# 高级搜索树

## 伸展树：分摊分析

10-A4

所谓物价，其实就是我称之为生命的那部分，必须在交换时支付：要么立即支付，要么以后支付

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# S的势能

❖ （任何时刻的）任何一棵伸展树$S$，都可以**假想地**被认为具有势能

$$\Phi(S) = \log\left(\prod_{v\in S} size(v)\right) = \sum_{v\in S}\log\left(size(v)\right) = \sum_{v\in S} rank(v)$$

❖ 直觉：越**平衡**/**倾侧**的树，势能越**小**/**大**

- 单链：$\Phi(S) = \log n! = \mathcal{O}(n\log n)$
- 满树：$\Phi(S) = \log\prod_{d=0}^{h}\left(2^{h-d+1}-1\right)^{2^d} \leq \log\prod_{d=0}^{h}\left(2^{h-d+1}\right)^{2^d}$

$$= \log\prod_{d=0}^{h}2^{(h-d+1)\cdot 2^d} = \sum_{d=0}^{h}(h-d+1)\cdot 2^d = (h+1)\cdot\sum_{d=0}^{h}2^d - \sum_{d=0}^{h}d\cdot 2^d$$

$$= (h+1)\cdot\left(2^{h+1}-1\right) - \left[(h-1)\cdot 2^{h+1} + 2\right] = 2^{h+2}-h-3 = \mathcal{O}(n)$$

# T的上界

- 考查对伸展树$S$ 的$m \gg n$次连续访问（不妨仅考查search(v)）
- 若记：$A^{(k)} = T^{(k)} + \Delta\Phi^{(k)},\ k = 0, 1, 2, \ldots, m$

  则有：$A - \mathcal{O}(n \log n) \leq T = A - \Delta\Phi \leq A$
- 故若能证明：$A = \mathcal{O}(m \log n)$

  则必有：$T = \mathcal{O}(m \log n)$
- 好消息是，尽管$T^{(k)}$ 的变化幅度可能很大

  我们却能证明，$A^{(k)}$ 都不致超过节点v的势能变化量，即

  $$\mathcal{O}(rank^{(k)}(v) - rank^{(k-1)}(v)) = \mathcal{O}(\log n)$$
- 事实上，每一个$A^{(k)}$ 都是若干次伸展操作（时间成本）的累积，这些操作无非**三种**情况...

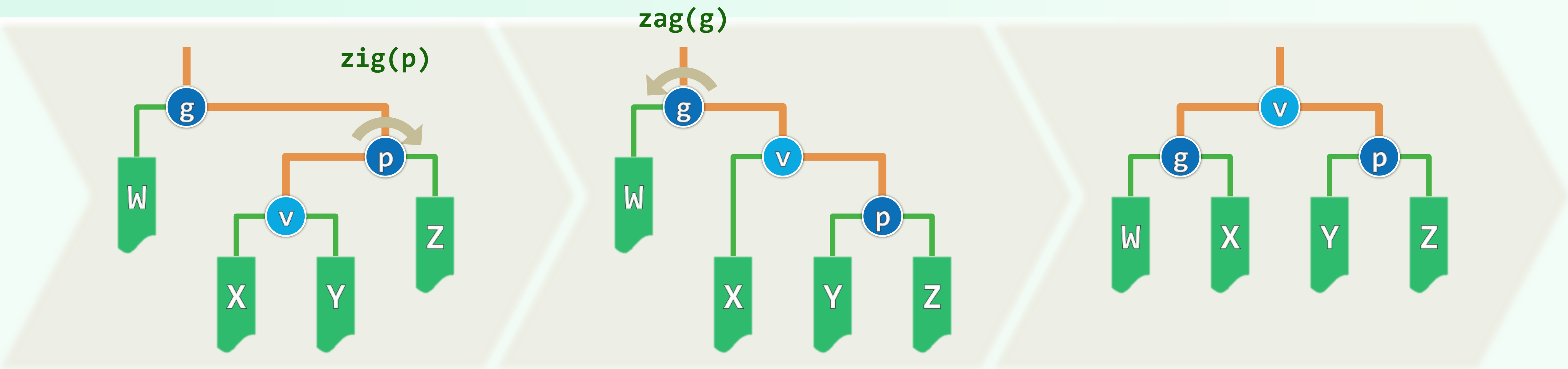
# Zig / Zag


$$A_i^{(k)} \;=\; T_i^{(k)} + \Delta\Phi(S_i^{(k)}) \;=\; 1 + \Delta rank_i(r) + \Delta rank_i(v)$$

$$=\; 1 + [\underline{rank_i(r) - rank_{i-1}(r)}] + [rank_i(v) - rank_{i-1}(v)]$$

$$\leq\; 1 + [rank_i(v) - rank_{i-1}(v)]$$

# zig-zag / zag-zig




$$A_i^{(k)} = T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 2 + \Delta rank_i(g) + \Delta rank_i(p) + \Delta rank_i(v)$$

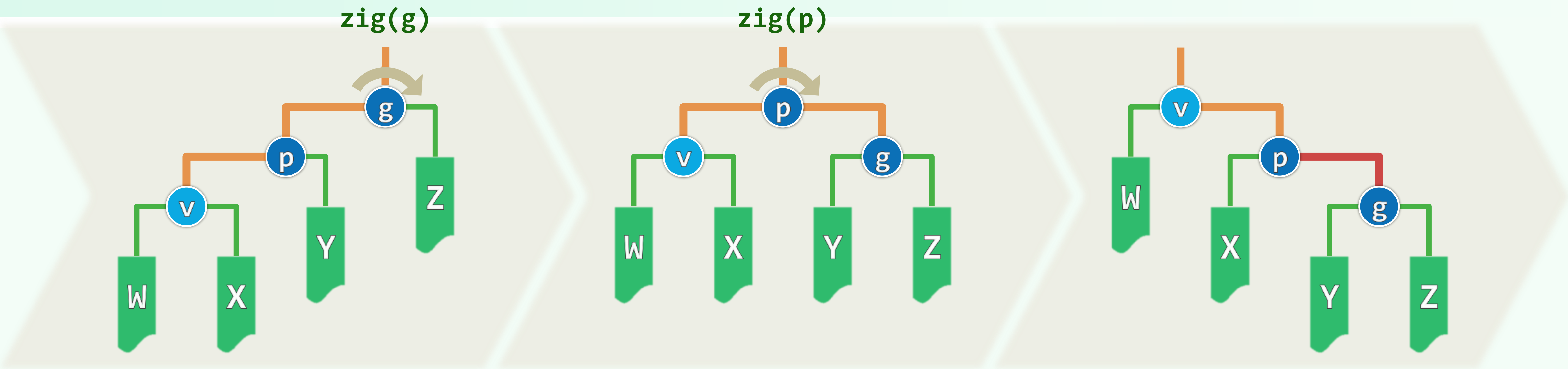
$$= 2 + [rank_i(g) - \cancel{rank_{i-1}(g)}] + [rank_i(p) - \underline{rank_{i-1}(p)}] + [\cancel{rank_i(v)} - \underline{rank_{i-1}(v)}]$$

$$\leq 2 + \underline{rank_i(g) + rank_i(p)} - 2 \cdot rank_{i-1}(v) \qquad (\text{since } rank_{i-1}(p) > rank_{i-1}(v))$$

$$\leq 2 + \underline{2 \cdot rank_i(v) - 2} - 2 \cdot rank_{i-1}(v) \qquad (\text{since } rank(v) = \log(size(v)) \text{ is concave})$$

$$= 2 \cdot (rank_i(v) - rank_{i-1}(v))$$

# zig-zig / zag-zag




$$
\begin{aligned}
A_i^{(k)} &= T_i^{(k)} + \Delta\Phi(S_i^{(k)}) \;=\; 2 + \Delta rank_i(g) + \Delta rank_i(p) + \Delta rank_i(v) \\
&= 2 + [rank_i(g) - \cancel{rank_{i-1}(g)}] + [rank_i(p) - \underline{rank_{i-1}(p)}] + [\cancel{rank_i(v)} - \underline{rank_{i-1}(v)}] \\
&\le 2 + rank_i(g) + \underline{rank_i(p)} - 2 \cdot rank_{i-1}(v) \qquad (\text{since } rank_{i-1}(p) > rank_{i-1}(v)) \\
&\le 2 + rank_i(g) + \underline{rank_i(v)} - 2 \cdot rank_{i-1}(v) \qquad (\text{since } rank_i(p) < rank_i(v)) \\
&\le 3 \cdot (rank_i(v) - rank_{i-1}(v)) \qquad (\text{since } rank_i(g) + rank_{i-1}(v) \le 2 \cdot rank_i(v) - 2)
\end{aligned}
$$