

科学计算引论作业(七)

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5.1 试分别利用中矩公式、梯形公式及 Simpson 公式计算定积分

$$I = \int_0^{\frac{1}{2}} \exp(3x) \cos 2x dx,$$

并比较其计算精度。

解：

中矩公式：

$$\int_0^{\frac{1}{2}} \exp(3x) \cos 2x dx \approx \left(\frac{1}{2} - 0\right) f\left(\frac{1}{4}\right) = 0.9289211490504564$$

梯形公式：

$$\int_0^{\frac{1}{2}} \exp(3x) \cos 2x dx \approx \left(\frac{1}{2} - 0\right) \frac{f\left(\frac{1}{2}\right) + f(0)}{2} = 0.8553667347219239$$

Simpson公式：

$$\int_0^{\frac{1}{2}} \exp(3x) \cos 2x dx \approx \frac{\frac{1}{2} - 0}{6} \left(f(0) + 4f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right)\right) = 0.9044030109409457$$

比较准确的解为 $I = 0.9082171883002617$, $\varepsilon = 1.0083236338112719e - 14$, 可以看出 Simpson 公式精度最好

5.2 试确定下面求积公式

$$\int_{-1}^1 f(x) dx \approx a[f(x_0) + f(x_1) + f(x_2)],$$

使其具有 3 次代数精度，并由该公式计算定积分：

$$\int_{-1}^1 \frac{x \sin x}{\sqrt{1+x^2}} dx$$

解:

由代数精度的定义可以得到以下方程:

$$\begin{cases} a(1+1+1)=2 \\ a(x_0+x_1+x_2)=0 \\ a(x_0^2+x_1^2+x_2^2)=\frac{2}{3} \\ a(x_0^3+x_1^3+x_2^3)=0 \\ a(x_0^4+x_1^4+x_2^4)\neq 0 \end{cases} \Rightarrow \begin{cases} a=\frac{2}{3} \\ x_0=-\frac{\sqrt{2}}{2} \\ x_1=0 \\ x_2=\frac{\sqrt{2}}{2} \end{cases}$$

计算结果如下:

```
1 def f(x):
2     return x * np.sin(x)/np.sqrt(1+x**2)
3
4 integrate.quad(f, -1, 1), 2/3 * (f(-np.sqrt(2)/2) + f(0) + f(np.sqrt(2)/2))
```

Executed at 2023.11.08 22:20:33 in 4ms

((0.4833545313354184, 9.03548815670577e-09), 0.5000907488711981)

5.5 试构造两点 Gauss 型求积公式

$$\int_{-1}^1 f(x)dx \approx A_0 f(x_0) + A_1 f(x_1),$$

并由此计算积分:

$$\int_0^1 \sqrt{1+2x}dx$$

解:

由代数精度的定义可以得到以下方程:

$$\begin{cases} A_0 + A_1 = 2 \\ A_0 x_0 + A_1 x_1 = 0 \\ A_0 x_0^2 + A_1 x_1^2 = \frac{2}{3} \\ A_0 x_0^3 + A_1 x_1^3 = 0 \end{cases} \Rightarrow \begin{cases} A_0 = 1 \\ A_1 = 1 \\ x_0 = -\frac{1}{\sqrt{3}} \\ x_1 = \frac{1}{\sqrt{3}} \end{cases}$$

对于原积分不能直接使用上述的得到的Gauss积分公式, 我们进行一个积分变换:

$$\int_0^1 \sqrt{1+2x}dx \stackrel{t=2x-1}{=} \frac{1}{2} \int_{-1}^1 \sqrt{2+t}dt$$

计算结果如下:

```
def f(x):  
    return np.sqrt(1+2 * x)  
  
integrate.quad(f, 0, 1), 0.5 * (np.sqrt(2+1/np.sqrt(3)) + np.sqrt(2-1/np  
.sqrt(3)))  
Executed at 2023.11.08 22:52:17 in 1ms
```

((1.3987174742355442, 1.5528883448418452e-14), 1.3990808081581056)

5.7 (实验题) 分别用变步长梯形求积公式和 Romberg 算法计算椭圆积分

$$\int_0^{\pi} \frac{\sqrt{2}}{(1 + \sin^2 x) \sqrt{2 - \sin^2 x}} dx$$

要求其逼近值 $T_k, T_k^{(0)}$ 的计算精度分别满足 $|T_k - T_{k-1}| < 10^{-12}$ 和 $|T_k^{(0)} - T_{k-1}^{(0)}| < 10^{-12}$.

解:

结果如下:

```
1 import numpy as np  
2 from scipy import integrate  
3  
4 def trapezoid(f,a,b,N):  
5     h = (b-a)/N  
6     xi = np.linspace(a,b,N+1)  
7     fi = f(xi)  
8     s = 0.0  
9     for i in range(1,N):  
10         s = s + fi[i]  
11     s = (h/2)*(fi[0] + fi[N]) + h*s  
12     return s  
13  
14 # romberg method  
15 def romberg(f,a,b,eps,nmax):  
16     Q = np.zeros((nmax,nmax),float)  
17     converged = 0  
18     for i in range(0,nmax):  
19         N = 2**i  
20         Q[i,0] = trapezoid(f,a,b,N)  
21         for k in range(0,i):  
22             n = k + 2  
23             Q[i,k+1] = 1.0/((4**(n-1)-1))*((4**(n-1)*Q[i,k] - Q[i-1,k])  
24             if (i > 0):  
25                 if (abs(Q[i,k+1] - Q[i,k]) < eps):  
26                     converged = 1  
27                     break  
28             print(Q[i,k+1],N,abs(Q[i,k+1] - Q[i,k]))  
29             return Q[i,k+1],N,converged  
30  
31 f = lambda x: np.sqrt(2) / ((1 + np.sin(x) ** 2) * np.sqrt(2-np.sin(x) ** 2))  
32 romberg(f, 0, np.pi, 1e-12, 12)  
33  
34 result = integrate.romberg(f, 0, np.pi, tol=1e-12, divmax=12, show=True)  
Executed at 2023.11.13 08:48:17 in 4ms  
2.546254733498455 128 3.339550858072471e-13  
Romberg integration of <function vectorize1.<locals>.vfunc at 0x0000027A239B2480> from [0, 3.141592653589793]  
  
Steps StepSize Results  
1 3.141593 3.141593  
2 1.570796 2.681517 2.528159  
4 0.785398 2.549958 2.506105 2.504635  
8 0.392699 2.546258 2.545024 2.547619 2.548301  
16 0.196350 2.546255 2.546254 2.546336 2.546315 2.546308  
32 0.098175 2.546255 2.546255 2.546255 2.546254 2.546253 2.546253  
64 0.049087 2.546255 2.546255 2.546255 2.546255 2.546255 2.546255  
128 0.024544 2.546255 2.546255 2.546255 2.546255 2.546255 2.546255 2.546255  
  
The final result is 2.546254733498458 after 129 function evaluations.
```