09-A1

BST Application

Range Query: 1D

邓俊辉

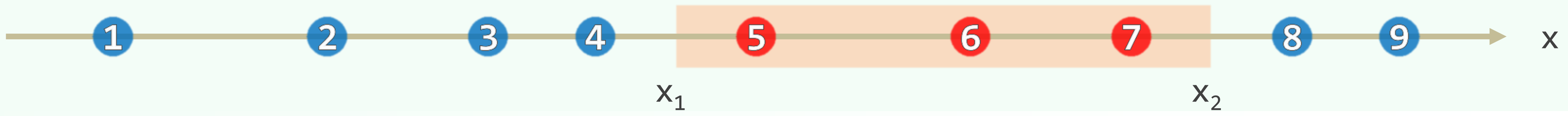
deng@tsinghua.edu.cn

你这个人太敏感了。这个社会什么都需要，唯独不需要敏感。

1D Range Query

- Let $P = \{ p_1, p_2, p_3, \ldots, p_n \}$ be a set of n points on the x-axis
- For any given interval $I = (x_1, x_2]$
  - COUNTING: how many points of P lies in the interval?
  - REPORTING: enumerate all points in $I \cap P$ (if not empty)
- [Online] P is fixed while I is randomly and repeatedly given
- How to PREPROCESS P into a certain data structure s.t.

  the queries can be answered efficiently?

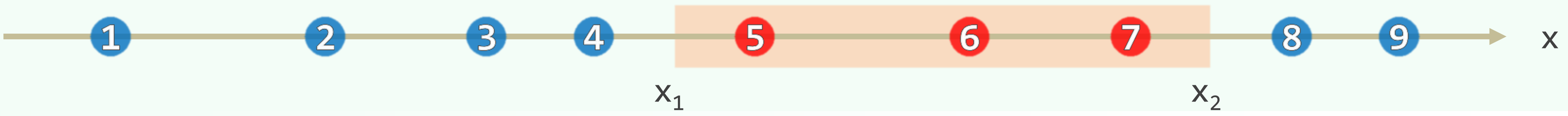


Brute-Force

- For each point p of P, test if $p \in (x_1, x_2]$
- Thus each query can be answered in LINEAR time
- Can we do it faster? It seems we can't, for ...
- In the worst case,

  the interval contains up to $\mathcal{O}$(n) points, which need $\mathcal{O}$(n) time to enumerate
- However, how if we

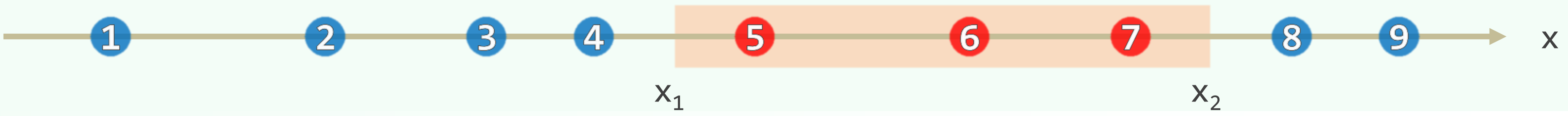
  ignore the time for enumerating and count only the searching time?



Binary Search

- Sort all points into a sorted vector and add an extra sentinel p[0] = $-\infty$
- For any interval $I = (x_1, x_2]$
  - Find t = search(x_2) = max{ i | p[i] $\leq$ x_2 } //$\mathcal{O}$(logn)
  - Traverse the vector BACKWARD from p[t] and report each point //$\mathcal{O}$(r)

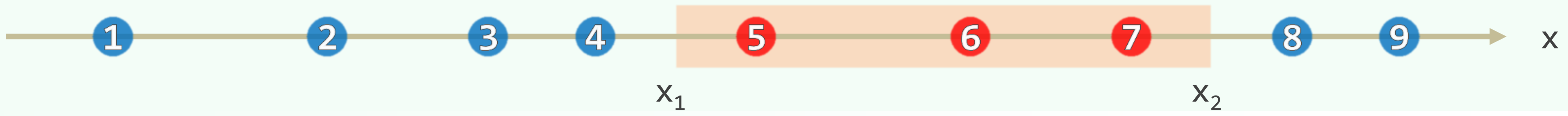
    until escaping from I at point p[s]
  - return r = t - s //output size



Output-Sensitivity

- An enumerating query can be answered in $\mathcal{O}(r + \log n)$ time
- p[s] can also be found by binary search in $\mathcal{O}(\log n)$ time
- Hence for COUNTING query, $\mathcal{O}(\log n)$ time is enough //independent to r
- Can this simple strategy be extended to PLANAR range query?

  TTBOMK, unfortunately, no!



09-A2

BST Application

Range Query: 2D

昔者明王必尽知天下良士之名；既知其名，又知其数；
既知其数，又知其所在。

邓俊辉
deng@tsinghua.edu.cn

Planar Range Query

❖ Let $P = \{ p_1, p_2, p_3, \ldots, p_n \}$ be a planar set

❖ Given $R = (x_1, x_2] \times (y_1, y_2]$

- COUNTING: $|R \cap P| = ?$
- REPORTING: $R \cap P = ?$

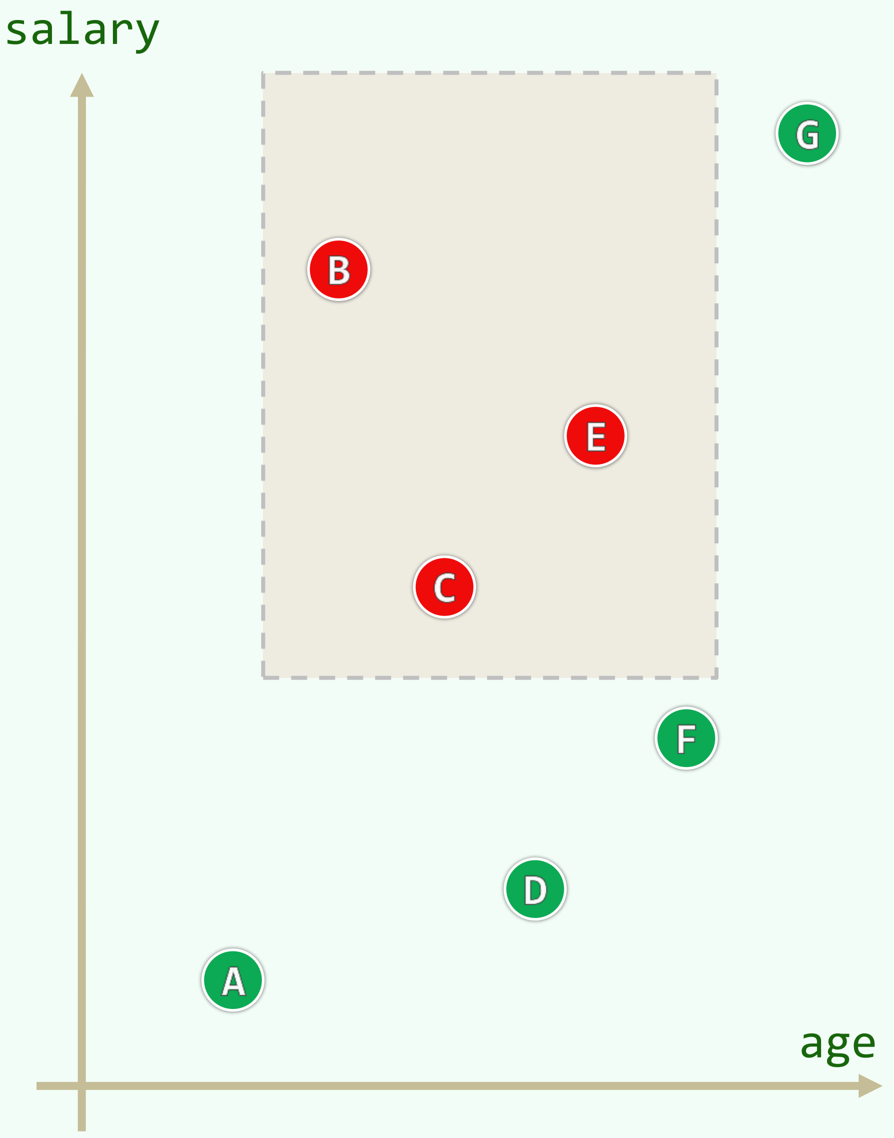
❖ Binary search

doesn't help this kind of query

❖ You might consider to

expand the counting method using

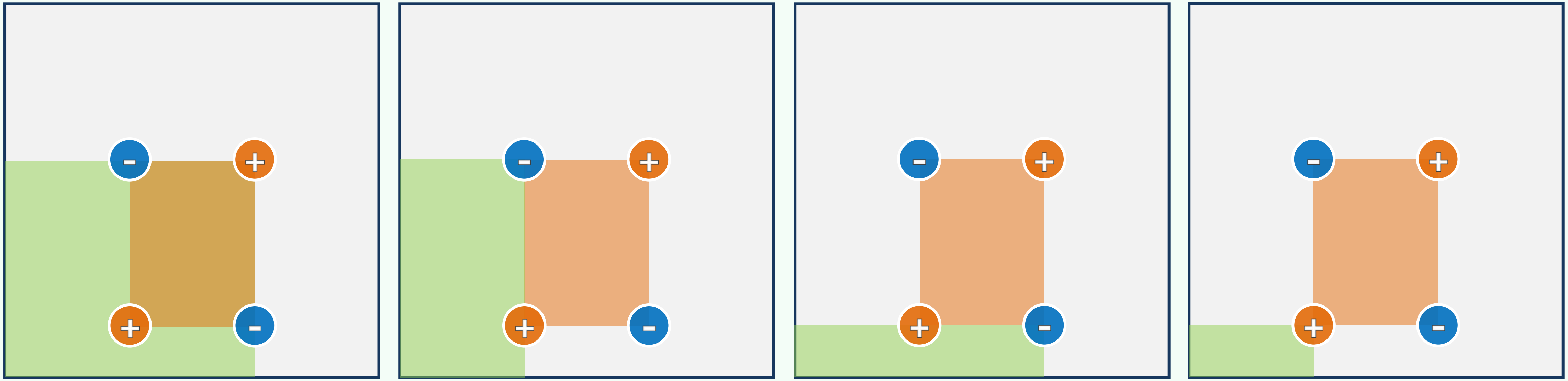
the Inclusion-Exclusion Principle



Preprocessing

❖ $\forall$ point (x, y), let $n(x, y) = \left|((0, x] \times (0, y]) \cap P\right|$

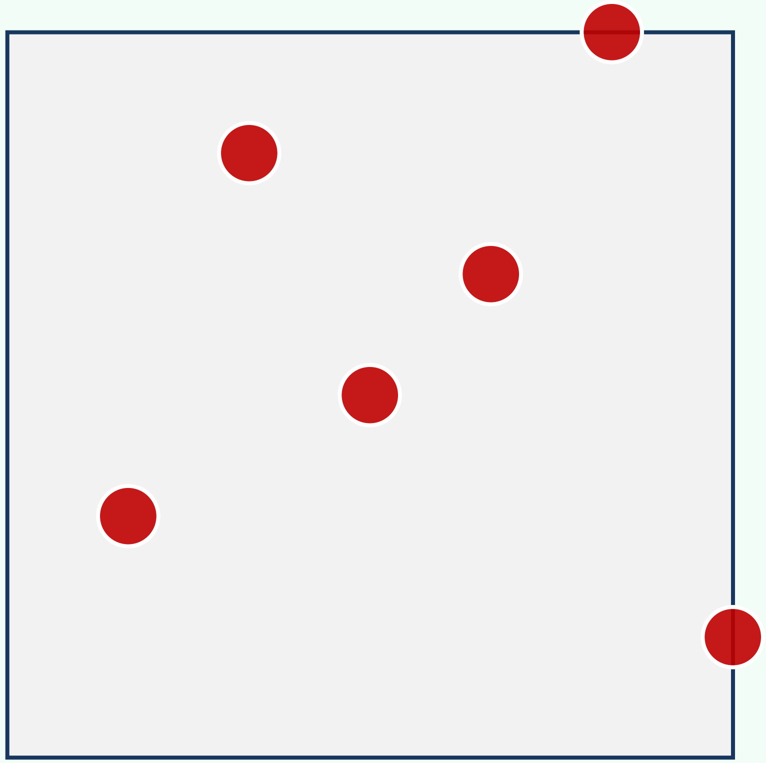
❖ This requires $\mathcal{O}(n^2)$ time/space

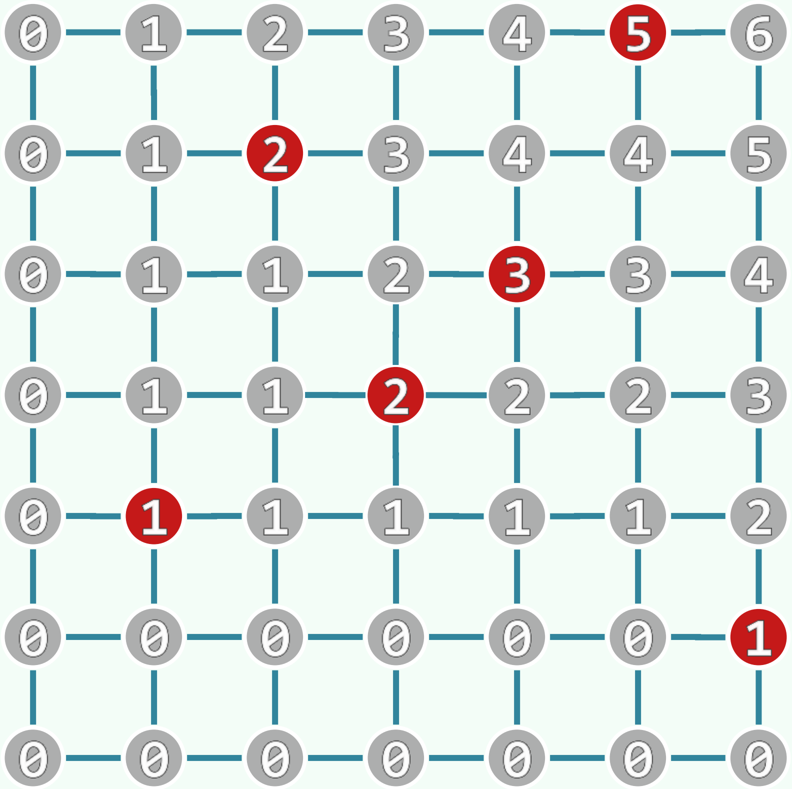


Domination

❖ A point (u, v) is called to be **DOMINATED** by point (x, y) if

$u \le x$ and $v \le y$

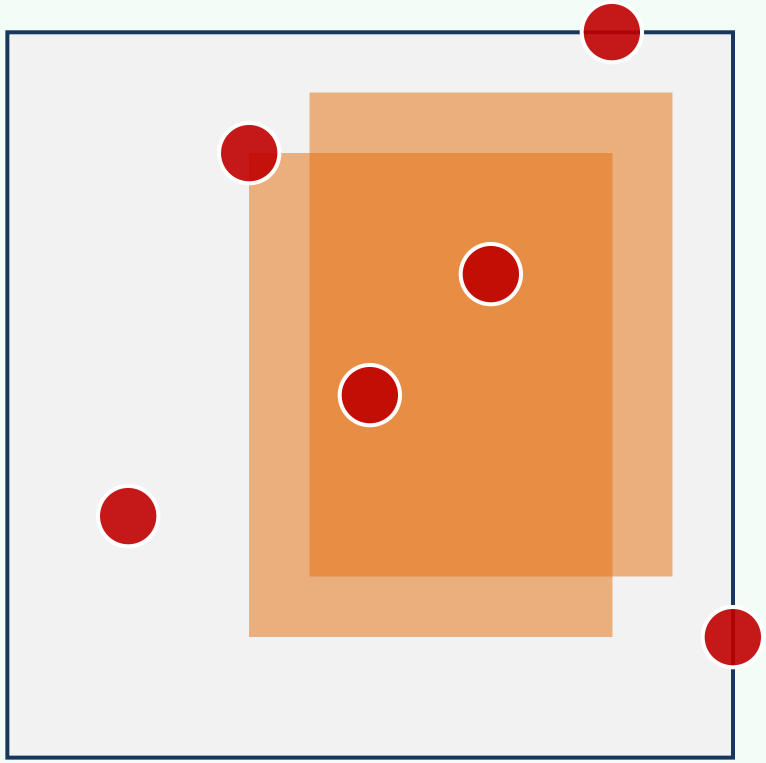


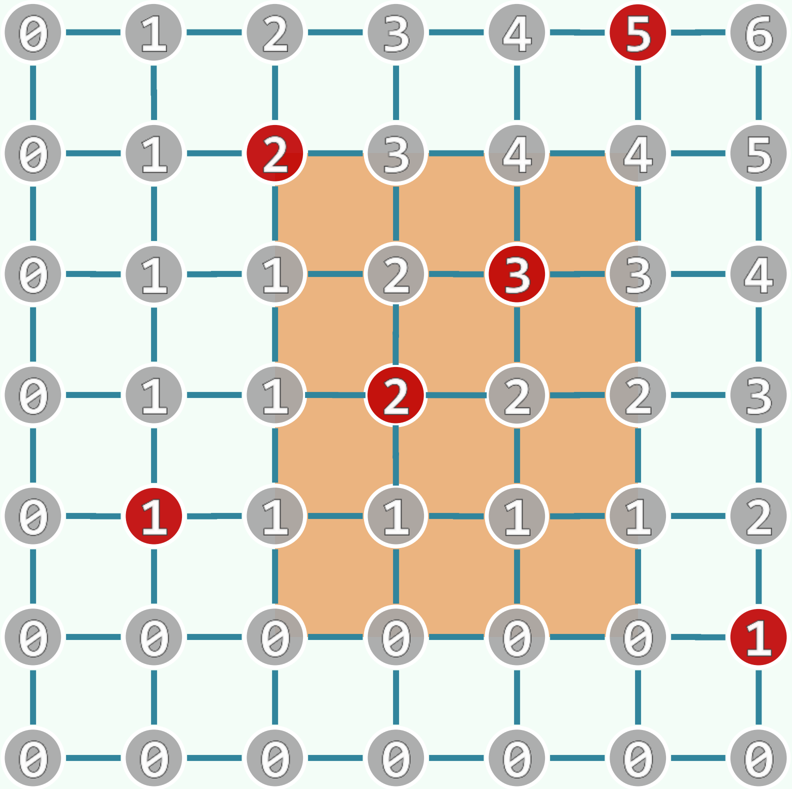


Inclusion-Exclusion Principle

❖ Then for any rectangular range $\mathcal{R} = (x_1, x_2] \times (y_1, y_2]$, we have

$$\left|\mathcal{R} \cap \mathcal{P}\right| \;=\; n(x_1, y_1) + n(x_2, y_2) - n(x_1, y_2) - n(x_2, y_1)$$



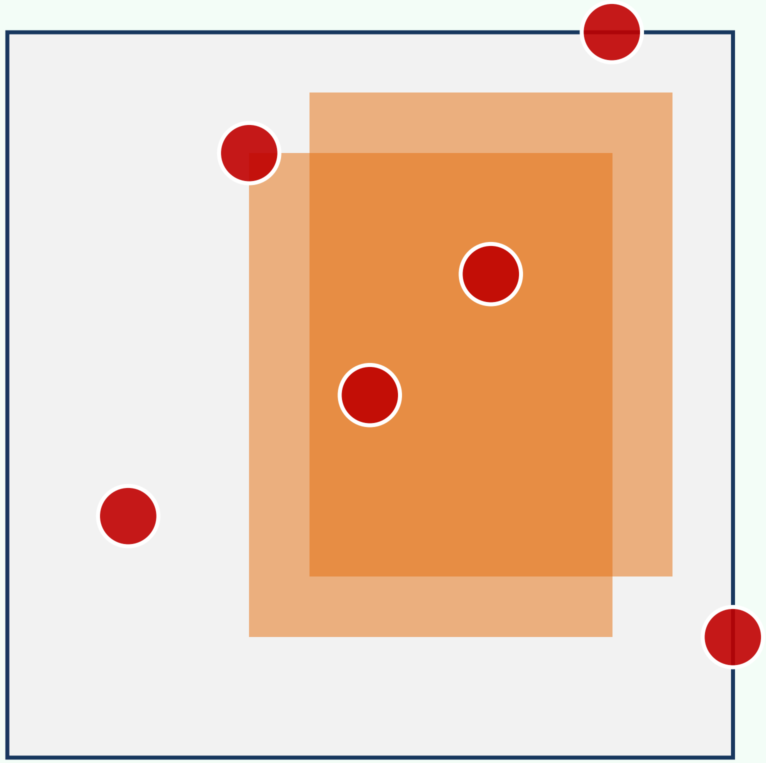


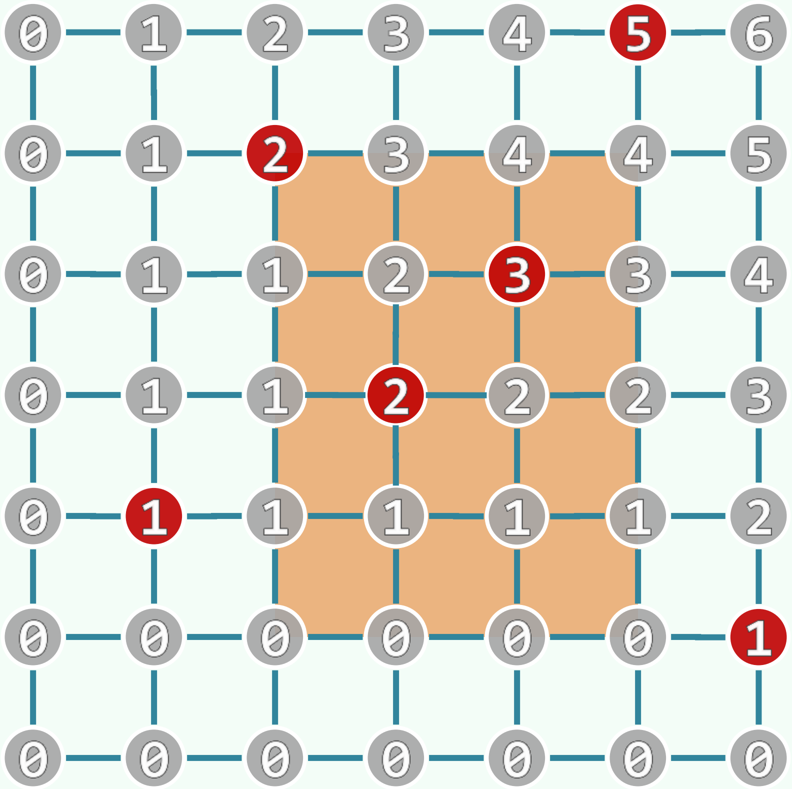
Performance

❖ Each query needs only $\mathcal{O}(\log n)$ time

❖ Uses $\Theta(n^2)$ storage and even more for higher dimensions

❖ To figure out a better solution, let's go back to the 1D case ...





09-B1

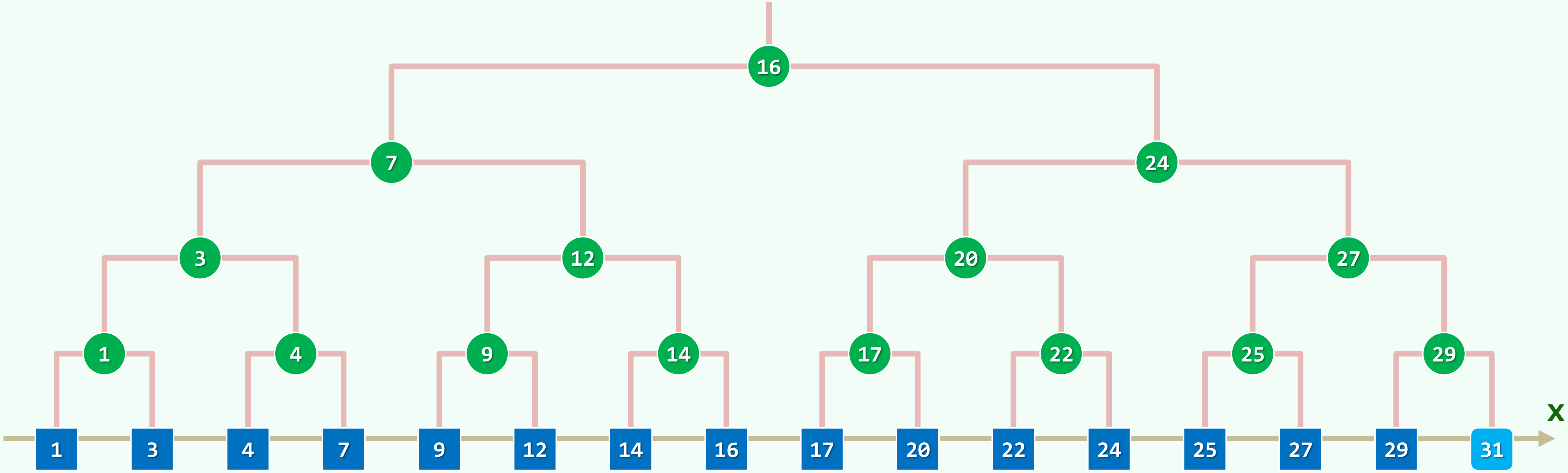
BST Application

kd-Tree: 1D

邓俊辉
deng@tsinghua.edu.cn

Structure

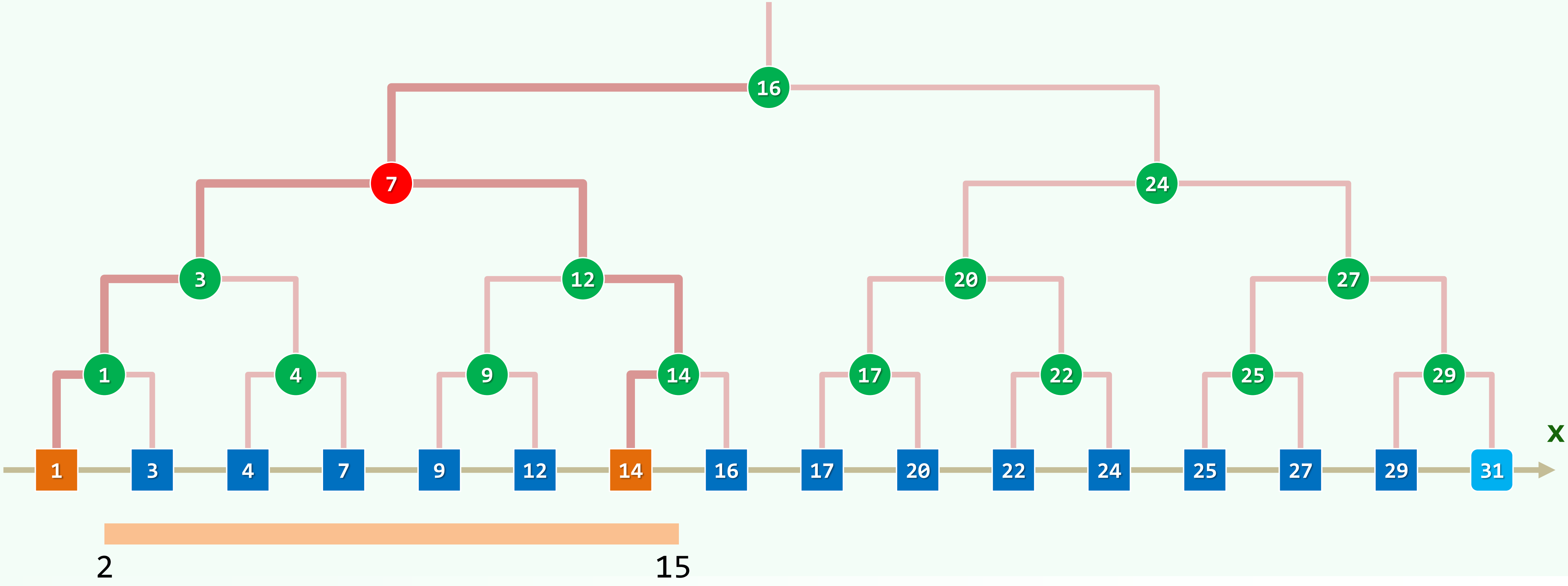
- For each v, v.key = max{ u.key | u ∈ L-Tree(v) } = v.pred().key
- For each u in L/R-Tree(v), x(u) ≤/> x(v)
- search(x) : returns the maximum key not greater than x



Lowest Common Ancestor

❖ Consider, as an example, the query for (2, 15] ...

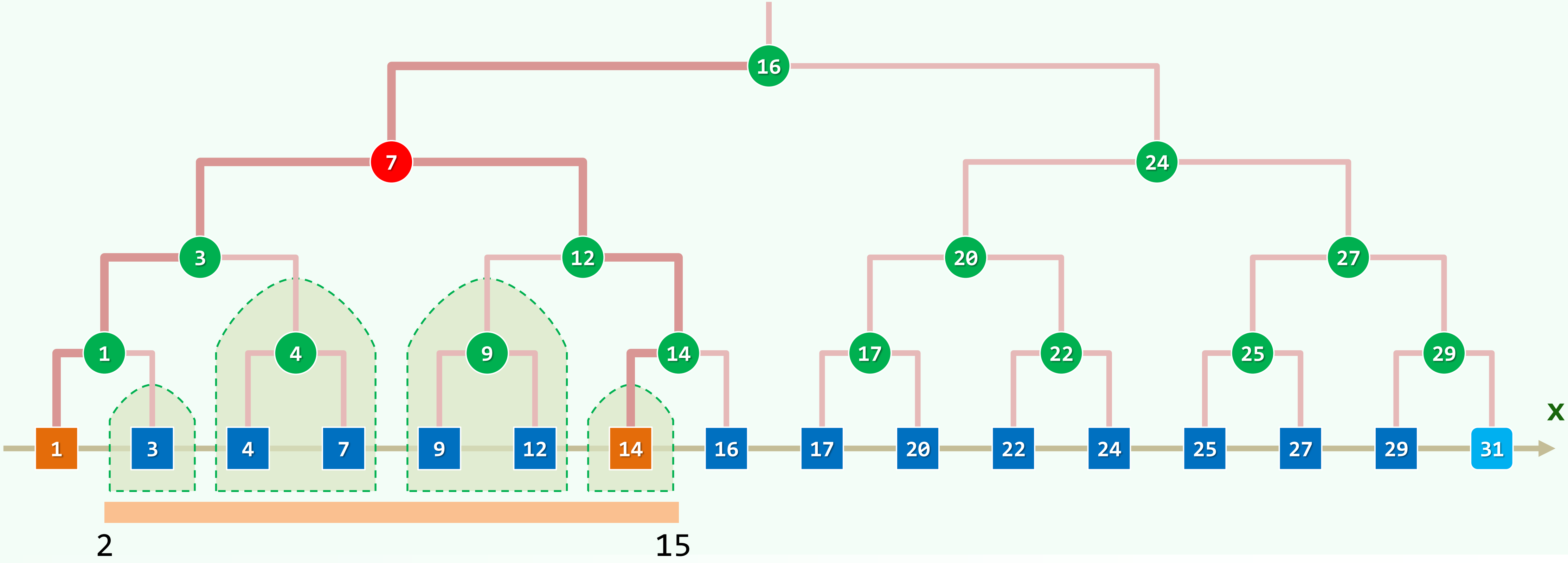
❖ search(2) = 1 , search(15) = 14 , LCA(1, 14) = 7



Traversal

❖ Starting from the LCA, traverse path(1) and path(14) once more resp.

- All Right/Left-turns along path(1/14) are ignored and
- the Right/Left subtree at each Left/Right-turn is reported



Complexity

❖ **Query:**

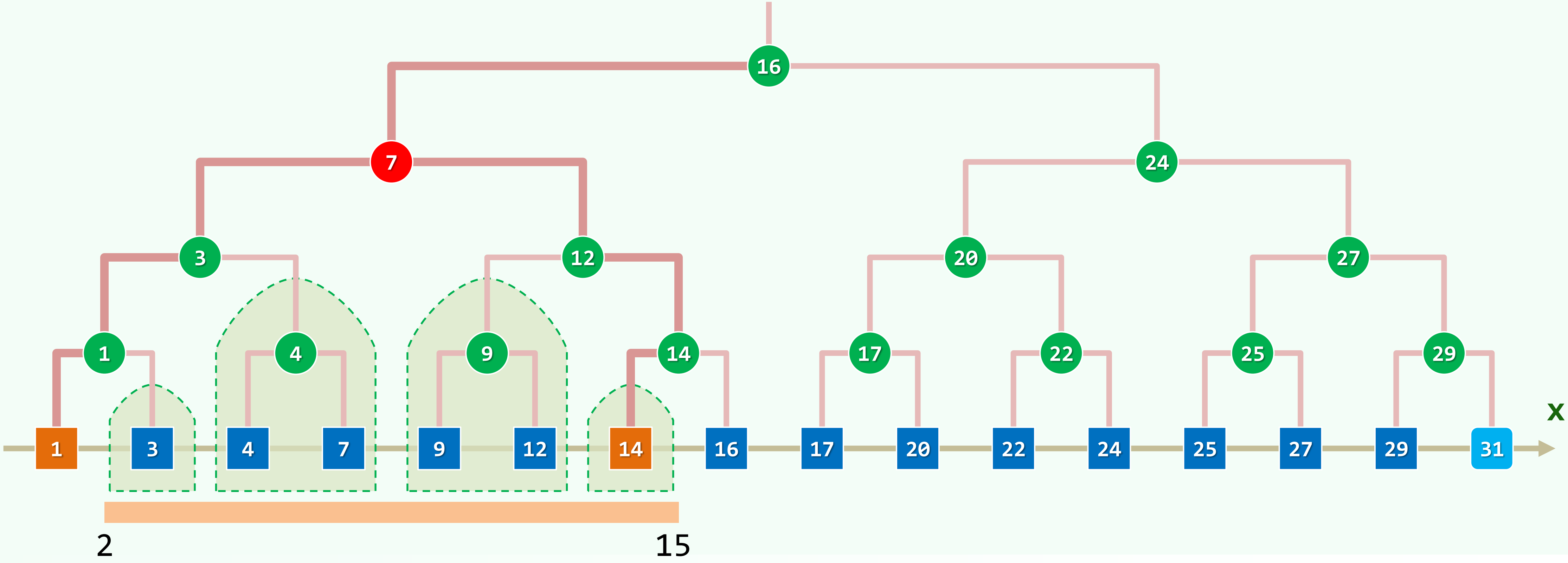
$\mathcal{O}(\log n)$

❖ **Preprocessing:**

$\mathcal{O}(n \log n)$

❖ **Storage:**

$\mathcal{O}(n)$



09-B2

BST Application

kd-Tree: 2D

凡见字数，如停匀，即平分一半为上卦，一半为下卦。如字数不均，即少一字为上卦，取天轻清之义，以多一字为下卦，取地重浊之义

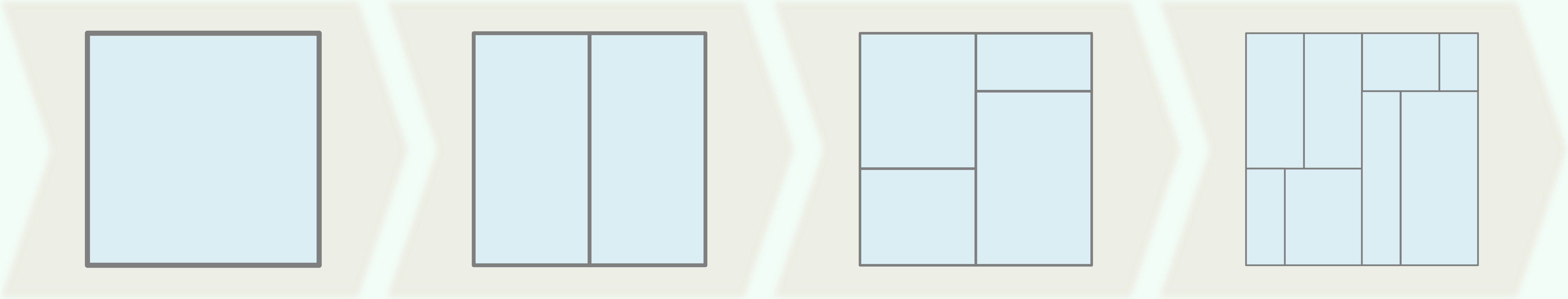
邓俊辉
deng@tsinghua.edu.cn

Divide-And-Conquer

- To extend the BBST method to planar GRS, we
  - **divide** the plane recursively and
  - **arrange** the regions into a kd-tree
- Start with a single region (the entire plane)

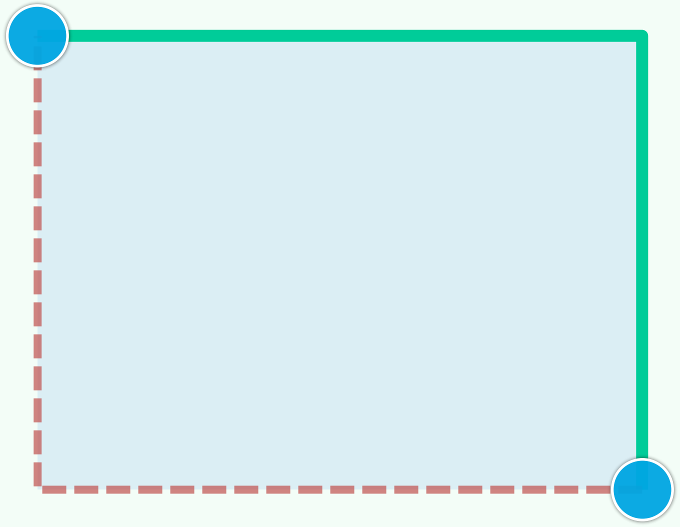
  Partition the region vertically/horizontally on each even/odd level

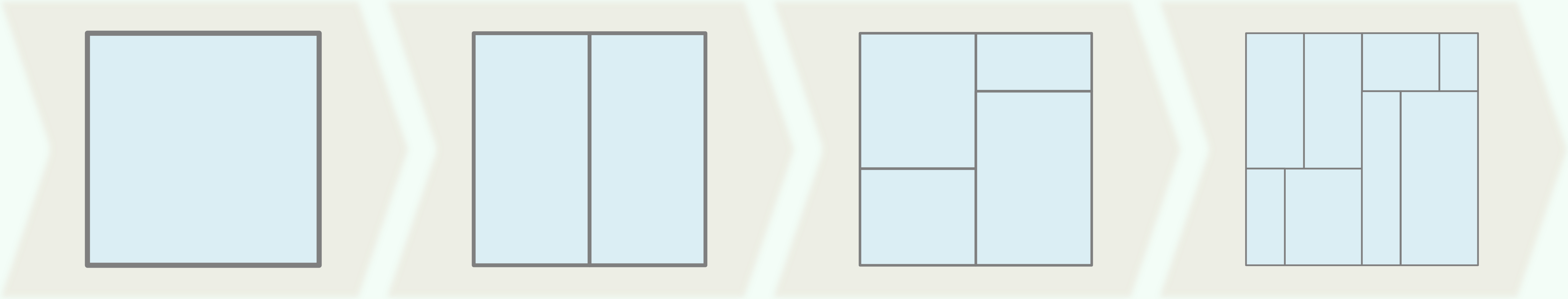
  Partition the sub-regions recursively



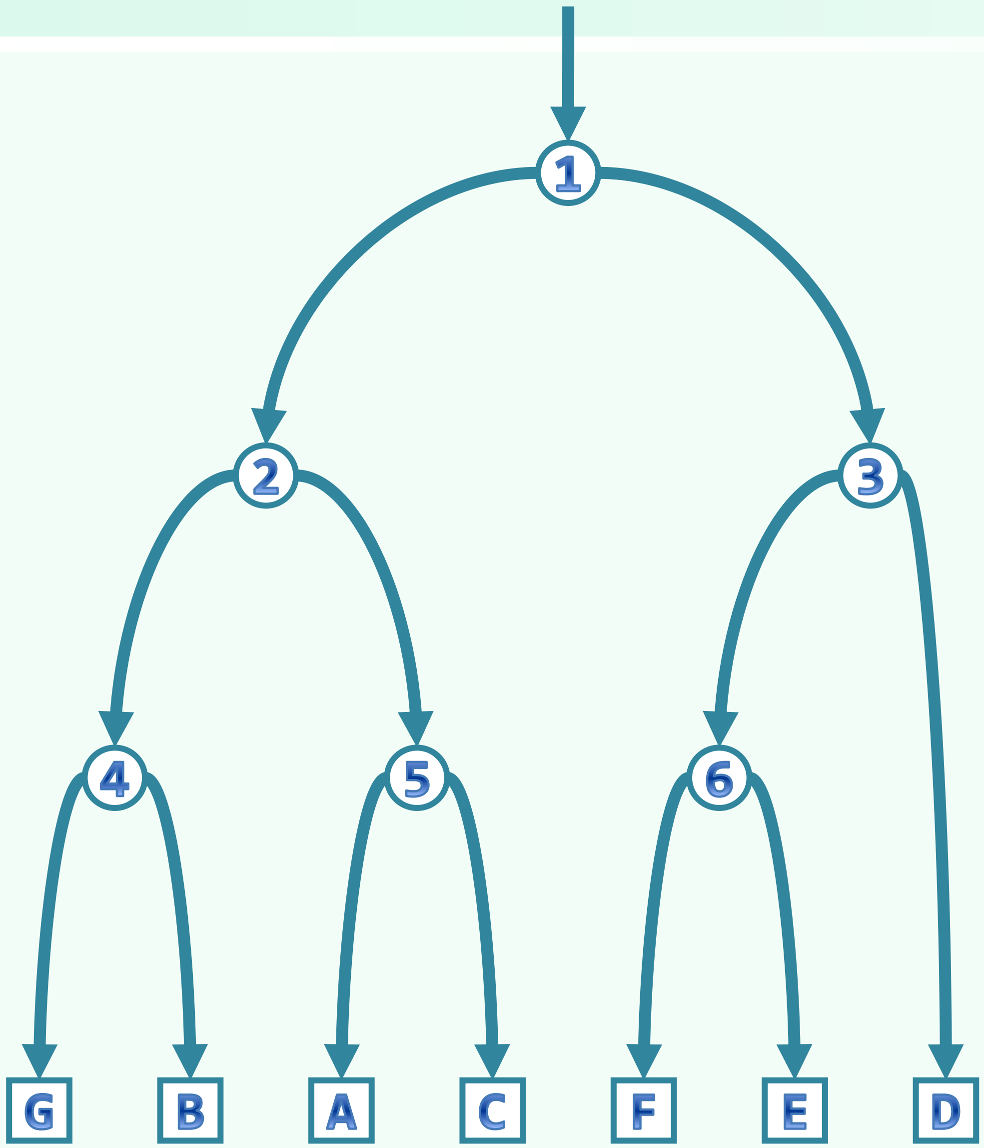
More Details

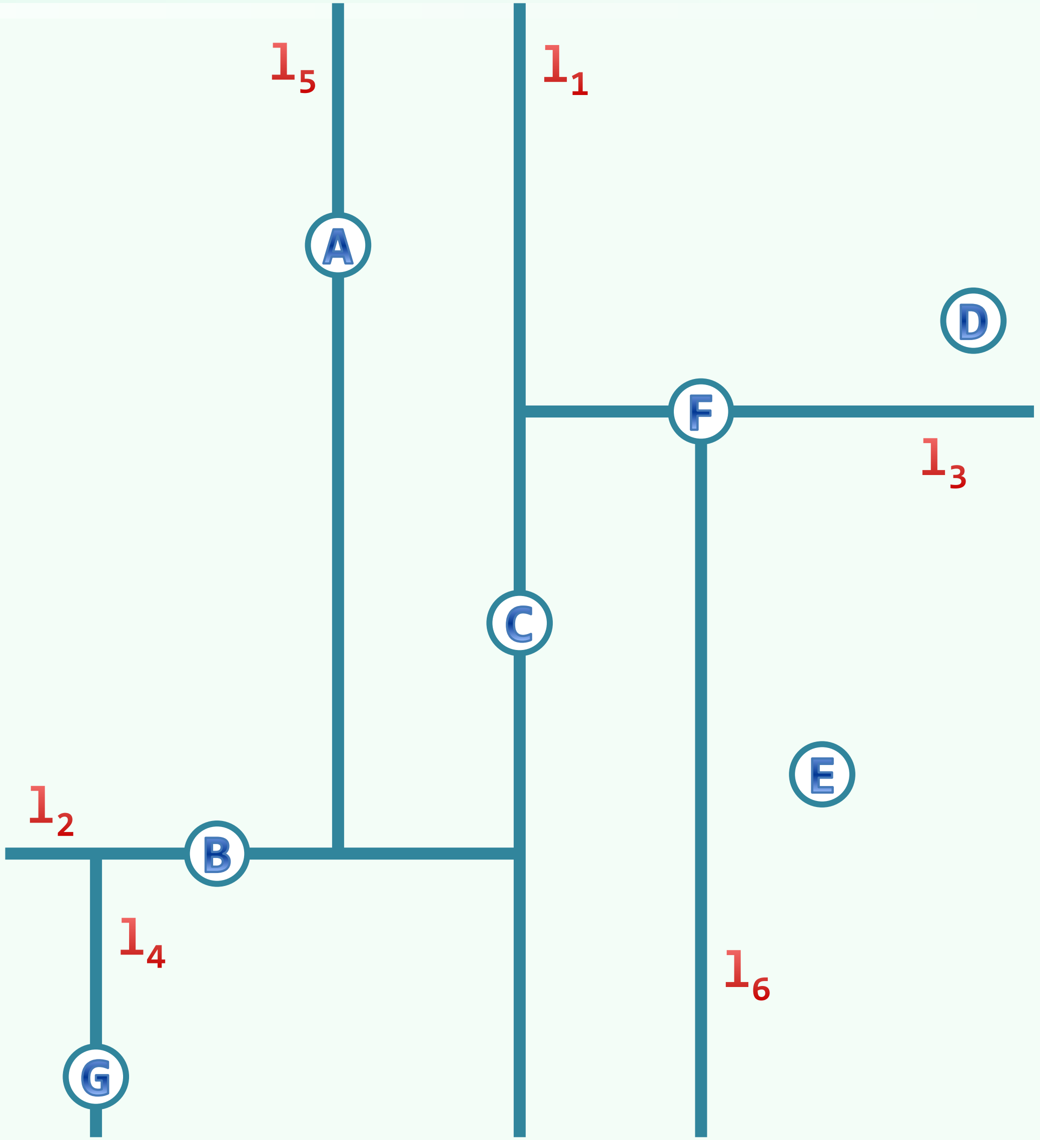
- To make it work,
  - each partition should be done as **evenly** as possible (at median)
  - each region is defined to be **open/closed** on the **left-lower/right-upper** sides



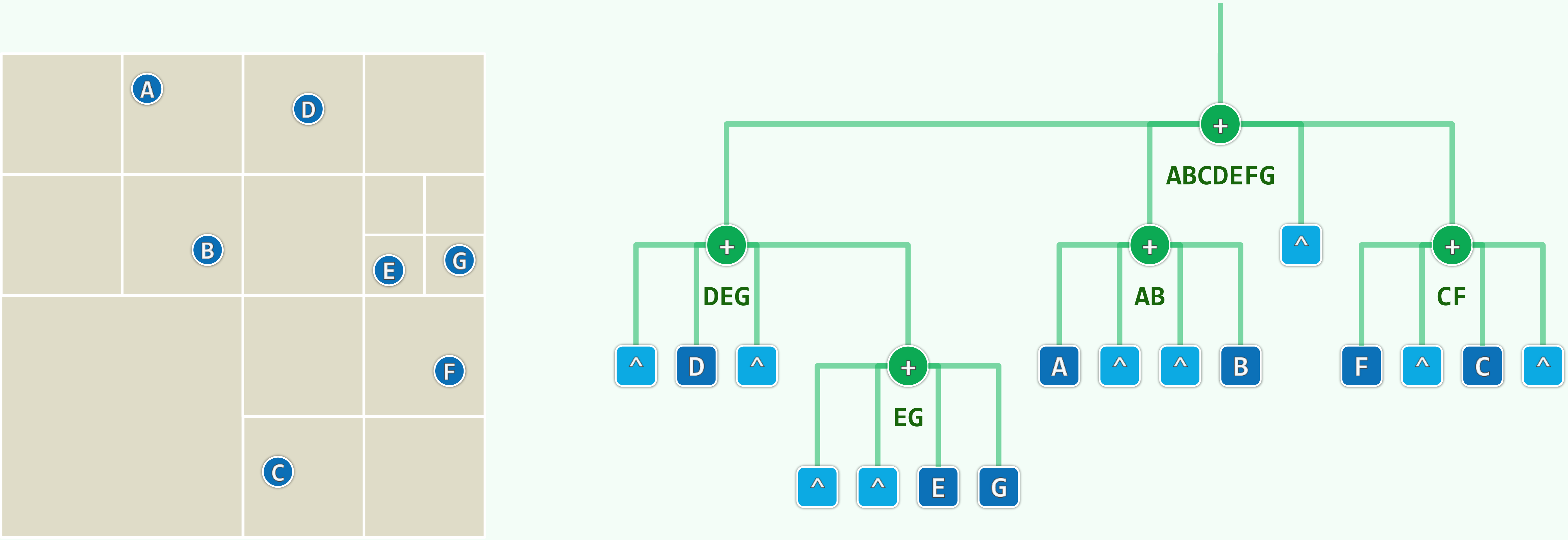


Example





Quadtree



09-B3

BST Application

kd-Tree: Construction

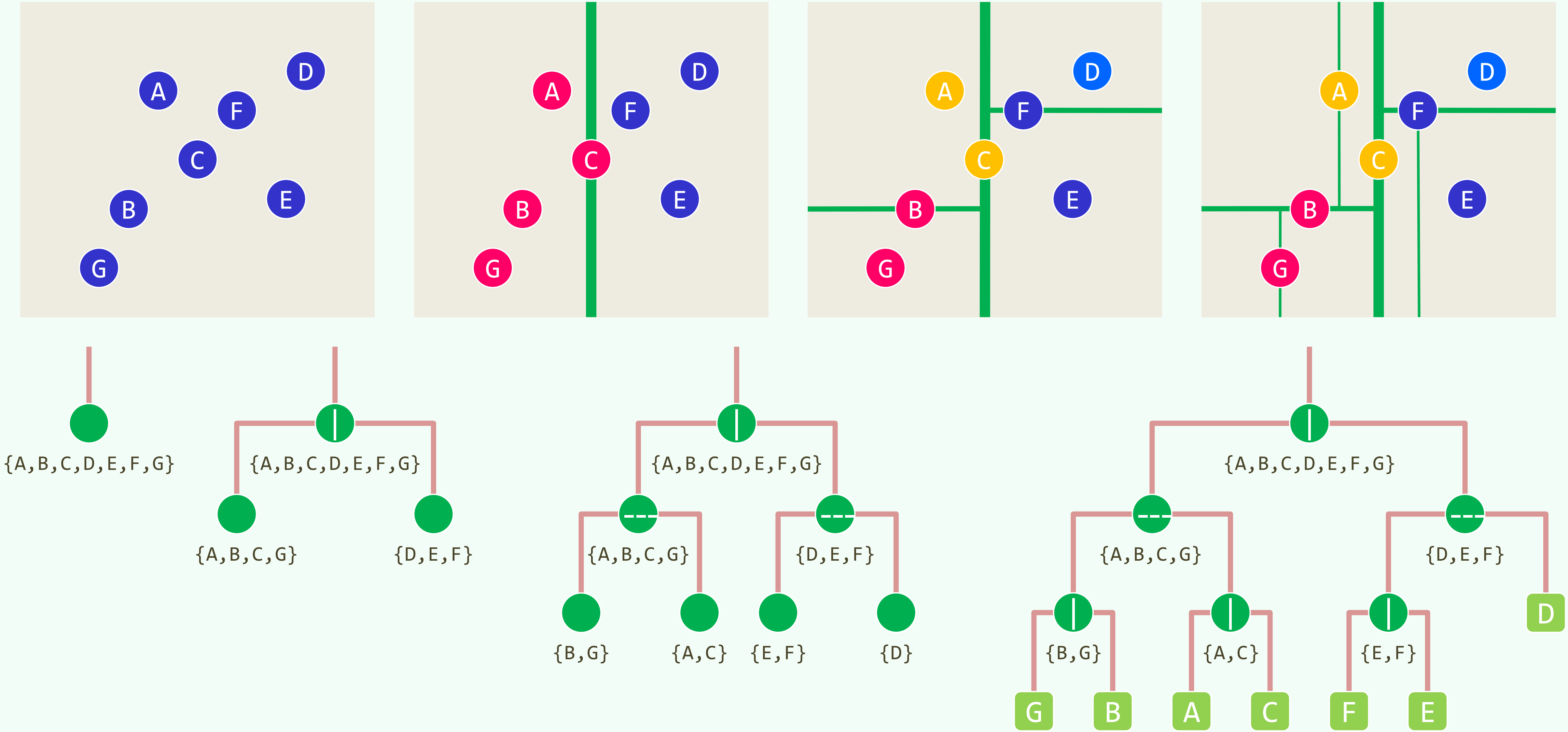
God found himself by creating.

邓俊辉
deng@tsinghua.edu.cn

buildKdTree(P,d)

```
// construct a 2d-(sub)tree for point (sub)set P at depth d
   if ( P == {p} ) return CreateLeaf( p )  //base
   root = CreateKdNode()
   root->splitDirection = Even(d) ? VERTICAL : HORIZONTAL
   root->splitLine = FindMedian( root->splitDirection, P )  //O(n)!
   ( P1, P2 ) = Divide( P, root->splitDirection, root->splitLine )  //DAC
   root->lChild = buildKdTree( P1, d + 1 )  //recurse
   root->rChild = buildKdTree( P2, d + 1 )  //recurse
   return( root )
```

Example



BST Application

kd-Tree: Canonical Subsets

09-B4

韦小宝跟著她走到桌边，只见桌上大白布上钉满了几千枚绣花针，几千块碎片已拼成一幅完整无缺的大地图，难得的是几千片碎皮拼在一起，既没多出一片，也没少了一片。

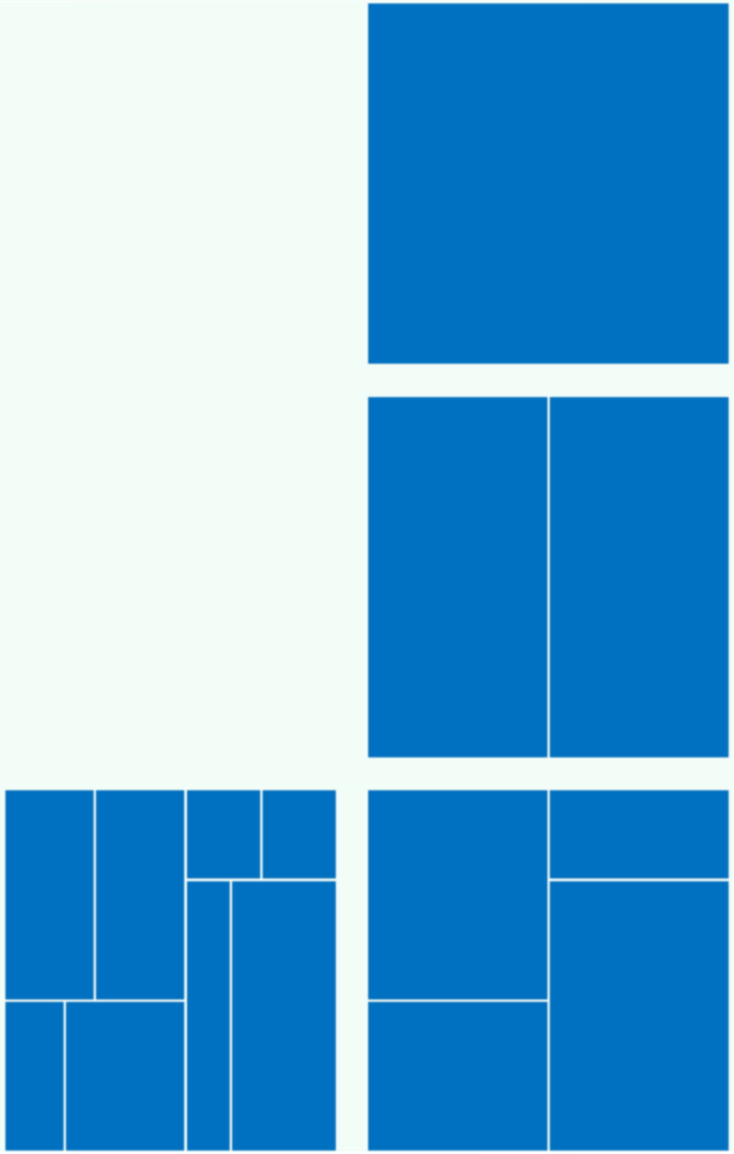
邓俊辉
deng@tsinghua.edu.cn

Canonical Subset

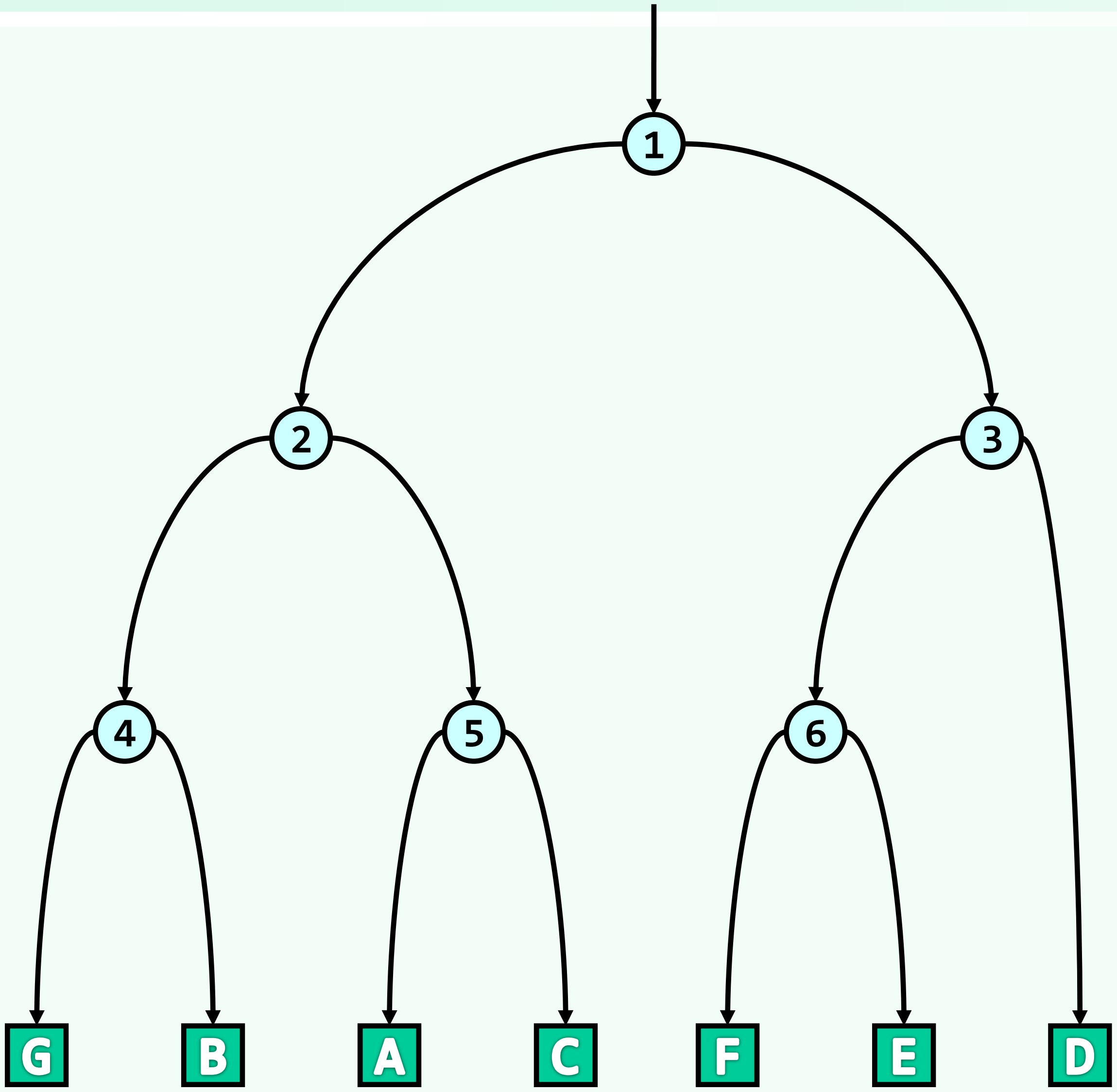
- Each node corresponds to
  - a rectangular sub-region of the plane, as well as
  - the subset of points contained in the sub-region
- Each of these subsets is called a **canonical subset**
- For each internal node X with children L and R,

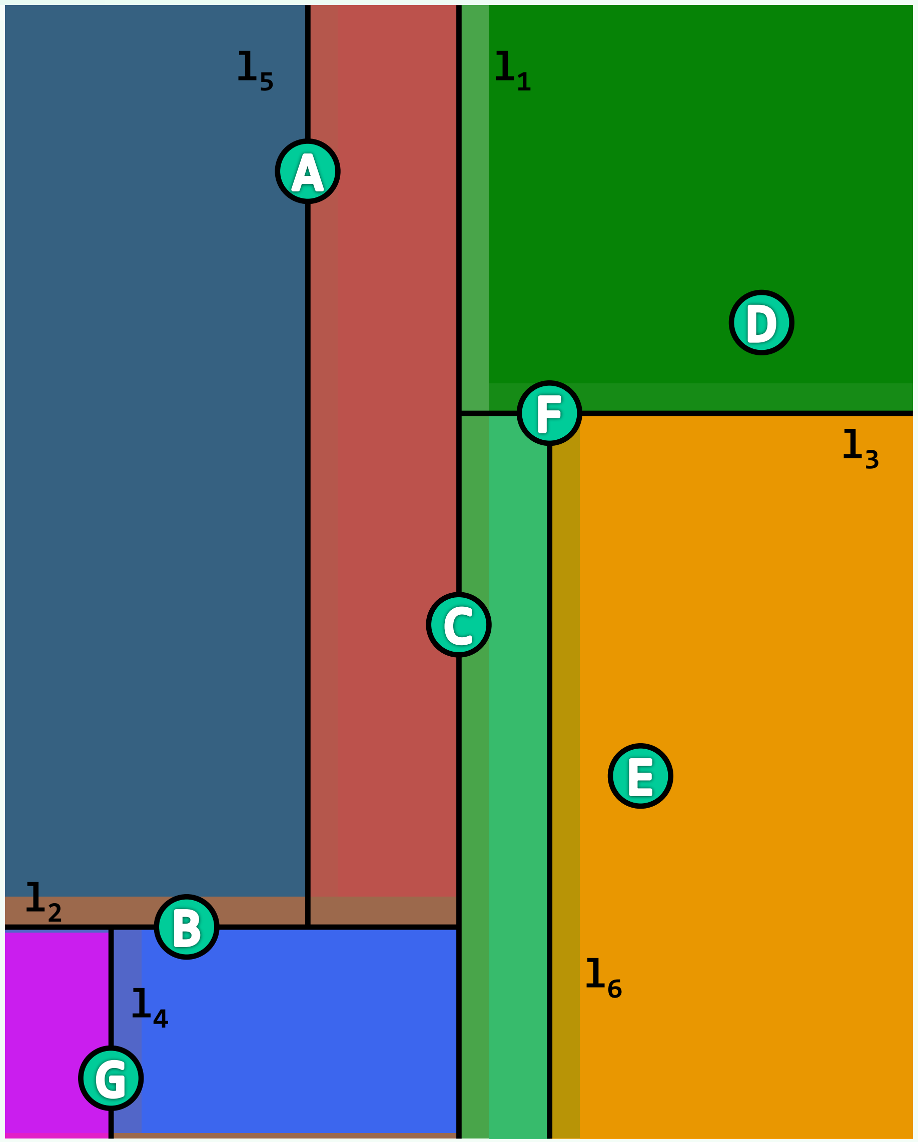
  region(X) = region(L) ∪ region(R)
- Sub-regions of nodes at a same depth
  - never intersect with each other, and
  - their union covers the entire plane
- We will see soon that each 2D GRS can be

  answered by the **union** of a number of CS's



Example





09-B5

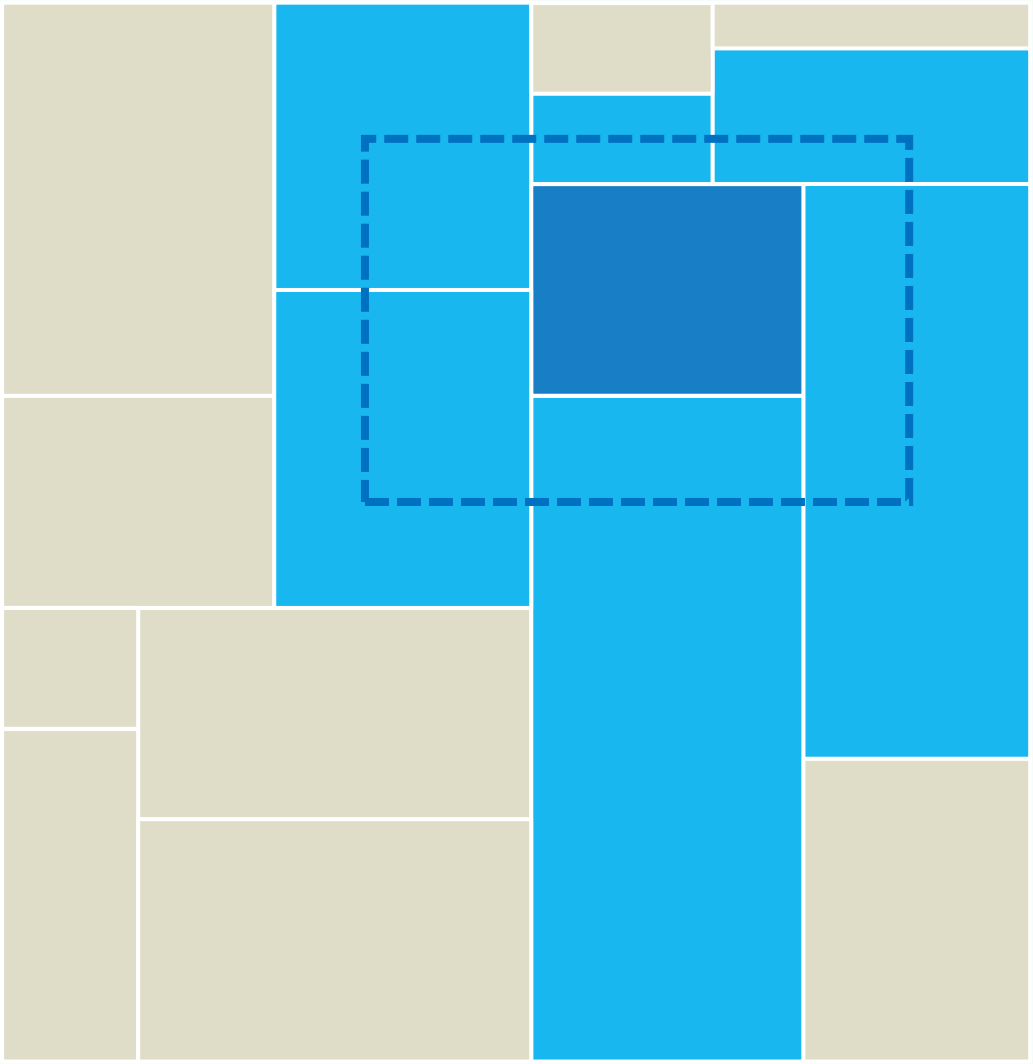
BST Application

kd-Tree: Query

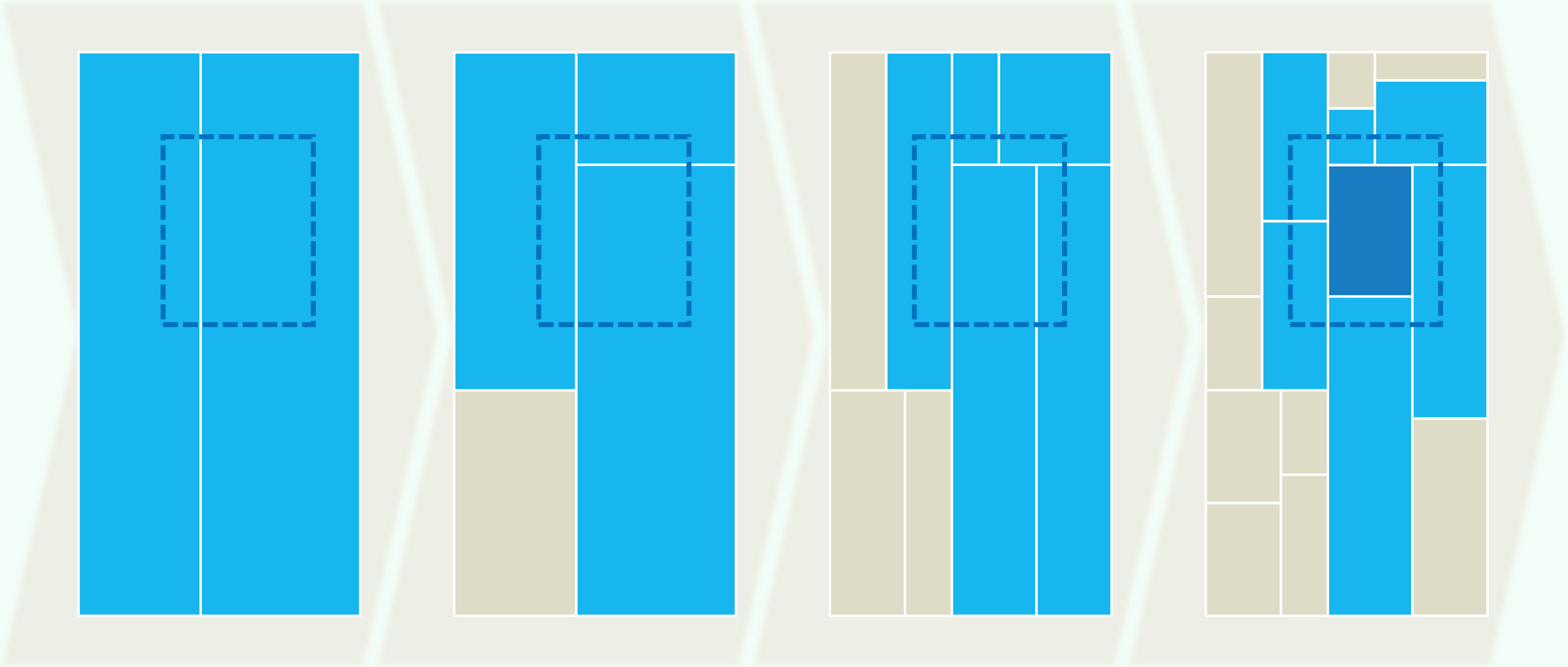
邓俊辉
deng@tsinghua.edu.cn

kdSearch(v,R)

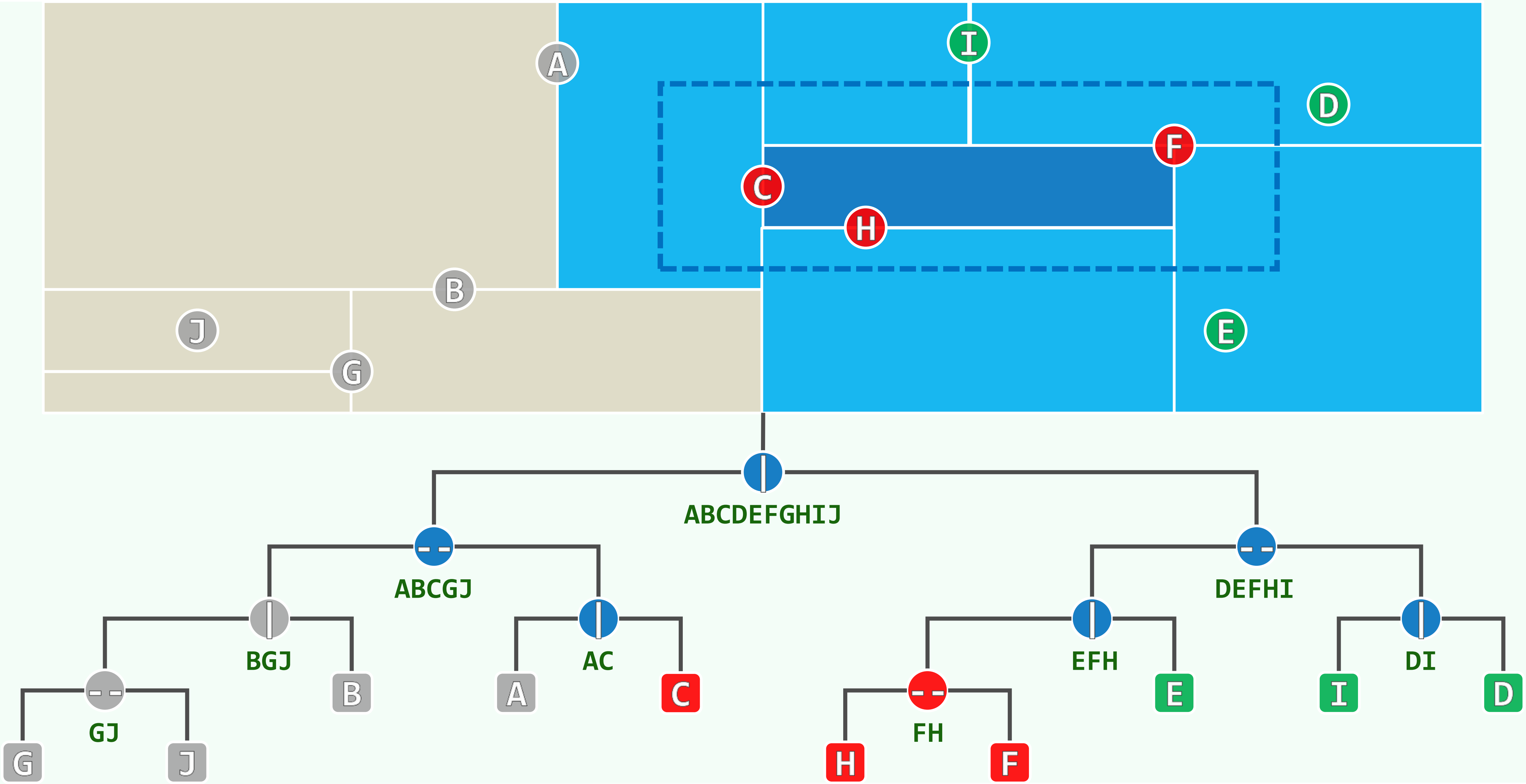
```
❖ if ( isLeaf( v ) )
      if ( inside( v, R ) ) report(v)
      return
❖ if ( region( v->lc ) ⊆ R )
      reportSubtree( v->lc )
  else if ( region( v->lc )∩R ≠ ∅ )
      kdSearch( v->lc, R )
❖ if ( region( v->rc ) ⊆ R )
      reportSubtree( v->rc )
  else if ( region( v->rc )∩R ≠ ∅ )
      kdSearch( v->rc, R )
```



Example



Example



A
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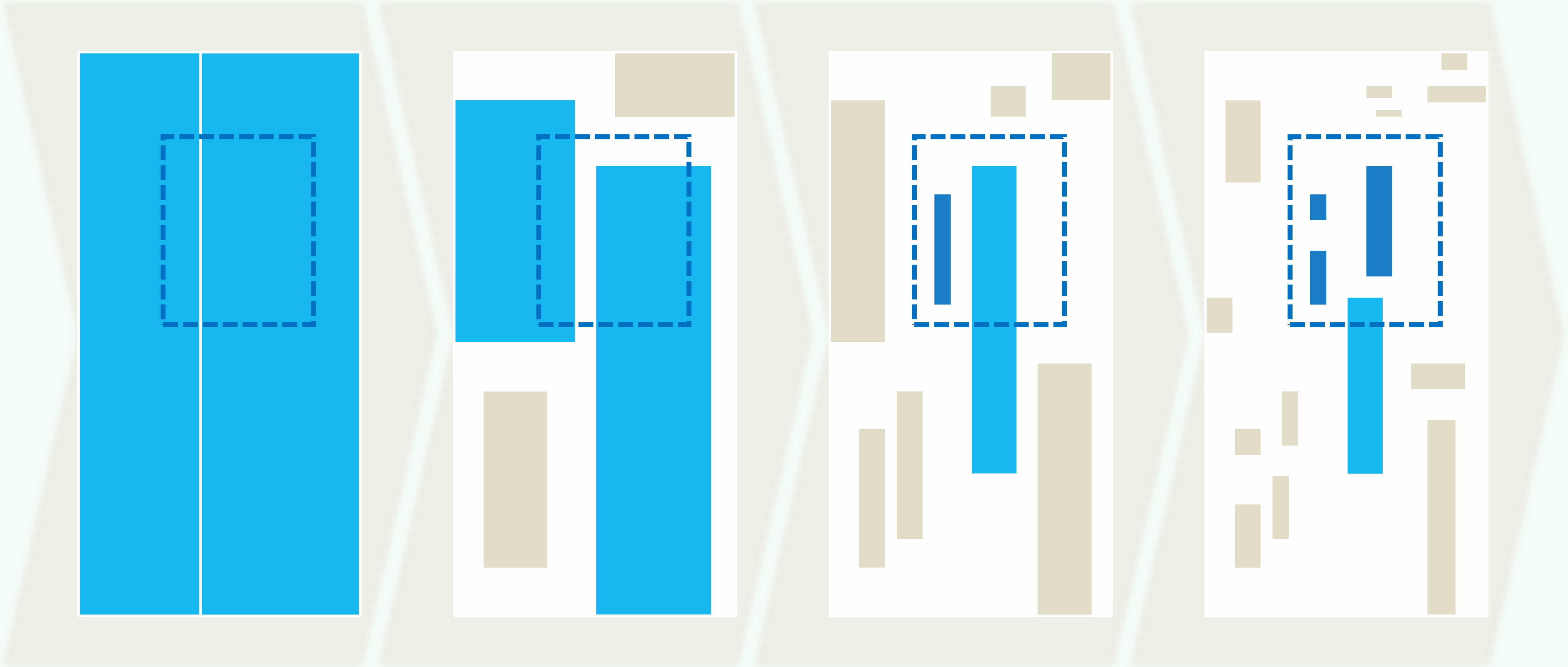
Bounding Box



09-B6

BST Application

kd-Tree: Complexity

肉眼看不清细节，但他们都知道那是木星所在的位置，这颗太阳系最大的行星已经坠落到二维平面上了。

有人嘲笑这种体系说：为了能发现这个比例中项并组成政府共同体，按照我的办法，只消求出人口数字的平方根就行了。

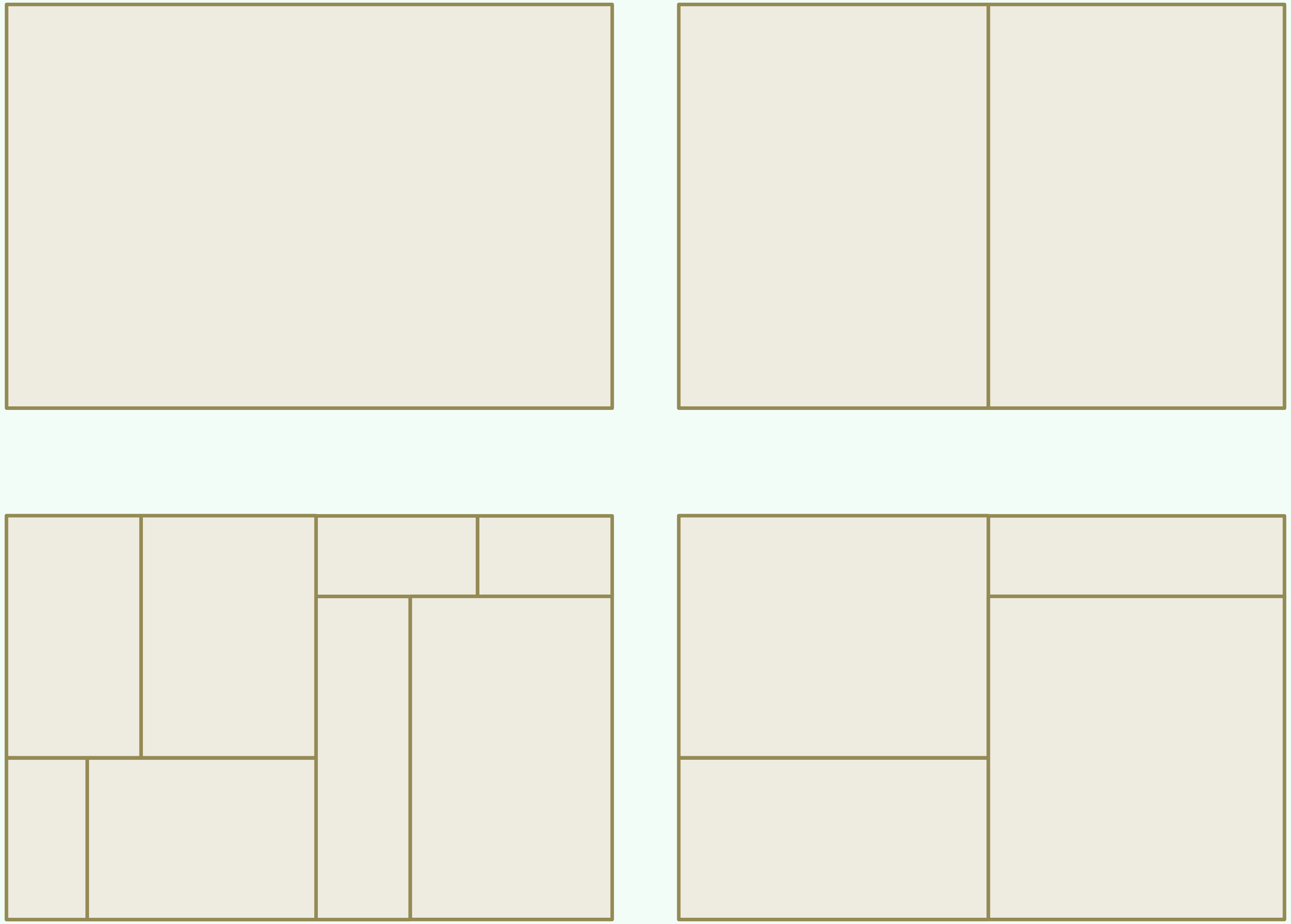
邓俊辉
deng@tsinghua.edu.cn

Preprocessing

- $T(n)$

  $= 2*T(n/2) + \mathcal{O}(n)$

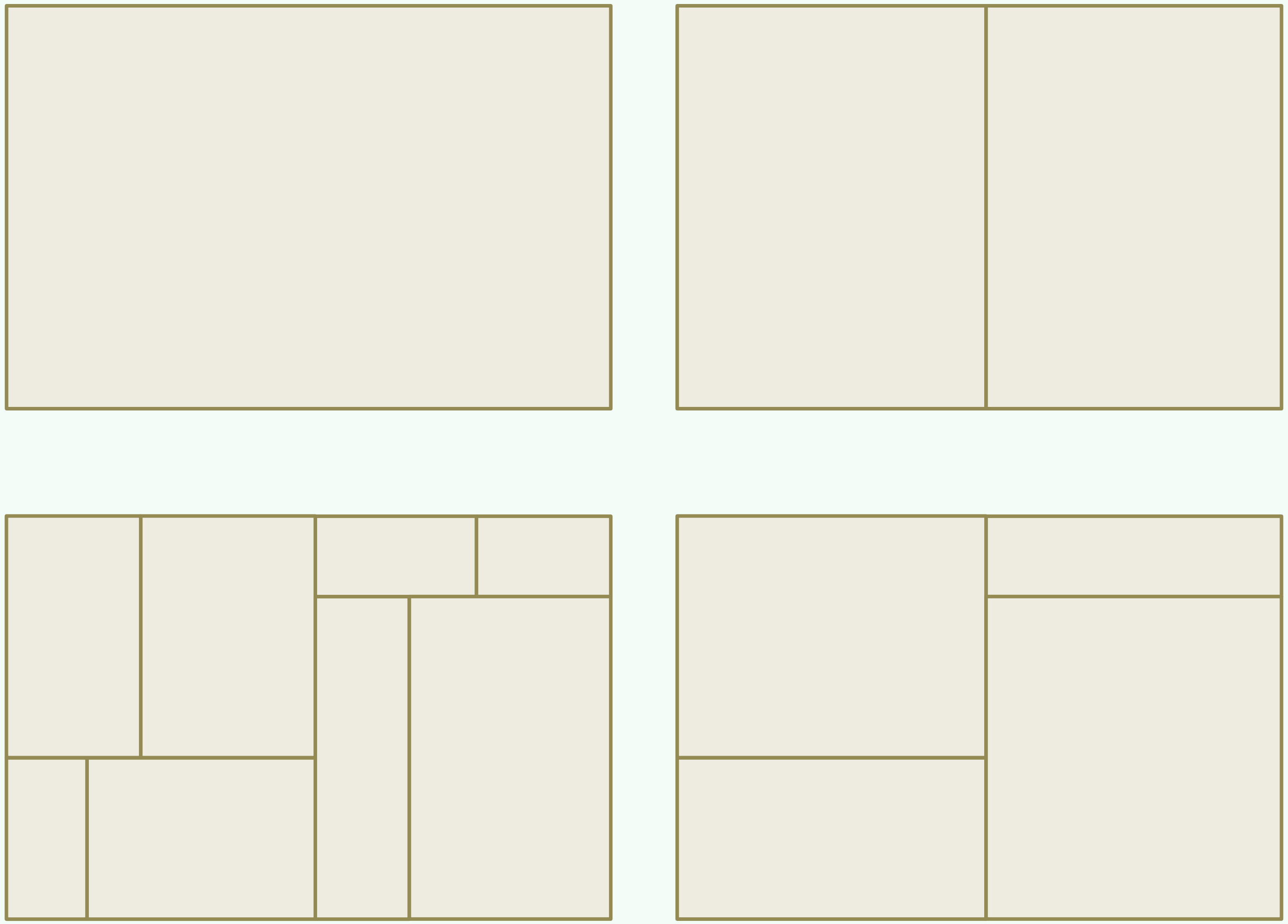
  $= \mathcal{O}(n\log n)$



Storage

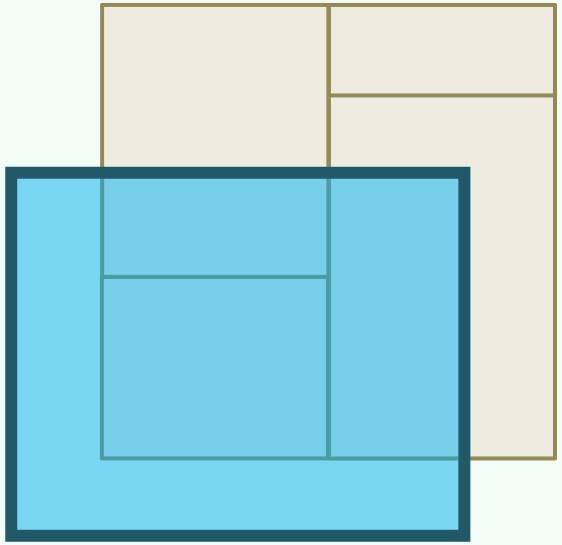
❖ The tree has a height of $\mathcal{O}(\log n)$

❖ $$\begin{aligned} & 1 \\ + & \ 2 \\ + & \ 4 \\ + & \ \ldots \\ + & \ \mathcal{O}(2^{\log n}) \\ = & \ \mathcal{O}(n) \end{aligned}$$

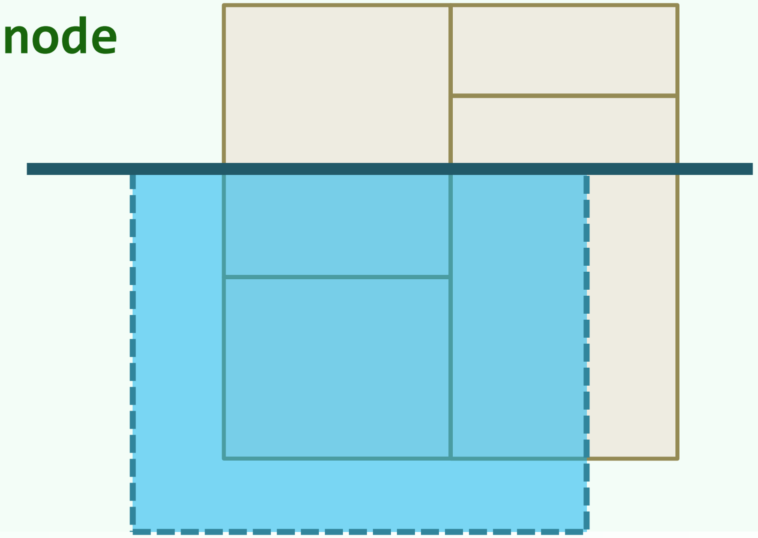


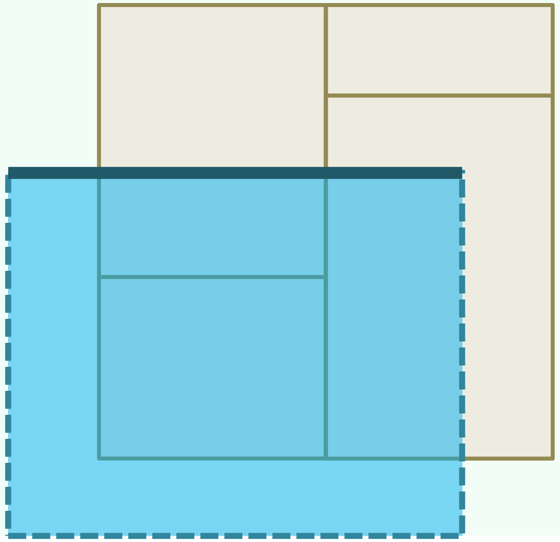
Query Time

- Claim: Report + Search = $\mathcal{O}(r+\sqrt{n})$
- The searching time depends on Q(n), the number of
  - recursive calls, or
  - sub-regions intersecting with R (at all levels)



- No more than 2 of the 4 grandchildren of each node will recurse
  - $Q(1) = \mathcal{O}(1)$
  - $Q(n) = 2 + 2 \cdot Q(n/4)$





- Solve to $Q(n) = \mathcal{O}(\sqrt{n})$

Beyond 2D

❖ Can 2d-tree be extended to kd-tree and help HIGHER dimensional GRS?

If yes, how efficiently can it help?

❖ A kd-tree in k-dimensional space is constructed by

recursively divide $\mathcal{E}^d$ along the $\boxed{1^{st},\ 2^{nd},\ \ldots,\ k\text{th}}$ dimensions

❖ An orthogonal range query on a set of n points in $\mathcal{E}^d$

- can be answered in $\mathcal{O}(r + n^{1-1/d})$ time,
- using a kd-tree of size $\mathcal{O}(n)$, which
- can be constructed in $\mathcal{O}(n \log n)$ time

09-C

BST Application

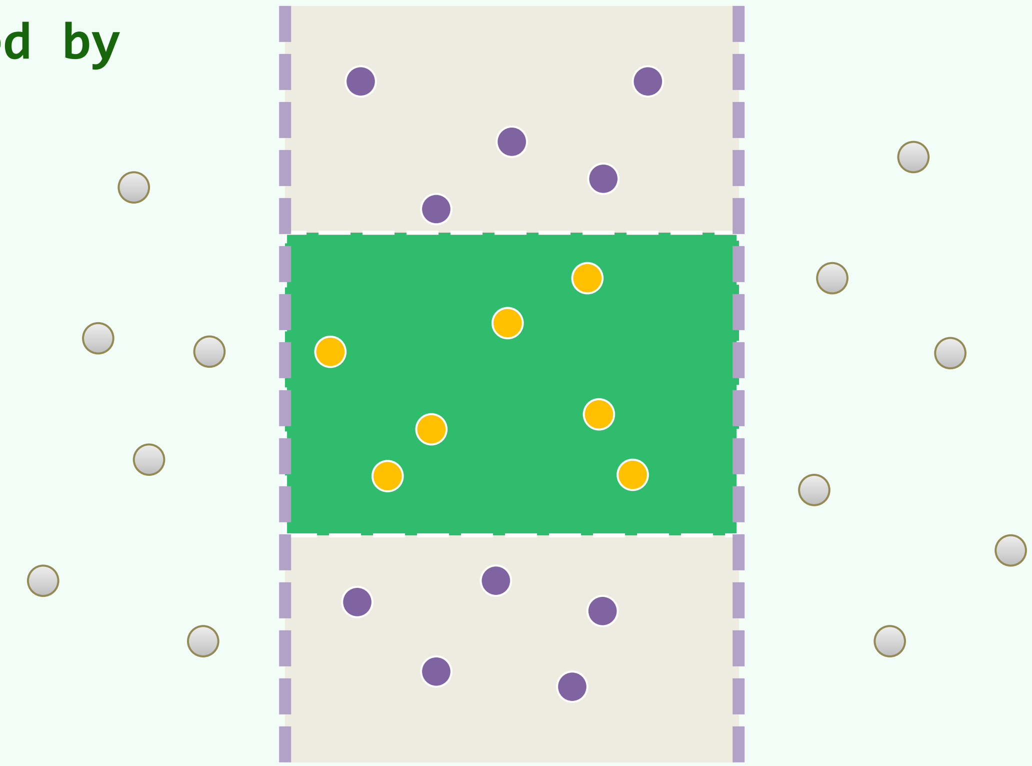
Multi-Level Search Tree

邓俊辉
deng@tsinghua.edu.cn

我们竟为这无用的找寻浪费了这么多天！我想找寻的心上人绝对不会在这里出现。

2D Range Query = x-Query + y-Query

- Is there any structure which answers range query FASTER than kd-trees?
- An m-D orthogonal range query can be answered by the INTERSECTION of m 1D queries
- For example, a 2D range query can be divided into two 1D range queries:
  - find all points in $[x_1, x_2]$; and then
  - find from these candidates those lying in $[y_1, y_2]$



Worst Cases

❖ Using kd-trees needs $\mathcal{O}(1 + \sqrt{n})$ time. But here ...

❖ The x-query returns

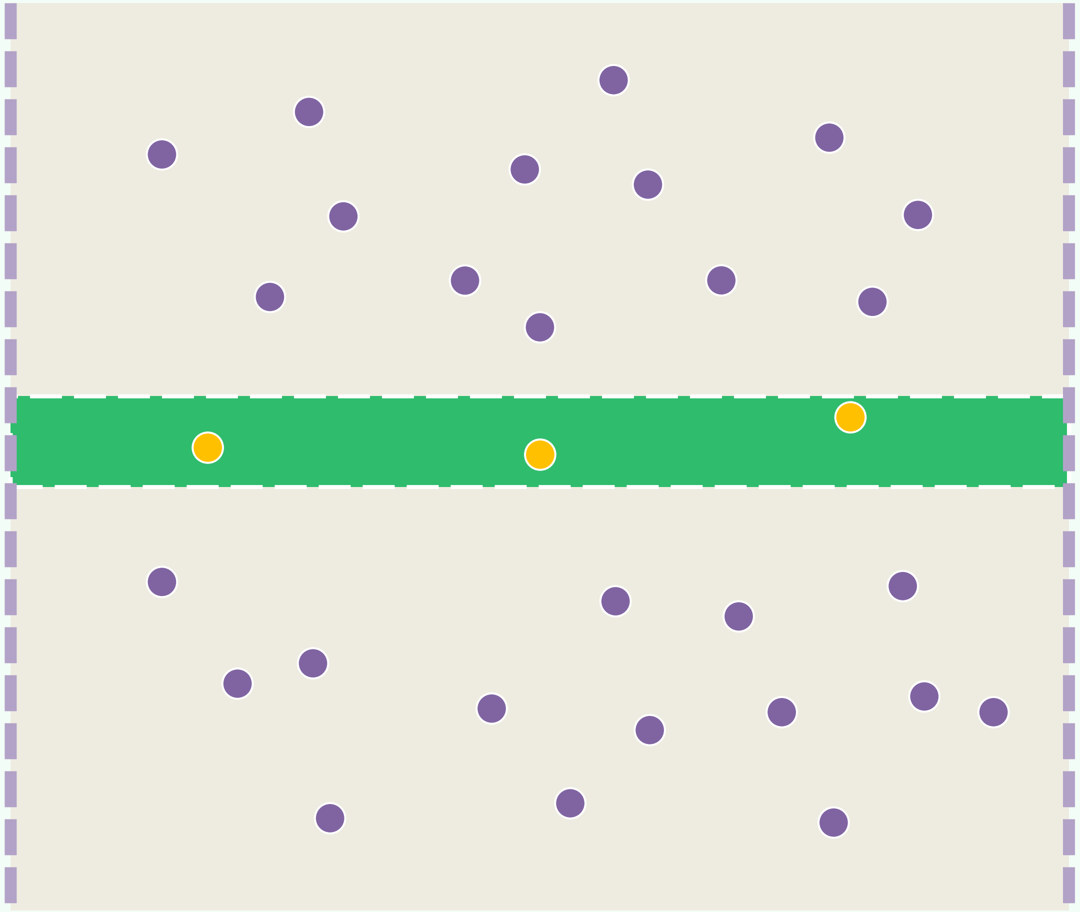
(almost) all points whereas

the y-query rejects

(almost) all

❖ We spent Ω(n) time

before getting r = 0 points



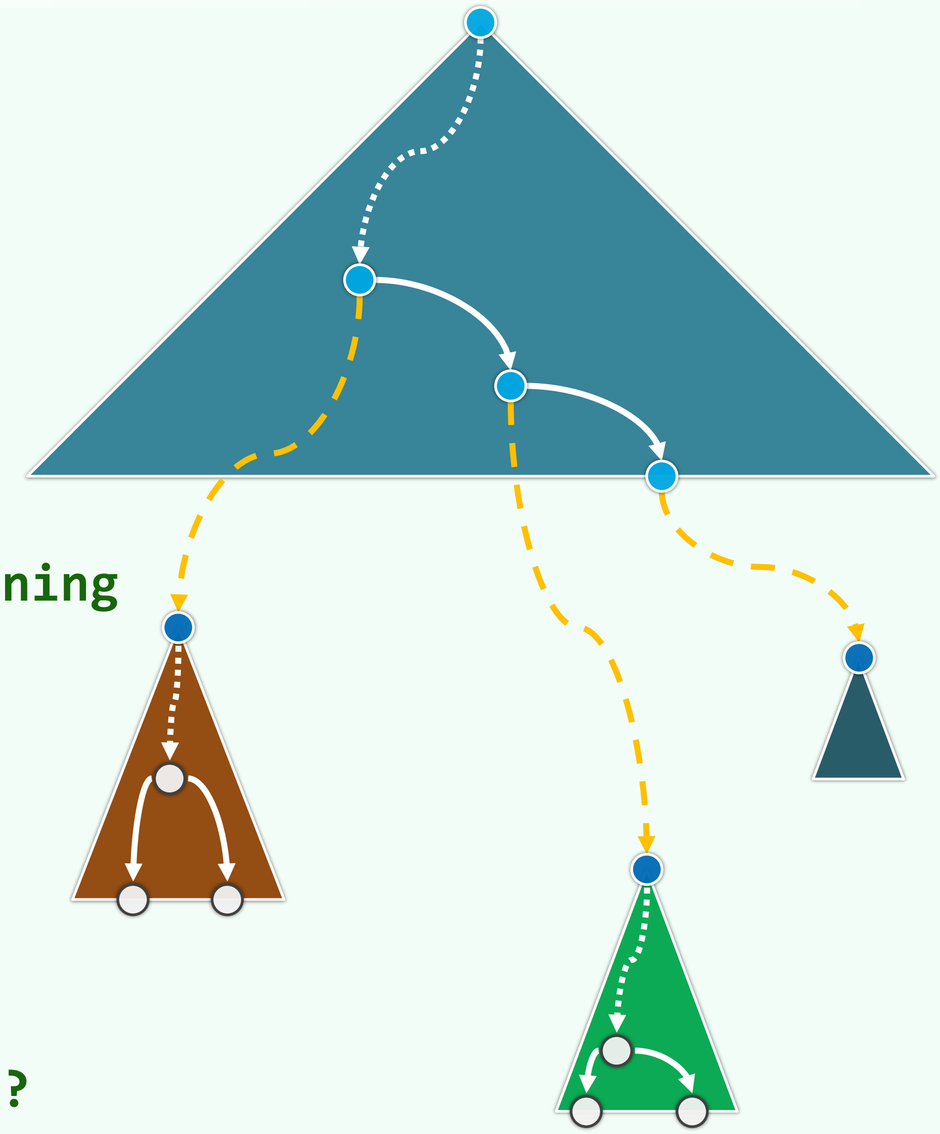
2D Range Query = x-Query * y-Query

❖ Tree of trees

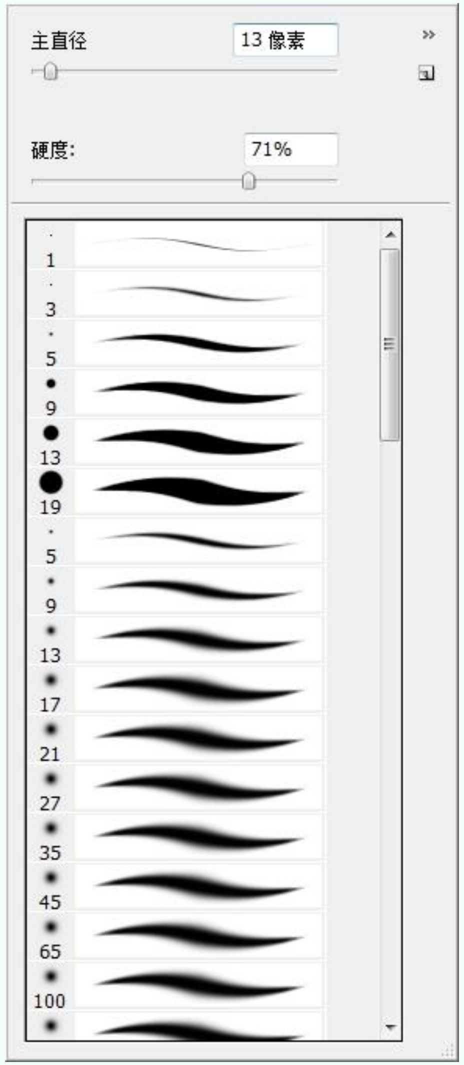
- build a 1D BBST (called x-tree) for the first range query (x-query);
- for each node v in the x-range tree, build a y-coordinate BBST (y-tree), containing the canonical subset associate with v

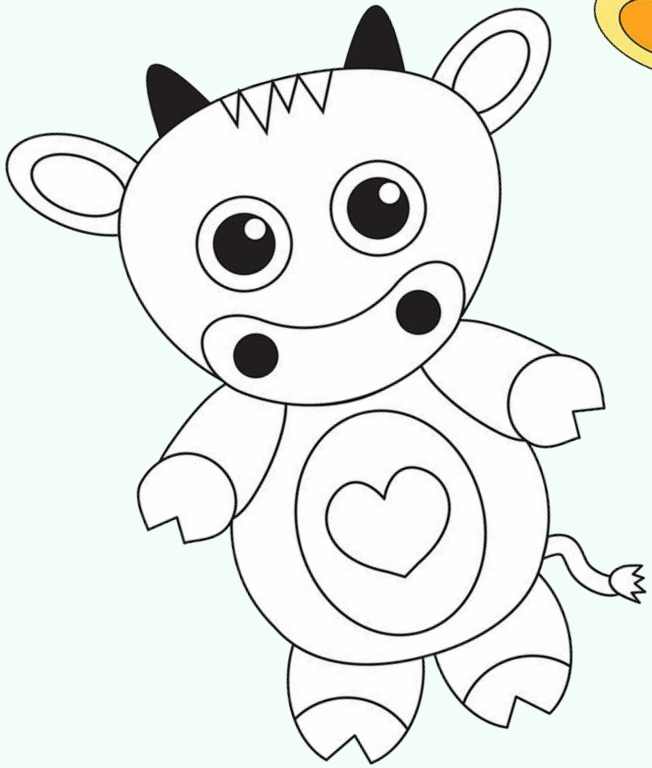
❖ It's an x-tree of (a number of) y-trees, called a Multi-Level Search Tree

❖ How to answer range queries with such an MLST?



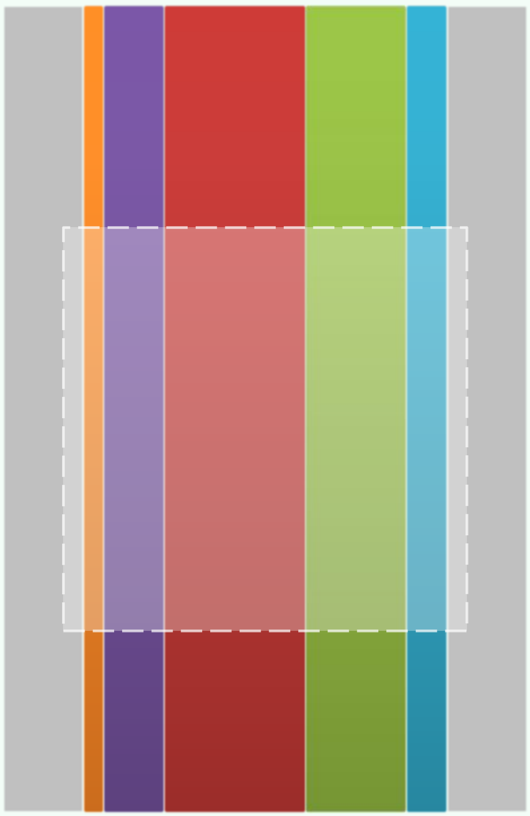
Painters' Strategy

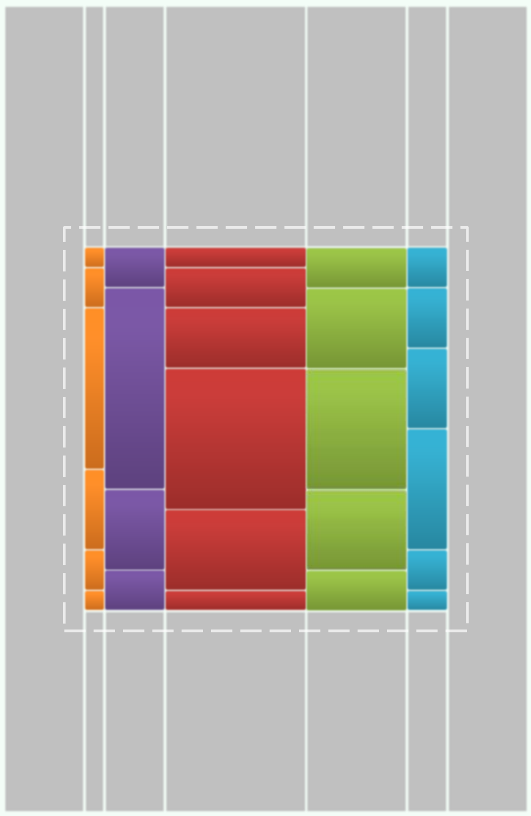


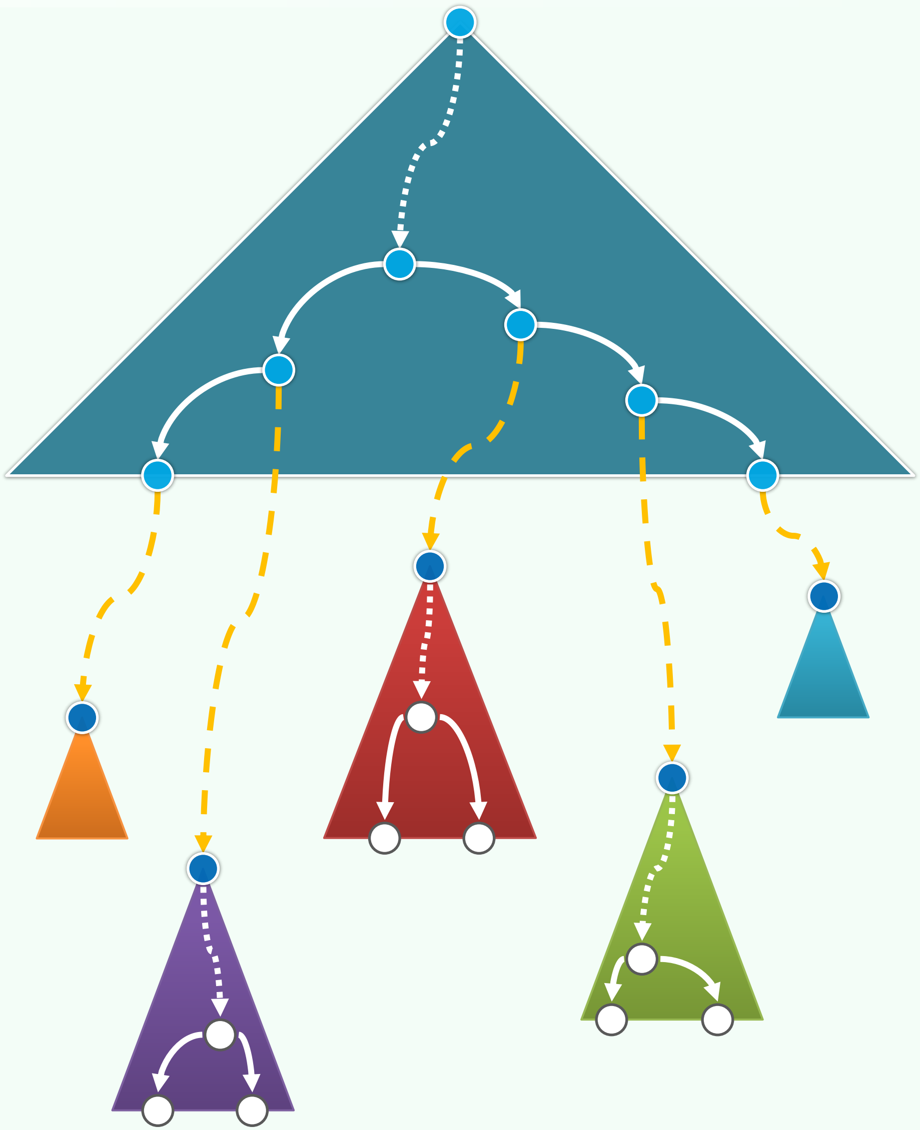


2D Range Query = x-Query * y-Queries

❖ Query Time = $\mathcal{O}(r + \log^2 n)$ ~ $\mathcal{O}(r + \log n)$







Query Algorithm

1. Determine the canonical subsets of points that satisfy the first query

   // there will be $\mathcal{O}(\log n)$ such canonical sets,

   //   each of which is just represented as

   //   a node in the x-tree

2. Find out from each canonical subset

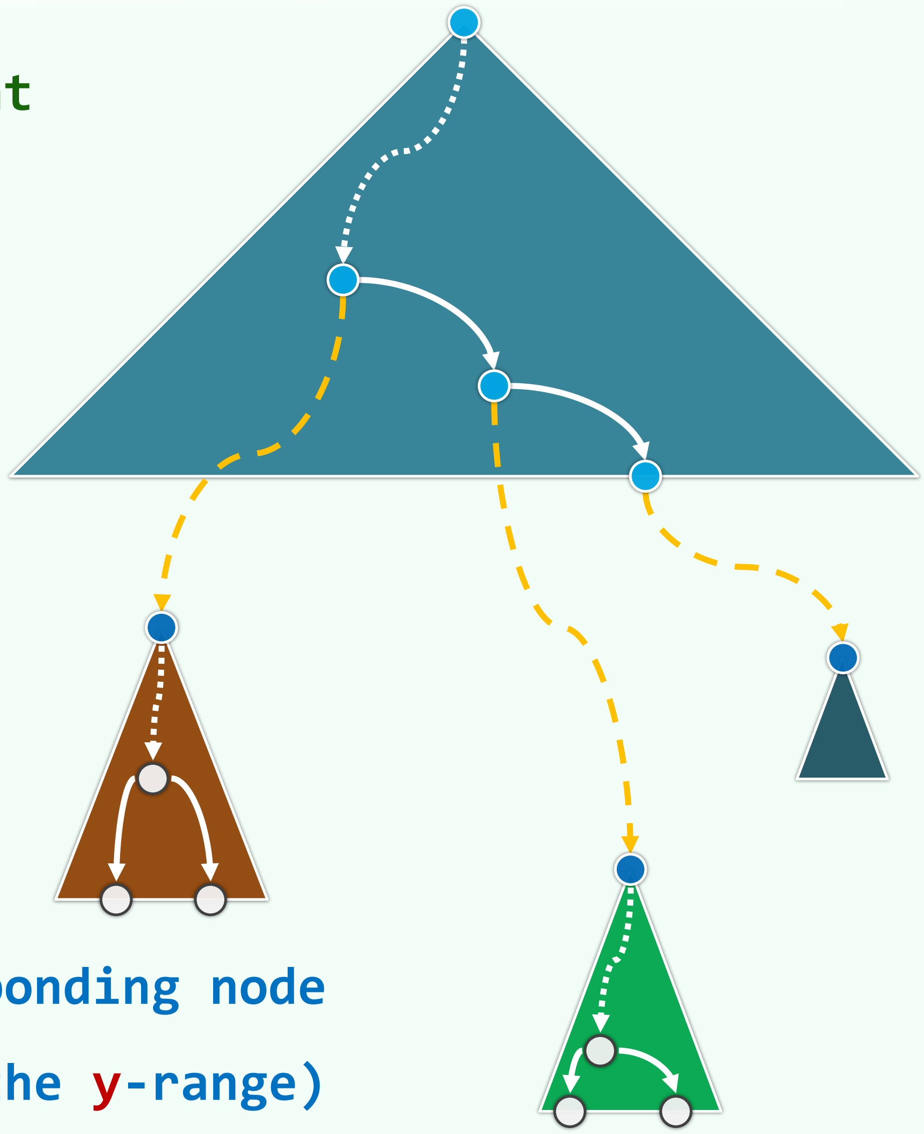
   which points lie within the y-range

   // To do this,

   //   for each canonical subset,

   //   we access the y-tree for the corresponding node

   // this will be again a 1D range search (on the y-range)



Complexity: Preprocessing Time + Storage

- A 2-level search tree

  for n points in the plane

  can be built

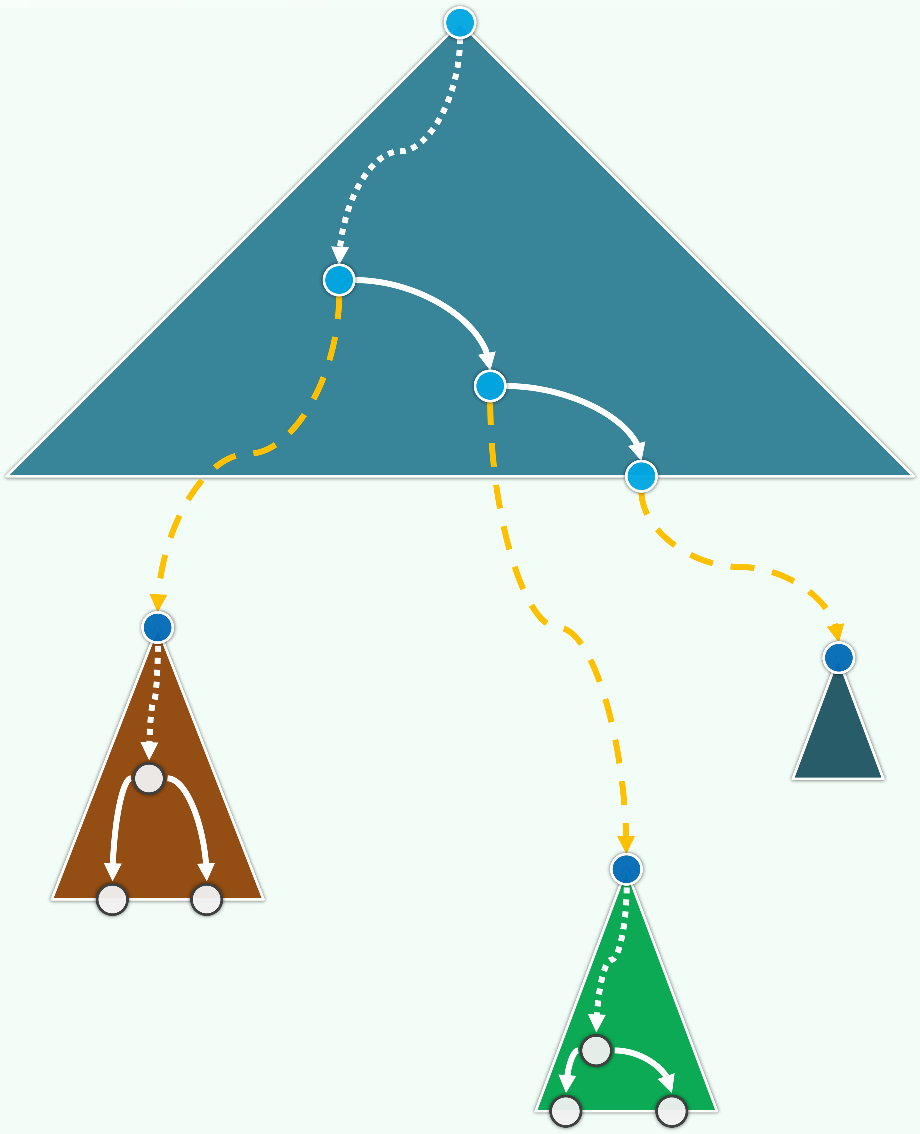
  in $\mathcal{O}(n \log n)$ time

- Each input point is stored in $\mathcal{O}(\log n)$ y-trees

- A 2-level search tree

  for n points in the plane

  needs $\mathcal{O}(n \log n)$ space



Complexity: Query Time

❖ Claim: A 2-level search tree for n points in the plane

answers each planar range query

in $\mathcal{O}(r + \log^2 n)$ time

❖ The x-range query needs $\mathcal{O}(\log n)$ time

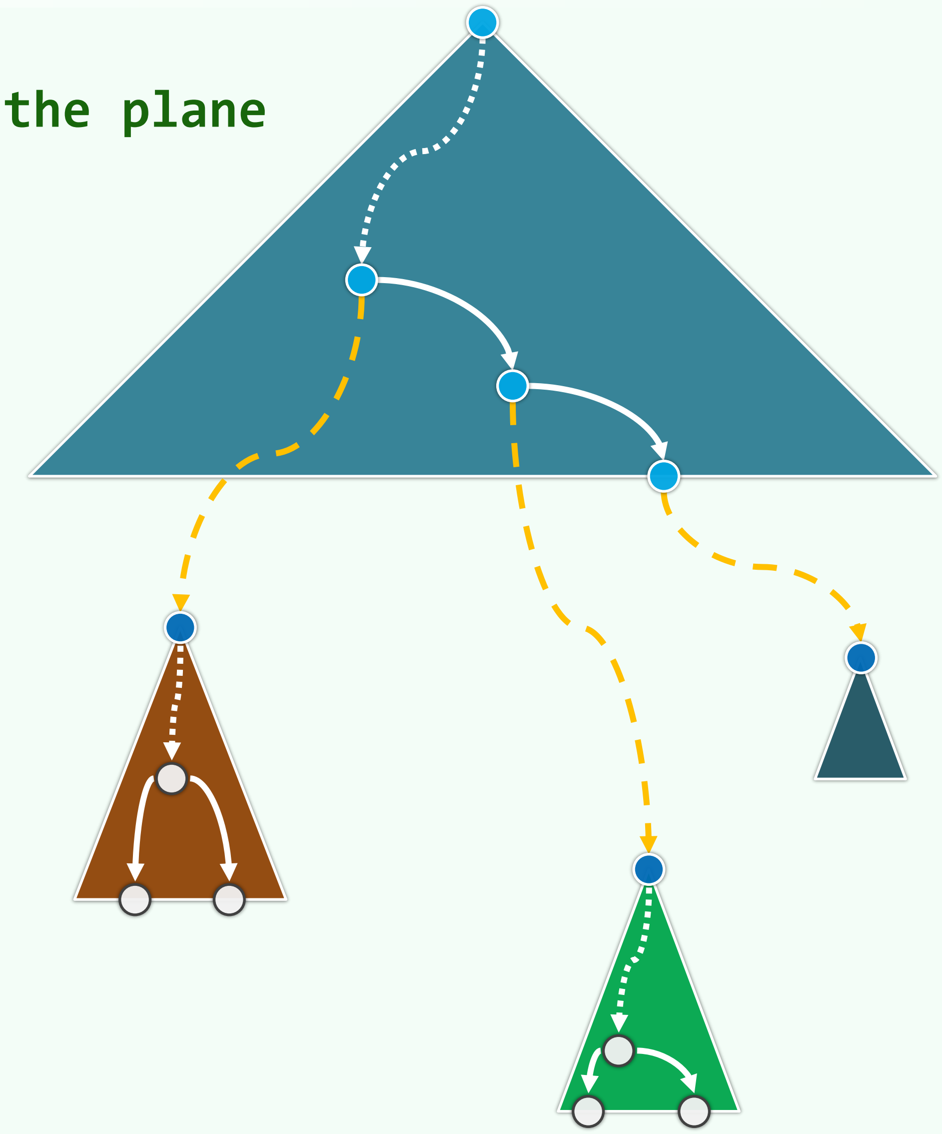
to locate the $\mathcal{O}(\log n)$ nodes

representing the canonical subsets

❖ Then for each of these nodes,

a y-range search is invoked,

which needs $\mathcal{O}(\log n)$ time

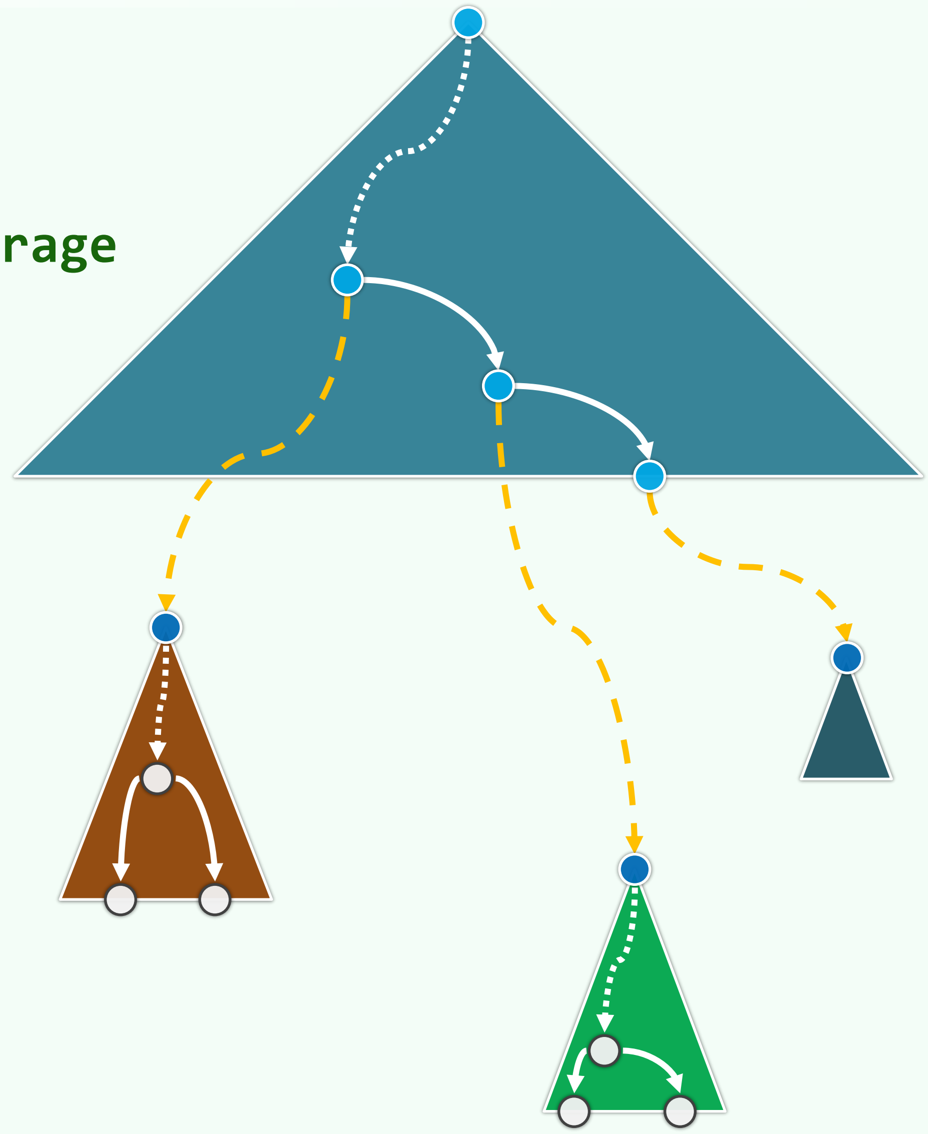


Beyond 2D

❖ Let S be a set of n points in $\mathcal{E}^d$, d ≥ 2

- A d-level tree for S uses $\mathcal{O}(n \cdot \log^{d-1} n)$ storage
- Such a tree can be constructed in $\mathcal{O}(n \cdot \log^{d-1} n)$ time
- Each orthogonal range query of S can be answered in $\mathcal{O}(r + \log^{d} n)$ time

❖ For planar case, can the query time be improved to, say, $\mathcal{O}(\log n)$?



09-XA

BST Application

Range Tree

邓俊辉
deng@tsinghua.edu.cn

顺藤摸瓜

BBST<BBST<T>> --> BBST<List<T>>

❖ Note that each y-search is just

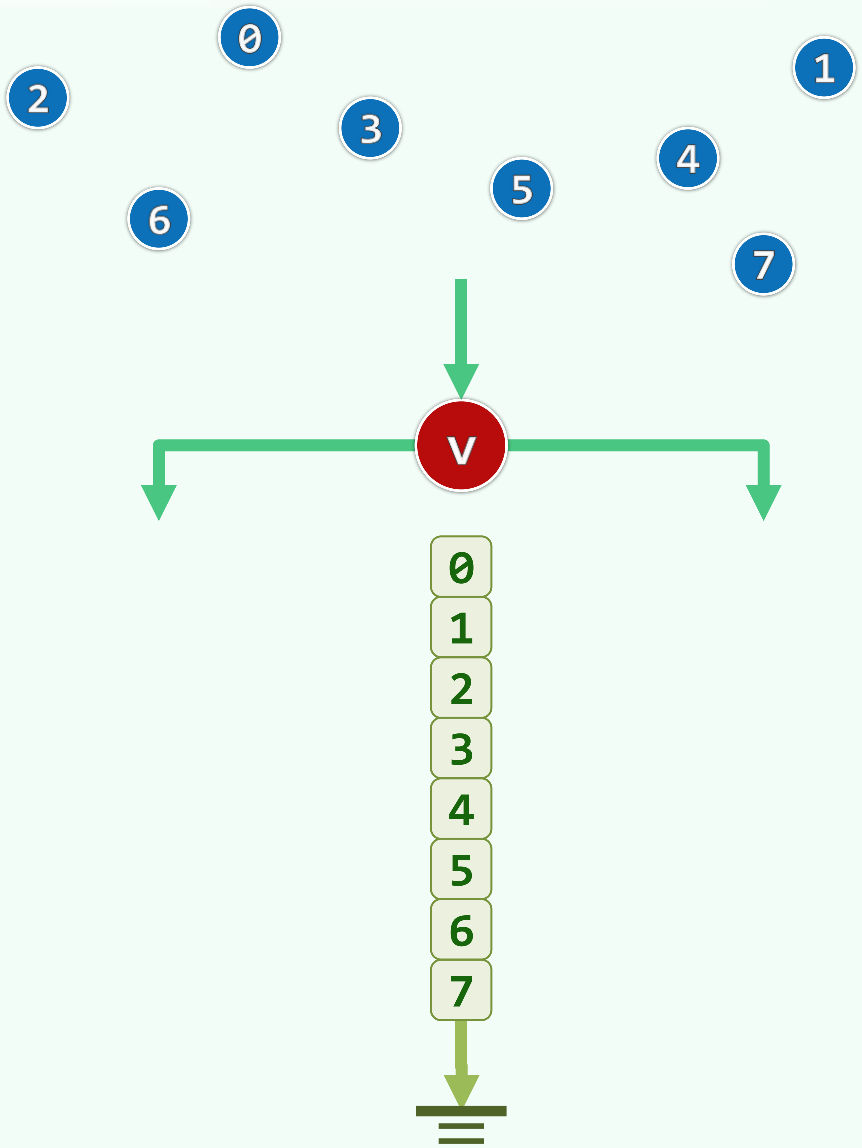
a 1D query without further recursions

❖ So it not necessary

to store each canonical subset

as a BBST

❖ Instead, a sorted y-list simply works



Coherence

❖ Observe further that, for each query

- we need to repeatedly search

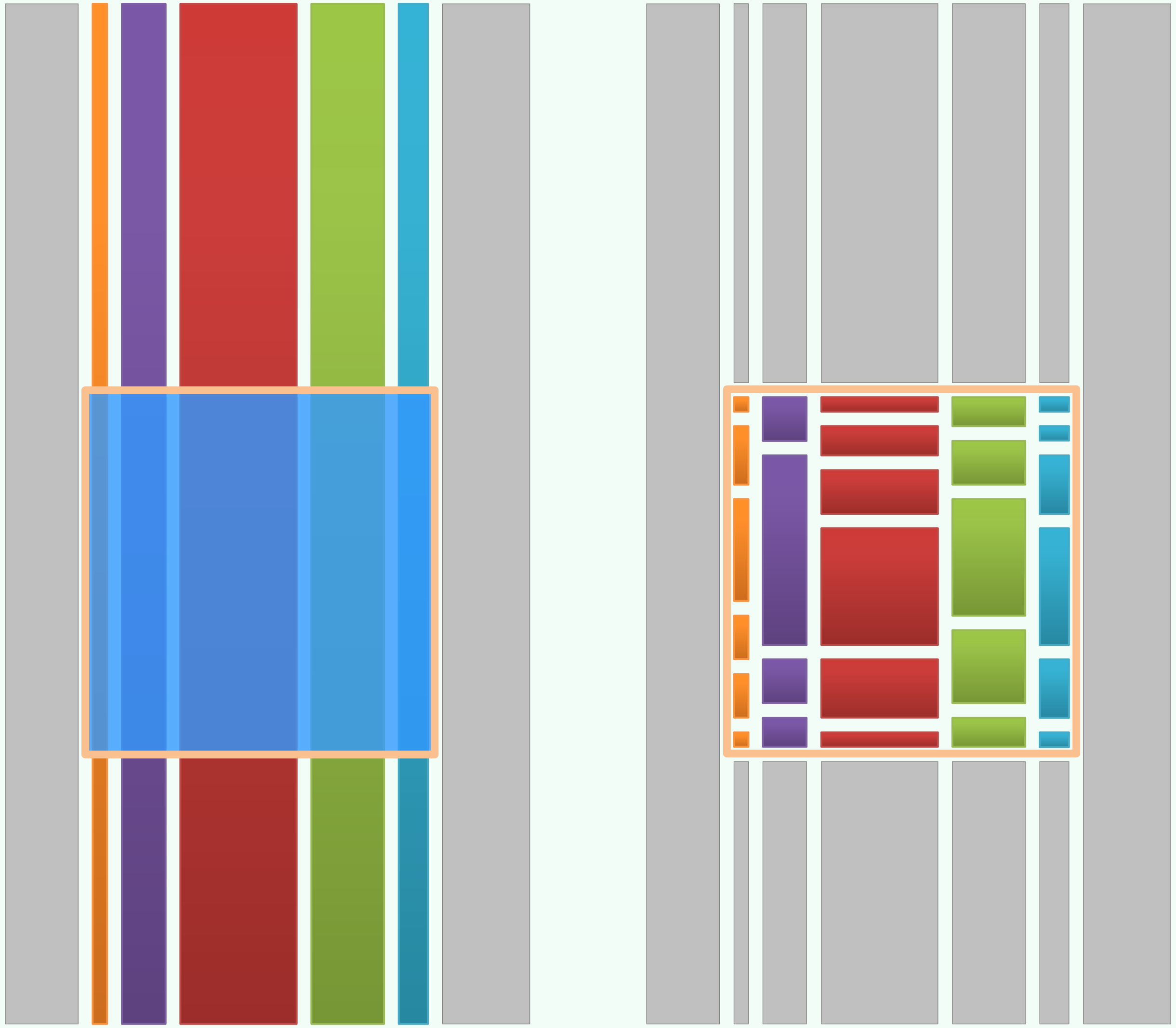
  DIFFERENT y-lists,

- but always with

  the SAME key

❖ However, such an essential fact

  is not used yet



Links Between Lists

❖ The idea for an improvement is that

we **CONNECT** all the different lists

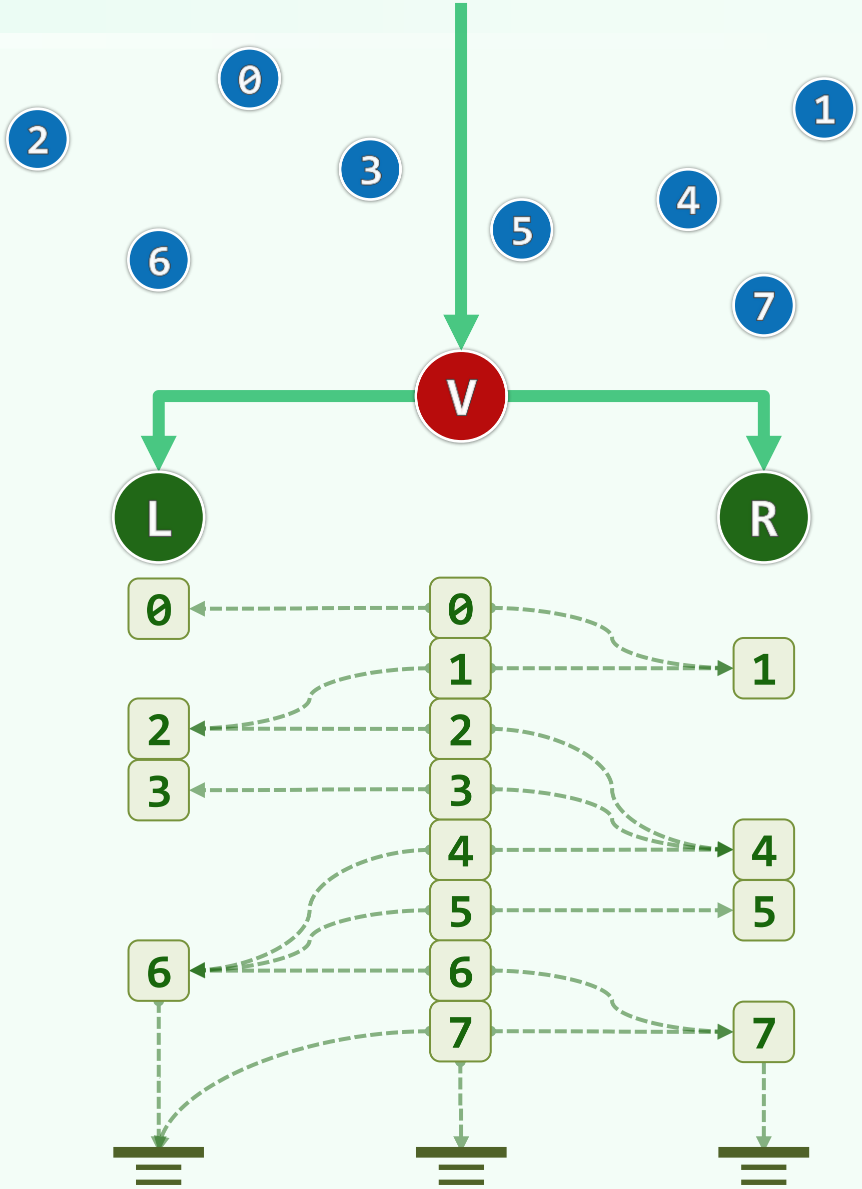
into a **SINGLE** massive list

❖ Thus, once a parent y-list is searched,

we can get, in $\mathcal{O}(1)$ time,

the entry for child y-list by

following the link between them

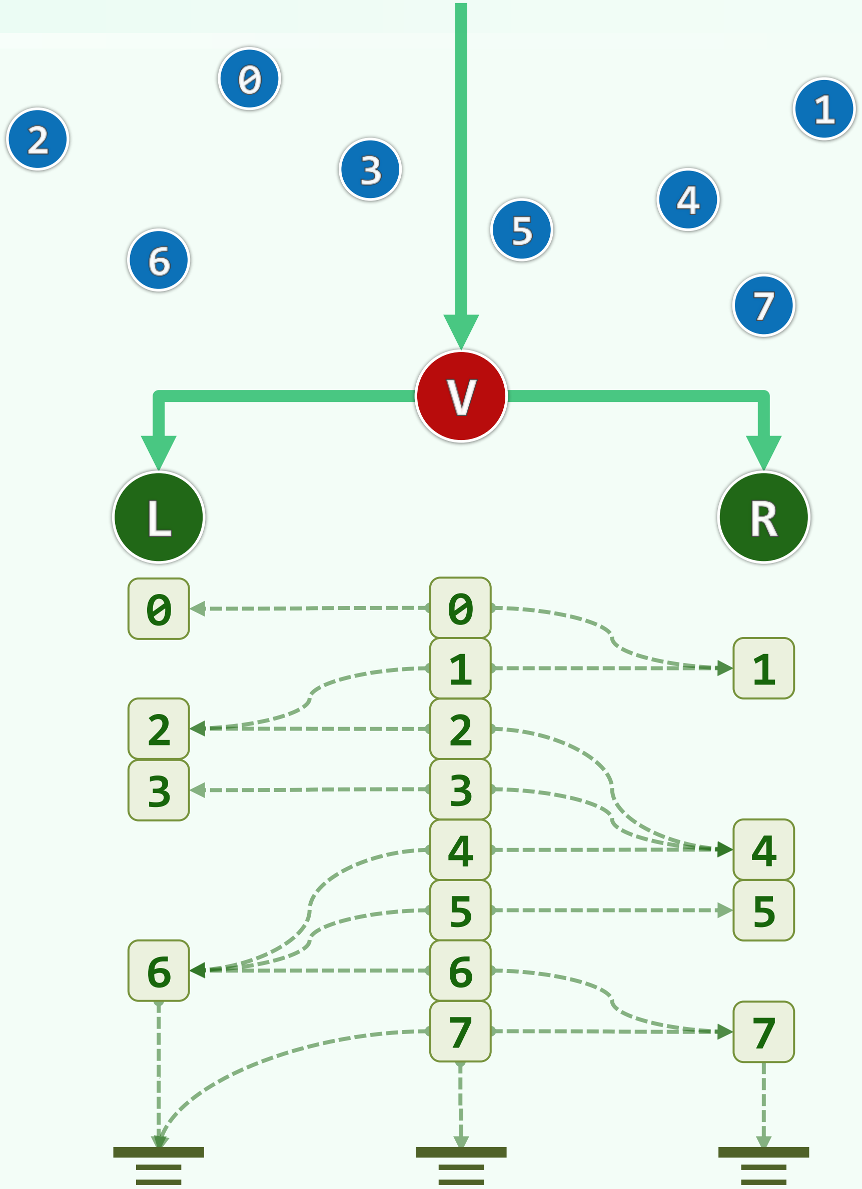


Using Coherence

❖ To answer a 2D range query, we will do an expensive ($\mathcal{O}(\log n)$) search on the y-list for the root

❖ Thereafter, while descending the x-tree, we can keep track of the position of y-range in each successive list in $\mathcal{O}(1)$ time

❖ This technique is called Fractional Cascading



Fractional Cascading

❖ For each item in A_v,

we store two pointers to

the item of NLT value

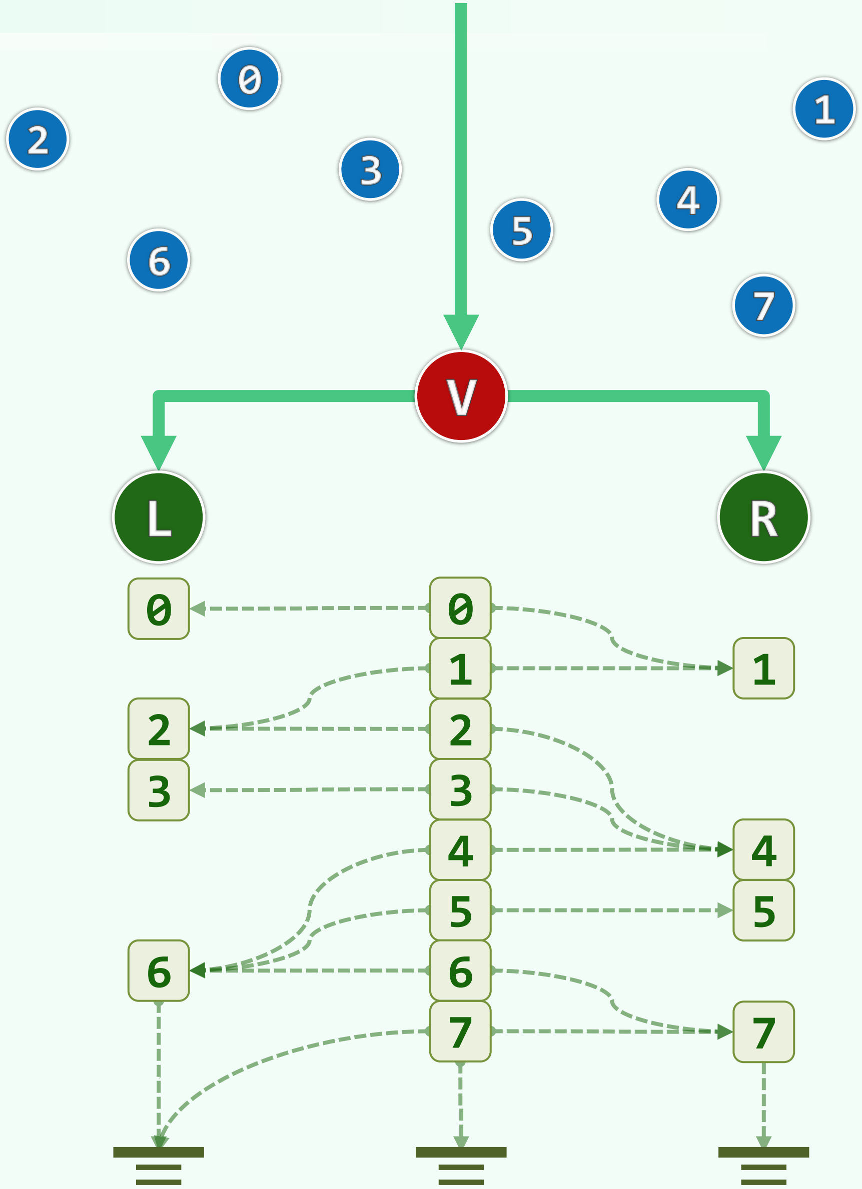
in A_L and A_R resp.

❖ Hence for any y-query with q_y,

once we know its entry in A_v, we can

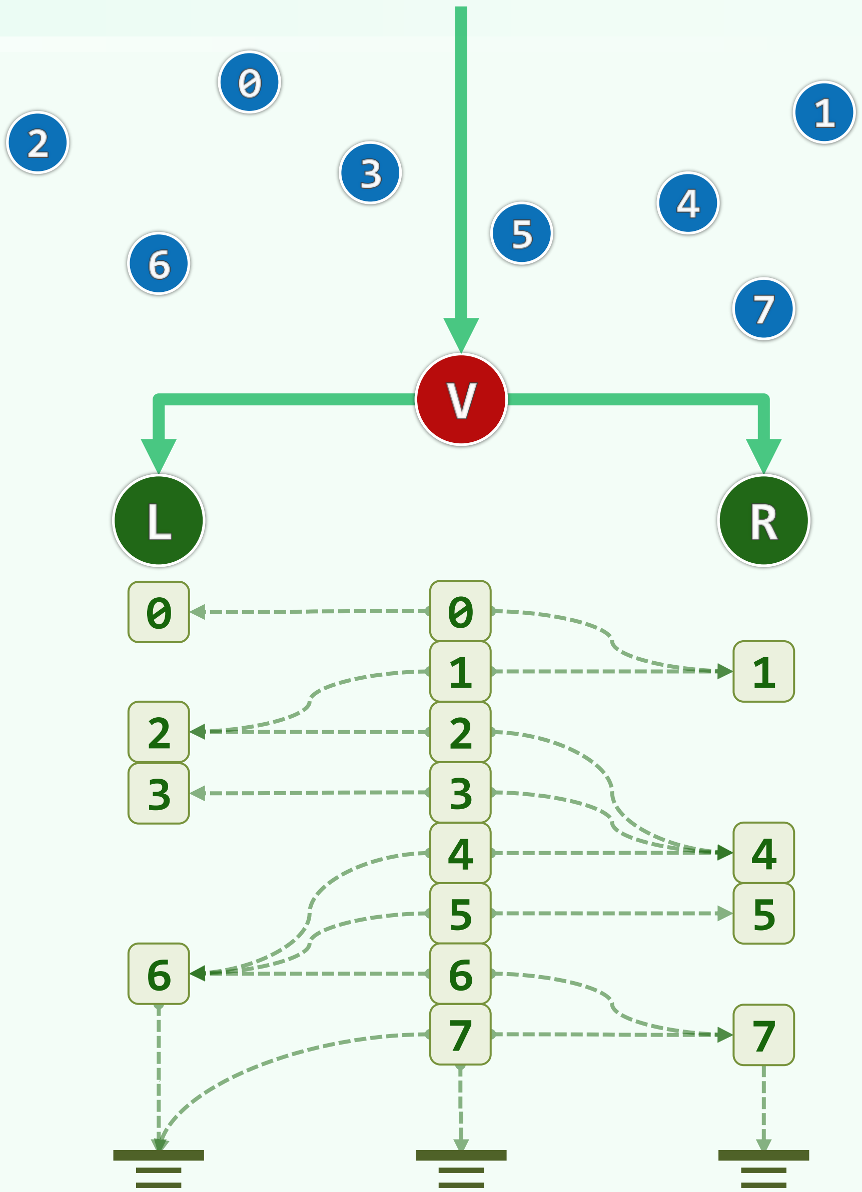
determine its entry in either A_L or A_R

in $\mathcal{O}(1)$ additional time



Construction By 2-Way Merging

- Let V be an internal node in the x-tree
  with L/R its left/right child resp.
- Let A_v be the y-list for v and
  A_L/A_R be the y-lists for its children
- Assuming no duplicate y-coordinates, we have
  - A_v is the disjoint union
    of A_L and A_R, and hence
  - A_v can be obtained by
    merging A_L and A_R (in linear time)



Complexity

❖ An MLST with fractional cascading is called a **range tree**

❖ A **y-search** for root is done in $\mathcal{O}(\log n)$ time

❖ To drop down to each next level, we can

get, in $\mathcal{O}(1)$ time, the current **y-interval**

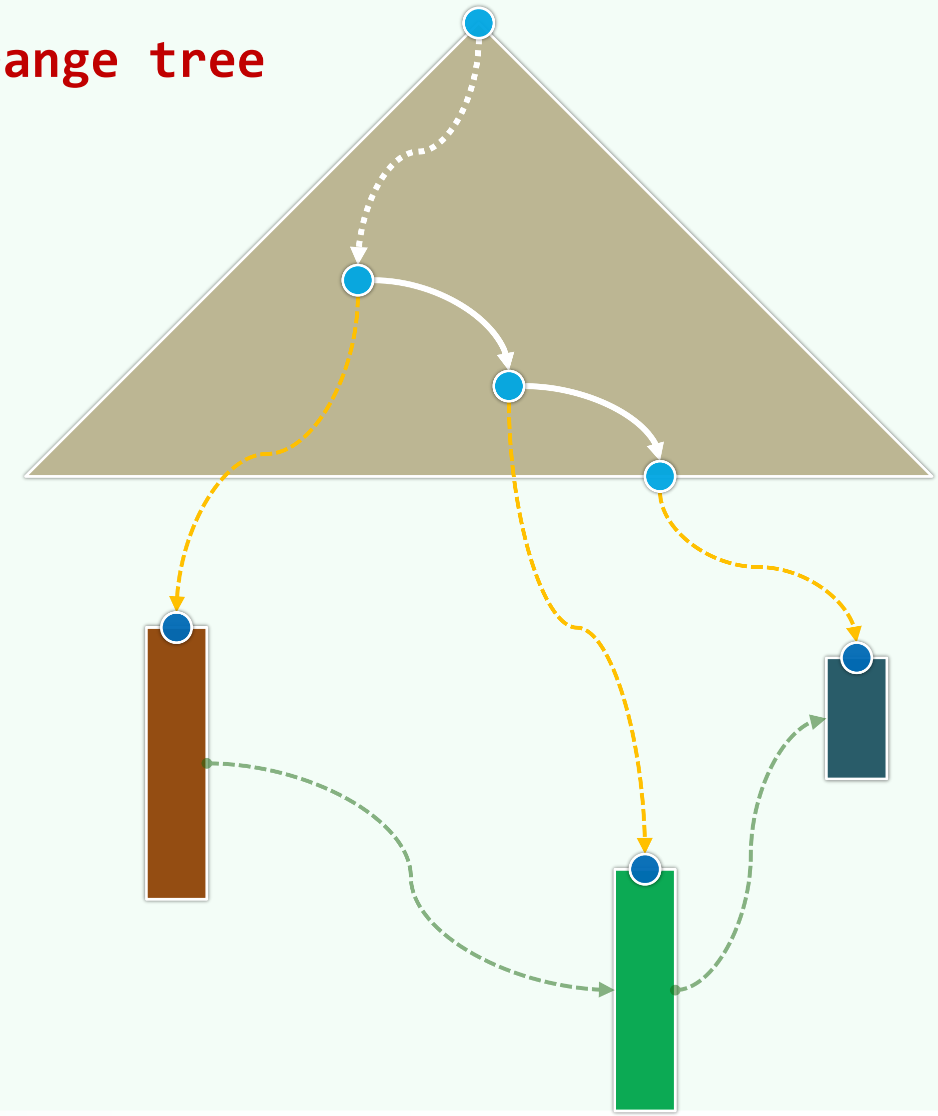
from that of the **prior** level

❖ Hence, each 2D orthogonal range query

- can be answered in $\mathcal{O}(r + \log n)$ time

- from a data structure of size $\mathcal{O}(n \cdot \log n)$,

- which can be constructed in $\mathcal{O}(n \cdot \log n)$ time



Beyond 2D

❖ Unfortunately, the trick of fractional cascading

can **ONLY** be applied to

the **LAST** level the search structure

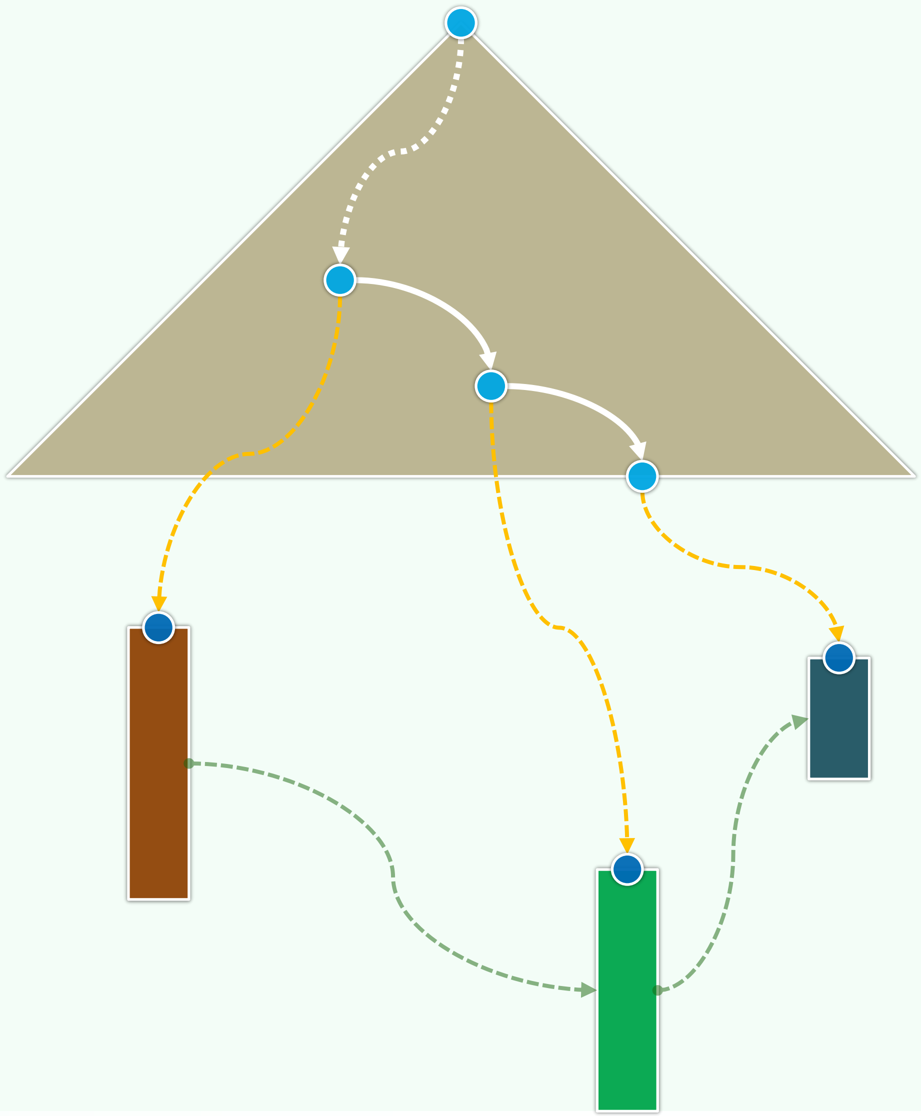
❖ Given a set of n points in $\mathcal{E}^d$,

an orthogonal range query

- can be answered in $\mathcal{O}(r + \log^{d-1} n)$ time

- from a data structure of size $\mathcal{O}(n \cdot \log^{d-1} n)$,

- which can be constructed in $\mathcal{O}(n \cdot \log^{d-1} n)$ time



09-XB

BST Application

Interval Tree

Your instinct, rather than precision stabbing, is
more about just random bludgeoning.

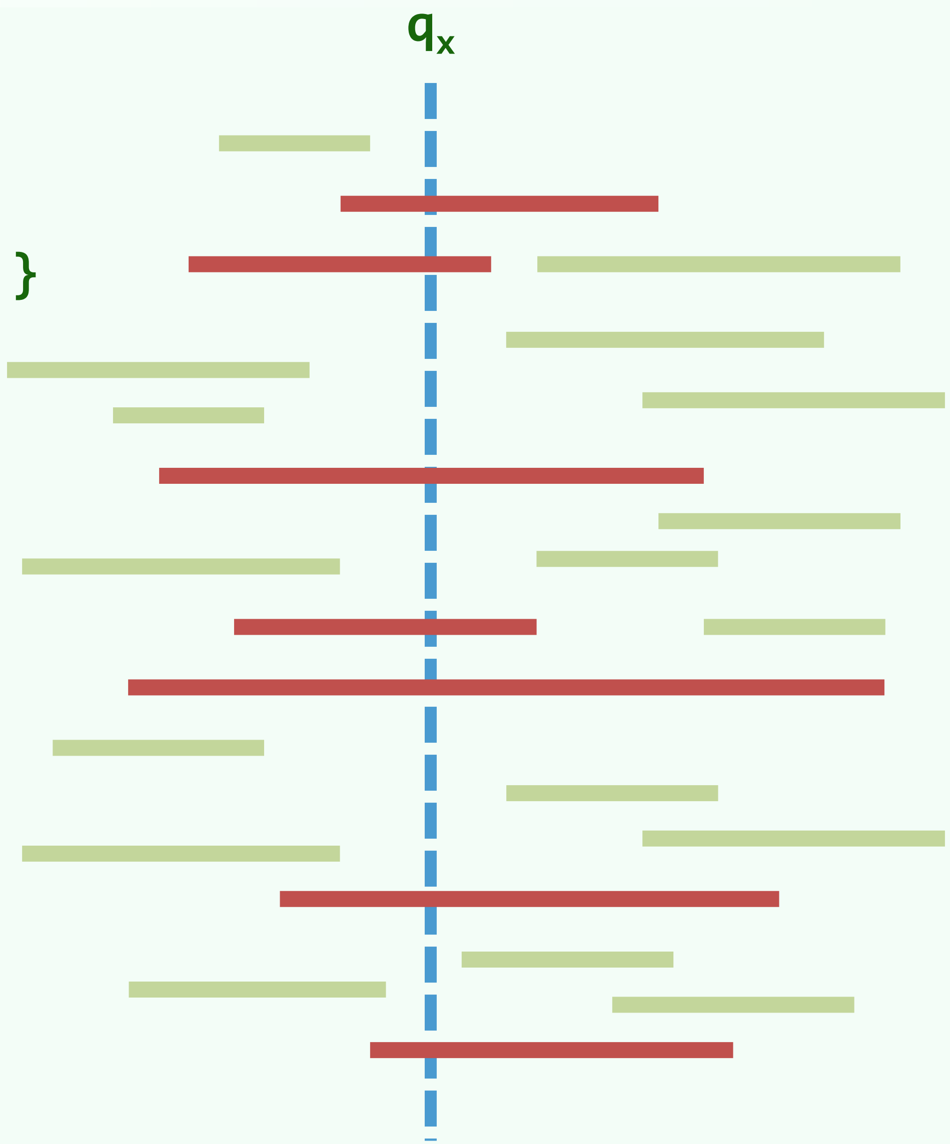
邓俊辉
deng@tsinghua.edu.cn

Stabbing Query

- Given a set of intervals in general position on the x-axis: $S = \{ s_i = [x_i, x_i'] \mid 1 \le i \le n \}$ and a query point q_x
- Find all intervals that contain q_x

  $\{ s_i = [x_i, x_i'] \mid x_i \le q_x \le x_i' \}$

- To solve this query, we will use the so-called interval tree ...

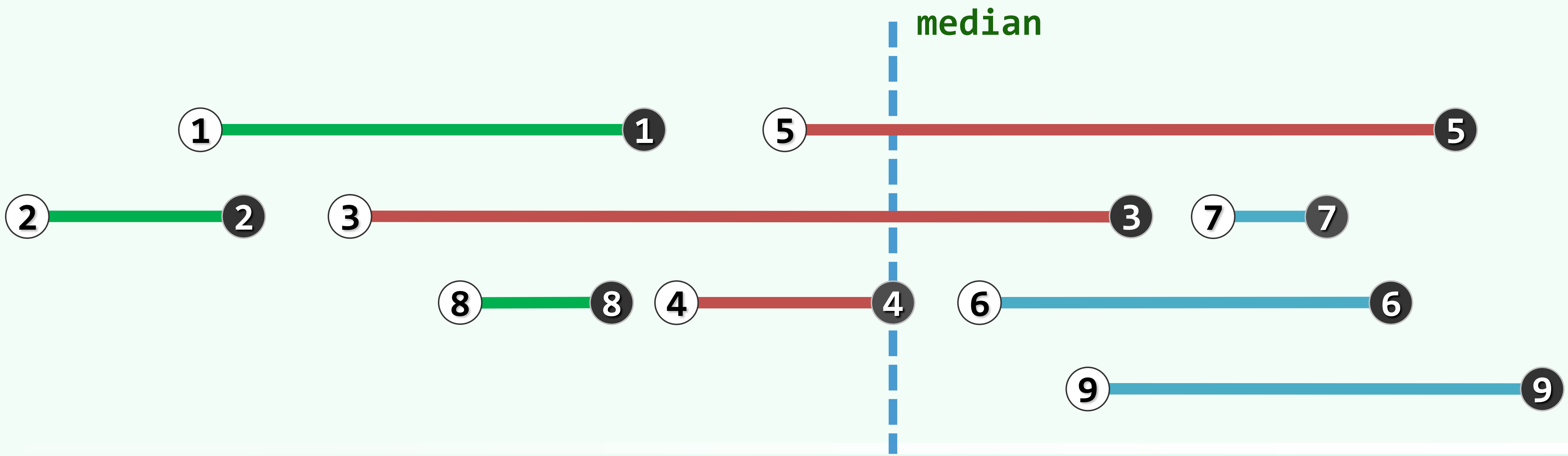


Median

- Let $S = \{ s_1, \ldots, s_n \}$ be the set of intervals
- Let $P = \partial S$ be the set of all endpoints

  // by general position assumption, $|P| = 2n$

- Let $\text{median}(P) = x_{mid}$ be the median of P

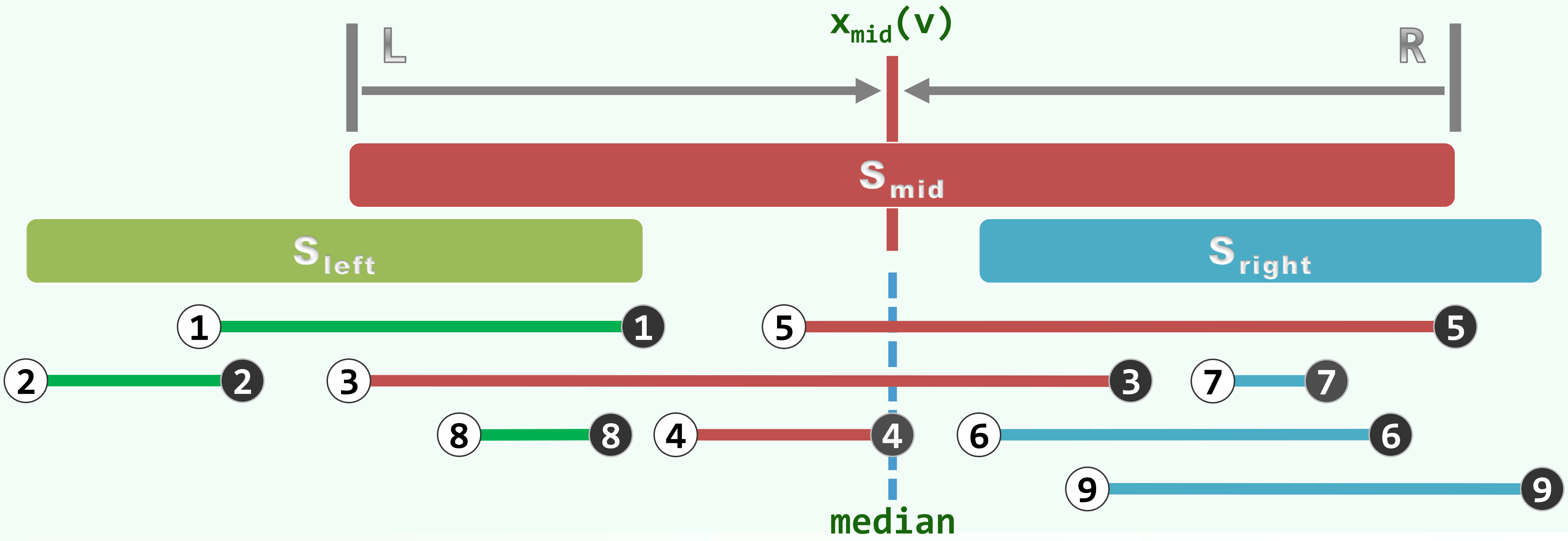


Partitioning

- All intervals can be then categorized into 3 subsets :

$$S_{left} = \{ S_i \mid x'_i < x_{mid} \} \quad S_{mid} = \{ S_i \mid x_i \le x_{mid} \le x'_i \} \quad S_{right} = \{ S_i \mid x_{mid} < x_i \}$$

- $S_{left/right}$ will be **recursively** partitioned until they are empty (leaves)

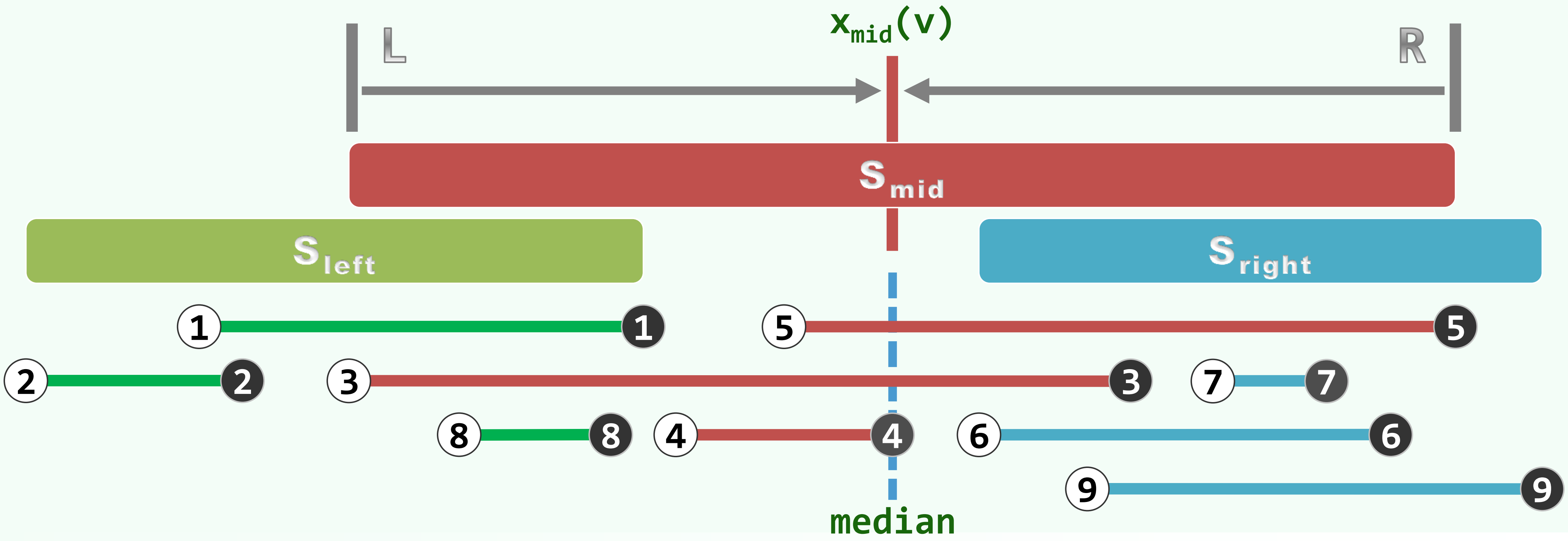


Balance

- $\max\{\ |S_{left}|, |S_{right}\ \}\ \leq\ n/2$

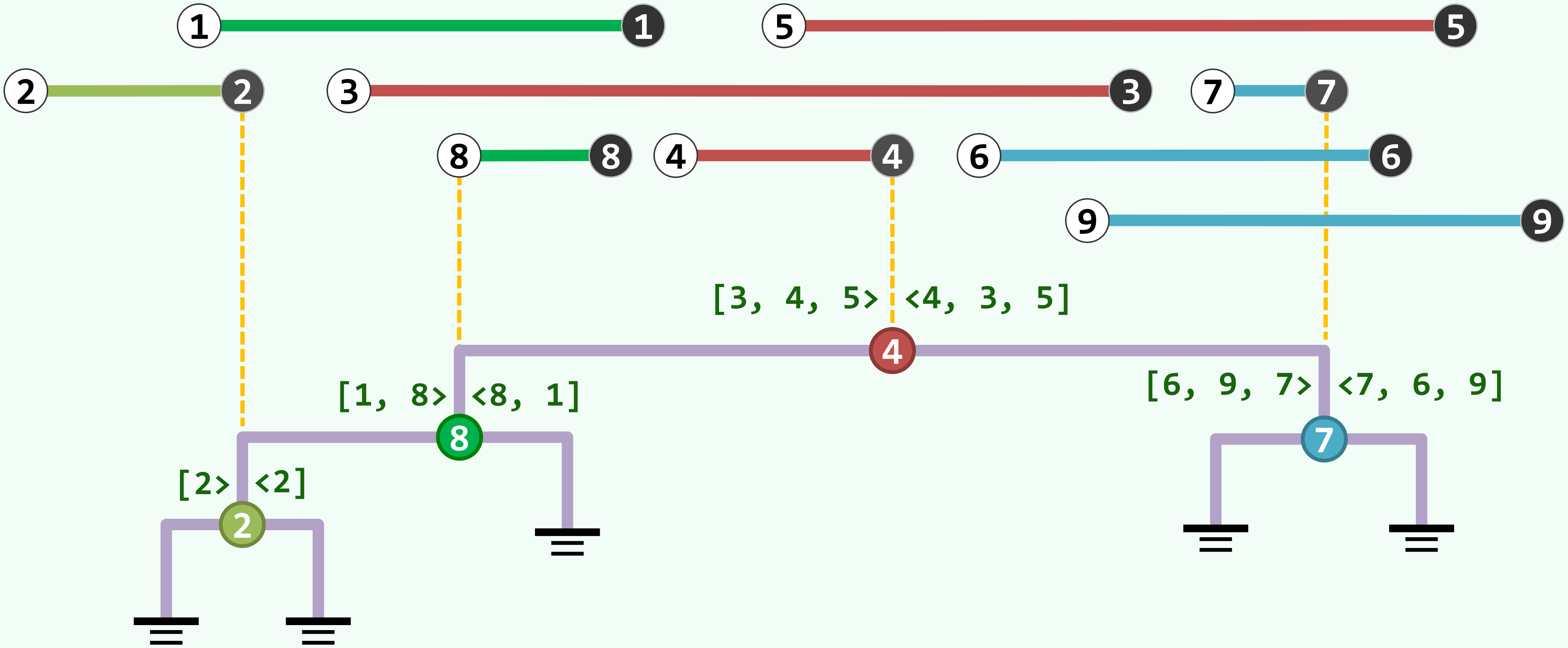
- **Best case:** $|S_{mid}|\ =\ n$

- **Worst case:** $|S_{mid}|\ =\ 1$



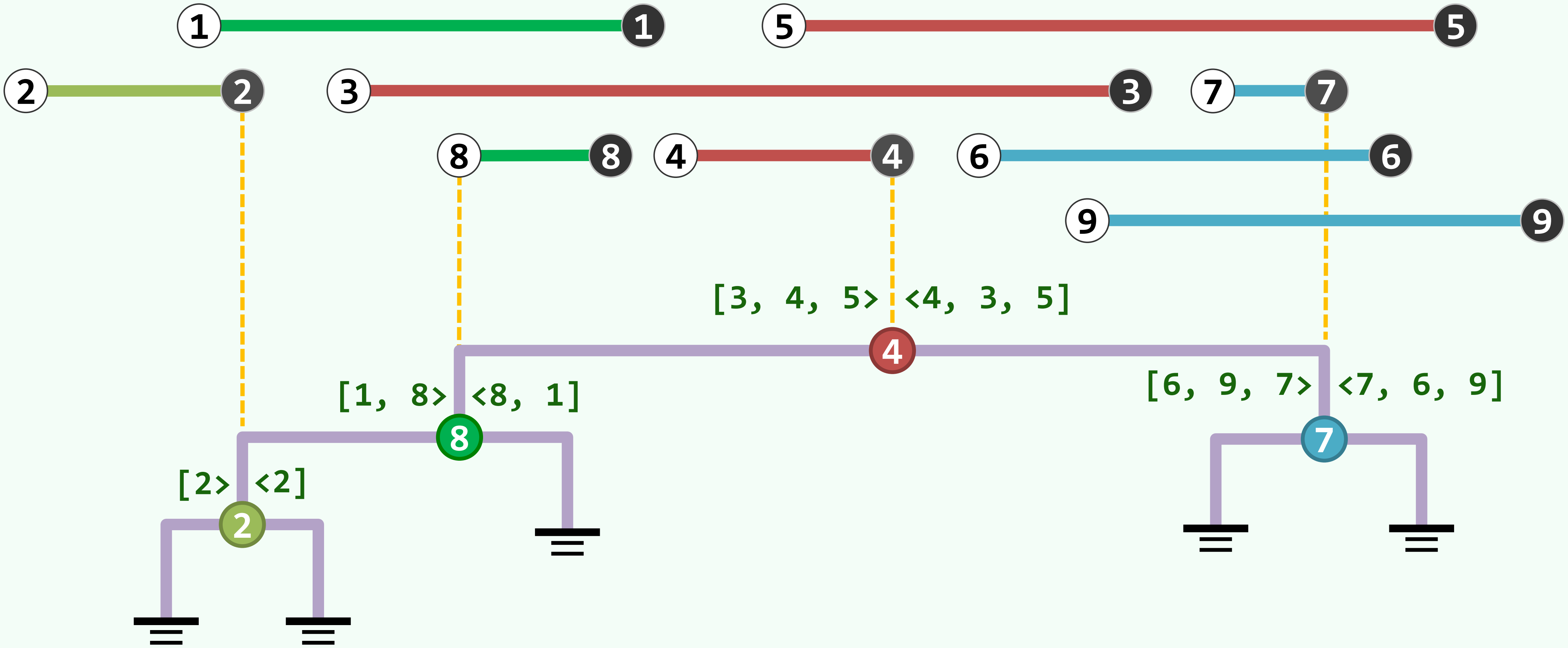
Associative Lists

❖ $L_{left/right}$ = all intervals of S_{mid} sorted by the LEFT/RIGHT endpoints



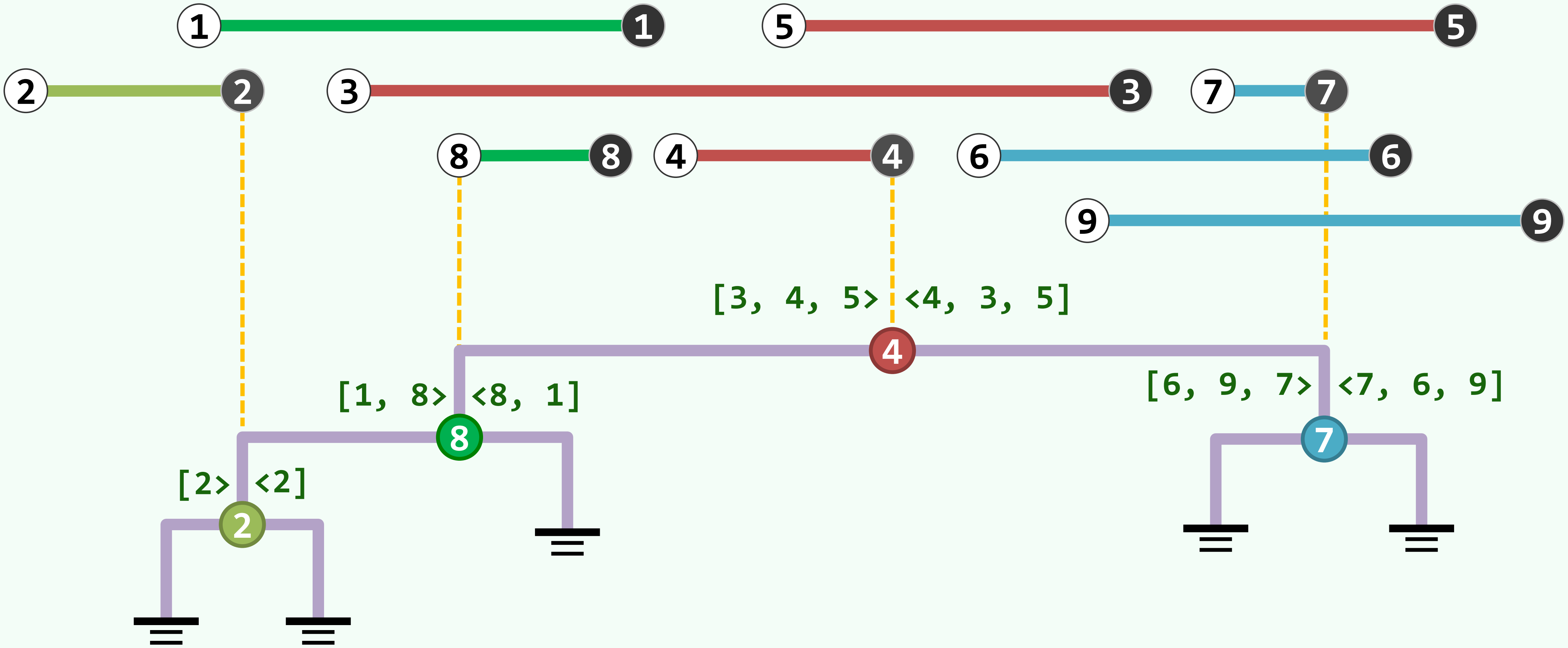
$\mathcal{O}$(n) Size

❖ Each segment appears twice



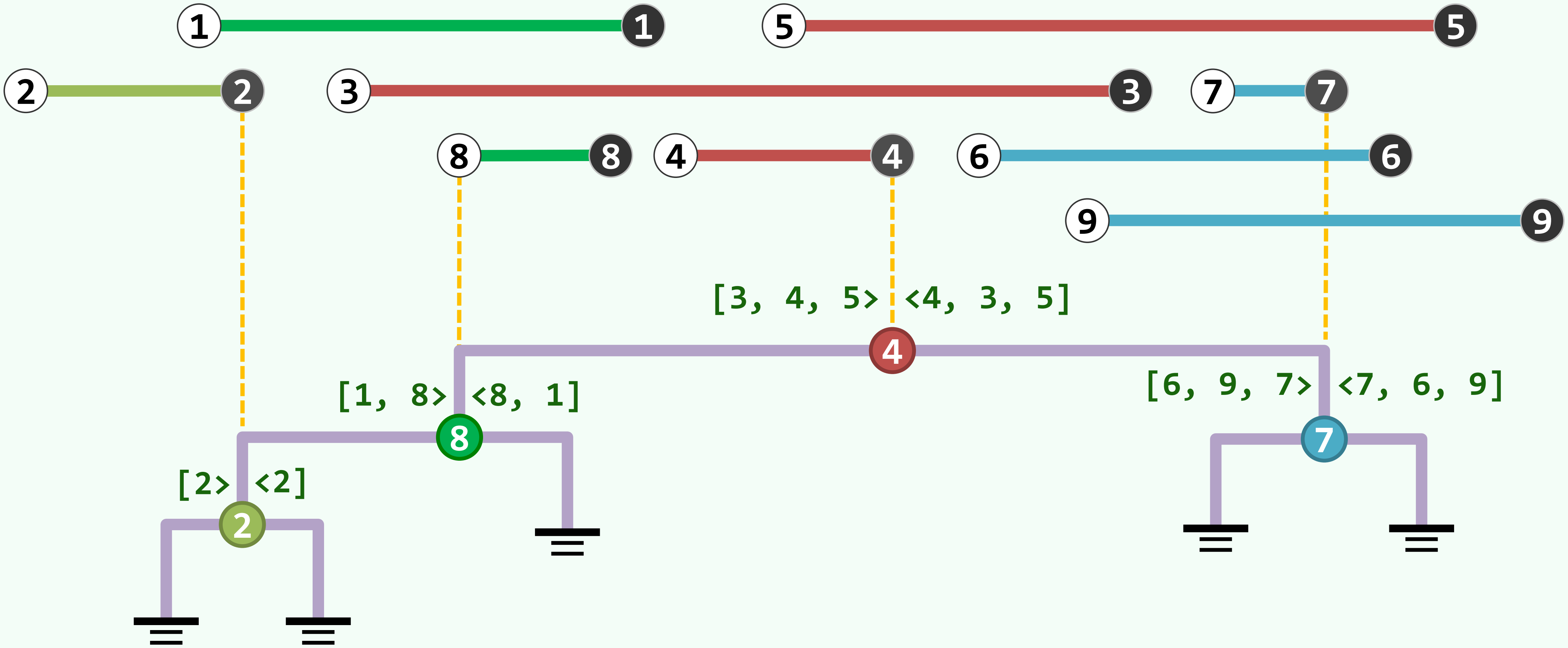
$\mathcal{O}(\log n)$ Depth

❖ Partitionings are done evenly



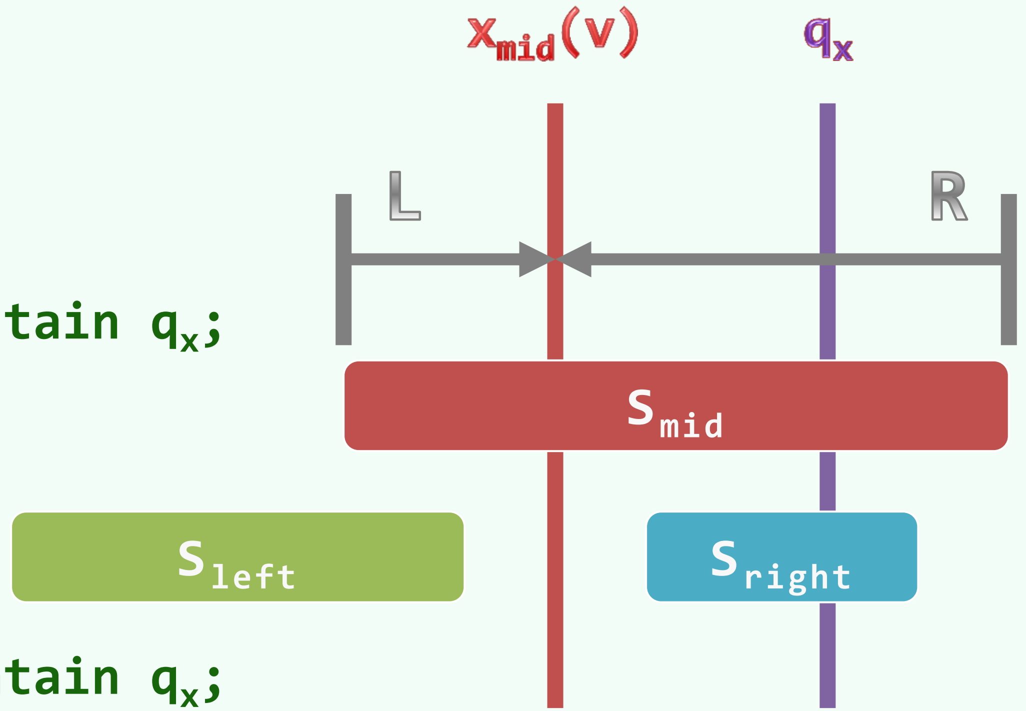
$\mathcal{O}$(nlogn) Construction Time

❖ Hint: avoid repeatedly sorting



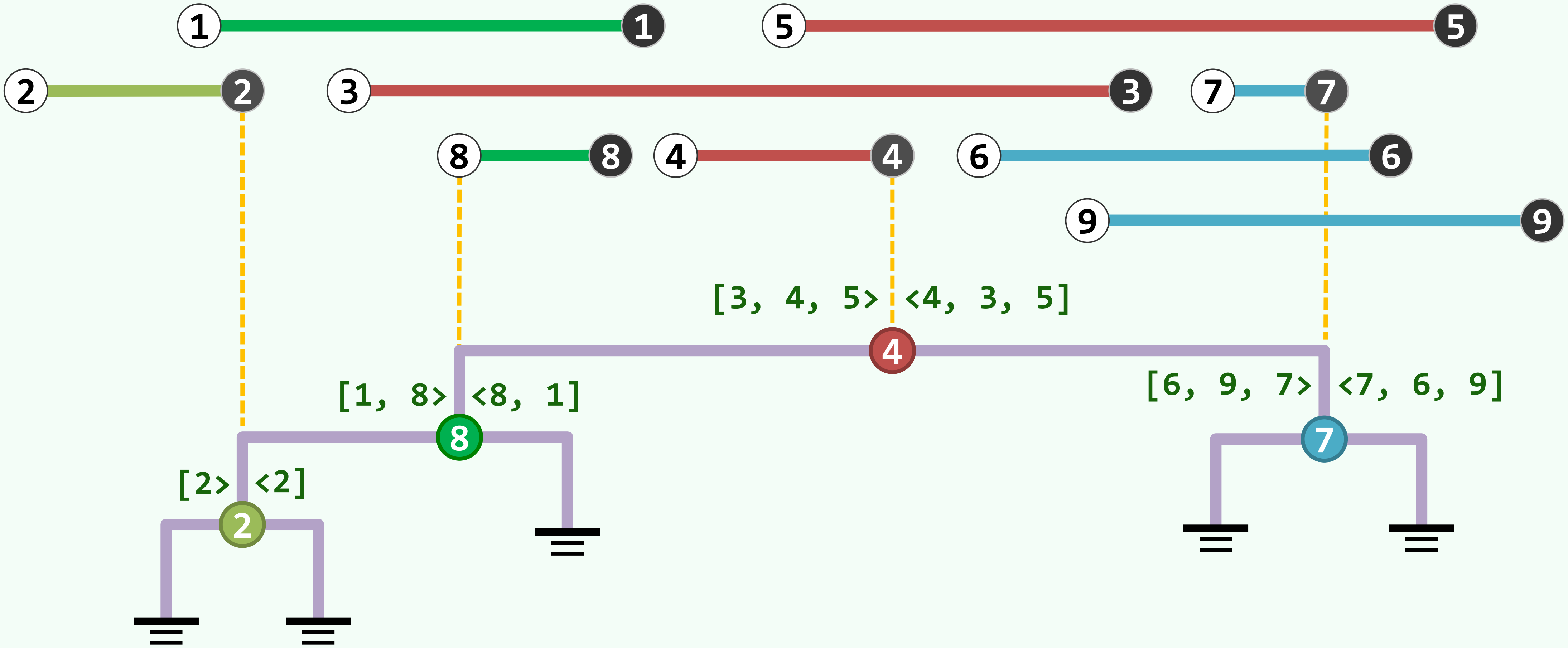
queryIntervalTree(v, q_x)

```
if ( ! v ) return;  //base

if ( qx < xmid(v) ) {

   report all segments of Smid(v) that contain qx;

   queryIntervalTree( lc(v), qx );

} else if ( xmid(v) < qx ) {

   report all segments of Smid(v) that contain qx;

   queryIntervalTree( rc(v), qx );

} else //with a probability ≈ 0

   report all segments of Smid( v ); //both rc(v) & lc(v) can be ignored
```



$\mathcal{O}$(r + logn) Query Time

❖ Each query visits $\mathcal{O}$(logn) nodes //LINEAR recursion



09-XC

BST Application

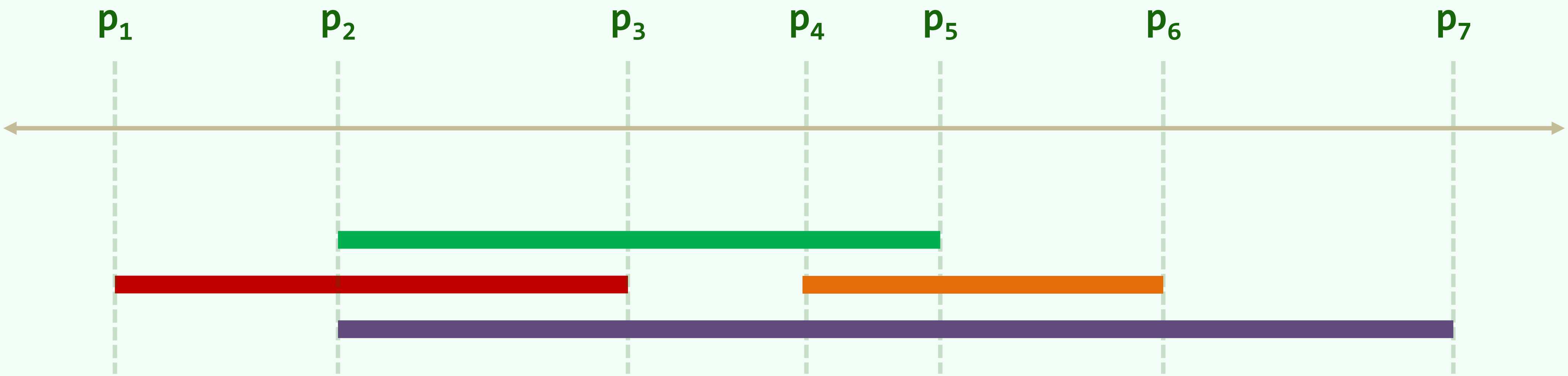
Segment Tree

邓俊辉
deng@tsinghua.edu.cn

Elementary Intervals

❖ Let $I = \{ [x_i, x_i'] \mid i = 1, 2, 3, \ldots, n \}$ be n intervals on the x-axis

❖ Sort all the endpoints into $\{ p_1,\ p_2,\ p_3,\ \ldots,\ p_m \}$, $m \leq 2n$



❖ m+1 elementary intervals are hence defined as:

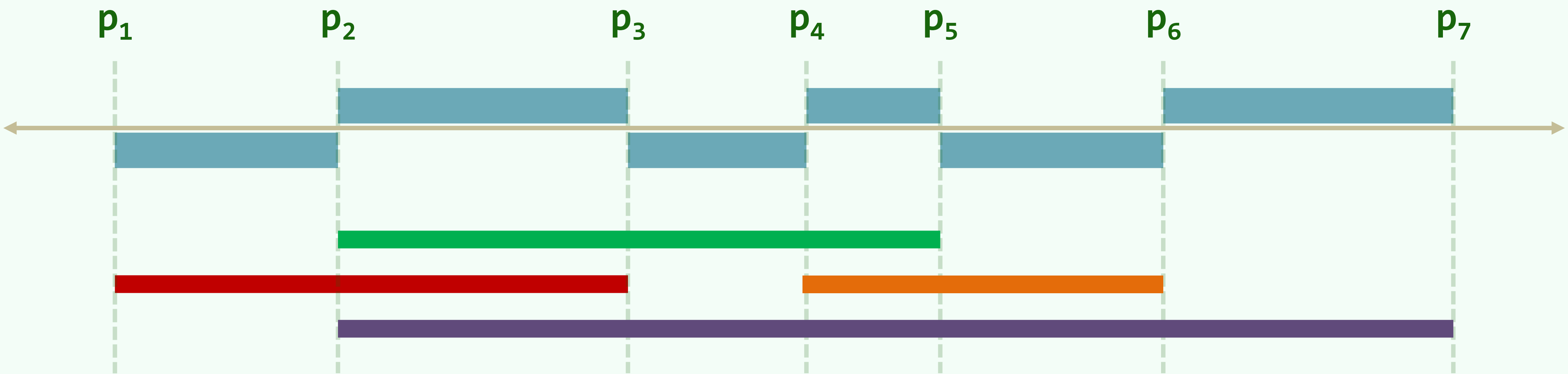
$$(-\infty, p_1],\ (p_1, p_2],\ (p_2, p_3],\ \ldots,\ (p_{m-1}, p_m],\ (p_m, +\infty]$$

Discretization

👁Within each EI, all stabbing queries share a same output

∴ If we **sort** all EI's into a vector and

store the corresponding **output** with each EI, then ...

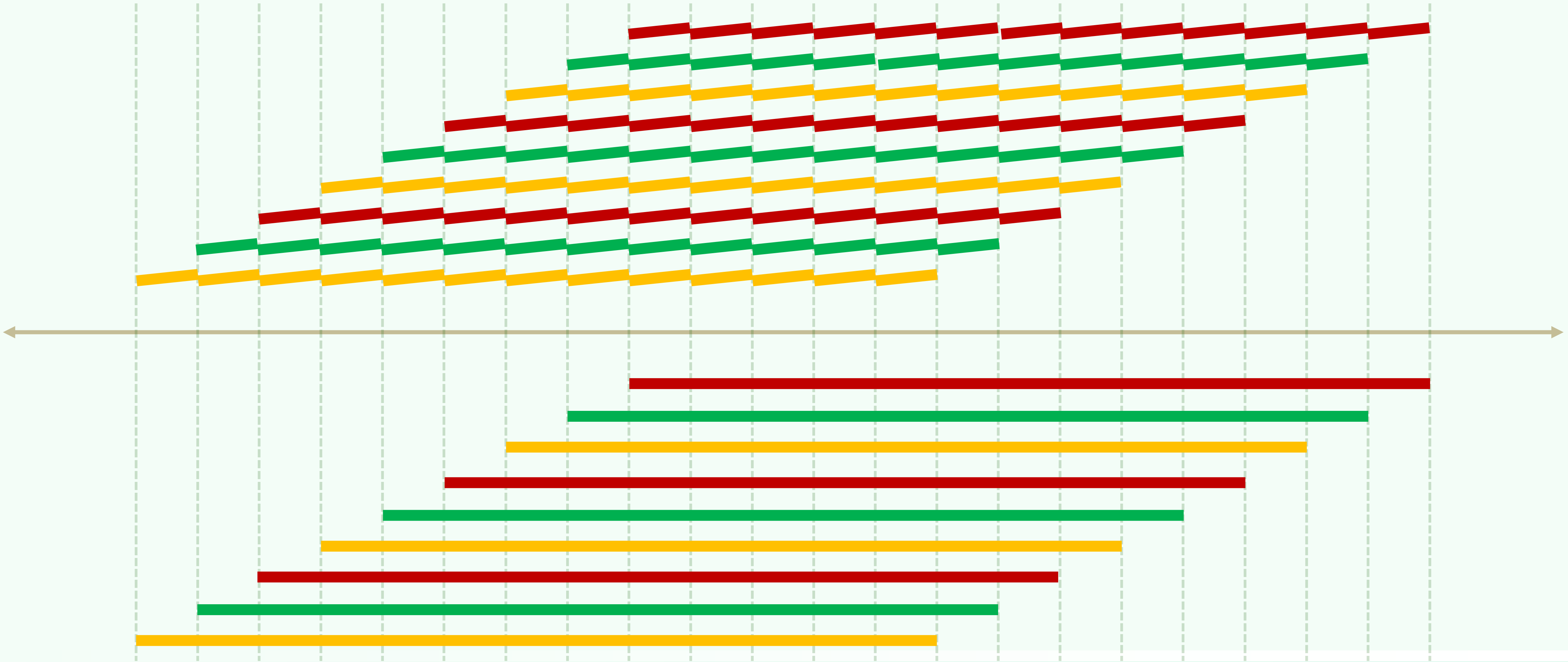


∴ Once a query position is determined, //by an $\mathcal{O}(\log n)$ time binary search

the output can then be returned directly //$\mathcal{O}(r)$

Worst Case

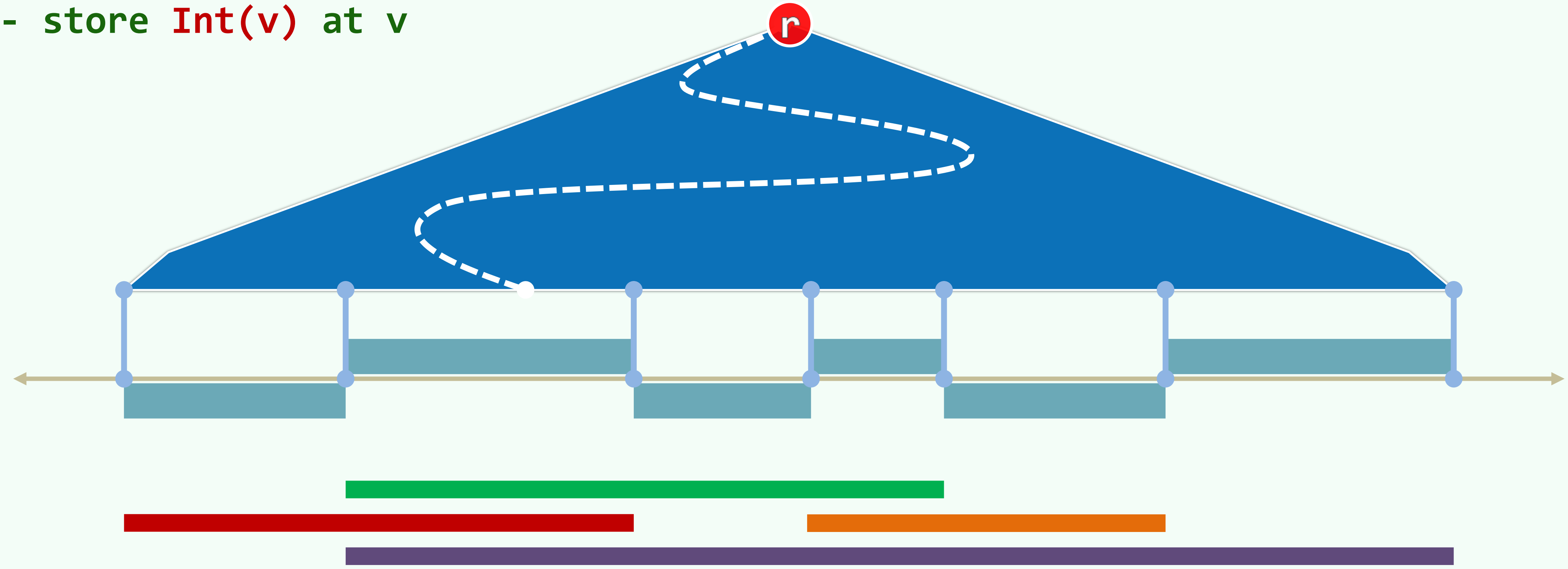
- Every interval spans $\Omega(n)$ EI's and a total space of $\Omega(n^2)$ is required



Sorted Vector --> BBST

❖ For each leaf v,

- denote the corresponding elementary interval as EI(v)
- denote the subset of intervals containing EI(v) as Int(v) and
- store Int(v) at v

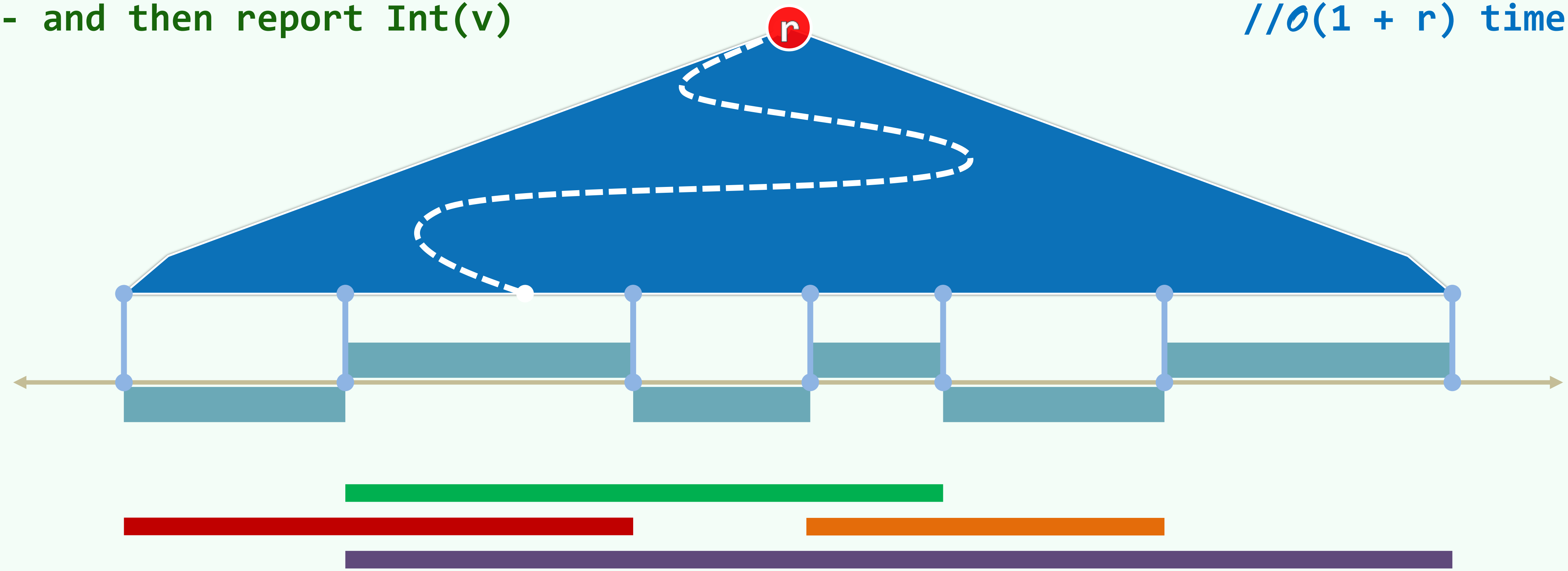


1D Stabbing Query with BBST

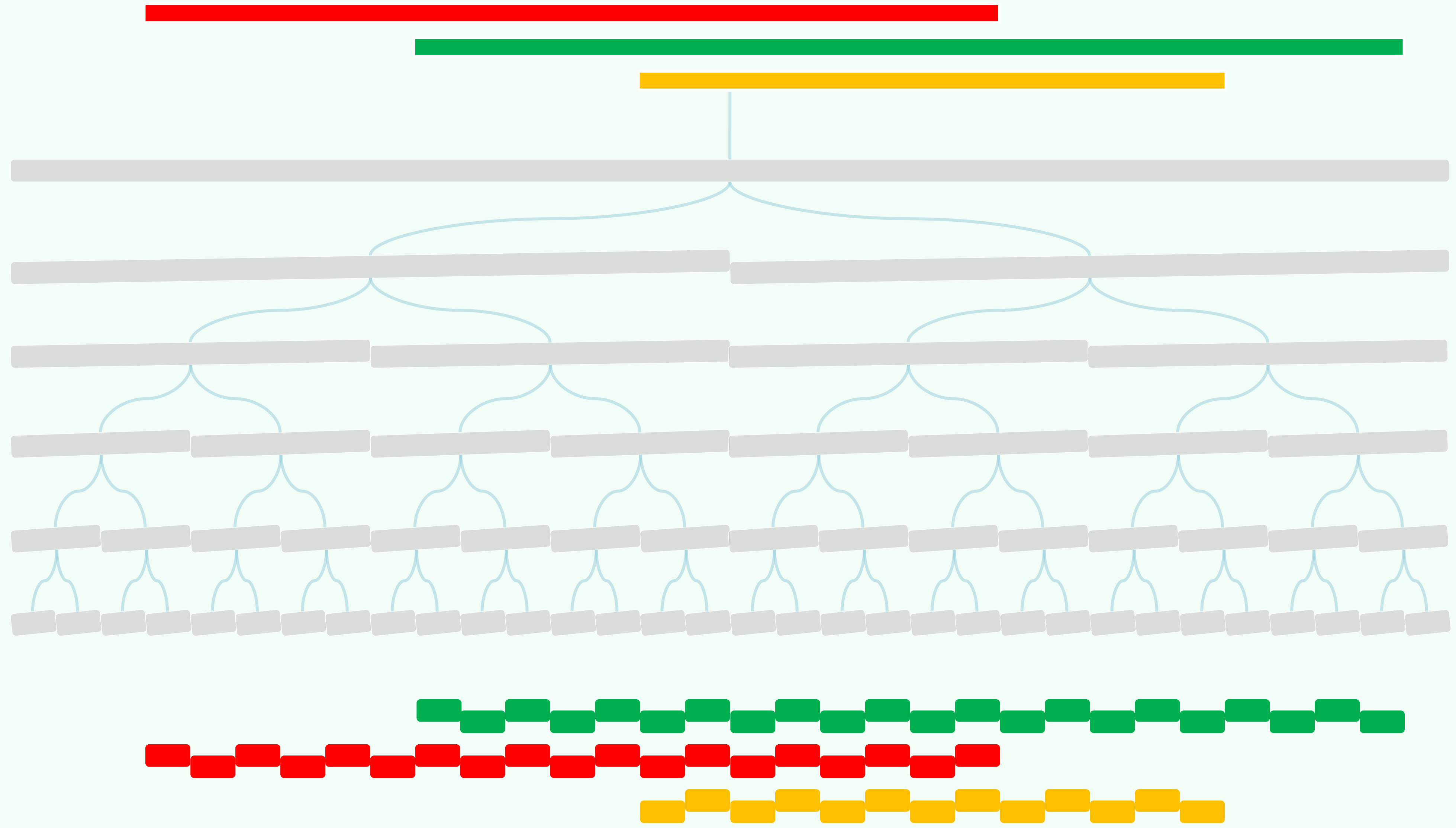
❖ To find all intervals containing q_x, we can

- find the EI(v) containing q_x //$\mathcal{O}(\log n)$ time for a BBST

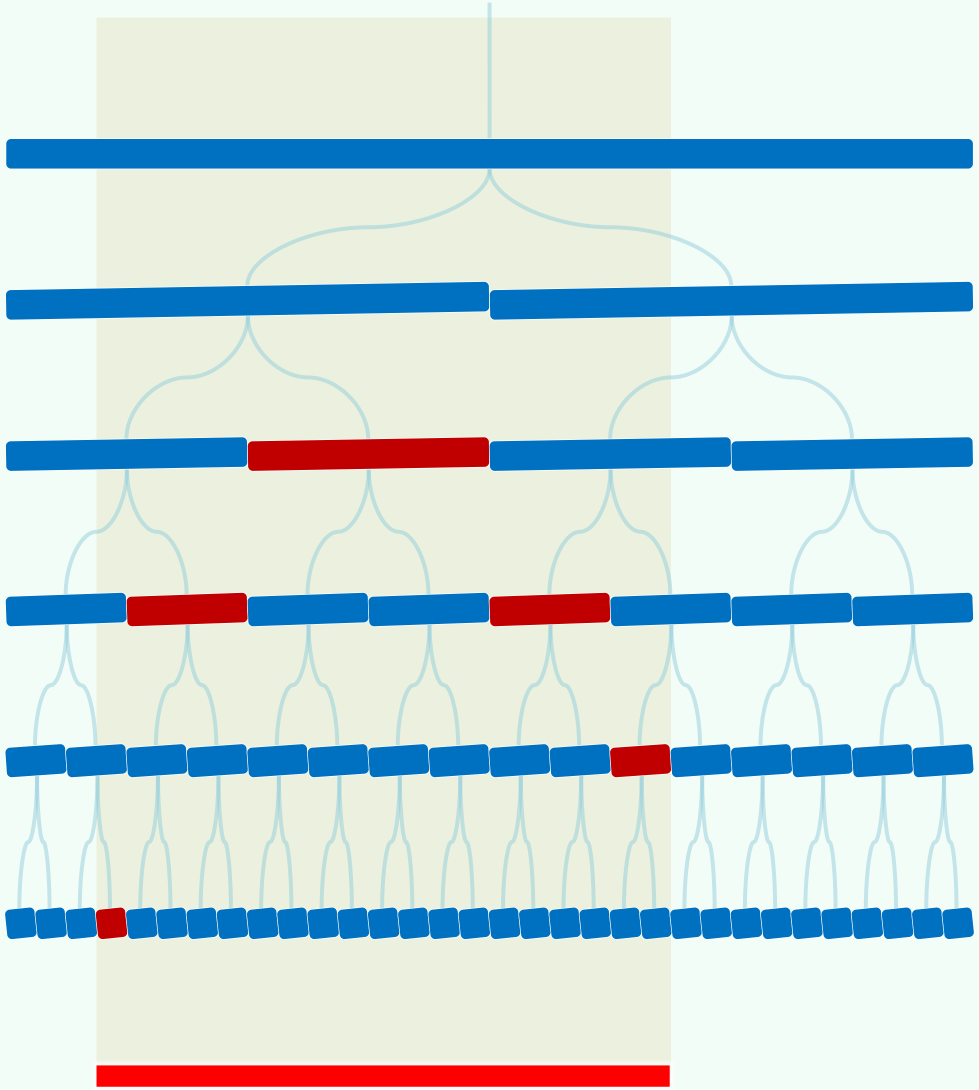
- and then report Int(v) //$\mathcal{O}(1 + r)$ time

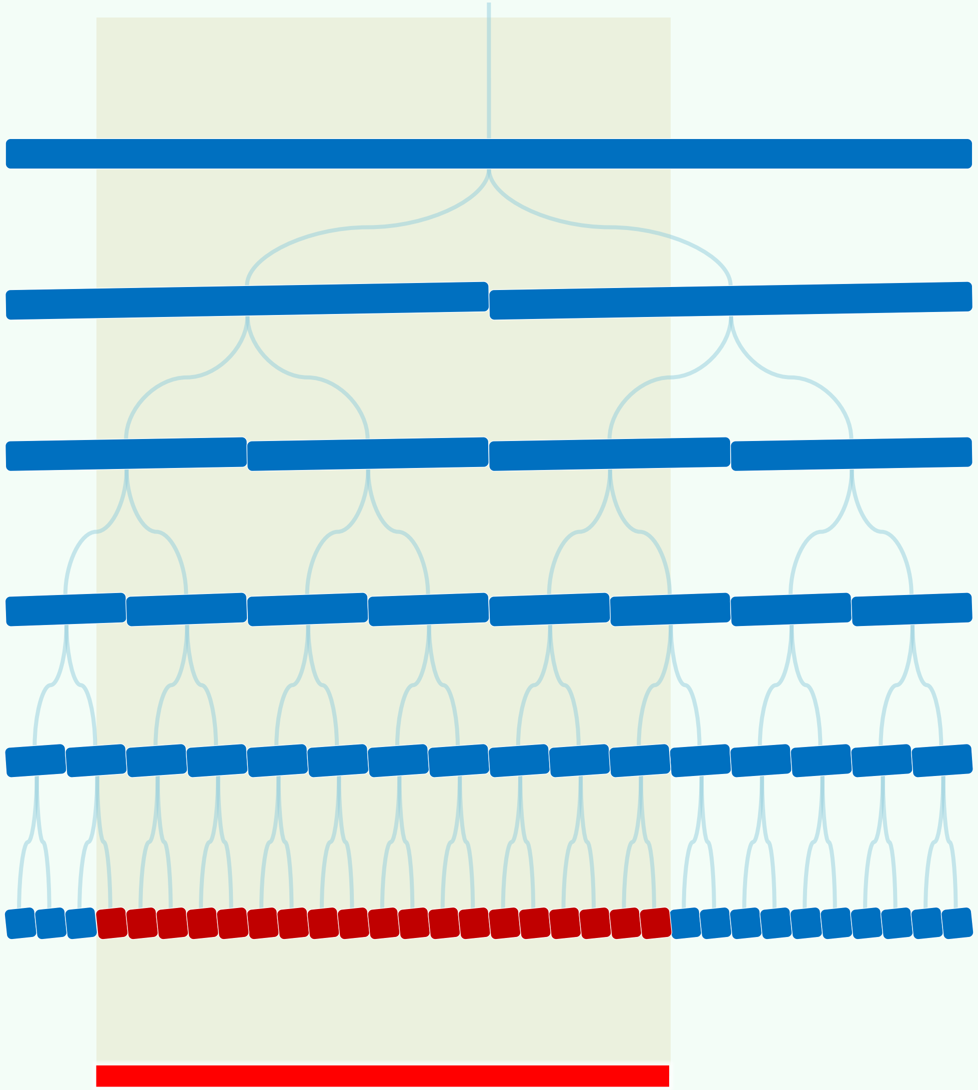


$\Omega(n^2)$ Total Space In The Worst Cases

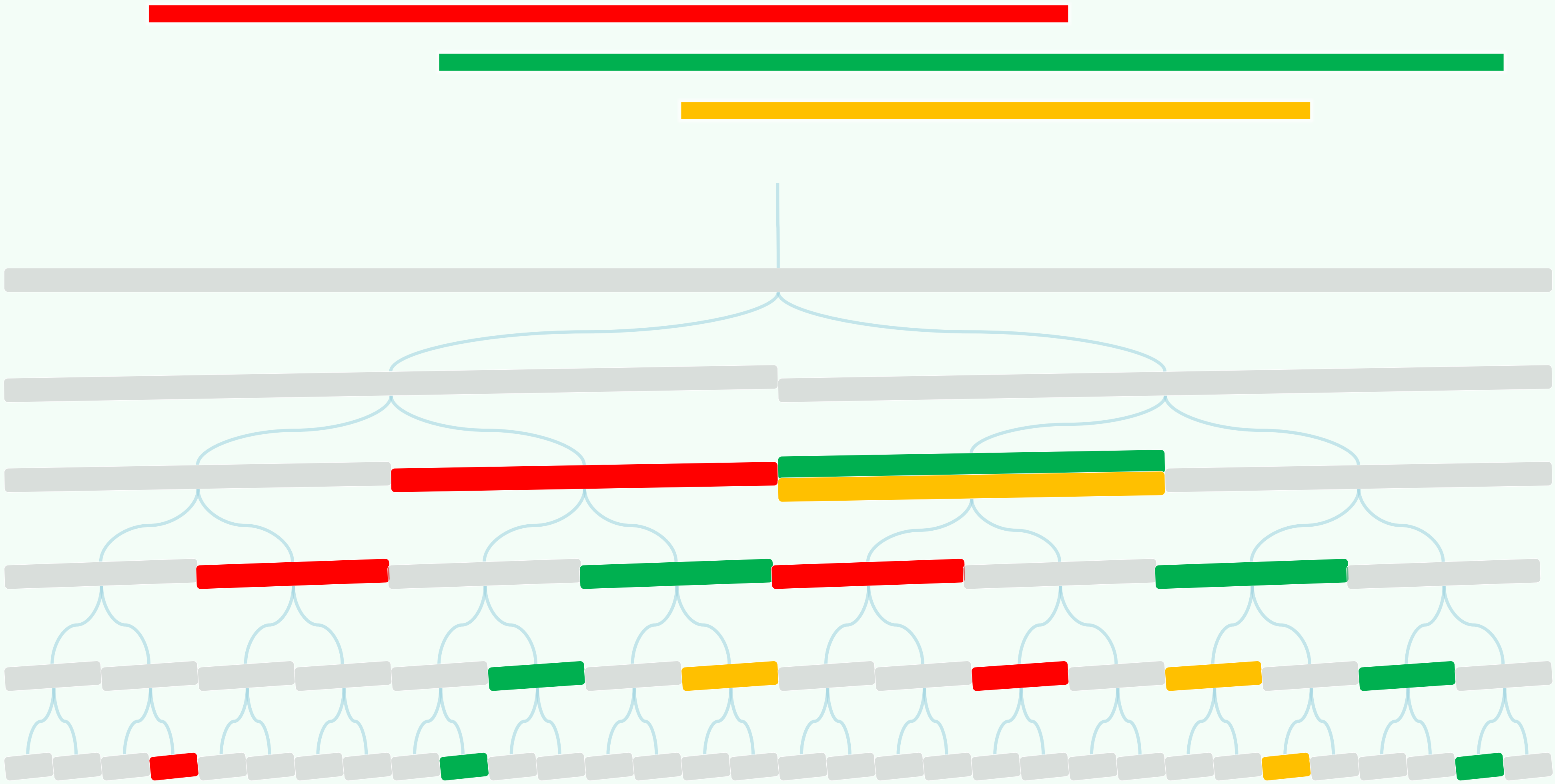


Merge At Common Ancestors



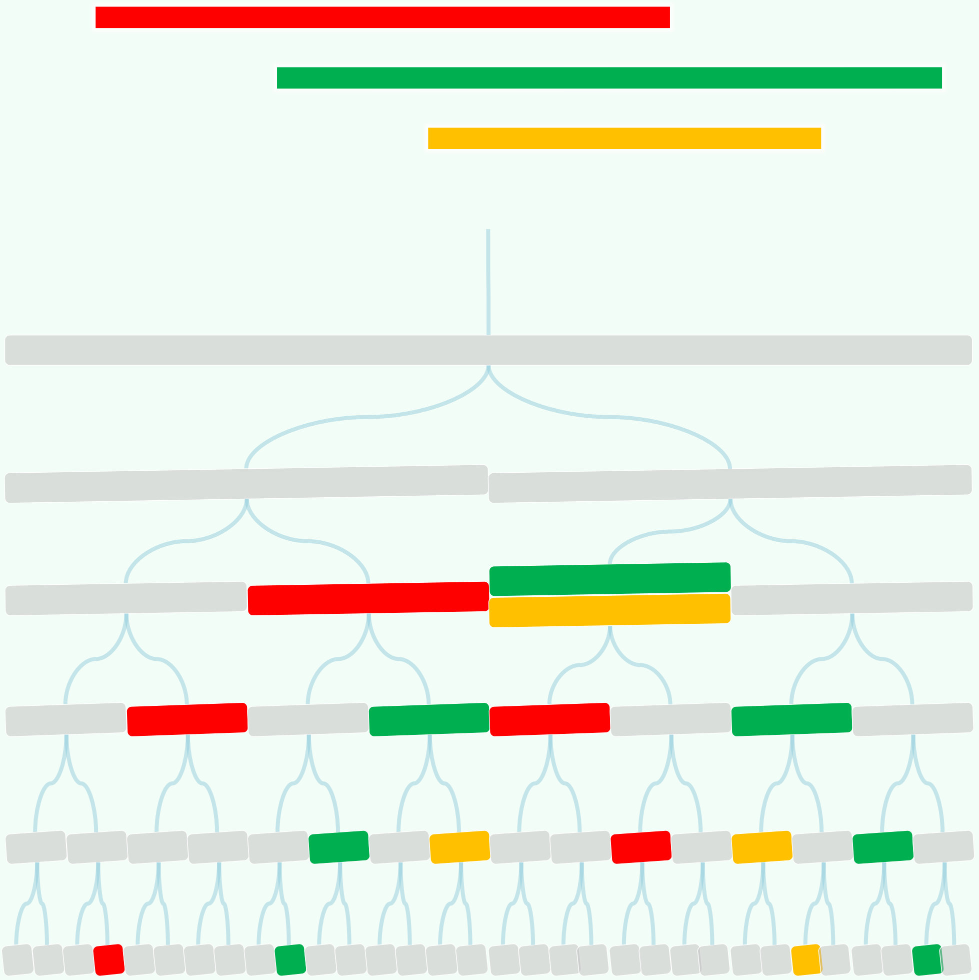


Canonical Subsets with $\mathcal{O}(n\log n)$ Space



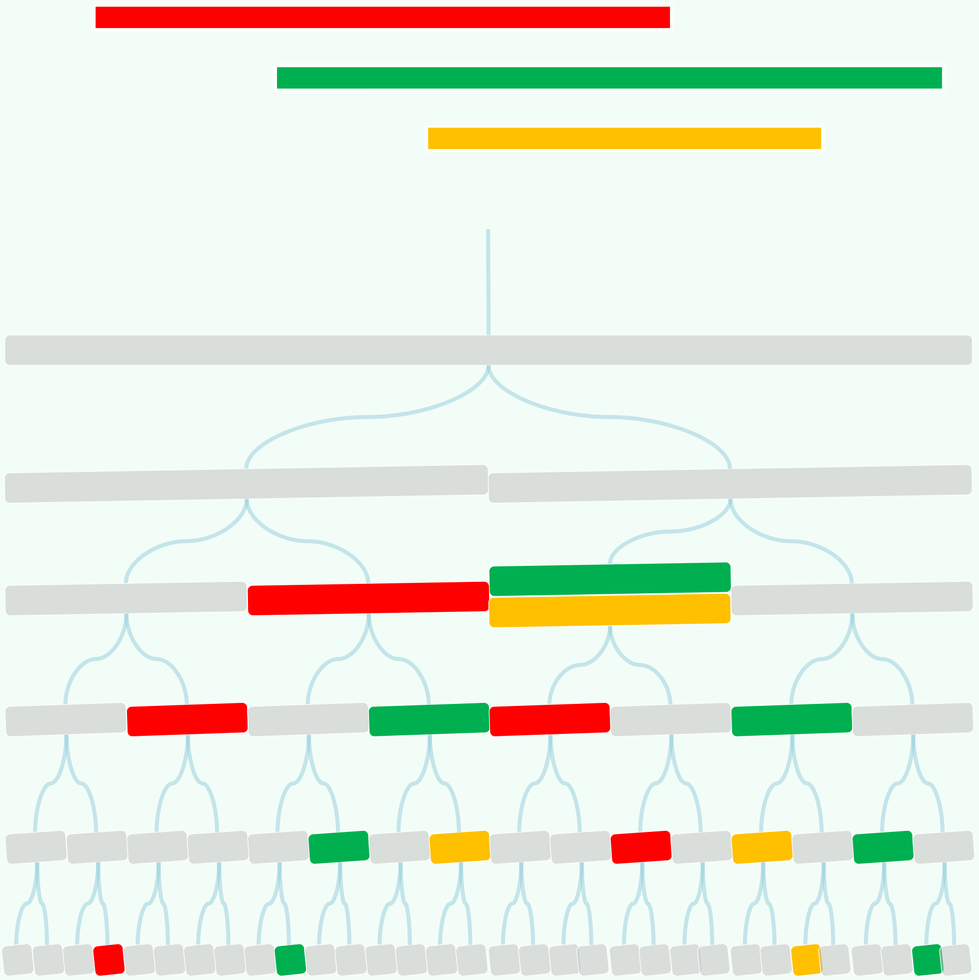
BuildSegmentTree(I)

```
❖ // Construct a segment tree on
  // a set I of n intervals
  Sort all endpoints in I before
     determining all EI's //O(nlogn)
  Create T a BBST on all the EI's //O(n)
  Determine Int(v) for each node v
     //O(n) if done in a bottom-up manner
  For each s of I
     call InsertSegmentTree( T.root , s )
```



InsertSegmentTree(v , s)

```
// Insert an interval s into
// a segment (sub)tree rooted at v
if ( Int(v) ⊆ s )
   store s at v and return;
if ( Int( lc(v) ) ∩ s ≠ ∅ ) //recurse
   InsertSegmentTree( lc(v), s );
if ( Int( rc(v) ) ∩ s ≠ ∅ ) //recurse
   InsertSegmentTree( rc(v), s );
```

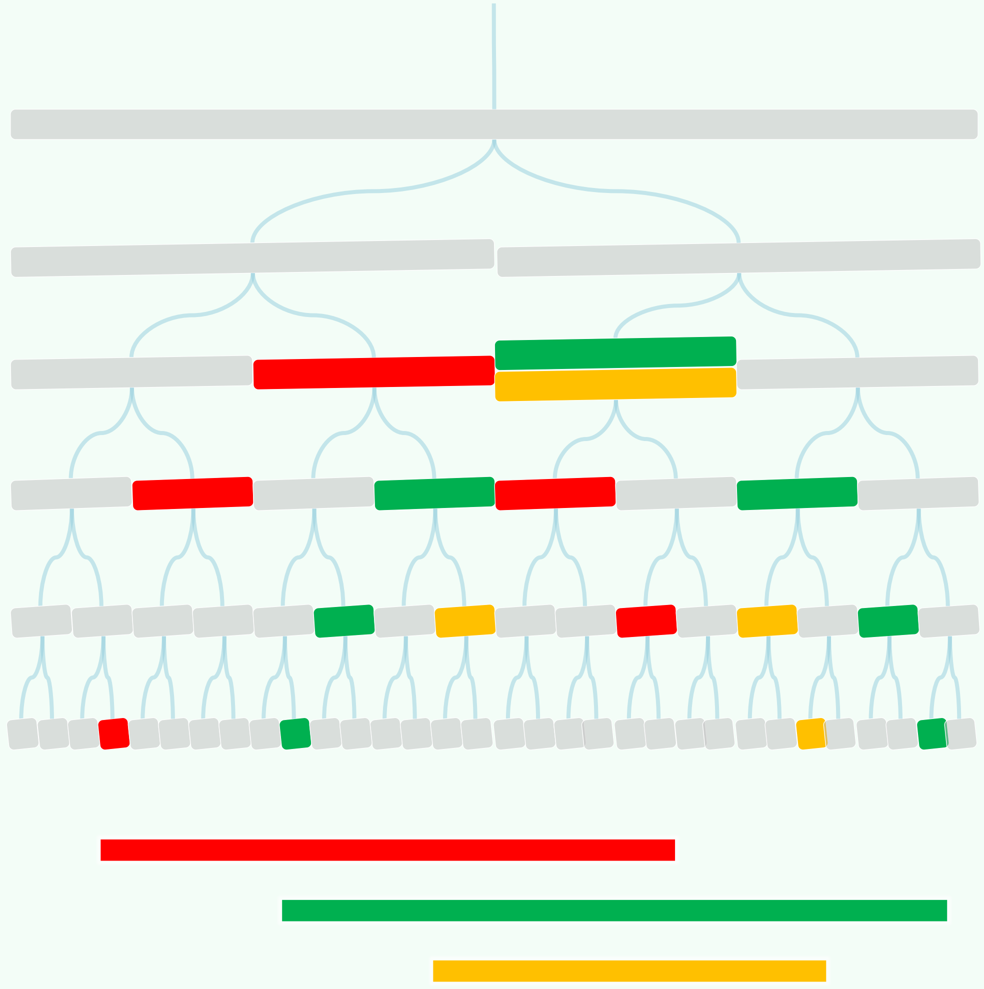


- At each level, ≤ 4 nodes are visited (2 stores + 2 recursions)

$\therefore \mathcal{O}(\log n)$ time

QuerySegmentTree(v , q_x)

```
// Find all intervals
// in the (sub)tree rooted at v
// that contain qx
report all the intervals in Int(v)
if ( v is a leaf )
   return
if ( qx ∈ Int( lc(v) ) )
   QuerySegmentTree( lc(v), qx )
else
   QuerySegmentTree( rc(v), qx )
```



$\mathcal{O}(r + \log n)$

- Only one node is visited per level,

  altogether $\mathcal{O}(\log n)$ nodes

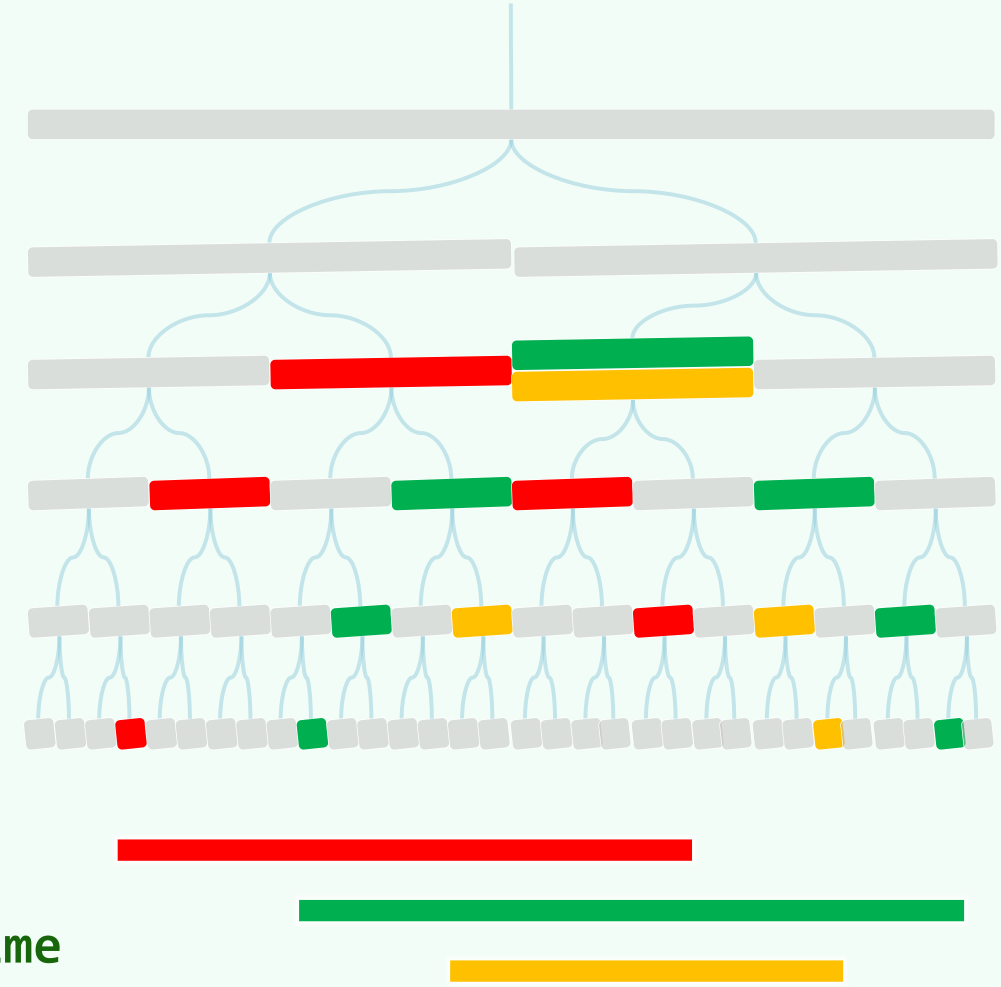
- At each node v

  - the CS Int(v) is reported

  - in time

    $$1 + |\mathrm{Int}(v)| = \mathcal{O}(1 + r_v)$$

$\therefore$ Reporting all the intervals costs $\mathcal{O}(r)$ time



Conclusion

❖ For a set of n intervals,

- a segment tree of size $\mathcal{O}(n\log n)$
- can be built in $\mathcal{O}(n\log n)$ time
- which reports all intervals

  containing a query point

  in $\mathcal{O}(r + \log n)$ time