第六章　常微分方程初值问题的数值解法

1. 微分方程

包含自变量、函数以及函数导数的方程称为微分方程，如关于函数$u(x,y,t)$的微分方程：

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=D\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+f(x,y,t;u)$$

偏微分

在微分方程中，如果自变量的个数只有一个，就称为常微分方程，如：

$$\frac{du}{dt}=f(t;u)$$

如果自变量个数两个及以上，就称为偏微分方程，如$a\neq 0,b\neq 0,D\neq 0$时

一阶常微分方程组初值问题：

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_m' \end{bmatrix}=\begin{bmatrix} f_1(t;u_1,u_2,\cdots,u_m) \\ f_2(t;u_1,u_2,\cdots,u_m) \\ \vdots \\ f_m(t;u_1,u_2,\cdots,u_m) \end{bmatrix},$$

简记为
$$\vec{u}'=f(t;\vec{u})$$
$$\vec{u}(0)=\vec{u}_0$$

$$u_1(0)=u_{1,0},\quad \cdots,u_m(0)=u_{m,0}$$

初始条件.

高阶常微分方程

$$a_m u^{(m)} + a_{m-1} u^{(m-1)} + \cdots + a_1 u' = f(t; u, u', \cdots u^{(m)})$$

可以化为一阶常微分方程组：

$u_0 = u$

令： $u_1 = u' \triangleq u_0', \ u_2 = u'' = u_1', \ \cdots, \quad u_m = u^{(m)} = u_{m-1}'$

$$\begin{bmatrix} u_0' \\ u_1' \\ \vdots \\ u_{m-1}' \\ \sum_{i=0}^{m} a_i u_i \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ f(t; u_0, u_1, \ldots, u_m) \end{bmatrix}$$

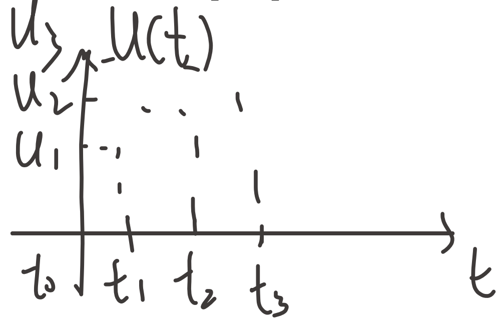
本章只考虑最简单的形式：一阶常微分方程初值问题

$$\frac{du}{dt} = f(t; u)$$

$$u(0) = u_0$$

如果$f(t;u)$在$t \in [a,b]$满足李普希兹（Lipschitz）条件，那么初值问题在区间[a,b]上具有唯一连续可微解

$$|f(t;u_1) - f(t;u_2)| \le L|u_1 - u_2|$$



微分方程的数值解法，就是寻找微分方程在离散节点：$t_0 < t_1 < t_2 \ldots < t_n$上的近似值$y_0, y_1, \ldots, y_n$。$h_n = t_{n+1} - t_n$称为步长。特别步长是定值时，那么$t_n = t_0 + nh$

常微分方程的数值解法的基本出发点就是将连续的问题离散化，并且采用“步进式”的解法，即求解过程顺着节点排序的次序一步一步向前推进:

$$u_{n+1} = \varphi(u_0, u_1, \ldots, u_{n-1})$$

常用的设计方法：数值微分、数值积分、泰勒展开等。

基于数值微分的方法

$$\frac{du}{dt} = f(t;u)$$

数值微分近似

向前差商：

$$\left.\frac{du}{dt}\right|_{t_n} \approx \frac{u_{n+1} - u_n}{h}$$

$$\frac{u_{n+1} - u_n}{h} = f(t_n, u_n) \to \boxed{u_{n+1} = u_n + hf(t_n, u_n)}$$

显式Euler格式

$$\left.\frac{du}{dt}\right|_{t_n} = f(t_n, u_n)$$

$u(t)$: $u_n = u(t_n)$

$u_n \to u_{n+1} \to u_{n+2}$.

向后差商：

$$\left.\frac{du}{dt}\right|_{t_{n+1}} \approx \frac{u_{n+1}-u_n}{h}$$

例 $f(t,u)=\sin t\, e^{u}$

$$\frac{u_{n+1}-u_n}{h} = f(t_{n+1},u_{n+1}) \to \boxed{u_{n+1} = u_n + hf(t_{n+1},u_{n+1})}$$

隐式Euler格式

每一步需要非线性方程求根

平均 $u_{n+1} = u_n + \frac{h}{2}\left[f(t_n,u_n)+f(t_{n+1},u_{n+1})\right]$

如何简化求根？

（1） $u^* = u_n + hf(t_n, u_n)$ → 预估

$u_{n+1} = u_n + hf(t_{n+1}, u^*)$ → 校正

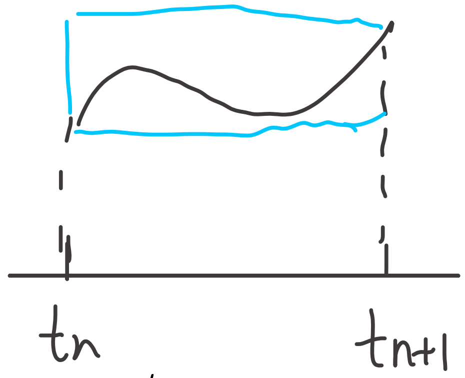
（2） $u^* = u_n + hf(t_n, u_n)$

$u^{**} = u_n + hf(t_{n+1}, u^*)$

取平均： $$\boxed{u_{n+1} = \frac{u^* + u^{**}}{2} = u_n + \frac{h}{2}\left[f(t_n, u_n) + f(t_{n+1}, u^*)\right]}$$

改进Euler法，又称“预估-校正”法

基于数值积分的方法



$$\frac{du}{dt} = f(t;u)$$

$$\int_{t_n}^{t_{n+1}} \frac{du}{dt}dt = \int_{t_n}^{t_{n+1}} f(t,u)dt \Rightarrow u_{n+1} - u_n = \int_{t_n}^{t_{n+1}} f(t,u)dt$$

$$U_{n+1} = U_n + \frac{h}{6}\left[f(t_n, U_n) + 4f(t_{n+\frac{1}{2}}, U_{n+\frac{1}{2}}) + f(t_{n+1}, U_{n+1})\right]$$

数值积分近似

左矩形积分： $u_{n+1} = u_n + hf(t_n, u_n)$ 显式Euler格式

右矩形积分： $u_{n+1} = u_n + hf(t_{n+1}, u_{n+1})$ 隐式Euler格式

梯形积分： $u_{n+1} = u_n + \frac{h}{2}\left[f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right]$

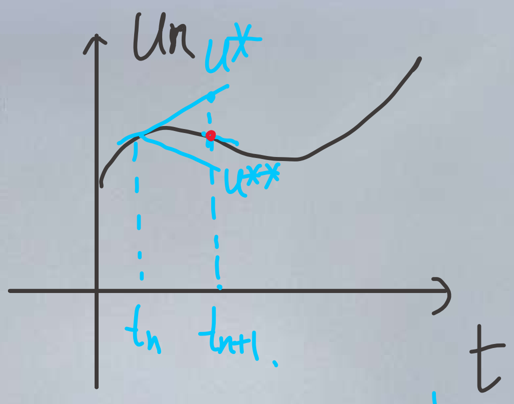
预估-校正格式 $u^* = u_n + hf(t_n, u_n)$

中点矩形积分：$u_{n+1} = u_n + hf(t_n + h / 2,\ u(t_n + h / 2))$

怎么预测？

$$u^* = u_n + \frac{h}{2} f(t_n,\ u_n)$$

Runge-Kutta格式



显式Euler格式

$$u_{n+1} = u_n + hf(t_n, u_n)$$

可以写为：

$$K_1 = f(t_n, u_n)$$
$$u_{n+1} = u_n + hK_1$$

改进Euler格式

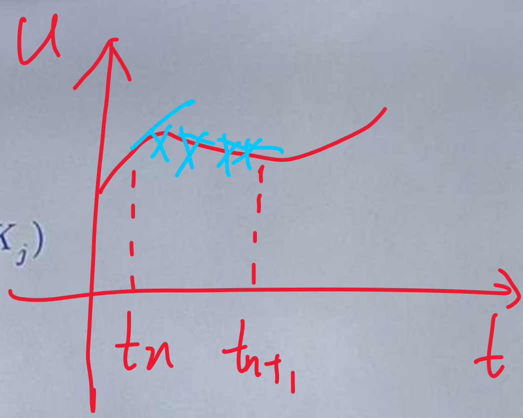
$$u^* = u_n + hf(t_n, u_n)$$
$$u^{**} = u_n + hf(t_{n+1}, u^*)$$
$$u_{n+1} = u_n + \frac{h}{2}\left[f(t_n, u_n) + f(t_{n+1}, u^*)\right]$$

可以写为：

$$K_1 = f(t_n, u_n)$$
$$K_2 = f(t_n + h, u_n + hK_1)$$
$$u_{n+1} = u_n + \frac{h}{2}(K_1 + K_2)$$

使用$f(t,u)$在某些点上的线性组合得到$u_{n+1} \approx u(t_{n+1})$

一般Runge-Kutta（RK）格式



$$K_1 = f(t_n, u_n)$$
$$K_i = f(t_n + c_i h, u_n + h\sum_{j=1}^{i-1} a_{ij} K_j)$$
$$u_{n+1} = u_n + h\sum_{i=1}^{p} b_i K_i$$

a_{ij}, b_i, c_i待定系数，确定原则是使得近似公式在(t_n, u_n)的Taylor展开与$y(t_n)$的Taylor展开尽可能多的符合

p: RK格式的阶数

显式Euler格式：1阶RK

改进Euler格式：2阶RK

$K_1 =$

$K_2 = (K_1)$

$K_3 = (K_1, K_2)$

$K_4 = (K_1, K_2, K_3)$.

$p=2$（二阶RK）：

$$K_1 = f(t_n, u_n)$$
$$K_2 = f(t_n + c_2 h, u_n + h a_{21} K_1)$$

$$u_{n+1} = u_n + h(b_1 K_1 + b_2 K_2)$$

$$\boxed{\begin{aligned} &\frac{du}{dt} = f(t,u) \\ &\to \frac{d^2u}{dt^2} = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = f_t + f f_u \end{aligned}}$$

$$\begin{aligned} K_2 &= f(t_n + c_2 h, u_n + h a_{21} K_1) \\ &= f(t_n, u_n) + c_2 h \frac{\partial f}{\partial t}(t_n, u_n) + h a_{21} f(t_n, u_n) \frac{\partial f}{\partial u}(t_n, u_n) + O(h^2) \end{aligned}$$

$$\begin{aligned} u_{n+1} &= u_n + h(b_1 K_1 + b_2 K_2) \\ &= u_n + b_1 h f(t_n, u_n) \\ &\quad + b_2 h \left[f(t_n, u_n) + c_2 h \frac{\partial f}{\partial t}(t_n, u_n) + h a_{21} f(t_n, u_n) \frac{\partial f}{\partial u}(t_n, u_n) + O(h^2) \right] \\ &= u_n + (b_1 + b_2) h f(t_n, u_n) + b_2 c_2 h^2 f_t(t_n, u_n) + b_2 a_{21} h^2 f(t_n, u_n) f_u(t_n, u_n) + O(h^3) \end{aligned}$$

$u_{n+1} \approx u(t_{n+1})$ 准确值.

$$u_{n+1} = u_n + (b_1 + b_2)hf(t_n, u_n) + b_2 c_2 h^2 f_t(t_n, u_n) + b_2 a_{21} h^2 f(t_n, u_n) f_u(t_n, u_n) + O(h^3)$$

$$\begin{aligned} u(t_{n+1}) &= u(t_n) + hu'(t_n) + \frac{h^2}{2} u''(t_n) + O(h^3) \\ &= u_n + hf(t_n, u_n) + \frac{h^2}{2}\left[\frac{\partial f}{\partial t}(t_n, u_n) + f(t_n, u_n)\frac{\partial f}{\partial u}(t_n, u_n)\right] + O(h^3) \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} b_1 + b_2 &= 1 \\ b_2 c_2 &= \frac{1}{2} \\ b_2 a_{21} &= \frac{1}{2} \end{aligned}}$$

$$K_1 = f(t_n, u_n)$$
$$K_2 = f(t_n + c_2 h, u_n + h a_{21} K_1)$$

$$u_{n+1} = u_n + h(b_1 K_1 + b_2 K_2)$$

$b_1 = b_2 = 1/2, c_2 = 1, a_{21} = 1$:

$$K_1 = f(t_n, u_n),\ K_2 = f(t_n + h, u_n + hK_1),\ u_{n+1} = u_n + \frac{h}{2}(K_1 + K_2)$$

改进Euler格式（梯形）

$b_1 = 0, b_2 = 1, c_2 = 1/2, a_{21} = 1/2$:

$$K_1 = f(t_n, u_n),\ K_2 = f(t_n + 0.5h, u_n + 0.5hK_1),\ u_{n+1} = u_n + hK_2$$

中点Euler格式

3阶RK格式

$$
\begin{aligned}
K_1 &= f(t_n, u_n)\\
K_2 &= f(t_n + c_2 h, u_n + h a_{21} K_1)\\
K_3 &= f(t_n + c_3 h, u_n + h(a_{31} K_1 + a_{32} K_2))\\
u_{n+1} &= u_n + h(b_1 K_1 + b_2 K_2 + b_3 K_3)
\end{aligned}
$$

6个方程，8个未知数

$$
\begin{gathered}
b_1 + b_2 + b_3 = 1\\
c_2 = a_{21}\\
c_3 = a_{31} + a_{32}\\
b_2 c_2 + b_3 c_3 = \frac{1}{2}\\
b_2 c_2^2 + b_3 c_3^2 = \frac{1}{3}\\
b_3 c_2 a_{32} = \frac{1}{6}
\end{gathered}
$$

常用的3阶RK格式:

$$
\begin{gathered}
b_1 = \frac{1}{6}, b_2 = \frac{4}{6}, b_3 = \frac{1}{6}\\
c_2 = \frac{1}{2}, c_3 = 1\\
a_{21} = \frac{1}{2}, a_{31} = -1, a_{32} = 2
\end{gathered}
$$

$$
\begin{aligned}
K_1 &= f(t_n, u_n)\\
K_2 &= f(t_n + \tfrac{1}{2} h, u_n + \tfrac{1}{2} h K_1)\\
K_3 &= f(t_n + h, u_n + h(2K_2 - K_1))\\
u_{n+1} &= u_n + \tfrac{h}{6}(K_1 + 4K_2 + K_3)
\end{aligned}
$$

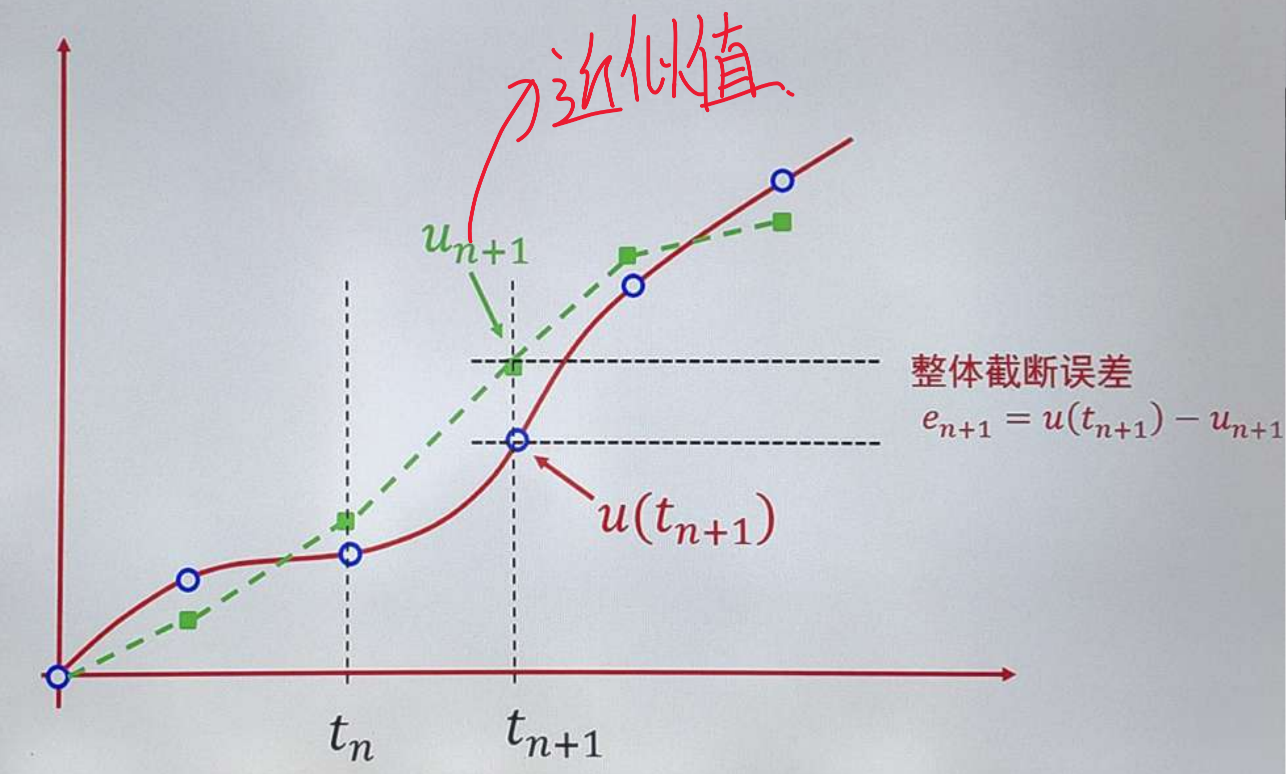
思考：和辛普森积分公式有什么关联？

格式的误差（精度）、收敛性、数值稳定性

（一步显式格式为例）

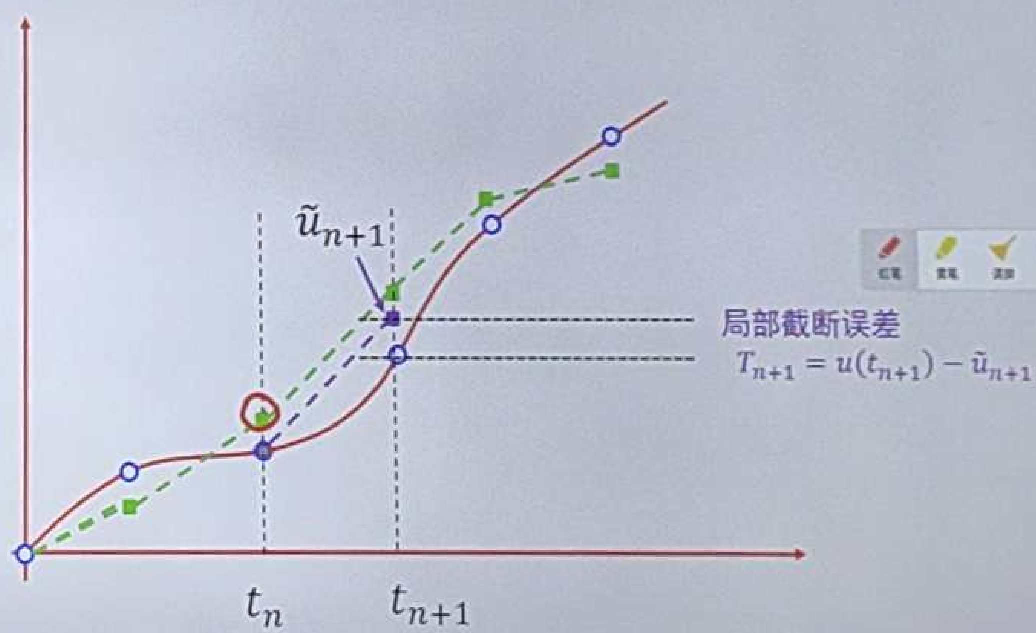
截断误差 $u' = f(t,u) \longrightarrow$ 精确解 $u(t)$

$$u_0 = u(0)$$
$$u_{n+1} = \varphi(u_n) \longrightarrow \text{数值解 } u_n \approx u(t_n)$$



假设 $u_n = u(t_n)$ →拉到准确值

$\tilde{u}_{n+1} = \varphi(u_n) \qquad (\neq u_{n+1})$



➢ 定义（相容性）：

如果局部截断误差 $T_{n+1} = O(h^{p+1})$，称格式具有 p 阶精度，与原微分方程是p阶相容的。特别的，如果 $= Ah^{p+1}$

$$\frac{1}{h}T_{n+1} \to 0, \qquad as\ h \to 0$$

则称格式是相容的

➢ 定义（收敛性）：

如果整体截断误差 $e_{n+1} = O(h^p)$，称格式 p 阶收敛。特别的，如果

$$e_{n+1} \to 0, \qquad as\ h \to 0$$

则称格式是收敛的

一般地，一个格式如果局部截断误差是$T_{n+1} = O(h^{p+1})$，则整体截断误差$e_{n+1} = O(h^p)$

稳定性：

假设 u_0 产生了扰动δ_0，其后的计算值都会产生扰动。我们得到两个序列：

$\bar{u}_0 = u_0 + \delta_0$

$$\bar{u}_{n+1} = \varphi(\bar{u}_n)$$
$$u_{n+1} = \varphi(u_n)$$

$\bar{u}_1 = \varphi(\bar{u}_0)$

如果 $|\bar{u}_k - u_k| < C|\delta_0|$, 则称格式稳定

绝对稳定：如果对试验方程 $f(t,u)=\lambda u$

$$u' = \lambda u, \qquad Re(\lambda) < 0$$

满足 $|\delta_{n+k}| < |\delta_n|$, 则称格式是绝对稳定的。稳定性依赖$\lambda$和$h$

例：显式 Euler 格式分析.

$$u' = f(t, u)$$

$$u_0 = u(0)$$

$$u_{n+1} = u_n + h f(t_n, u_n)$$

1. 局部截断误差

$$T_{n+1} = u(t_{n+1}) - u_{n+1}$$

$$u_{n+1} = u(t_n) + h f(t_n, u_n)$$

$$u(t_{n+1}) = u(t_n) + h \frac{du}{dt} + \frac{h^2}{2} \frac{d^2u}{dt^2} + o(h^2)$$

$$= u(t_n) + h f(t_n, u_n) + \frac{h^2}{2} \frac{d^2u}{dt^2} + o(h^3)$$

$$T_{n+1} = \frac{h^2}{2} \frac{d^2u}{dt^2} + o(h^3) = O(h^2)$$

$$= o(h^{1+1}).$$

一阶相容.

$$\lim_{h \to 0} \frac{T_{n+1}}{h} = \lim_{h \to 0} o(h) = 0.$$

$$e = o(h)$$

稳定性: $u_{n+1} = u_n + h\lambda u_n = (1+h\lambda)u_n$.

$\overline{u_{n+1}} = (1+h\lambda)\overline{u_n}$.

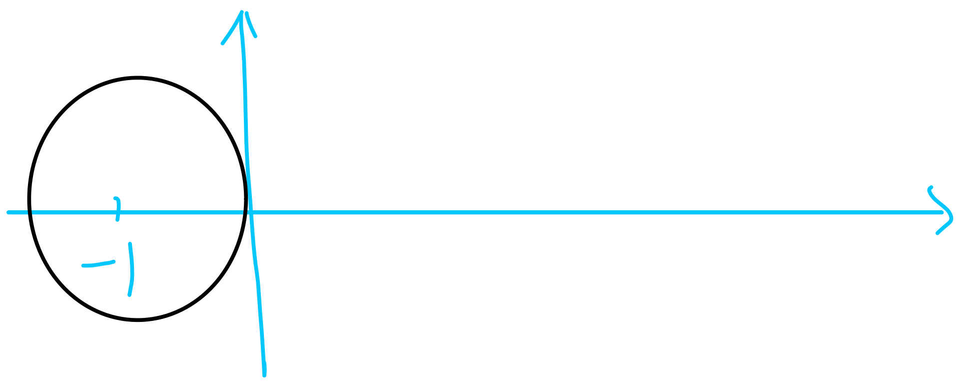
$\varepsilon_{n+1} = u_{n+1} - \overline{u_{n+1}} = (1+h\lambda)(u_n - \overline{u_n})$

$= (1+h\lambda)^2(u_{n-1} - \overline{u_{n-1}})$

$= (1+h\lambda)^{n+1}(u_0 - \overline{u_0})$

$|\varepsilon_{n+1}| = |(1+h\lambda)^{n+1}\varepsilon_0|$

$|1+h\lambda| < 1$



CH-1 误差—有效数字.