1.1 A sequence of binary digits is transmitted in a certain communication system. Any given digit is received erroneously with probability p and received correctly with probability $1-p$. Errors occur independently from digit to digit. Out of a sequence of n digits transmitted, what is the probability that no more than j digits are received erroneously?

Solution:

Assume the number of erroneous digits is X. Obviously, $X \sim B(n,p)$

$$P(X \leq j) = \sum_{i=0}^{j} P(X=i) = \sum_{i=0}^{j} \mathrm{C}_n^i p^i (1-p)^{n-i}$$

1.2 Suppose that X_1 and X_2 are Gaussian random variables with means μ_1 and μ_2, respectively. Assume that $\mu_1 \neq \mu_2$. The variance for each of the two random variables is σ^2. Find the value of x for which $f_{X_1}(x) = f_{X_2}(x)$.

Solution:

As $X_1 \sim N(\mu_1, \sigma), X_2 \sim N(\mu_2, \sigma)$, their $p.d.fs$ is

$$f_{X_1}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right), f_{X_2}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu_2)^2}{2\sigma^2}\right)$$

Let $f_{X_1}(x) = f_{X_2}(x) \Rightarrow (x-\mu_1)^2 = (x-\mu_2)^2 \Rightarrow x = \dfrac{\mu_1+\mu_2}{2}$

What is the characteristic function of an exponential random variable X? Find $E[X^3]$.

Solution:

The $p.d.fs$ of exponential random variable is $f(x) = \begin{cases} \lambda e^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$

It's characteristic function: $\phi(t) = \displaystyle\int_0^{\infty} e^{itx} \lambda e^{-\lambda x} \mathrm{d}x = \frac{\lambda}{\lambda - it}$

$$E[X^3] = \frac{1}{i^3}\phi^{(3)}(0) = \frac{1}{i^3}\left(\frac{\lambda}{\lambda - it}\right)^{(3)}\Bigg|_{t=0} = \frac{6}{\lambda^3}$$

What is the probability generating function of an binomial random variable?

Solution:

Assume an binomial random variable $X \sim B(n,p)$

The probability generating function can be calculated as follow:

$$P_X(z) = a^g(z) = \sum_{k=0}^{n} P(X=ki) z^k = \sum_{k=0}^{n} \mathrm{C}_n^k p^k (1-p)^{n-k} z^k$$

Consider a clock powered by a single AAA battery. When a battery fails, the owner replaces the failed battery by one chosen randomly from two different brands. With probability p one of brand 1 is chosen, and with probability $1-p$ one of brand 2 is chosen. Let X_i denote the lifetime of a brand i battery. For $i=1$ and 2, we assume that$E\{X_i\} = \mu_i$ and $Var\{X_i\} = \sigma_i^2$. Let X denote the lifetime of the battery in use. Find $E[X]$ and $Var[X]$.

Solution:

$X = pX_1 + (1-p)X_2$

$$E[X] = pE[X_1] + (1-p)E[X_2] = p\mu_1 + (1-p)\mu_2$$

$$Var[X] = p^2 Var[X_1] + (1-p)^2 Var[X_2] = p^2\sigma_1^2 + (1-p)^2\sigma_2^2$$