第三章课后习题参考答案

3.1 n维Euclid空间

3.1.1

证若 $E' \subset E$, 则 $E = E' \cup E = E$; 反之, 若 $E = E$, 则由 $E = E' \cup E$, 得到 $E' \subset E$.

3.1.2.

因为 $E, F \subset \mathbf{R}^n$ 为有界闭集, 所以 E, F 为有界闭集, 于是可知 $E \cap F$ 和 $E\cup F$ 也都是有界闭集.

3.1.3.

由于 F 为闭集, 所以 F^c 为开集, 而 $E \backslash F = E \cap F^c$ 也是开集。由于 E 为开集, 所以 E^c 为闭集, 从而 $F \backslash E = F \cap E^c$ 也是闭集.

3.1.4

(1) $E' = \left\{(x, y) : x^2 + y^2 \geq 3\right\} = \bar{E}, E$ 非闭. (2) $E' = \mathbf{R}^2 = \bar{E}, E$ 非闭. (3) $E' = \emptyset, \bar{E} = E, E$ 闭. (4) $E' = \left\{(x, y) : y^2 - x^2 + 1 \leq 0\right\}, \bar{E} = E, E$ 闭. (5) $E' = \{(x, y) : y = \sin(1/x), x \in (0, 1]$ or $x = 0, y \in [-1, 1]\}. \bar{E} = E'$, E非闭。

3.1.5

(1) 由 $\forall k \in \{1, 2, \cdots, n\}$, 有

$$|x_k - y_k| \leq \left(\sum_{j=1}^{n} |x_j - y_j|^p\right)^{1/p}$$

得证 $\rho_1(\vec{x}, \vec{y}) \leq \rho_2(\vec{x}, \vec{y})$. (2) 因为

$$|x_1 - y_1|^p + \cdots + |x_n - y_n|^p \leq n \cdot \max_{1 \leq j \leq n} |x_j - y_j|^p = n \cdot \left(\max_{1 \leq j \leq n} |x_j - y_j|\right)^p,$$

所以 $\rho_2(\vec{x}, \vec{y}) \leq n^{1/p} \rho_1(\vec{x}, \vec{y})$. 3.1.6.

用反证法. 假设 $\forall a \in (0, 1)$, 存在 $\vec{x}_a, \vec{y}_a \in A, \vec{x}_a \neq \vec{y}_a$, 使得

$$|F\vec{x}_a - F\vec{y}_a| \geq a\,|\vec{x}_a - \vec{y}_a|$$

取 $a = 1 - \dfrac{1}{n}$ 时, 相应有 $\vec{x}_n, \vec{y}_n \in A$, 使得

$$|F\vec{x}_n - F\vec{y}_n| \geq \left(1 - \frac{1}{n}\right)|\vec{x}_n - \vec{y}_n|$$

由 A 为 n 维欧氏空间的有界闭集可知, $\{\vec{x}_n\}$ 和 $\{\vec{y}_n\}$ 相应有收玫了列 $\{\vec{x}_{n_k}\}$ 和 $\{\vec{y}_{n_k}\}$, 记它们的极限依次为 $\vec{x}_0, \vec{y}_0$, 则 $\vec{x}_0 \in A, \vec{y}_0 \in A$. 最后由

$$|F(\vec{x}_{n_k}) - F(\vec{y}_{n_k})| \geq \left(1 - \frac{1}{n_k}\right) |\vec{x}_{n_k} - \vec{y}_{n_k}|$$

和映射 F 的条件(推出 F 连续), 让 $k \to \infty$, 得到 $|F\vec{x}_0 - F\vec{y}_0| \geq |\vec{x}_0 - \vec{y}_0|$. 这与已知矛盾。

3.2 多元函数的极限与连续性

(A)

3.2.1

(1)$D = \left\{(x,y) | -1 \leqslant \dfrac{y}{x} \leqslant 1\right\}$

(2)$D = \left\{(x,y,z) | x^2 + y^2 > 1\right\}$

(3)$D = \left\{(x,y) | (2x - x^2 - y^2)(x^2 + y^2 - x) \geqslant 0 \text{且} x^2 + y^2 - x \neq 0\right\}$

(4)$D = \left\{(x,y) | x^2 + y^2 > 1\right\}$

3.2.2

由于初等多元函数在定义域内连续，所以平凡地：

(1) $\lim\limits_{(x,y)\to(0,1)} \dfrac{x + e^y}{x^2 + y^2} = e$ (2) $\lim\limits_{(x,y)\to(2,0)} = \dfrac{1}{4}$

对于(3)(4):

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x} = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} \cdot y = 0$$

(4) $\lim\limits_{(x,y)\to(+\infty,+\infty)} (x^2 + y^2) e^{-(x+y)} = \lim\limits_{(x,y)\to(+\infty,+\infty)} \dfrac{x^2}{e^x} \cdot e^{-y} + \lim\limits_{(x,y)\to(+\infty,+\infty)} \dfrac{y^2}{e^y} \cdot e^{-x} = 0$

3.2.3

易知$f(x,y)$在除去点$(0,0)$处均连续，因此考虑$f(x,y)$在$(0,0)$处的连续性：

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} y \cdot \frac{\sin xy}{xy} \cdot \frac{x}{\sqrt{x^2 + y^2}} = 0 = f(0,0)$$

故$f(x,y)$在$(0,0)$处连续，也即$f(x,y)$在$\mathbf{R}^2$内是连续的

3.2.4

证: $\lim\limits_{t\to0} f(t\cos\alpha, t\sin\alpha) = \lim\limits_{t\to0} \dfrac{t\cos^2\alpha\sin\alpha}{t^2\cos^4\alpha + \sin^2\alpha} = 0 \left(\dfrac{t\cos^2\alpha\sin\alpha}{t^2\cos^4\alpha + \sin^2\alpha} < \dfrac{t\cos^2\alpha}{\sin\alpha} \to 0\right)$

取$y = x^2$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4 + y^2} = \dfrac{1}{2} \neq f(0,0)$，故$f(x,y)$在$(0,0)$不连续

3.2.5

$f(x,y)$在D内对x连续$\Leftrightarrow \forall\varepsilon > 0, \exists\delta_1 > 0, |x - x_1| < \delta_1, |f(x,y) - f(x_1,y)| < \dfrac{\varepsilon}{2}$

$\forall\varepsilon > 0, \exists\delta = \min\left\{\delta_1, \dfrac{\varepsilon}{2L}\right\}, |x - x_1| < \delta, |y - y_1| < \delta,$

$|f(x,y)-f(x_1,y_1)| \leqslant |f(x_1,y)-f(x,y)|+|f(x_1,y)-f(x_1,y_1)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

3.2.6　提示：

(1)取$y=kx^2-x$(2)取$y^2=x^2$

(3)取$y=kx^3$(4)取$x^3+y^3=kx^4$

(B)

3.2.1

取两点列:$(\sqrt{n+1},\sqrt{n+1}),(\sqrt{n},\sqrt{n})$,且$d((\sqrt{n+1},\sqrt{n+1}),(\sqrt{n},\sqrt{n}))\to 0(n\to\infty)$

由于$f(\sqrt{n+1},\sqrt{n+1})-f(\sqrt{n},\sqrt{n})=\frac{1}{2}\nrightarrow 0(n\to\infty)$

由$Heine$定理知：$f(x,y)$在$\mathbf{R}^2$上不一致收敛

3.2.2

(1)$\varepsilon>0$,取$\delta=\frac{\varepsilon}{6}$,$|x-1|<\delta,|y-1|<\delta$,

$|x^2+y^2-2| \leqslant |x^2-1|+|y^2-1| = |x+1||x-1|+|x-1||y-1| < 3|x-1|+3|y-1| < \varepsilon$

(2)(注本题有误，改为：证$\lim\limits_{(x,y)\to(0,0)}\frac{1}{\sqrt{xy+1}+1}=\frac{1}{2}$)

$\varepsilon>0$,取$\delta=\sqrt{\varepsilon}$,$\sqrt{x^2+y^2}<\delta$,

$|\frac{1}{\sqrt{xy+1}+1}-\frac{1}{2}|=|\frac{1-\sqrt{xy+1}}{2(\sqrt{xy+1}+1)}|=|\frac{xy}{2(\sqrt{xy+1}+1)^2}|=\frac{1}{2}|xy|<x^2+y^2<\varepsilon$

3.2.3

不存在，取$y=-\frac{3}{2}x+k\sqrt{|x|}$，该极限不存在是显然的

3.3　多元函数的偏导数和全微分

(A)

3.3.1

(1) $z_x=2xy-y^2, z_y=x^2-2yx;$

(2) $z_x=y^{x+1}x^{y-1}+x^yy^x\ln y, z_y=x^{y+1}y^{x-1}+y^xx^y\ln x;$

(3) $f_x=y+\frac{1}{y}, f_y=x-\frac{x}{y^2};$

(4) $z_x=\frac{1}{y^2}, z_y=-\frac{2x}{y^3};$

(5) $z_x=-\frac{2x\sin x^2}{y}, z_y=-\frac{\cos x^2}{y^2};$

(6) $z_x=\frac{2x}{y}\sec^2\frac{x^2}{y}, z_y=-\frac{x^2}{y^2}\sec^2\frac{x^2}{y};$

(7) $z_x=yx^{y-1}, z_y=x^y\ln x;$

(8) $g_x = \dfrac{x}{x^2+y^2}, g_y = \dfrac{y}{x^2+y^2}$;

(9) $u_x = \dfrac{-2x}{\left(x^2+y^2+z^2\right)^2}, u_y = \dfrac{-2y}{\left(x^2+y^2+z^2\right)^2}, u_x = \dfrac{-2z}{\left(x^2+y^2+z^2\right)^2}$;

(10) $u_x = yz^{2y}\ln z, u_y = xz^{2y}\ln z, u_z = xyz^{2y-1}$;

(11) $u_x = yz(xy)^{x-1}, u_y = xz(xy)^{x-1}, u_z = (xy)^z\ln(xy)$;

(12) $u_x = \dfrac{y}{z}x^{\frac{y}{z}-1}, u_y = \dfrac{1}{z}x^{\frac{y}{z}}\ln x, u_z = -\dfrac{y}{z^2}x^{\frac{y}{z}}\ln x$.

(13)$z_x = -\dfrac{|y|}{x^2+y^2}, z_y = \dfrac{xy}{|y|(x^2+y^2)}$

(14)$u_x = ye^{\sin yz}, u_y = x(e^{\sin yz} + yze^{\sin yz}\cos yz), u_z = xy^2e^{\sin yz}\cos yz$

(15)$u_x = -\dfrac{y}{x^2} - \dfrac{1}{z}, u_y = \dfrac{1}{x} - \dfrac{z}{y^2}, u_x = -\dfrac{1}{y} + \dfrac{x}{z^2}$

3.3.2

(1) $z_x(1,0) = z_y(1,0) = z_y(0,1) = 0, z_x(0,1) = 1$;

(2) $z_x\left(0,\dfrac{\pi}{4}\right) = -1, z_y\left(0,\dfrac{\pi}{4}\right) = 0$;

(3) $f_x(1,1,1) = 1, f_y(1,1,1) = -1, f_z(1,1,1) = 0$;

(4) $z_x(0,0) = -1, z_y(0,0) = 0$.

3.3.3

(1) $\mathrm{d}z = -\mathrm{e}^{-x}\cos y\,\mathrm{d}x - \mathrm{e}^{-x}\sin y\,\mathrm{d}y$;

(2) $\mathrm{d}f = \cos(xy)y\,\mathrm{d}x + \cos(xy)x\,\mathrm{d}y$;

(3) $\mathrm{d}g = (2u+v)\mathrm{d}u + u\,\mathrm{d}v$;

(4) $\mathrm{d}u = yzx^{yz-1}\,\mathrm{d}x + zx^{yz}\ln x\,\mathrm{d}y + yx^{yz}\ln x\,\mathrm{d}z$;

(5) $\mathrm{d}z = -\dfrac{xy}{\left(x^2+y^2\right)^{3/2}}\,\mathrm{d}x + \dfrac{x^2}{\left(x^2+y^2\right)^{3/2}}\,\mathrm{d}y$;

(6) $\mathrm{d}z = \left(2xy + \dfrac{1}{y}\right)\mathrm{d}x + \left(x^2 - \dfrac{x}{y^2}\right)\mathrm{d}y$.

3.3.4

(1) $\mathrm{d}f(1,0) = \mathrm{d}x - \mathrm{d}y$;

(2) $\mathrm{d}g\left(2,\dfrac{\pi}{4}\right) = 4\,\mathrm{d}x$;

(3) $\mathrm{d}F(100,10) = \dfrac{G}{5}\left(\dfrac{1}{20}\,\mathrm{d}m - \mathrm{d}r\right)$;

(4) $\mathrm{d}f(1,2) = \dfrac{1}{3}\,\mathrm{d}x + \dfrac{2}{3}\,\mathrm{d}y$.

3.3.5

(1)$f_x(0,0) = \lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0}\sin\dfrac{1}{x^2}$,此极限显然不存在，故$f_x(0,0)$不存在

$f_y(0,0) = \lim\limits_{y\to 0}\dfrac{f(0,0)-f(0,y)}{y} = 0$

(2)$f_x(0,0) = \lim\limits_{\Delta x\to 0}\dfrac{f(0+\Delta x,0)-f(0,0)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{|\Delta x|}{\Delta x}g(\Delta x,0)$. 要使 $f_x(0,0)$ 存在,则必

然有 $g(0,0)=0$, 此时 $f_x(0,0)=0$.
$f_y(0,0)=\lim\limits_{\Delta y\to 0}\dfrac{|\Delta y|}{\Delta y}g(0,\Delta y)$, 同理只有当 $g(0,0)=0$ 时$f_y(0,0)$存在, 且 $f_y(0,0)=0$. 故当 $g(0,0)=0$ 时, $f(x,y)$ 可偏导.
令 $\rho=\sqrt{x^2+y^2}$, 若 $g(0,0)\neq 0$, 则一定不可微（因为$f(x,y)$不可偏导则一定不可微）.
而 $g(0,0)=0$ 时, 有：

$$\frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\rho}=|x-y|g(x,y)$$

由于$\dfrac{|x-y|}{\rho}$ 有界($0\leqslant\dfrac{|x-y|}{\rho}\leqslant\dfrac{|x|+|ay|}{\rho}\leqslant 2$), 又因为 $\lim\limits_{(x,y)\to(0,0)}g(x,y)=0$.
故 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\rho}=0$.
即当 $g(0,0)=0$ 时 $f(x,y)$ 在 $(0,0)$ 处可微.

3.3.6

(1) 令 $f(x,y)=(1+x)^m(1+y)^n$.
当 x,y 绝对值很小时. $f(x,y)-f(0,0)\approx f_x(0,0)(x-0)+f_y(0,0)(y-0)=mx+ny$.
故 $f(x,y)\approx f(0,0)+mx+ny=1+mx+ny$.
(2) 令 $f(x,y)=\arctan\dfrac{x+y}{1+xy}$.
当 $|x|,|y|$ 很小时, $f(x,y)\approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=x+y$.

3.3.7

(1) $f(x,y)=x^y, x=1, y=1, \Delta x=-0.03, \Delta y=0.05$
$(0.97)^{1.05}\approx f(1,1)+f_x(1,1)\cdot\Delta x+f_y(1,1)\cdot\Delta y=1.021$;
(2) $f(x,y)=\sin x\tan y, x=\dfrac{\pi}{6}, y=\dfrac{\pi}{4}, \Delta x=-\dfrac{\pi}{180}, \Delta y=\dfrac{\pi}{180}$
$\sin 29^\circ\tan 46^\circ\approx f(\dfrac{\pi}{6},\dfrac{\pi}{4})+f_x(\dfrac{\pi}{6},\dfrac{\pi}{4})\cdot\Delta x+f_y(\dfrac{\pi}{6},\dfrac{\pi}{4})\cdot\Delta y=0.502$.

3.3.8

$\dfrac{\partial g}{\partial x}=-f'\left(\dfrac{1}{r}\right)\dfrac{1}{r^2}\dfrac{x}{r}=-\dfrac{x}{r^3}f'\left(\dfrac{1}{r}\right)$,
$\dfrac{\partial^2 g}{\partial x^2}=\dfrac{x^2}{r^6}f''\left(\dfrac{1}{r}\right)-\dfrac{r^2-3x^2}{r^5}f'\left(\dfrac{1}{r}\right)$,
易知x,y等价，由对称性可得： $\dfrac{\partial^2 g}{\partial y^2}=\dfrac{y^2}{r^6}f''\left(\dfrac{1}{r}\right)-\dfrac{r^2-3y^2}{r^5}f'\left(\dfrac{1}{r}\right)$
故 $\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial y^2}=\dfrac{1}{r^4}f''\left(\dfrac{1}{r}\right)+\dfrac{1}{r^3}f'\left(\dfrac{1}{r}\right)$.

3.3.9

(1) $z_{xx}=\mathrm{e}^x(\cos y+x\sin y+2\sin y)$,
$z_{xy}=\mathrm{e}^x(x\cos y+\cos y-\sin y), z_{yy}=-\mathrm{e}^x(\cos y+x\sin y)$;
(2) $z_{xxy}=0, z_{xyy}=-\dfrac{1}{y^2}$;
(3)$z_{xx}=y^4f_{11}+4xy^3f_{12}+4x^2y^2f_{22}+2yf_2$,

$z_{xy} = 2xy^3 f_{11} + 5x^2y^2 f_{12} + 2x^3 y f_{22} + 2y f_1 + 2x f_2,$
$z_{yy} = 4x^2y^2 f_{11} + 4x^3 y f_{12} + x^4 f_{22} + 2x f_1$;
(4) $u_{xx} = 2f' + 4x^2 f'', u_{yy} = 2f' + 4y^2 f'', u_{zz} = 2f' + 4z^2 f'',$
$u_{xy} = 4xyf'', u_{yz} = 4yzf'', u_{xz} = 4xzf''$;
(5)$z_{xy} = f_1 - \frac{1}{y^2} f_2 + xy f_{11} - \frac{x}{y^3} f_{22} - \frac{1}{x^2} g'\left(\frac{y}{x}\right) - \frac{y}{x^3} g''\left(\frac{y}{x}\right).$
(6)$z_{xx} = \frac{1}{y} f''(\frac{x}{y}) + \frac{y^2}{x^3} g''(\frac{y}{x})$
$z_{xy} = -\frac{x}{y^2} f''(\frac{x}{y}) - \frac{y}{x^2} g'(\frac{y}{x})$
3.3.10
(1)$\frac{\partial z}{\partial x} = y\varphi'(xy) + \frac{1}{y}\varphi'\left(\frac{x}{y}\right), \frac{\partial z}{\partial y} = x\varphi'(xy) - \frac{x}{y^2}\varphi'\left(\frac{x}{y}\right)$
$dz = \left[y\varphi'(xy) + \frac{1}{y}\varphi'\left(\frac{x}{y}\right)\right] dx + \left[x\varphi'(xy) - \frac{x}{y^2}\varphi'\left(\frac{x}{y}\right)\right] dy$
(2) $\frac{\partial z}{\partial x} = ye^{xy}\sin(x+y) + e^{xy}\cos(x+y), \frac{\partial z}{\partial y} = xe^{xy}\sin(x+y) + e^{xy}\cos(x+y)$
$dz = [ye^{xy}\sin(x+y) + e^{xy}\cos(x+y)]\, dx + [xe^{xy}\sin(x+y) + e^{xy}\cos(x+y)]\, dy$
(3) $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2+z^2}, \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2+z^2}, \frac{\partial u}{\partial z} = \frac{z}{x^2+y^2+z^2}$
$du = \frac{xdx}{x^2+y^2+z^2} + \frac{ydy}{x^2+y^2+z^2} + \frac{zdz}{x^2+y^2+z^2}$
(4) $\frac{\partial u}{\partial x} = 2xf_1' + ye^{xy}f_2', \frac{\partial u}{\partial y} = -2yf_1' + xe^{xy}f_2', \frac{\partial u}{\partial z} = f_3'$
$du = \left[2xf_1' + ye^{xy}f_2'\right] dx + \left[-2yf_1' + xe^{xy}f_2'\right] dy + f_3' dz$
3.3.11
显然我们要找$\begin{cases} \xi = x + ay \\ \eta = x + by \end{cases}$的逆变换$\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases}$
该存在逆变换的条件是$\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$，解得$a \neq b$
从而可以解出逆变换：$\begin{cases} x = \frac{1}{a-b}(a\eta - b\xi) \\ y = \frac{1}{a-b}(\xi - \eta) \end{cases}$
$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\cdot\frac{a}{a-b} + \frac{\partial u}{\partial y}\cdot\frac{-1}{a-b} = \frac{1}{a-b}\left(a\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$
$\frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{\partial}{\partial\xi}\left(\frac{1}{a-b}\left(a\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)\right)$
$= \frac{1}{a-b}\left[a\left(\frac{\partial^2 u}{\partial x^2}\frac{-b}{a-b} + \frac{\partial^2 u}{\partial y\partial x}\frac{1}{a-b}\right) - \left(\frac{\partial^2 u}{\partial x\partial y}\frac{-b}{a-b} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{1}{a-b}\right)\right]$
$= -\frac{ab}{(a-b)^2}\left[\frac{1}{ab}\frac{\partial^2 u}{\partial x^2} - \frac{a+b}{ab}\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\right]$

易知 $\begin{cases} \dfrac{1}{ab}=3 \\ -\dfrac{a+b}{ab}=4 \end{cases}$ 解得 $\begin{cases} a=-1 \\ b=-\dfrac{1}{3} \end{cases}$ 或 $\begin{cases} b=-1 \\ a=-\dfrac{1}{3} \end{cases}$.

3.3.12

$F(1)=f[1,f(1,f(1,1))]=f[1,f(1,1)]=f(1,1)=1,$

$F'(1)=f_1[1,f(1,f(1,1))]+f_2[1,f(1,f(1,1))]\cdot\left.\dfrac{\mathrm{d}f(x,f(x,x))}{\mathrm{d}x}\right|_{x=1}$

$=f_1[1,f(1,1)]+f_2[1,f(1,1)]\left[f_1(1,f(1,1))+f_2(1,f(1,1))\left.\dfrac{\mathrm{d}f(x,x)}{\mathrm{d}x}\right|_{x=1}\right]$

$=f_1(1,1)+f_2(1,1)\left[f_1(1,1)+f_2(1,1)\left(f_1(1,1)+f_2(1,1)\right)\right]$

$=a+b[a+b(a+b)].$

3.3.13

证明: 由逆变换定理可以保证存在逆变换： $\begin{cases} x=u \\ y=\dfrac{u}{1+uv} \end{cases}$

则 $\varphi=\dfrac{1}{z}-\dfrac{1}{x}$ 是 u,v 的复合函数，故有:

$\dfrac{\partial\varphi}{\partial u}=-\dfrac{1}{z^2}\left(\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}\right)+\dfrac{1}{u^2}$

$=-\dfrac{1}{z^2}\left(\dfrac{\partial z}{\partial x}+\dfrac{1}{(1+uv)^2}\dfrac{\partial z}{\partial y}\right)+\dfrac{1}{u^2}$

$=-\dfrac{1}{z^2}\left(\dfrac{\partial z}{\partial x}+\dfrac{y^2}{x^2}\dfrac{\partial z}{\partial y}\right)+\dfrac{1}{u^2}$

$=-\dfrac{1}{x^2z^2}\left(x^2\dfrac{\partial z}{\partial x}+y^2\dfrac{\partial z}{\partial y}\right)+\dfrac{1}{u^2}$

$=\dfrac{1}{u^2}-\dfrac{1}{x^2}=0$

3.3.14

不妨设 $\rho=\sqrt{x^2+y^2}$, 则 $\dfrac{\partial u}{\partial x}=u'(\rho)\dfrac{x}{\rho},\dfrac{\partial u}{\partial y}=u'(\rho)\dfrac{y}{\rho},$

$\dfrac{\partial^2 u}{\partial x^2}=u''(\rho)\cdot\dfrac{x^2}{\rho^2}+u'(\rho)\cdot\dfrac{1}{\rho}-u'(\rho)\dfrac{x}{\rho^2}\cdot\dfrac{x}{\rho},$

$\dfrac{\partial^2 u}{\partial y^2}=u''(\rho)\dfrac{y^2}{\rho^2}+u'(\rho)\dfrac{1}{\rho}-u'(\rho)\dfrac{y}{\rho^2}\cdot\dfrac{y}{\rho}.$

代入原方程可得: $u''(\rho)+u(\rho)=\rho^2.$

对于齐次方程：$u''(\rho)+u(\rho)=0$

其特征方程为：$\lambda^2+1=0$,解得特征根$\lambda_1=i,\lambda_2=-i$

对应齐次解:$u(\rho)=C_1\cos\rho+C_2\sin\rho$

对于非齐次方程：$u''(\rho)+u(\rho)=\rho^2$，$\alpha=0$非特征根

故设其特解为：$u^*=A\rho^2+B\rho+C$,带入原微分方程解得特解：$u^*=\rho^2-2$

故$u(\rho)=C_1\cos\rho+C_2\sin\rho+\rho^2-2,$

也即:$u=u\left(\sqrt{x^2+y^2}\right)=C_1\cos\sqrt{x^2+y^2}+C_2\sin\sqrt{x^2+y^2}+x^2+y^2-2.$

3.3.15

不妨设$\rho=\sqrt{x^2+y^2+z^2}$, $\dfrac{\partial u}{\partial x}=f'(\rho)\cdot\dfrac{x}{\rho}$

$$\frac{\partial u^2}{\partial^2 x}=\frac{f'(\rho)}{\rho}+x\left(\frac{f''(\rho)}{\rho}-\frac{f'(\rho)}{\rho^2}\right)\cdot\frac{x}{\rho}=\frac{f'(\rho)}{\rho}+\frac{x^2f''(\rho)}{\rho^2}-\frac{x^2f'(\rho)}{\rho^3}$$

易知x,y,z三者是相互等价的，所以对称地有：

$$\frac{\partial u^2}{\partial^2 y}=\frac{f'(\rho)}{\rho}+\frac{y^2f''(\rho)}{\rho^2}-\frac{y^2f'(\rho)}{\rho^3},\frac{\partial u^2}{\partial^2 z}=\frac{f'(\rho)}{\rho}+\frac{z^2f''(\rho)}{\rho^2}-\frac{z^2f'(\rho)}{\rho^3}$$

$$\frac{\partial u^2}{\partial^2 x}+\frac{\partial u^2}{\partial^2 y}+\frac{\partial u^2}{\partial^2 z}=f''(\rho)+2\frac{f'(\rho)}{\rho}=0$$

$$\frac{f''(\rho)}{f'(\rho)}=-\frac{2}{\rho}\Rightarrow\ln f'(\rho)=\ln\frac{1}{\rho^2}+C=\ln\frac{C_1}{\rho^2}$$

$$f'(\rho)=\frac{C_1}{\rho^2}\Rightarrow f(\rho)=-\frac{C_1}{\rho}+C_2$$

带入初始条件$f'(1)=1,f(1)=0$，解得:$f(\rho)=-\dfrac{1}{\rho}$

3.3.16

由于 $f(x,y)=h(r)=h\left(\sqrt{x^2+y^2}\right)$, 得 $\dfrac{\partial f}{\partial x}=h'(r)\dfrac{x}{r}$

$$\frac{\partial^2 f}{\partial x\partial y}=h''(r)\frac{xy}{r^2}-h'(r)\frac{xy}{r^3}=0\Rightarrow h''(r)-\frac{1}{r}h'(r)=0$$

此微分方程的处理方法与上题类似，可解得： $h(r)=C_1r^2+C_2$,

即：$f(x,y)=C_1\left(x^2+y^2\right)+C_2$

3.3.17

$u=f(xyz)$

$$\frac{\partial y}{\partial x}=f'(xyz)\cdot yz$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial\left(\frac{\partial u}{\partial x}\right)}{\partial y}=f''(xyz)\cdot xz\cdot yz+f'(xyz)\cdot z=xyz^2f''(xyz)+zf'(xyz)$$

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=\frac{\partial\left(\frac{\partial^2 u}{\partial x\partial y}\right)}{\partial z}=2xyzf''(xyz)+xyz^2\cdot f'''(xyz)\cdot xy+1\cdot f'(xyz)+zf''(xyz)\cdot xy$$

$$=3xyzf''(xyz)+x^2y^2z^2f'''(xyz)+f'(xyz)=x^2y^2z^2f'''(xyz)$$

即 $3xyzf''(xyz)+f'(xyz)=0\Leftrightarrow$ 设$t=xyz$ ，$3tf''(t)+f'(t)=0$

该微分方程的处理方法与3.3.15类似，带入初始条件$f(0)=0,f'(1)=1$:

$\Rightarrow u=\dfrac{3}{2}(xyz)^{\frac{2}{3}}$

3.3.18

记 $r=\sqrt{x^2+y^2},v=lnr$, 则有 $\dfrac{\partial u}{\partial x}=\dfrac{x}{r^2}\cdot f'(v),\dfrac{\partial^2 u}{\partial x^2}=\dfrac{x^2}{r^4}\cdot f''(v)+\dfrac{1}{r^2}\cdot f'(v)-\dfrac{2x^2}{r^4}\cdot f'(v)$,

易知x,y是等价的，所以对称地有：$\dfrac{\partial^2 u}{\partial y^2}=\dfrac{y^2}{r^4}\cdot f''(v)+\dfrac{1}{r^2}\cdot f'(v)-\dfrac{2y^2}{r^4}\cdot f'(v).$

$\because \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\left(x^2+y^2\right)^{\frac{3}{2}}$
$\Rightarrow f''(v)=e^{5v}$
$\Rightarrow f(v)=\dfrac{1}{25}e^{5v}+C_1v+C_2$ 其中 C_1,C_2 是任意常数.

3.3.19

$\dfrac{\partial(F,G)}{\partial(u,v)}=x^2-y^2$ 由隐函数定理，在$x^2-y^2\neq 0$的条件下可将u,v 看作 x,y 的隐函数,

两端分别对 x 求导, 得 $\begin{cases} u+xu_x+yv_x=0, \\ yu_x+v+xv_x=0, \end{cases}$ 解得 $u_x=\dfrac{\partial u}{\partial x}=\dfrac{yv-ux}{x^2-y^2}$.

同理，原方程组两端对 y 求偏导得 $\begin{cases} xu_y+yv_y+v=0, \\ yu_y+xv_y+u=0, \end{cases}$ 解得 $v_y=\dfrac{\partial v}{\partial y}=\dfrac{yv-ux}{x^2-y^2}$.

注：在方程组两边取全微分，相应的解出$\mathrm{d}u,\mathrm{d}v$亦可

3.3.20

先验证隐函数的存在条件：

$$J=\frac{\partial(F,G,H)}{\partial(u,v,w)}=\begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}=(u-v)(u-w)(v-w)$$

在$J\neq 0$的条件下可保证隐函数$u(x,y,z),v(x,y,z),z(x,y,z)$的存在性，故在方程两端求全微分,得:

$$\begin{cases} \mathrm{d}u+\mathrm{d}v+\mathrm{d}w=\mathrm{d}x, \\ (u+w)\mathrm{d}v+(v+w)\mathrm{d}u+(v+u)\mathrm{d}w=\mathrm{d}y \\ vw\,\mathrm{d}u+uw\,\mathrm{d}v+uv\,\mathrm{d}w=\mathrm{d}z. \end{cases}$$

解得 $\begin{cases} \mathrm{d}u=\dfrac{v-w}{J}\left(u^2\,\mathrm{d}x-u\,\mathrm{d}y+\mathrm{d}z\right) \\ \mathrm{d}v=\dfrac{u-w}{J}\left(-v^2\,\mathrm{d}x+v\,\mathrm{d}y-\mathrm{d}z\right) \\ \mathrm{d}w=\dfrac{u-v}{J}\left(w^2\,\mathrm{d}x-w\,\mathrm{d}y+\mathrm{d}z\right) \end{cases}$

故 $\dfrac{\partial u}{\partial x}=\dfrac{u^2}{(u-v)(u-w)},\dfrac{\partial u}{\partial y}=-\dfrac{u}{(u-v)(u-w)},\dfrac{\partial u}{\partial z}=\dfrac{1}{(u-v)(u-w)}$

注：若能记住公式：$\dfrac{\partial u}{\partial x}=-\dfrac{1}{J}\dfrac{\partial(F,G,H)}{\partial(x,v,w)},\dfrac{\partial u}{\partial y}=-\dfrac{1}{J}\dfrac{\partial(F,G,H)}{\partial(y,v,w)}$
$\dfrac{\partial u}{\partial z}=-\dfrac{1}{J}\dfrac{\partial(F,G,H)}{\partial(z,v,w)}$自然更好

3.3.21

依题意有, y 是函数, x,z 是自变量。将方程 $z=f(x,y)$ 两边同时对 x 求导, $0=f_x+f_y\dfrac{\partial y}{\partial x}$,

则 $\dfrac{\partial y}{\partial x}=-\dfrac{f_x}{f_y}$, 于是有：

$\dfrac{\partial^2 y}{\partial x^2}=\dfrac{\partial}{\partial x}\left(-\dfrac{f_x}{f_y}\right)$

$$= -\frac{f_y\left(f_{xx}+f_{yx}\frac{\partial y}{\partial x}\right)-f_x\left(f_{yx}+f_{yy}\frac{\partial y}{\partial x}\right)}{f_y^2}$$
$$= -\frac{f_y\left(f_{xx}-f_{yx}\frac{f_x}{f_y}\right)-f_x\left(f_{yx}-f_{yy}\frac{f_x}{f_y}\right)}{f_y^2}$$
$$= -\frac{f_x^2f_{yy}-2f_xf_yf_{xy}+f_y^2f_{yy}}{f_y^3}$$
$$= \frac{f_xf_{xy}}{f_y^2}$$

(B)

3.3.1

证:求函数 $z=x^nf\left(\frac{y}{x^2}\right)$ 的偏导数:

$$\frac{\partial z}{\partial x}=nx^{n-1}f\left(\frac{y}{x^2}\right)+x^nf'(\frac{y}{x^2})\cdot\left(-\frac{2y}{x^3}\right)=nx^{n-1}f\left(\frac{y}{x^2}\right)-2x^{n-3}yf'\left(\frac{y}{x^2}\right)$$
$$\frac{\partial z}{\partial y}=x^nf\left(\frac{y}{x^2}\right)\cdot\frac{1}{x^2}=x^{n-2}f'\left(\frac{y}{x^2}\right)$$

所以有：

$$x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial y}=x\left[nx^{n-1}f\left(\frac{y}{x^2}\right)-2x^{n-3}yf'\left(\frac{y}{x^2}\right)\right]+2y\left[x^{n-2}f'\left(\frac{y}{x^2}\right)\right]$$
$$=nx^nf\left(\frac{y}{x^2}\right)-2x^{n-2}yf'\left(\frac{y}{x^2}\right)+2x^{n-2}yf'\left(\frac{y}{x^2}\right)=nz$$

3.3.2

不妨设$G(x,y,z)=F(z+\frac{1}{x},z-\frac{1}{y})=0$,则由隐函数求导定理，得：

$$z_x=-\frac{G_x}{G_z}=\frac{F_1\cdot(-\frac{1}{x^2})}{F_1+F_2}=-\frac{1}{2x^2},z_{xx}=\frac{1}{x^3},z_{xy}=0$$
$$z_y=-\frac{G_y}{G_z}=\frac{F_1\cdot\frac{1}{y^2}}{F_1+F_2}=\frac{1}{2y^2},z_{yy}=-\frac{1}{y^3}$$
$$\Rightarrow x^2z_x+y^2z_y=0,x^3z_{xx}+xy(x+y)z_{xy}+y^3z_{yy}=0$$

3.3.3

证明:

“$\Rightarrow$”: 由 $f(x,y)$ 是 n 次齐函数知, $\forall t>0$, 有$f(tx,ty)=t^nf(x,y)$.

等式两端对 t 求导： $xf_1(tx,ty)+yf_2(tx,ty)=nt^{n-1}f(x,y)$.

取 $t=1$ 得: $xf_1(x,y)+yf_2(x,y)=nf(x,y)$,

即: $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$.

“$\Leftarrow$” :不妨设 $F(t)=f(tx,ty)(t>0)$, 则有：

$$\frac{\mathrm{d}F}{\mathrm{d}t}=xf_1(tx,ty)+yf_2(tx,ty).$$
$$\Rightarrow t\frac{\mathrm{d}F}{\mathrm{d}t}=txf_1(tx,ty)+tyf_2(tx,ty)=nf(tx,ty)=nF(t)\ .$$

即：$\dfrac{\mathrm{d}F}{F}=\dfrac{n}{t}\,\mathrm{d}t$. $\Rightarrow F(t)=Ct^n$, 取 $t=1$, 得 $F(1)=C$.
又 $F(t)=f(tx,ty)$, 取 $t=1$, 得 $F(1)=f(x,y)$, 则$C=f(x,y)$.
故$f(tx,ty)=t^n f(x,y)$
3.3.4
不妨设$\dfrac{f_x}{x}=\dfrac{f_y}{y}=\dfrac{f_y}{y}=\lambda$，$\lambda$为常数
对$u=f(x,y,z)$取全微分得：

$$\mathrm{d}u=f_x\mathrm{d}x+f_y\mathrm{d}y+f_z\mathrm{d}z=\lambda x\mathrm{d}x+\lambda x\mathrm{d}x+\lambda x\mathrm{d}x$$
$$=\frac{\lambda}{2}\mathrm{d}(x^2+y^2+z^2)=\frac{\lambda}{2}\mathrm{d}r^2=\lambda r\mathrm{d}r$$

$\Rightarrow u=\dfrac{1}{2}\lambda r^2+C$，其中$C$为任意常数.

3.4 方向导数和梯度

(A)

3.4.1 易得$\nabla f(P_0)=\{3,2,1\}$,又由于$f(x,y,z)=x^3y^2z$在$P_0(1,1,1)$处可微，可以使用定理3.4.1,故有：
(1)$\dfrac{\partial f}{\partial \mathbf{l}}\Big|_{(1,1,1)}=3$ (2)$\dfrac{\partial f}{\partial \mathbf{l}}\Big|_{(1,1,1)}=2$
(3)$\dfrac{\partial f}{\partial \mathbf{l}}\Big|_{(1,1,1)}=1$ (4)$\dfrac{\partial f}{\partial \mathbf{l}}\Big|_{(1,1,1)}=2\sqrt{3}$
3.4.2 (注：本题疑似有误，因为$f(a,b,c)$的可微性未知，故不可直接使用定理3.4.1）
设$\mathbf{l}$上的单位向量为$\{x_1,x_2,x_3\}$,添上可微性再利用定理3.4.1则有：
$\dfrac{\partial f}{\partial \mathbf{l}}\Big|_{(a,b,c)}=\nabla f(a,b,c)\cdot\mathbf{l}=2x_1+3x_2+x_3=0$
(1)解方程组 $\begin{cases}2x_1+3x_2+x_3=0\\ x_1^2+x_2^2+x_3^2=1\end{cases}$
很容易得到三个解（找到满足第一个方程的解后单位化）:
$\left\{\dfrac{1}{\sqrt{5}},0,\dfrac{-2}{\sqrt{5}}\right\}$,$\left\{0,\dfrac{1}{\sqrt{10}},\dfrac{-3}{\sqrt{10}}\right\}$, $\left\{\dfrac{-3}{\sqrt{13}},\dfrac{2}{\sqrt{13}},0\right\}$
(2)上述方程组显然有无穷组解，故这样的单位向量是无穷多的
3.4.3
(1)$\nabla f(2,5)=\{20,4\}$, $\nabla f(3,1)=\{6,9\}$
(2)$\nabla f(1,2)=\left\{\dfrac{1}{5\sqrt{5}},-\dfrac{2}{5\sqrt{5}}\right\}$, $\nabla f(3,0)=\left\{-\dfrac{1}{\sqrt{3}},0\right\}$

3.4.4 由梯度定义3.4.2可知：当沿着梯度方向时，函数值增长最快
(1)$\mathbf{l}=\nabla f(0,0)=\{1,1\}$ (2)$\mathbf{l}=\nabla f(2,0,1)=\{2,0,1\}$

$(3)\mathbf{l} = \nabla f(\frac{1}{3}, \frac{1}{2}, \pi) = \left\{-\frac{\pi}{4}, -\frac{\pi}{6}, -\frac{1}{12}\right\}$ $\qquad (4)\mathbf{l} = \nabla f(-1, 1) = \{-6, 8\}$

3.4.5　同3.4.4，当沿着梯度的反方向时，函数值减小最快

$(1)\mathbf{l} = \nabla f(\frac{1}{2}, \frac{2}{3}) = \left\{\frac{\pi}{3}, \frac{\pi}{4}\right\}$ $\qquad (2)\mathbf{l} = \nabla f(-1, 1, 3) = \left\{-\frac{1}{4}, -\frac{1}{4}, 0\right\}$

3.4.6 (本题略微超纲，需要利用3.5节的$Lagrange$乘数法)

$f(x, y, z) = x^2 + y^2 + z^2$显然可微，可以利用定理3.4.1:

$\frac{\partial f}{\partial \mathbf{l}} = \nabla f(x, y, z) \cdot \mathbf{e_l} = \sqrt{2}x - \sqrt{2}y$

即求$g(x, y) = \sqrt{2}(x - y)$ 在条件 $2x^2 + 2y^2 + z^2 = 1$ 下的最大值. 设 $F(x, y, z, \lambda) = \sqrt{2}(x - y) + \lambda\left(2x^2 + 2y^2 + z^2 - 1\right)$, 则由方程组

$$\begin{cases} \frac{\partial f}{\partial x} = \sqrt{2} + 4\lambda x = 0 \\ \frac{\partial f}{\partial y} = -\sqrt{2} + 4\lambda y = 0 \\ \frac{\partial f}{\partial z} = 2\lambda z = 0 \\ \frac{\partial f}{\partial \lambda} = 2x^2 + 2y^2 + z^2 - 1 = 0 \end{cases}$$

解得 $z = 0$ ， $x = -y = \pm\frac{1}{2}$, 故驻点为 $M_1\left(\frac{1}{2}, -\frac{1}{2}, 0\right)$ 与 $M_2\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$. 由于最大值必然存在, 因此只需比较 $\left.\frac{\partial f}{\partial l}\right|_{M_1} = \sqrt{2}$, $\left.\frac{\partial f}{\partial l}\right|_{M_2} = -\sqrt{2}$ 的大小. 所以 $\left.\frac{\partial f}{\partial l}\right|_{M_2} = -\sqrt{2}$ 为所求最大值.

3.4.7　由于$f(x, y)$可微，可以利用定理3.4.1：

$\frac{\partial f}{\partial \mathbf{l}} = \nabla f(x, y) \cdot \mathbf{e_l}$

故有方程组：

$$\begin{cases} \frac{\partial f}{\partial \mathbf{u}}\big|_p = \frac{3}{5}f_x - \frac{4}{5}f_y = -6 \\ \\ \frac{\partial f}{\partial \mathbf{v}}\big|_p = \frac{3}{5}f_x + \frac{4}{5}f_y = 17 \end{cases}$$

解得 $\left.\frac{\partial f}{\partial x}\right|_p = 10$, $\left.\frac{\partial f}{\partial y}\right|_p = 15$,

故有：$df|_p = 10dx + 15dy$.

(B)

3.4.1　证: 由于$f(x, y)$在P_0处有连续的偏导数，故$f(x, y)$在P_0处是可微的，在 $\mathbf{R}^2$ 中利

用定理3.4.1,

$$\sum_{j=1}^{n}\frac{\partial f\left(x_0,y_0\right)}{\partial \mathbf{l_j}}=\sum_{j=1}^{n}[\nabla f\left(x_0,y_0\right)\cdot \mathbf{l_j}]$$
$$=|\nabla f(x_0,y_0)|\sum_{j=1}^{n}\cos<\mathbf{l_j},\nabla f(x_0,y_0)>.$$

不妨设 $\nabla f(x_0,y_0)$与l_1 的夹角为 α, 则 $l_1,l_2,\cdots,l_n$ 与 $\nabla f(x_0,y_0)$ 的夹角顺次为

$$\alpha,\alpha+\frac{2\pi}{n},\cdots,\alpha+(n-1)\frac{2\pi}{n},$$

因此

$$\begin{aligned}\sum_{j=1}^{n}\cos<\mathbf{l_j},\nabla f(x_0,y_0)>&=\cos\alpha+\cos\left(\alpha+\frac{2\pi}{n}\right)+\cdots+\cos\left[\alpha+(n-1)\frac{2\pi}{n}\right]\\&=\frac{1}{2\sin\frac{2\pi}{n}}\sum_{i=1}^{n-1}2\cos\left(\alpha+i\frac{2\pi}{n}\right)\sin\frac{2\pi}{n}\\&=\frac{1}{2\sin\frac{2\pi}{n}}\sum_{i=1}^{n-1}\left\{\sin\left[\alpha+(i+1)\frac{2\pi}{n}\right]-\sin\left[\alpha+(i-1)\frac{2\pi}{n}\right]\right\}\\&=0.\end{aligned}$$

证毕

3.4.2　依题意知, 温度函数为 $T(x,y)=\dfrac{k}{\sqrt{x^2+y^2}}(k>0)$, 易知沿梯度的反方向温度下降最快，于是有：

$$\operatorname{grad}T(x,y)=\frac{-k}{\sqrt{(x^2+y^2)^3}}\{x,y\},\quad \operatorname{grad}T(3,2)=\frac{-k}{13\sqrt{13}}\{3,2\},$$

故蚂蚁应朝方向 $\left\{\dfrac{3}{\sqrt{13}},\dfrac{2}{\sqrt{13}}\right\}$ 爬行, 才能最快达到凉爽处.

3.4.3　同上题，沿梯度反方向z下降最快，故登山者应沿$\mathbf{l}=-\nabla z(\frac{1}{2},-\frac{1}{2})=\{1,-2\}$可最快到达山底

3.5　多元函数的极值问题

(A)

3.5.2. 求 $f(x,y)=\sin x\sin y$ 在点 $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$ 的二阶Taylor公式.
解因为 $f_x(x,y)=\cos x\sin y, f_y(x,y)=\sin x\cos y,$

$$f_{xx}(x,y)=-\sin x\sin y,\quad f_{xy}=\cos x\cos y,\quad f_{yy}(x,y)=-\sin x\sin y$$

所以

$$f\left(\frac{\pi}{4},\frac{\pi}{4}\right)=f_x\left(\frac{\pi}{4},\frac{\pi}{4}\right)=f_y\left(\frac{\pi}{4},\frac{\pi}{4}\right)=-f_{xx}\left(\frac{\pi}{4},\frac{\pi}{4}\right)=f_{xy}\left(\frac{\pi}{4},\frac{\pi}{4}\right)=-f_{yy}\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\frac{1}{2}$$

故

$$f(x,y)=\frac{1}{2}+\frac{1}{2}\left(x-\frac{\pi}{4}\right)+\frac{1}{2}\left(y-\frac{\pi}{4}\right)-\frac{1}{4}\left[\left(x-\frac{\pi}{4}\right)^2-2\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)+\left(y-\frac{\pi}{4}\right)^2\right]+o\left(\rho^2\right),$$

其中 $\rho=\sqrt{\left(x-\frac{\pi}{4}\right)^2+\left(y-\frac{\pi}{4}\right)^2}$.
3.5.4. 求下列函数的极值: (2) $f(x,y)=e^x\left(x+y^2+2y\right)$; (4) $f(x,y)=\sin x+\sin y+\sin(x+y), 0<x<\pi, 0<y<\pi$.
解这两个函数的可能极值点只有驻点.
(2) 由 $\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases}$ 知: $\begin{cases} x+y^2+2y+1=0 \\ x+y^2+4y+2=0 \end{cases}$, 从而得驻点 $P\left(-\frac{1}{4},-\frac{1}{2}\right)$.
经计算得到 $A=f_{xx}(P)=\exp\left(-\frac{1}{4}\right)>0, B=f_{xy}(P)=\exp\left(-\frac{1}{4}\right), C=f_{yy}(P)=3\exp\left(-\frac{1}{4}\right), \Delta=AC-B^2=2\exp\left(-\frac{1}{2}\right)>0$. 因此点 P 为函数的极小值点, 极小值为 $f(P)=-2\exp\left(-\frac{1}{4}\right)$.
(4) 由

$$\begin{cases} f_x(x,y)=\cos x+\cos(x+y)=0 \\ f_y(x,y)=\cos y+\cos(x+y)=0 \end{cases}$$

解得 $x=y$, 进而得到唯一驻点 $P\left(\frac{\pi}{3},\frac{\pi}{3}\right)$. 经计算得到 $A=C=-\sqrt{3}<0, B=-\frac{\sqrt{3}}{2}$,
从而有 $\Delta=AC-B^2=3-\frac{3}{4}>0$.
因此点 P 为函数的极大值点, 极大值为 $f(P)=\frac{3\sqrt{3}}{2}$.

3.5.5. 求下列函数在指定区域 D 上的最大值与最小值:

(1) $z=x^2y(4-x-y), \quad D=\{(x,y)\mid x\geq 0, y\geq 0, x+y\leq 4\}$; (3) $z=x^2+y^2-12x+16y, \quad D=\left\{(x,y)\mid x^2+y^2\leq 25\right\}$. 解 (1) 函数在区域 D 内仅一个驻点 $P(1,2)$, 得函数值 $f(P)=4$.

在两坐标轴上函数恒为 0 . 在直线段 $y=4-x(0\leq x\leq 4)$ 上, $f(x,4-x)=0(0\leq x\leq 4)$.

故, 函数在 D 上的最大值为 4, 最小值为 0 .

(3) 由 $\begin{cases} f_x(x,y)=2x-12=0 \\ f_y(x,y)=2y+16=0 \end{cases}$ 解得 $x=6, y=-8$, 点 $(6,-8)\notin D$. 下面考虑函数在边界 ∂D 的取值情况.

$L=x^2+y^2-12x+16y+\lambda\left(x^2+y^2-25\right)$ 或 $L=25-12x+16y+\mu\left(x^2+y^2-25\right)$.

则由 $\nabla L=\overrightarrow{0}$, 得到 $\lambda+1=\pm 2$ (或 $\mu=\pm 2$), $x=\dfrac{6}{\lambda+1}=\pm 3$ (或 $x=\dfrac{6}{\mu}=\pm 3$), 同时 $y=-\dfrac{8}{\lambda+1}=\mp 4$ (或 $\left.y=-\dfrac{8}{\mu}=\mp 4\right)$. 因为 $f(3,-4)=-75, f(-3,4)=125$, 所以函数在 D 上的最大值为 125, 最小值为 -75 .

3.5 .6 求原点到曲线 $\begin{cases} x^2+y^2=z, \\ x+y+z=1 \end{cases}$ 的最长和最短距离.

解设目标函数 $d^2=f(x,y,z)=x^2+y^2+z^2$, 令 $L=x^2+y^2+z^2+\lambda\left(x^2+y^2-z\right)+\mu(x+y+z-1)$.

由 $\begin{cases} L_x=2x+2\lambda x+\mu=0, \\ L_y=2y+2\lambda y+\mu=0, \text{ 的前两式相减, 得到 } (\lambda+1)(x-y)=0 \\ L_z=2z-\lambda+\mu=0 \end{cases}$

由于 $\lambda\neq -1$ (否则有 $\mu=0, z=-1/2<0$, 不合约束条件), 所以 $x=y$.

再联立约束条件 $z=x^2+y^2$ 与 $x+y+z=1$,

可解出 $x=y=\dfrac{1}{2}(-1\pm\sqrt{3}), z=2x^2=2\mp\sqrt{3}$.

于是得到 $d^2=9\mp 5\sqrt{3}$. 故, 最长距离 $=\sqrt{9+5\sqrt{3}}$, 最短距离 $=\sqrt{9-5\sqrt{3}}$.

注: 可由 $\nabla f, \nabla g, \nabla h$ 的混合积 $2(2z+1)(x-y)=0$ 得到 $x=y$,

其中 $g(x,y,x)=x^2+y^2-z, h(x,y,z)=x+y+z-1$.

3.5.9. 求函数 $f(x,y,z)=x+2y+3z$ 在圆柱 $x^2+y^2=2$ 与平面 $y+z=1$ 的交线椭圆上的最大值与最小值.

解目标函数为 $f(x,y,z)=x+2y+3z$, 此时有两个约束条件 $g_1=x^2+y^2-2=0$ 与 $g_2=y+z-1=0$. 作Lagrange函数

$$L(x,y,z,\lambda,\mu)=x+2y+3z+\lambda\left(x^2+y^2-2\right)+\mu(y+z-1).$$

由方程组 $\begin{cases} L_x=1+2\lambda x=0 \\ L_y=2+2\lambda y+\mu=0 \\ L_z=3+\mu=0 \\ L_\lambda=x^2+y^2-2=0 \\ L_\mu=y+z-1=0 \end{cases}$

解得Lagrange函数 L 有两个驻点 $(1,-1,2)$ 和 $(-1,1,0)$.

由于函数最大值和最小值存在, 故最大值为 $f(1,-1,2)=5$, 最小值为 $f(-1,1,0)=1$.

(B)

3.5.1. 求曲线 $\begin{cases} z=\sqrt{x} \\ y=0 \end{cases}$ 与曲线 $\begin{cases} x+2y-3=0, \\ z=0 \end{cases}$ 之间的距离.

解 :在第一条曲线上任取一点 $(x,0,\sqrt{x})$, 在第二条曲线上任取一点 $(3-2v,v,0)$, 设它们距离的平方为日标函数 $f(x,v)=(x+2v-3)^2+v^2+x(x\geq 0)$.

由 $\begin{cases} f_x(x,v)=2(x+2v-3)+1=0 \\ f_v(x,v)=4(x+2v-3)+2v=0 \end{cases}$ 解得唯一驻点 $x=1/2,v=1$.

由几何意义知 f 的最小值存在, 故在两曲线上对应点 $(1/2,0,1/\sqrt{2})$ 与 $(1,1,0)$ 的距离最小, 其值为 $\sqrt{7}/2$.

3.5.2. 设椭球面 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ 被通过原点的平面 $lx+my+nz=0$ 截成一个椭圆, 求这个椭圆的面积. 解设目标函数 $f(x,y,z)=x^2+y^2+z^2$, 则考虑函数 $f(x,y,z)$ 在约

束条件

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \\ \\ lx+my+nz=0 \end{cases}$$

下的极值. 令

$$L=x^2+y^2+z^2-\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\mu(lx+my+nz).$$

由 $(L_x,L_y:L_Z)=(0,0,0)$ 得到

$$\begin{cases} L_x=2x-\dfrac{2\lambda x}{a^2}+l\mu=0, & (1) \\ L_y=2y-\dfrac{2\lambda y}{b^2}+m\mu=0, & (2) \\ L_z=2z-\dfrac{2\lambda z}{c^2}+n\mu=0, & (3) \end{cases}$$

$(1)\times x+(2)\times y+(3)\times z$, 并利用约束条件, 得到 $x^2+y^2+z^2=\lambda$. 联立 (1), (2), (3)与 $lx+my+nz=0$, 由于 (x,y,z,μ) 为方程组的非零解, 所以系数行列式为 0 , 即

$$0=\begin{vmatrix} 2-\dfrac{2\lambda}{a^2} & 0 & 0 & l \\ 0 & 2-\dfrac{2\lambda}{b^2} & 0 & m \\ 0 & 0 & 2-\dfrac{2\lambda}{c^2} & n \\ l & m & n & 0 \end{vmatrix}$$

$$=-4m^2\left(1-\frac{\lambda}{a^2}\right)\left(1-\frac{\lambda}{c^2}\right)-4n^2\left(1-\frac{\lambda}{a^2}\right)\left(1-\frac{\lambda}{b^2}\right)-4l^2\left(1-\frac{\lambda}{b^2}\right)\left(1-\frac{\lambda}{c^2}\right).$$

因此, 有

$$\left(\frac{m^2}{a^2c^2}+\frac{n^2}{a^2b^2}+\frac{l^2}{b^2c^2}\right)\lambda^2-\left(\frac{a^2+c^2}{a^2c^2}m^2+\frac{a^2+b^2}{a^2b^2}n^2+\frac{b^2+c^2}{b^2c^2}l^2\right)\lambda+m^2+n^2+l^2=0.$$

得到: $\lambda_1\lambda_2=\dfrac{m^2+n^2+l^2}{\frac{m^2}{a^2c^2}+\frac{n^2}{a^2b^2}+\frac{l^2}{b^2c^2}}$, 故所求面积

$$S=\pi\sqrt{\lambda_1\lambda_2}=\pi\sqrt{\frac{m^2+n^2+l^2}{\frac{m^2}{a^2c^2}+\frac{n^2}{a^2b^2}+\frac{l^2}{b^2c^2}}}=\pi abc\sqrt{\frac{m^2+n^2+l^2}{b^2m^2+c^2n^2+a^2l^2}}.$$

3.5.5. 设函数 $f(x)$ 在 $[1+\infty)$ 内有二阶连续导数, $f(1)=0, f'(1)=1$ 且 $z=\left(x^2+y^2\right) f\left(x^2+y^2\right)$ 满足 $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$, 求 $f(x)$ 在 $[1+\infty)$ 上的最大值.
解 $z_x=2xf+2x\left(x^2+y^2\right)f'$, 于是

$$z_{xx}=2f+4x^2f'+2\left(x^2+y^2\right)f'+4x^2f'+4x^2\left(x^2+y^2\right)f''=2f+2\left(5x^2+y'f'+4x^2\left(x^2+y^2\right)f''\right.$$

同理可得 $z_{yy}=2f+2\left(x^2+5y^2\right)f'+4y^2\left(x^2+y^2\right)f''$. 因此

$$0=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=4\left(x^2+y^2\right)^2 f''\left(x^2+y^2\right)+12\left(x^2+y^2\right)f'\left(x^2+y^2\right)+4f\left(x^2+y^2\right).$$

若记 $t=x^2+y^2$, 则得到Euler方程

$$t^2f''(t)+3tf'(t)+f(t)=0$$

其对应的常系数方程为 $z''+2z'+z=0$ (令 $s=\ln t$, 则 $z=f\left(e^s\right)$), 通解为 $z=\left(C_1+C_2 s\right)e^{-s}$. 故, $f(t)=\left(C_1+C_2\ln t\right)t^{-1}$.
由初值.条件解得 $C_1=0, C_2=1$. 因此 $f(t)=\dfrac{\ln t}{t}$, 有唯一驻点 $t=e$, 且 $1<t<e$ 时 $f'(t)>0; t>e$ 时 $f'(t)<0$, 从而知 $f(e)=e^{-1}$ 为极大值.
又 $\lim\limits_{t\to+\infty}f(t)=0, f(1)=0$, 故 $f(e)=e^{-1}$ 是 $f(t)$ 在 $[1+\infty)$ 上的最大值.

3.6 多元函数微分学在几何上的简单应用

(A)

3.6.1. 求下列曲线在给定点的切线和法平面方程: (1) $\vec{r}=\left(t, 2t^2, t^2\right)$, 在 $t=1$ 处; (2) $\vec{r}=(3\cos\theta, 3\sin\theta, 4\theta)$, 在点 $\left(\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}, \pi\right)$ 处.
解 (1) 切线方程: $\dfrac{x-1}{1}=\dfrac{y-2}{4}=\dfrac{z-1}{2}$; 法平面方程: $x+4y+2z=11$. (2) 切线方程: $\dfrac{x-\frac{3}{\sqrt{2}}}{-3}=\dfrac{y-\frac{3}{\sqrt{2}}}{3}=\dfrac{z-\pi}{4\sqrt{2}}$;
法平面方程: $3x-3y-4\sqrt{2}z=-4\pi\sqrt{2}$ (或 $\dfrac{3}{\sqrt{2}}x-\dfrac{3}{\sqrt{2}}y-4z=-4\pi$).
3.6.2. 求下列平面曲线的弧长: (1) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},(a>0)$ 的全长; (2) $\rho=a(1+\cos\theta)$ 的全长. 解 (1) $6a$; (2) $8a$.
3.6.3. 求下列空间曲线的弧长: (1) $\vec{r}=\left(e^t\cos t, e^t\sin t, e^t\right)$ 介于点 $(1,0,1)$ 与点 $\left(0, e^{\frac{\pi}{2}}, e^{\frac{\pi}{2}}\right)$

之间的弧长; (3) $\begin{cases} x^2 = 3y, \\ 2xy = 9z \end{cases}$ 介于点 $(0,0,0)$ 与点 $(3,3,2)$ 之间的弧长.

解 (1) 弧长为: $\sqrt{3}\int_0^{\frac{\pi}{2}} e^t \, \mathrm{d}t = \sqrt{3}\left(e^{\frac{\pi}{2}} - 1\right)$. (3) 曲线以 x 为参数, 得到 $y = \frac{x^2}{3}, z = \frac{2x^3}{27}$. 所以弧长为:

$$\int_0^3 \sqrt{1 + \left(\frac{2}{3}x\right)^2 + \left(\frac{2}{9}x^2\right)^2} dx = \int_0^3 \left(1 + \frac{2}{9}x^2\right) \mathrm{d}x = 5$$

3.6.4. 求下列曲面在给定点的切平面与法线方程: (2) $z^2 = \frac{x^2}{4} + \frac{y^2}{9}$ 在 $(6,12,5)$ 处; (3) $x^3 + y^3 + z^3 + xyz - 6 = 0$ 在 $(1,2,-1)$ 处. 解对曲面 $F(x,y,z) = 0$, 法向量 $\vec{n} = (F_x, F_y, F_z)|_{p_0}$ 。 (2) 令 $F(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} - z^2$, 得法向量 $\vec{n} = (3, 8/3, -10)$. 切平面方程: $9x + 8y - 30z = 0$; 法线方程: $\frac{x-6}{9} = \frac{y-12}{8} = \frac{z-5}{-30}$. (3) 令 $F(x,y,z) = x^3 + y^3 + z^3 + xyz - 6$, 法向量 $\vec{n} = (1, 11, 5)$. 切平面方程: $x + 11y + 5z = 18$; 法线方程: $\frac{x-1}{1} = \frac{y-2}{11} = \frac{z+1}{5}$.

3.6.6. (1) 求曲面 $x^2 + y^2 + z^2 = x$ 的切平面, 使它垂直于平面 $x - y - \frac{1}{2}z = 2$ 和切平面, 求此切平面的方程.

解 (1) 依题意, 所求切平面的法向量为 $(1,-1,-1/2) \times (1,-1,-1) = (1/2, 1/2, 0)$, 取 $\vec{n} = (1,1,0)$. 另一方面, 曲面 $x^2+y^2+z^2 = x$ 上点 (x_0, y_0, z_0) 处的法向量为 $(2x_0 - 1, 2y_0, 2z_0)$, 从而有 $2x_0 - 1 = 2y_0, z_0 = 0$, 将其代入曲面方程 $x^2 + y^2 + z^2 = x$, 得到 $x_0 = \frac{2 \pm \sqrt{2}}{4}, y_0 = \frac{\pm\sqrt{2}}{4}, z_0 = 0$. 故所求切平面为 $\left(x - \frac{2+\sqrt{2}}{4}\right) + \left(y - \frac{\sqrt{2}}{4}\right) = 0$ 和 $\left(x - \frac{2-\sqrt{2}}{4}\right) + \left(y + \frac{\sqrt{2}}{4}\right) = 0$, 即 $x + y = \frac{1}{2}(1+\sqrt{2})$ 和 $x + y = \frac{1}{2}(1-\sqrt{2})$. (2) $9x + y - z = 27$ 与 $9x + 17y - 17z = -27$. 解依题意, 所求法线方向向量为 $(1,3,1) \times (1,1,0) = (-1,1,-2)$, 取 $\vec{n} = (1,-1,2)$. 另一方面, 曲面 $x^2 + 2y^2 + z^2 = 22$ 上点 (x_0, y_0, z_0) 处的法向量为 $(x_0, 2y_0, z_0)$, 所以 $x_0 = -2y_0 = z_0/2$. 将其代入曲面方程 $x^2 + 2y^2 + z^2 = 22$, 得到 $x_0 = \pm 2, y_0 = \mp 1, z_0 = \pm 4$. 故所求法线为

$$\frac{x \pm 2}{1} = \frac{y \mp 1}{-1} = \frac{z \pm 4}{2}$$

3.6.9. 求旋转抛物面 $S: z = x^2 + y^2$ 和平面 $\pi: x + y - 2z = 2$ 平行的切平面的方程. 解

设 S 上点 $P_0\,(x_0,y_0,z_0)$ 处的切平面与平面 π 平行. 由于 S 上点 P_0 处法向量为

$$\vec{n}|_{P_0}=(2x_0,2y_0,-1)\,,$$

平面 π 的法向量为 $(1,1,-2)$, 按照平面平行的条件, 应该有

$$\frac{2x_0}{1}=\frac{2y_0}{1}=\frac{-1}{-2}=\frac{1}{2},$$

从而求得 $P_0(1/4,1/4,1/8)$. 因此, 所求切平面方程是 $(x-1/4)+(y-1/4)-2(z-1/8)=0$, 或 $x+y-2z=1/4$.

(B)

3.6.3. 设函数 $f(u,v)$ 在全平面上有连续的偏导数, 取 S 由方程 $f\left(\dfrac{x-a}{z-c},\dfrac{y-b}{z-c}\right)=0$ 确定.

证明: 该曲面的所有切平面都过点 (a,b,c). 证记 $F(x,y,z)=f\left(\dfrac{x-a}{z-c},\dfrac{y-b}{z-c}\right)$, 则

$$(F_x,F_y,F_z)=\left(\frac{f_1}{z-c},\frac{f_2}{z-c},-\frac{(x-a)f_1+(y-b)f_2}{(z-c)^2}\right).$$

取曲面 S 的法向量

$$\vec{n}=((z-c)f_1,(z-c)f_2,-(x-a)f_1-(y-b)f_2)\,.$$

记 (x,y,z) 为曲面 S 上的点, (X,Y,Z) 为切平面上的点, 则曲面 S 上过点 x,y,z 的切平面为

$$(z-c)f_1(X-x)+(z-c)f_2(Y-y)-[(x-a)f_1+(y-b)f_2]\,(Z-z)=0.$$

对应任意的 $(x,y,z)(z\neq c),(X,Y,Z)=(a,b,c)$ 都满足切平面方程. 证.毕.

3.7 空间的曲率

(A)

3.7.1. 求下列平面曲线在给定点的曲率: (2) $y=\sin x$, 在点 $\left(\dfrac{\pi}{2},1\right)$ 处. 解 $\kappa=\left.\dfrac{|\sin x|}{(1+\cos^2 x)^{3/2}}\right|_{\frac{\pi}{2}}=1.$

3.7.2. 求下列平面曲线的曲率: (1) $y=ax^2$; (3) $\vec{r}=(a\cosh t,a\sinh t)$. 解 (1) $\kappa=\dfrac{2|a|}{(1+4a^2x^2)^{3/2}}$. (3) $\kappa=\dfrac{1}{a(\cosh(2t))^{3/2}}$. 3.7.3. 求下列曲线的曲率 $(a>0)$: (1) $\vec{r}=$

$(a\cosh t, a\sinh t, bt)$;　(3) $\vec{r}=(a(1-\sin t), a(1-\cos t), bt)$. 解 (1) $\kappa=\dfrac{a\sqrt{b^2\cosh(2t)+a^2}}{\left(a^2\cosh(2t)+b^2\right)^{3/2}}$.
(2) $\kappa=\dfrac{1}{a^2\left(a^2+b^2\right)}$.

3.7.4. 曲线 $y=\ln x$ 上哪一点处的曲率半径最小? 求出该点处的曲率半径. 解曲率 $\kappa(x)=\dfrac{x}{\left(1+x^2\right)^{3/2}}$. 由 $\kappa'(x)=0$ 得到驻点 $x_0=1/\sqrt{2}$, 经检验它也是 $\kappa(x)$ 的最大值点. 因此, 曲线 $y=\ln x$ 上点 $(1/\sqrt{2}, \ln(1/\sqrt{2}))$ 处的曲率半径最小, 该点处的曲率半径为 $1/\kappa\left(x_0\right)=3\sqrt{3}/2$.

(B)

3.7.1. 求曲率 $\kappa(s)=\dfrac{a}{a^2+s^2}$ 的平面曲线. (s 是弧长参数) 解

$$\kappa(s)=\frac{d\theta}{\mathrm{d}s}\Rightarrow d\theta=\kappa(s)\mathrm{d}s=\frac{a}{a^2+s^2}\,\mathrm{d}s,$$
$$\theta(s)=\int\kappa(s)\mathrm{d}s=\int\frac{a}{a^2+s^2}\,\mathrm{d}s=\arctan\frac{s}{a}+C.$$

设曲线的参数方程为

$$\begin{cases} x=x(s) & \\ & s\in[0,l] \\ y=y(s), & \end{cases}$$

由弧微分公式

$$\mathrm{d}s=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}$$

得

$$\mathrm{d}x=\cos\theta\mathrm{d}s,\ \mathrm{d}y=\sin\theta\mathrm{d}s.$$

不防设

$$x(0)=0, y(0)=0, \theta(0)=0,$$

则

$$\begin{aligned} x(s)&=x(0)+\int_0^s\cos(\theta(s))\mathrm{d}s=\int_0^s\cos\left(\arctan\frac{s}{a}\right)\mathrm{d}s\\ &=\int_0^s\frac{a}{\sqrt{a^2+s^2}}\,\mathrm{d}s=a\left[\ln\left(s+\sqrt{a^2+s^2}\right)-\ln a\right],\\ y(s)&=y(0)+\int_0^s\sin(\theta(s))\mathrm{d}s=\int_0^s\sin\left(\arctan\frac{s}{a}\right)\mathrm{d}s\\ &=\int_0^s\frac{s}{\sqrt{a^2+s^2}}\,\mathrm{d}s=\sqrt{a^2+s^2-a}, \end{aligned}$$

所以曲线的方程为

$$\mathbf{r}(s)=\left(a\ln\left(s+\sqrt{a^2+s^2}\right)-a\ln a,\sqrt{a^2+s^2}-a\right).$$

消去 s, 可得: $y=a\left(\cosh\dfrac{x}{a}-1\right)$. 事实上, 由 $x=a\ln\left(s+\sqrt{a^2+s^2}\right)-a\ln a$ 知

$$\frac{x}{a}=\ln\frac{s+\sqrt{a^2+s^2}}{a}=\ln\frac{a}{\sqrt{a^2+s^2}-s},$$

所以

$$e^{\frac{x}{a}}=\frac{s+\sqrt{a^2+s^2}}{a},\quad e^{-\frac{z}{a}}=\frac{\sqrt{a^2+s^2}-s}{a}.$$

于是, 注意到 $y=\sqrt{a^2+s^2}-a$, 我们有

$$\frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2}=\frac{\sqrt{a^2+s^2}}{a}\Rightarrow y=a\cosh\frac{x}{a}-a$$

3.7.3. 设 $\vec{r}(t)$ 是空间曲线, 曲率为 $\kappa(t)$. 求曲线 $\vec{r}=\vec{r}(-t)$ 的曲率. 解因为 $\vec{r}'(t)=-\vec{r}'(-t),\vec{r}''(t)=\vec{r}''(-t),\vec{r}'''(t)=-\vec{r}'''(-t)$, 所以曲线 $\vec{r}=\vec{r}(-t)$ 的曲率为

$$\tilde{\kappa}(t)=\frac{\|-\vec{r}'(-t)\times\vec{r}''(-t)\|}{\|-\vec{r}'(-t)\|^3}=\frac{\|\vec{r}'(-t)\times\vec{r}''(-t)\|}{\|\vec{r}'(-t)\|^3}=\kappa(-t).$$

3.8 多元向量值函数的导数和微分

(A)

3.8.1. 求下列向量值函数的Jacobi矩阵: (1) $\vec{f}(x,y)=\left(x^2+\sin y,2xy\right)^T$ (3) $\vec{f}(x,y,z)=(x\cos y,ye^x:\sin(xz))^T$. 解 (1) $\mathrm{D}\vec{f}=\begin{pmatrix}2x&\cos y\\2y&2x\end{pmatrix}$. (3) $\mathrm{D}\vec{f}=\begin{pmatrix}\cos y&-x\sin y&0\\ye^x&e^x&0\\z\cos(xz)&0&x\cos(xz)\end{pmatrix}$.

3.8.3. 求向量值函数 $\vec{f}(x,y)=(\arctan x,e^{xy})^T$ 的导数 $D\vec{f}(x,y)$. 解 $\mathrm{D}\vec{f}=\begin{pmatrix}\dfrac{1}{1+x^2}&0\\ye^{xy}&xe^{xy}\end{pmatrix}$.

3.8.6. 设向量值函数 $\vec{f}:\mathbb{R}^3\to\mathbb{R}^2$ 定义 $\vec{f}(x,y,z)=\left(e^x\cos y+e^yz^2, 2x\sin y-\ 3yz^3\right)^T$, 求 $\mathrm{D}\vec{f}\left(0,\dfrac{\pi}{2},1\right)$. 解原式

$$=\begin{pmatrix} e^x\cos y & -e^x\sin y+e^yz^2 & 2ze^y \\ \\ 2\sin y & 2x\cos y-3z^3 & -9yz^2 \end{pmatrix}_{\left(0,\frac{\pi}{2},1\right)} = \begin{pmatrix} 0 & -1+e^{\frac{\pi}{2}} & 2e^{\frac{\pi}{2}} \\ \\ 2 & -3 & -\dfrac{9}{2}\pi \end{pmatrix}.$$

(B)