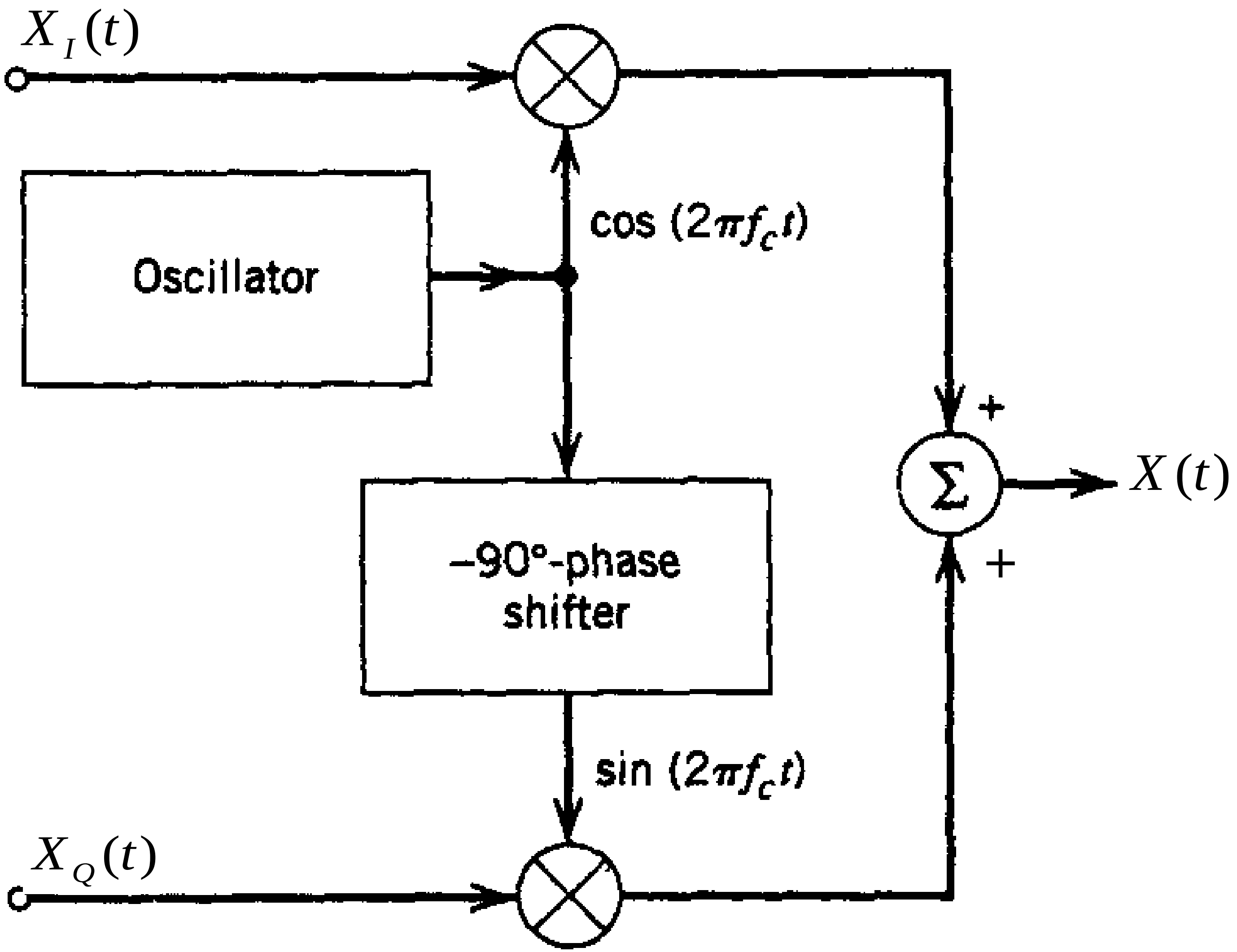
$$A(t) = \sqrt{X_I^2(t) + X_Q^2(t)}$$

$$\phi(t) = \arctan\frac{X_Q(t)}{X_I(t)}$$

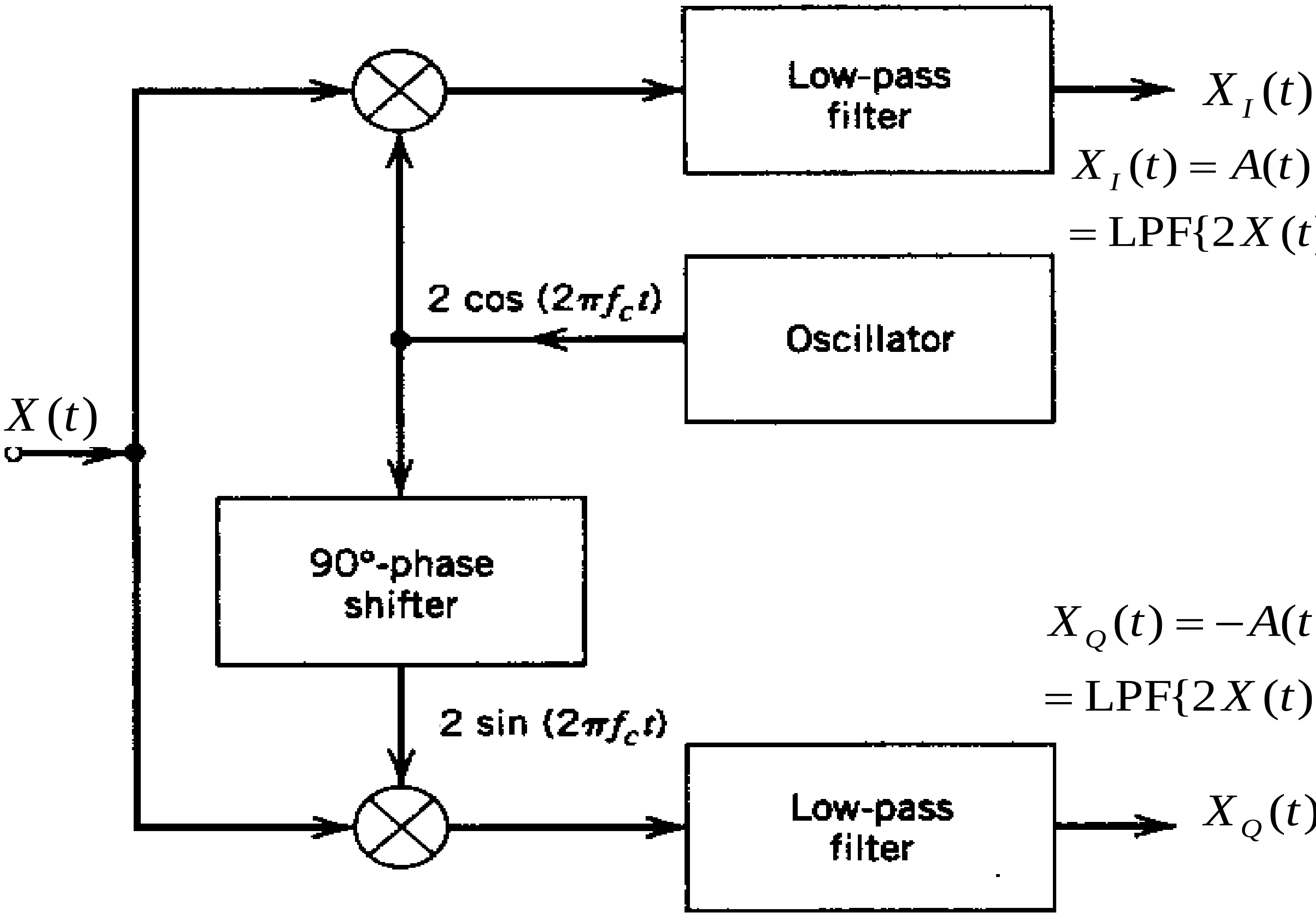
- $X_I(t)$ and $X_Q(t)$ are low-frequency bandlimited signals, their max frequencies are far lower than f_c
- $X_I(t)$ and $X_Q(t)$ can be seen as the *baseband signals*.
- Define $\boxed{a(t) = X_I(t) + jX_Q(t) = A(t)e^{-j\phi(t)}}$

  is the *complex envelop* of the baseband signal.

Block Diagram for recover the narrow-band signal (*Modulation*)



Block Diagraph for decompose a narrow-band signal (*Demodulation*)



$$X_I(t) = A(t)\cos\phi(t)$$
$$= \mathrm{LPF}\{2X(t)\cos(2\pi f_c t)\}$$

$$X_Q(t) = -A(t)\sin\phi(t)$$
$$= \mathrm{LPF}\{2X(t)\sin(2\pi f_c t)\}$$

4.4.3 Representation of Narrow-Band Signal (Canonic form)

$$\boxed{X(t) = A(t)\cos[2\pi f_c t + \phi(t)]} \qquad \boxed{X(t) = X_I(t)\cos 2\pi f_c t + X_Q(t)\sin 2\pi f_c t}$$

- Make Hilbert Transform on both side, we get:

$$\boxed{\hat{X}(t) = X_I(t)\sin 2\pi f_c t - X_Q(t)\cos 2\pi f_c t}$$

- Then:

$$\boxed{\begin{aligned} X_I(t) &= X(t)\cos 2\pi f_c t + \hat{X}(t)\sin 2\pi f_c t \\ X_Q(t) &= X(t)\sin 2\pi f_c t - \hat{X}(t)\cos 2\pi f_c t \end{aligned}}$$

- The analytical signal $\tilde{X}(t)$ ： $\tilde{X}(t) = X(t) + j\hat{X}(t)$
- The relationship between complex envelop $a(t)$ and $\tilde{X}(t)$

$$\boxed{\tilde{X}(t) = a(t)e^{-j2\pi f_c t}} \qquad \boxed{a(t) = X_I(t) + jX_Q(t) = A(t)e^{-j\phi(t)}}$$

$$X(t) = \mathrm{Re}[a(t)e^{-j2\pi f_c t}] = \mathrm{Re}[A(t)e^{-j[2\pi f_c t + \phi(t)]}]$$

$$\hat{X}(t) = \mathrm{Im}[a(t)e^{-j2\pi f_c t}] = \mathrm{Im}[A(t)e^{-j[2\pi f_c t + \phi(t)]}]$$

4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of Band

4.4.2 Hilbert Transform

4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

Facts about Narrow-band R.P.(1)

- Narrow Band Random Processes in **Canonic form**

$$X(t) = A(t)\cos[2\pi f_c t + \phi(t)]$$
$$= X_I(t)\cos 2\pi f_c t + X_Q(t)\sin 2\pi f_c t$$

$$\boxed{X_I(t) = A(t)\cos\phi(t)\ ,\ \ X_Q(t) = -A(t)\sin\phi(t)}$$

- A(t) is a random process representing the amplitude , $\emptyset(t)$ is a random process representing the phase,
- A(t) and $\emptyset(t)$ are *independent,*
- f_c is constant.

Review: Random Amplitude Processes

$$Y(t) = \sqrt{2}A(t)\cos(w_c t + \theta), \quad t > 0 \qquad R_{YY}(\tau) = R_{AA}(\tau)\cos(w_c\tau)$$

Facts about Narrow-band R.P.(1)

$$X(t) = A(t)\cos[2\pi f_c t + \phi(t)] = X_I(t)\cos 2\pi f_c t + X_Q(t)\sin 2\pi f_c t$$

$$\boxed{X_I(t) = A(t)\cos\phi(t) \ , \ X_Q(t) = -A(t)\sin\phi(t)}$$

- If X(t) is a *WSS narrow-band R.P. with zero mean*, then the following facts are established:

a) $X_I(t)$ and $X_Q(t)$ are stationary and jointly stationary

$$\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau \\ &= R_{X_Q X_Q}(\tau)\cos\omega_c\tau - R_{X_I X_Q}(\tau)\sin\omega_c\tau \end{aligned}$$

b) $X_I(t)$ and $X_Q(t)$ has zero mean, and $\sigma_{X_I}^2 = \sigma_{X_Q}^2 = \sigma_X^2$

c) $X_I(t)$ and $X_Q(t)$ has the same autocorrelation functions, and are *orthogonal* and uncorrelated at the same time *t* :

$$\boxed{R_{X_I}(\tau) = R_{X_Q}(\tau)} \quad \boxed{R_{X_I X_Q}(\tau) = R_{X_Q X_I}(-\tau) = -R_{X_Q X_I}(\tau)} \quad \boxed{R_{X_I X_Q}(0) = 0}$$

Facts about Narrow-band R.P

Prove:

a) $X_I(t)$ and $X_Q(t)$ are stationary and joint stationary processes

$$
\begin{aligned}
R_{XX}(\tau) &= E[X(t+\tau)X(t)] \\
&= E\{\ [X_I(t+\tau)\cos\omega_c(t+\tau) + X_Q(t+\tau)\sin\omega_c(t+\tau)] \\
&\quad [X_I(t)\cos(\omega_c t) + X_Q(t)\sin(\omega_c t)]\} \\
&= R_{X_I X_I}(t+\tau,t)\cos\omega_c(t+\tau)\cos(\omega_c t) \\
&\quad + R_{X_Q X_Q}(t+\tau,t)\sin\omega_c(t+\tau)\sin(\omega_c t) \\
&\quad + R_{X_I X_Q}(t+\tau,t)\cos\omega_c(t+\tau)\sin(\omega_c t) \\
&\quad + R_{X_Q X_I}(t+\tau,t)\sin\omega_c(t+\tau)\cos(\omega_c t)
\end{aligned}
$$

Facts about Narrow-band R.P

$t_1 = 0,$

$$R_{XX}(\tau) = R_{X_I X_I}(t_1+\tau, t_1)\cos\omega_c\tau + R_{X_Q X_I}(t_1+\tau, t_1)\sin\omega_c\tau$$

$$\boxed{\therefore \quad R_{X_I X_I}(t+\tau, t) = R_{X_I X_I}(\tau),\ R_{X_Q X_I}(t+\tau, t) = R_{X_Q X_I}(\tau)}$$

$t_2 = \dfrac{\pi}{2\omega_c},$

$$R_{XX}(\tau) = R_{X_Q X_Q}(t_2+\tau, t_2)\cos\omega_c\tau - R_{X_I X_Q}(t_2+\tau, t_2)\sin\omega_c\tau$$

$$\boxed{\therefore \quad R_{X_Q X_Q}(t+\tau, t) = R_{X_Q X_Q}(\tau),\ R_{X_I X_Q}(t+\tau, t) = R_{X_I X_Q}(\tau)}$$

So, $X_I(t)$ and $X_Q(t)$ are stationary and joint stationary

$\therefore$

$$\boxed{\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau \\ &= R_{X_Q X_Q}(\tau)\cos\omega_c\tau - R_{X_I X_Q}(\tau)\sin\omega_c\tau \end{aligned}}$$

Facts about Narrow-band R.P

b) Mean values and variances of $X_I(t)$ and $X_Q(t)$

$$X(t) = X_I(t)\cos\omega_c t + X_Q(t)\sin\omega_c t$$

$$t_1 = 0, \qquad E[X(t_1)] = E[X_I(t_1)] = 0 \qquad \therefore E[X_I(t)] = 0$$

$$t_2 = \frac{\pi}{2\omega_c}, \quad E[X(t_2)] = E[X_Q(t_2)] = 0 \qquad \therefore E\left[X_Q(t)\right] = 0$$

$$\boxed{\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau \\ &= R_{X_Q X_Q}(\tau)\cos\omega_c\tau - R_{X_I X_Q}(\tau)\sin\omega_c\tau \end{aligned}}$$

$$\therefore \quad R_{XX}(0) = R_{X_I X_I}(0) = R_{X_Q X_Q}(0) = \sigma_X^2$$

$$\sigma_{X_I}^2 = \sigma_{X_Q}^2 = \sigma_X^2$$

Facts about Narrow-band R.P

c) $X_I(t)$ and $X_Q(t)$ has the same autocorrelation functions,and are *orthogonal* and uncorrelated at same times t .

$$\boxed{\begin{aligned} R_{XX}(\tau) &= R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau \\ &= R_{X_Q X_Q}(\tau)\cos\omega_c\tau - R_{X_I X_Q}(\tau)\sin\omega_c\tau \end{aligned}}$$

$$\therefore\ R_{X_I X_I}(\tau) = R_{X_Q X_Q}(\tau)$$

$$\left.\begin{aligned} R_{X_I X_Q}(\tau) &= -R_{X_Q X_I}(\tau) \qquad (1) \\ R_{X_I X_Q}(\tau) &= R_{X_Q X_I}(-\tau) \qquad (2) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &\boxed{R_{X_Q X_I}(-\tau) = -R_{X_Q X_I}(\tau)} \\ &\boxed{R_{X_I X_Q}(-\tau) = -R_{X_I X_Q}(\tau)} \end{aligned}$$

$$\boxed{\therefore\quad R_{X_I X_Q}(0) = R_{X_Q X_I}(0) = 0}$$

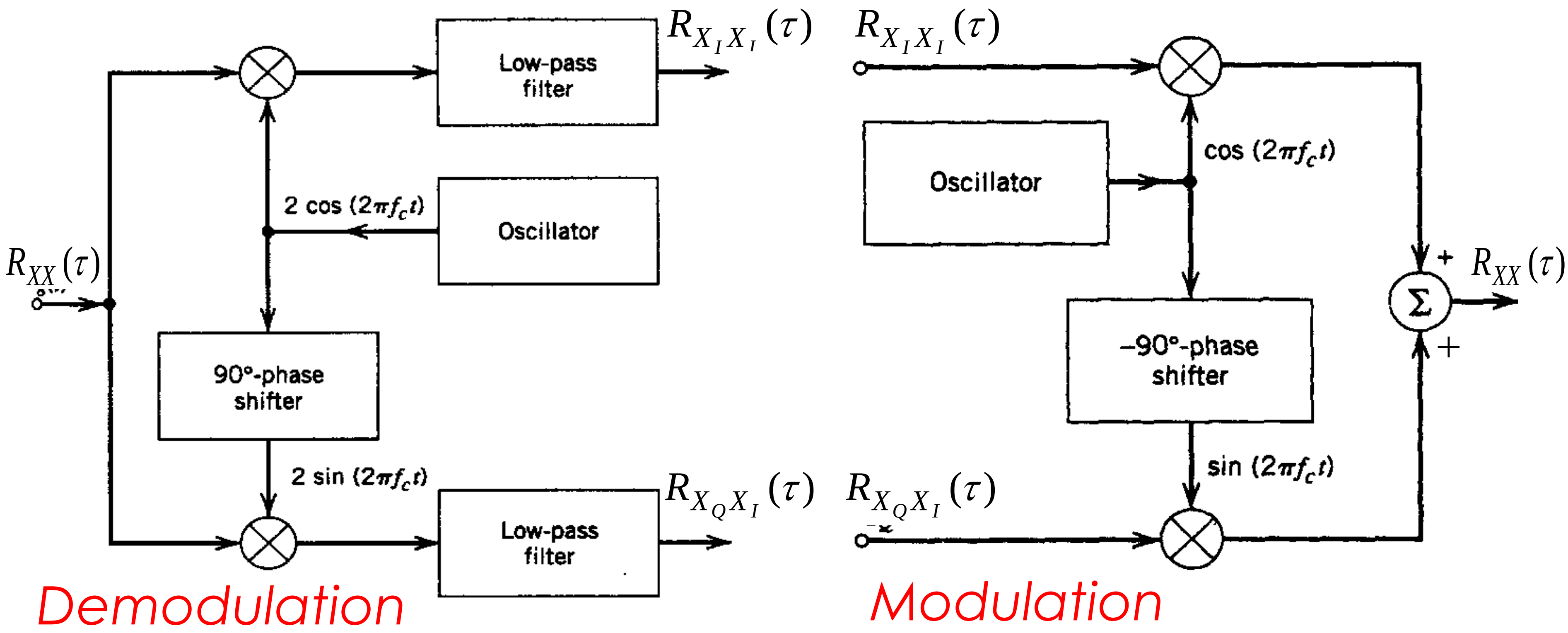
$X_I(t)$ and $X_Q(t)$ has zero mean, then they are *orthogonal* and uncorrelated at the same time *t*

Facts about Narrow-band R.P.(2)

$$X(t) = X_I(t)\cos\omega_c t + X_Q(t)\sin\omega_c t$$

$$\boxed{R_{XX}(\tau) = R_{X_I X_I}(\tau)\cos\omega_c\tau + R_{X_Q X_I}(\tau)\sin\omega_c\tau}$$

- The relationship between correlation functions is similar to that of between *X*(*t*), *XI*(*t*) and *XQ*(*t*).



Facts about Narrow-band R.P.(3)

- Relationship between power spectrum:

$$\boxed{R_{X_I}(\tau) = \text{LPF}\{2R_{XX}(\tau)\cos(\omega_c \tau)\}}$$

$$\boxed{R_{X_Q X_I}(\tau) = \text{LPF}\{2R_{XX}(\tau)\sin(\omega_c \tau)\}}$$

- $S_{X_I}(\omega), S_{X_Q}(\omega)$ concentrates in $|\omega| < \Delta\omega/2$, so they are low-frequency R.P. , then:

$$S_{X_I}(\omega) = S_{X_Q}(\omega) = S_X(\omega + \omega_c) + S_X(\omega - \omega_c) \quad |\omega| < \Delta\omega/2$$

$$S_{X_Q X_I}(\omega) = -S_{X_I X_Q}(\omega) = j[S_X(\omega + \omega_c) - S_X(\omega - \omega_c)] \quad |\omega| < \Delta\omega/2$$

- If the S.D.F of X(t) is *symmetry* about wc , then:

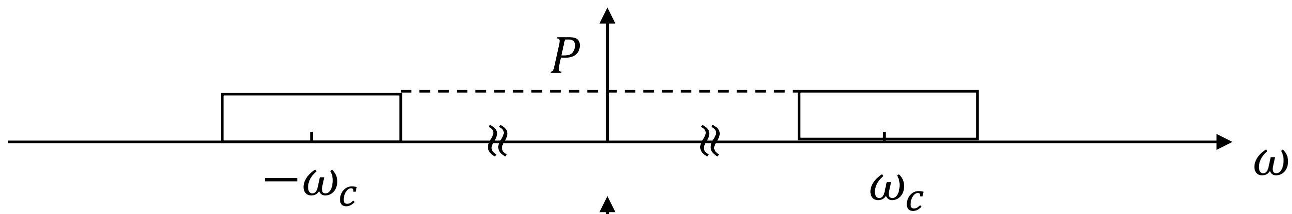
$$S_{X_I X_Q}(\omega) = S_{X_Q X_I}(\omega) = 0$$

Relationship between power spectrum:

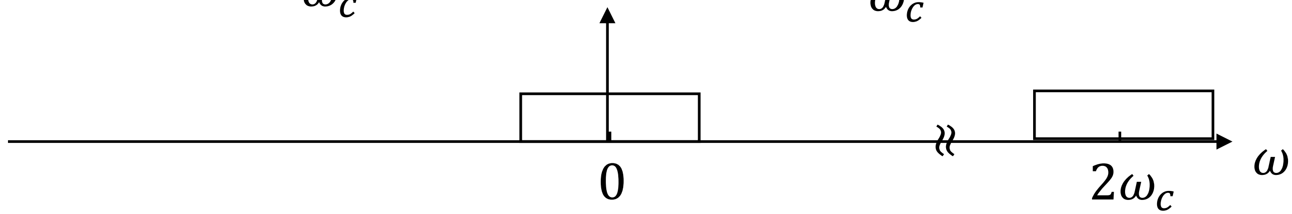
Demodulation

The S.D.F of X(t) is symmetry about ω_c

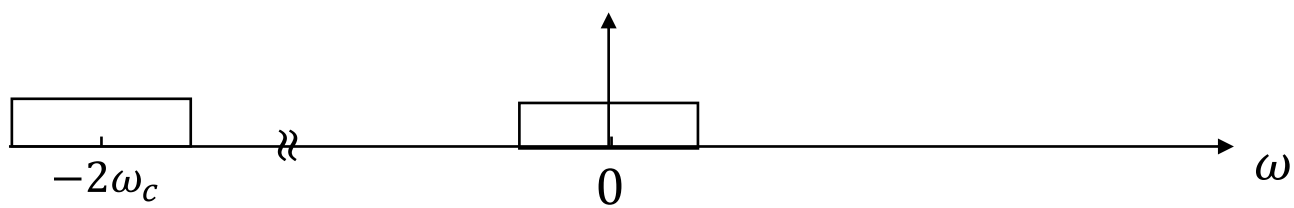
$S_X(\omega)$



$S_X(\omega - \omega_c)$



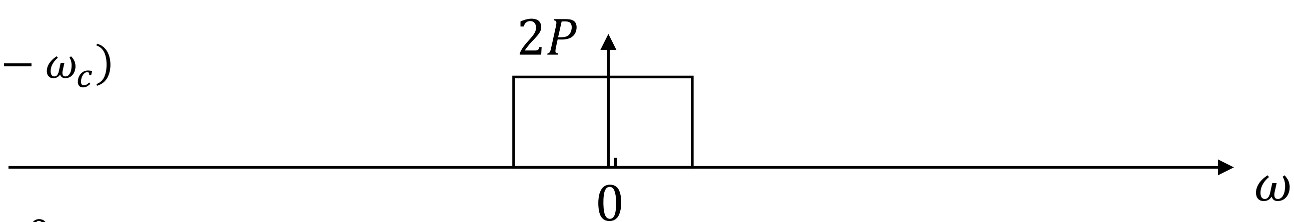
$S_X(\omega + \omega_c)$



$S_{X_I}(\omega) = S_{X_Q}(\omega)$

$= S_X(\omega + \omega_c) + S_X(\omega - \omega_c)$

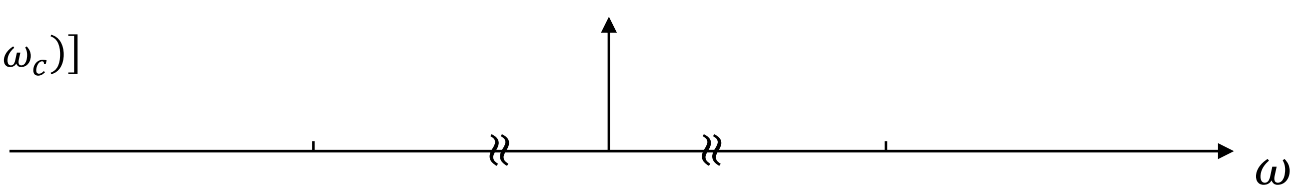
$|\omega| < {}^{\Delta\omega}/_{2}$



$S_{X_Q X_I}(\omega) = -S_{X_I X_Q}(\omega) = 0$

$j[S_X(\omega + \omega_c) - S_X(\omega - \omega_c)]$

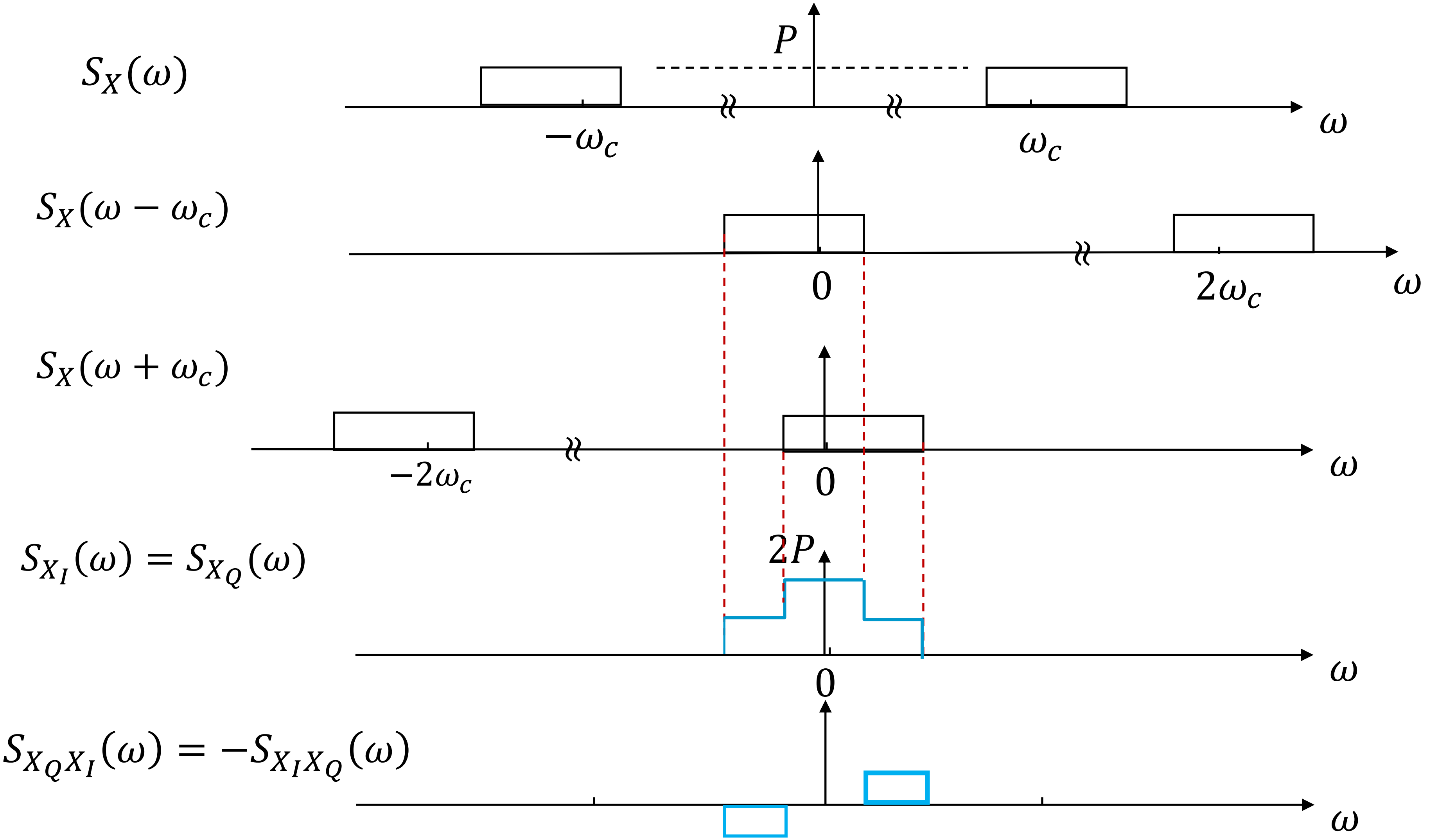
$|\omega| < {}^{\Delta\omega}/_{2}$



Relationship between power spectrum:

Demodulation

The S.D.F of X(t) **is not symmetry** about ω_c



4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of Band

4.4.2 An introduction of Hilbert Transform

4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

References

1. Distribution of Amplitudes and Phases for Narrow-Band Processes
   a. Peyton Z.Peebles, Probability, Random variables, and Random Signal Principles, O211 W58/2，

      Section 8.6，Bandpass,Band-limited and Narrowband Processes
   b. 樊昌信，通信原理，TN91 20/4,

      Section 2.6 窄带随机过程

2. Distribution of a Sinusoidal Signal Plus Noise
   a. Section 10.6, Envelop and Phase of a sinusoidal signal plus noise
   b. Section 2.7 正弦波加窄带高斯过程

4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

Suppose: $$X(t) = A(t)\cos[\omega_0 t + \varphi(t)]$$
$$= X_I(t)\cos\omega_0 t + X_Q(t)\sin\omega_c t$$

(1) X(t) is a narrow-band Gaussian Random Process with zero mean;

(2) The variance of X(t) is σ_X^2;

(3) The P.D.F of X(t) is symmetry about ω_c.

Obtain: Distribution of Envelope and Phase of *X*(*t*)

$$A(t) = \sqrt{X_I^2(t) + X_Q^2(t)}$$
$$\phi(t) = \arctan\frac{X_Q(t)}{X_I(t)}$$

$$X_I(t) = X(t)\cos\omega_0 t + \hat{X}(t)\sin\omega_0 t$$
$$X_Q(t) = X(t)\sin\omega_0 t - \hat{X}(t)\cos\omega_0 t$$

4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- $X_I(t)$ and $X_Q(t)$ are all linear combination of X(t), so, if X(t) is a Gaussian random variable, then , $X_I(t)$ and $X_Q(t)$ are also *Gaussian* random variable with zero mean and variance σ_X^2 .

Step1: Find the distribution of $X_I(t)$ and $X_Q(t)$

Step2: Using Jacobian transformation to find the distribution of

$A(t)$ and $\phi(t)$

Step1:

(1) Since the P.D.F of X(t) is symmetry about ω_c , $X_I(t)$ and $X_Q(t)$ are uncorrelated. $X_I(t)$ and $X_Q(t)$ are Gaussian random variable, then they are independent;

(2) $$f_{X_I X_Q}(x_i, x_q) = f_{X_I}(x_i) f_{X_Q}(x_q) = \frac{1}{2\pi\sigma_X^2} \exp[-\frac{x_{it}^2 + x_{qt}^2}{2\sigma_X^2}]$$

4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- Let A_t, ϕ_t denote $A(t)$ 、$\phi(t)$ respectively, then:

$$f_{A\phi}(a_t, \varphi_t) = |J|\, f_{X_I X_Q}(x_{it}, x_{qt})$$

- And:

$$X_{it} = A_t \cos\phi_t, \qquad X_{qt} = A_t \sin\phi_t$$

$$0 \le A_t < \infty, \quad 0 \le \phi_t < 2\pi$$

- So:

$$\begin{aligned} f_{A\phi}(a_t, \varphi_t) &= |J|\, f_{X_I X_Q}(x_{it}, x_{qt}) \\ &= a_t f_{X_I X_Q}(x_{it}, x_{qt}) \\ &= \frac{a_t}{2\pi\sigma_X^2} \exp[-\frac{a_t^2}{2\sigma_X^2}] \quad a_t \ge 0, 0 \le \phi_t < 2\pi \end{aligned}$$

4.4.5 Distribution of Envelope and Phases for Gaussian Narrow-Band Processes

- Thus:

$$f_A(a_t) = \int_0^{2\pi} f_{A\phi}(a_t, \varphi_t)\,d\varphi_t = \frac{a_t}{\sigma_X^2}\exp\left(-\frac{a_t^2}{2\sigma_X^2}\right) \quad a_t \geq 0$$

$$f_\phi(\varphi_t) = \int_0^{\infty} f_{A\varphi}(a_t, \varphi_t)\,da_t = \frac{1}{2\pi} \quad 0 \leq \phi_t < 2\pi$$

- That is: The envelope follows a *Reyleigh distribution*; the phase follows a uniform distribution.
- And: $f_{A\phi}(a_t, \varphi_t) = f_A(a_t) \cdot f_\phi(\varphi_t)$ illustrates that the envelope and phase are independent.

The mean of Reyleigh:

$$\sigma_X \sqrt{\frac{\pi}{2}} \approx 1.253\sigma_X$$

The variance of Reyleigh:

$$\sigma_X^2 \frac{4-\pi}{2} \approx 0.429\sigma_X^2$$

4.4 Narrow-band Gaussian Processes

4.4.1 The Definition of Band

4.4.2 An introduction of Hilbert Transform

4.4.3 Representation of Narrow-Band Signals

4.4.4 Narrow Band Random Processes

4.4.5 Gaussian Narrow-Band Random Processes

4.4.6 Sine Wave Plus Narrow-Band Noise

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

Given: $r(t)$ is a sinusoidal signal with random phase plus a zero-mean, stationary Gaussian random process with a narrow-band spectrum,

$$r(t) = s(t) + n(t)$$

$$s(t) = A\cos[\omega_0 t + \theta]$$

$$n(t) = n_c(t)\cos(\omega_0 t) - n_s(t)\sin(\omega_0 t)$$

where A is constant, θ is uniformly distributed on $[0, 2\pi]$,

$n(t)$ is the narrow-band Gaussian noise with zero mean and variance σ_n^2 .

Obtain: Probability density function of amplitude and phase of $r(t)$.

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

Sln: $$r(t) = A\cos[\omega_0 t + \theta] + n_c(t)\cos(\omega_0 t) - n_s(t)\sin(\omega_0 t)$$
$$= [A\cos\theta + n_c(t)]\cos(\omega_0 t) - [A\sin\theta + n_s(t)]\sin(\omega_0 t)$$
$$= Z_c(t)\cos(\omega_0 t) - Z_s(t)\sin(\omega_0 t)$$
$$= Z(t)\cos[\omega_0 t + \phi(t)]$$

Where,

$$Z_c(t) = Z(t)\cos\phi(t)$$
$$Z_s(t) = Z(t)\sin\phi(t)$$

$$Z(t) = \sqrt{Z_c^2(t) + Z_s^2(t)}$$
$$\phi(t) = ac\tan\frac{Z_s(t)}{Z_c(t)}$$

Step 1: find the joint distribution of $Z_c(t)$ and $Z_s(t)$

Step 2: Using Jacobian transformation to find the distribution of $Z(t)$ and $\phi(t)$

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

- Modified Bessel function of order zero

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp[x\cos\theta]d\theta$$

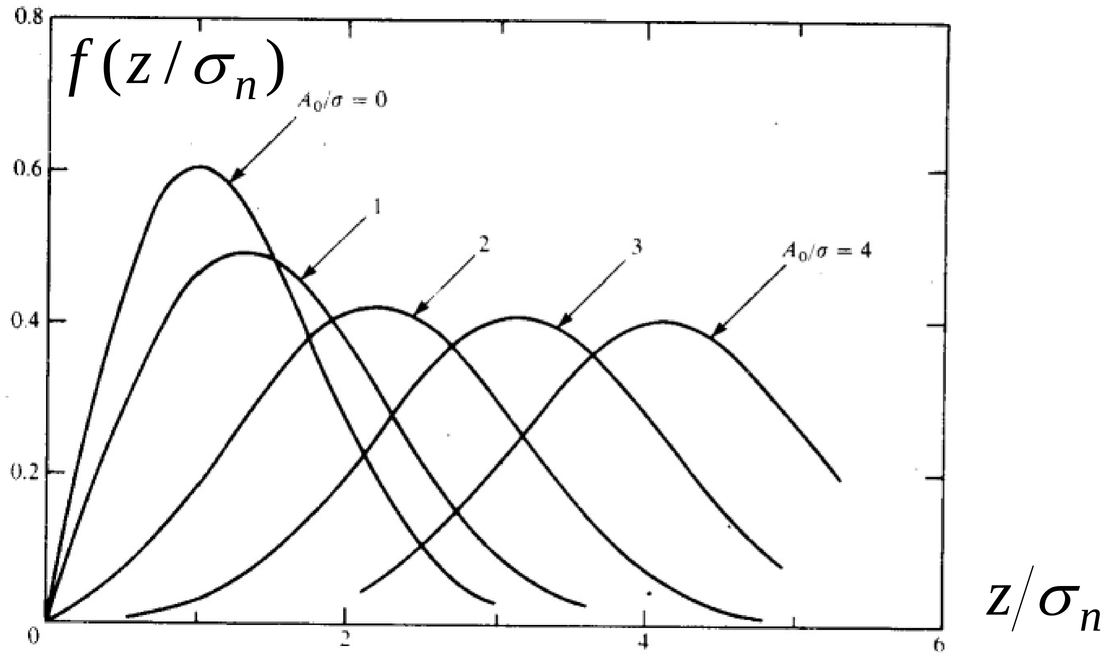
$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}} \qquad x >> 1$$

- Probability densities of the envelope follows Race distribution:

$$f(z) = \frac{z}{\sigma_n^2}\exp[-\frac{1}{2\sigma_n^2}(z^2 + A^2)]I_0(\frac{Az}{\sigma_n^2}) \qquad z \geq 0$$

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

Probability densities of the envelope for various ratios $\dfrac{A}{\sigma_n}$



$\boxed{\dfrac{A}{\sigma_n} \ll 1}$, $f(z/\sigma_n)$ is similar to Reyleigh distribution;

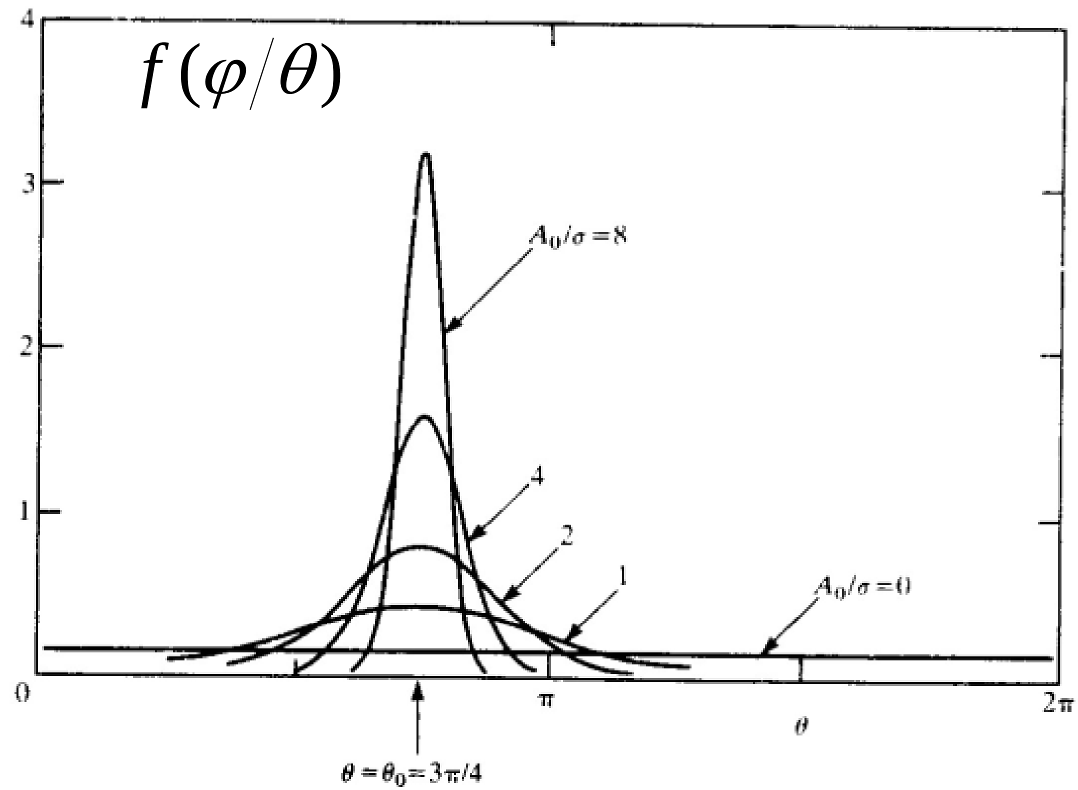
$\boxed{\dfrac{A}{\sigma_n} \gg 1}$, $f(z/\sigma_n)$ is similar to standard Gaussian distribution.

4.4.6 Distribution of a Sinusoidal Signal Plus Noise

Probability densities of the phase for various ratios $\dfrac{A}{\sigma_n}$.

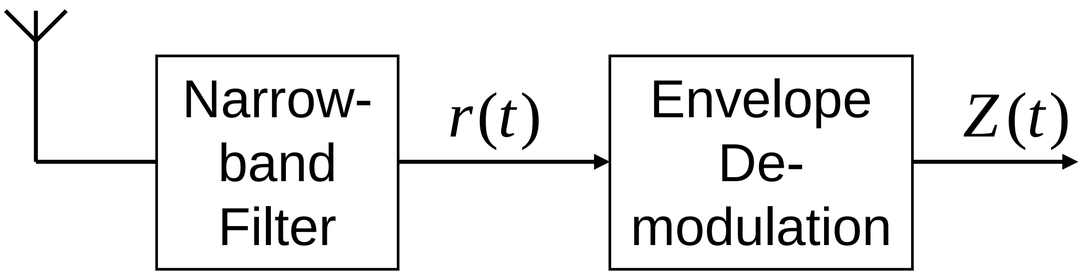
$$f(\varphi/\theta)=\frac{1}{2\pi}e^{-\frac{\alpha^2}{2}}+\frac{\alpha\cos(\varphi-\theta)}{\sqrt{2\pi}}e^{-\frac{\alpha^2\sin^2(\varphi-\theta)}{2}}\Phi\left[\frac{\alpha\cos(\varphi-\theta)}{\sigma_n}\right]$$

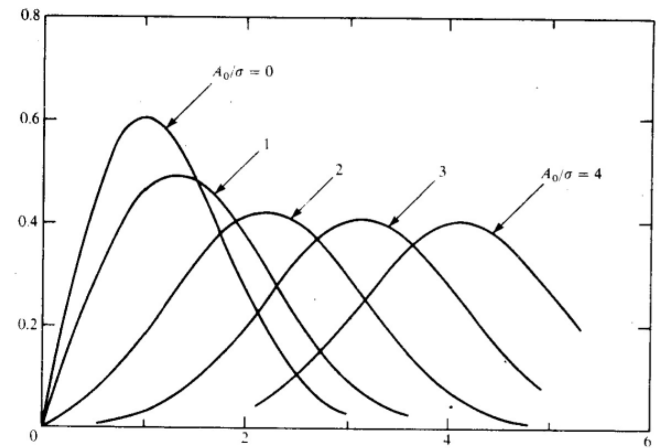
$$\alpha=\frac{A}{\sigma_n}$$



Application

- From the amplitude Z(t) of the received signal r(t), we may detect whether the s(t) exists.





- If s(t) exists, Z(t) follows Race distribution. More large of s(t) than noise, the peak wave is further from y-axis.
- If s(t) is not exist, Z(t) follows Reyleigh distribution which is near to y-axis.

Homework

4.14,

4.16

$$x(t) = \operatorname{sgn}(t) = \begin{cases} +1 & (t > 0) \\ -1 & (t < 0) \end{cases} \qquad X(j\omega) = \lim_{a \to 0} X_1(j\omega) = \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$

$$x(t) = \sin \omega_0 t \qquad X(j\omega) = \pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$$

$$x(t) = \cos \omega_0 t \qquad X(j\omega) = \pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$$