例题部分

极限

> **证明:**
>
> 1. 证明:$\lim_{x\to x_0} x = x_0$
> 2. 证明:$\lim_{x\to x_0} c = c$
> 3. 证明:$\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$
> 4. 证明:$\lim_{x\to\infty} \frac{1}{x} = 0$
> 5. 证明:$\lim_{x\to x_0} f(x) = \infty$为无穷大, $\lim_{x\to x_0} \frac{1}{f(x)} = 0$为无穷小
> 6. 证明:$\lim[f(x) \pm g(x)] = \lim f(x) \pm \lim g(x).$
> 7. 证明:$\lim[f(x) \cdot g(x)] = \lim f(x) \cdot \lim g(x).$
> 8. 证明:$\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}, (\lim g(x) \neq 0)$
> 9. 证明:$\lim_{x\to 0} \frac{\sin x}{x} = 1, \lim_{x\to 0} \cos x = 1$
> 10. 证明:$\lim_{x\to\infty} (1+\frac{1}{x})^x = e$
> 11. 证明:当$x \to 0,\quad \sin x\text{\textasciitilde}x,\ \tan x\text{\textasciitilde}x\ \ \arcsin x\text{\textasciitilde}x,\quad 1-\cos x\text{\textasciitilde}\frac{1}{2}x^2$
> 12. 证明:当$x \to 0, \sqrt[n]{1+x} - 1\text{\textasciitilde}\frac{1}{n}x$
> 13. 证明:当$x \to 0, \ln(1+x)\text{\textasciitilde}\ x, e^x - 1\text{\textasciitilde}\ x, (1+x)^\alpha - 1\text{\textasciitilde}\ \alpha x.$

> **极限:**
>
> 1. 求:$\lim_{x\to 0} \frac{\tan x}{x} = 1$
> 2. 求:$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
> 3. 求:$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$
> 4. 求:$\lim_{x\to\infty} (1-\frac{1}{x})^x = \frac{1}{e}$
> 5. 求:$\lim_{x\to 3} \sqrt{\frac{x-3}{x-9}} = \frac{\sqrt{6}}{6}.$
> 6. 求:$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\ln a}$
> 7. 求:$\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$
> 8. 求:$\lim_{x\to 0} \frac{(1+x)^\alpha-1}{x} = \alpha$
> 9. 求:$\lim_{x\to 0} (1+2x)^{\frac{3}{\sin x}} = e^6$

> **例题:**
>
> - 证明:$\lim_{x\to 1} (2x-1) = 1$
> - 证明:$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$
> - 求左右极限:
> $$f(x) = \begin{cases} x-1, & x<0 \\ 0, & x=0 \\ x+1, & x>0 \end{cases}, \to$$
> $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x-1) = -1, \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x-1) = 1$
> - 证明$\lim_{x\to 1} \frac{1}{x-1} = \infty$
> - 求:$\lim_{x\to 1} (2x-1) = 1$
> - 求:$\lim_{x\to 2} \frac{x^3-1}{x^2-5x+3} = -\frac{7}{3}$
> - 求:$\lim_{x\to 3} \frac{x-3}{x^2-9} = \frac{1}{6}$
> - 求:$\lim_{x\to 1} \frac{2x-3}{x^2-5x+4} = \infty$
> - 求:$\lim_{x\to\infty} \frac{3x^3+4x^2+2}{7x^3+5x^2-3} = \frac{3}{7}$

- 求:$\lim_{x\to\infty}\frac{3x^2-2x-1}{2x^3-x^2+5}=0$
- 求:$\lim_{x\to\infty}\frac{2x^3-x^2+5}{3x^2-2x-1}=\infty$
- 求:$\lim_{x\to\infty}\frac{\sin x}{x}=0$
- 求:$\lim_{x\to 0}\frac{\tan 2x}{\sin 5x}=\frac{2}{5}$
- 求:$\lim_{x\to 0}\frac{\sin x}{x^3+3x}=\frac{1}{3}$
- 求:$\lim_{x\to 0}\frac{(1+x^2)^{\frac{1}{3}}-1}{\cos x-1}=-\frac{2}{3}$.

导数

证明:

1. 证明:$[u(x)\pm v(x)]'=u'(x)\pm v'(x)$
2. 证明:$[u(x)v(x)]'=u'(x)v(x)+u(x)v'(x)$
3. 证明:$[(\frac{u(x)}{v(x)})]'=\frac{u'(x)v(x)-u(x)v'(x)}{v^2(x)}(v(x)\neq 0)$
4. 证明:$(Cu)'=Cu'$
5. 证明:$[f^{-1}(x)]'=\frac{1}{f'(y)},\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$.
6. 证明:$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dt}$
7. 证明:$(u\pm v)^{(n)}=u^{(n)}\pm v^{(n)}$
8. 证明:$(uv)^{(n)}=\sum_{k=0}^{n}C_n^k u^{(n-k)}v^{(k)}$.
9. 证明:$\begin{cases}x=\varphi(t)\\y=\psi(t)\end{cases},\rightarrow\frac{dy}{dx}=\frac{\psi'(t)}{\varphi'(t)}$.

导数:

1. 求:$f(x)=C\rightarrow(C)'=0$
2. 求:$f(x)=x^n(n\in N_+)\rightarrow(x^n)'=\begin{cases}1,n=1\\nx^{n-1},n>1\end{cases}$
3. 求:$f(x)=x^\mu(\mu\in R)\rightarrow(x^\mu)'=\mu x^{\mu-1}$
4. 求:$f(x)=\sin x\rightarrow(\sin x)'=\cos x,(\cos x)'=-\sin x$
5. 求:$f(x)=a^x(a>0,a\neq 1)\rightarrow(a^x)'=a^x\ln a,(e^x)'=e^x$
6. 求:$f(x)=\log_a x(a>0,a\neq 1)\rightarrow(\log_a x)'=\frac{1}{x\ln a},(\ln x)'=\frac{1}{x}$
7. 求:$y=\frac{1}{x}$,在$(\frac{1}{2},2)$处的切线斜率$y'=-\frac{1}{x^2}$
8. 求:$y=ax+b,\rightarrow y'=a,y''=0$
9. 求:$s=sin\omega t,\rightarrow s'=\omega cos\omega t,s''=-\omega^2\sin\omega t$.
10. 求:$y=e^x,\rightarrow(e^x)^{(n)}=e^x$
11. 求:
    $y=\sin x,\rightarrow y'=\cos x=\sin(\frac{x+\pi}{2}),(\sin x)^{(n)}=sin(x+n\cdot\frac{\pi}{2}),(\cos x)^{(n)}=cos(x+n\cdot\frac{\pi}{2})$
12. 求:$y=\ln(x+1),\rightarrow y'=\frac{1}{1+x},y''=-\frac{1}{(1+x)^2},y'''=\frac{1\cdot 2}{(1+x)^3},y^{(n)}=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$.
13. 求:
    $y=x^\mu,\rightarrow y'=\mu x^{\mu-1},y''=\mu(\mu-1)x^{\mu-2},y^{(n)}=\mu(\mu-1)(\mu-2)(\mu-3)\ldots(\mu-n+1)x^{\mu-n}$.
14. 求:$y=x^2e^{2x},\rightarrow$求$y^{(20)}\rightarrow$
    $u^{(k)}=2^ke^{2x}(k=1,2,3,\ldots),v'=2x,v''=2,v^{(k)}=0(k=3,4,\ldots)$
    $y^{(20)}=2^{20}e^{2x}(x^2+20x+95)$
15. 求:$e^y+xy-e=0\rightarrow\frac{dy}{dx}=-\frac{y}{x+e^y}$
16. 求:$y=x^{\sin x}(x>0),\rightarrow y'=x^{\sin x}(\cos x\cdot\ln x+\frac{\sin x}{x})$.
17. 求:$y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}},\rightarrow y'=\frac{y}{2}(\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4})$.
18. 求:$\begin{cases}x=a\cos t\\y=b\sin t\end{cases},\rightarrow\frac{dy}{dx}=-\frac{b}{a}$.

19. 求:$\begin{cases} x = v_1 t \\ y = v_2 - \frac{1}{2}gt^2 \end{cases}, \rightarrow \frac{dy}{dx} = \frac{v_2 - gt}{v_1}.$
20. 求:$\begin{cases} x = a(t - \sin t) \\ y = b(1 - \cos t) \end{cases}, \rightarrow \frac{dy}{dx} = \cot\frac{t}{2}, \rightarrow \frac{d^2y}{dx^2} = -\frac{1}{a(1-\cos t)^2}.$
21. 求:$\tan\alpha = \frac{h}{500}$, 对$t$求导, $\rightarrow \sec^2\alpha \cdot \frac{d\alpha}{dt} = \frac{1}{500} \cdot \frac{dh}{dt}.$

例题:

- 求:$f(x) = |x|$在$x = 0$处的导数.$\begin{cases} \lim_{h\to 0^-}\frac{|h|}{h} = -1 \\ \lim_{h\to 0^+}\frac{|h|}{h} = 1 \end{cases}$左右导数不相等,故在$x = 0$处不可导.
- 求:$f(x) = \sqrt[3]{x}$在区间$(-\infty, +\infty)$内连续,但在,点$x = 0$处不可导(或者称导数为无穷大其实为几何图形中垂直于x轴的切线$x = 0$),$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}} = +\infty.$
- 求:$y = 2x^3 - 5x^2 + 3x - 7 \rightarrow y' = 6x^2 - 10x + 3$
- 求:$y = x^3 + 4\cos x - sin\frac{\pi}{2} \rightarrow y' = 3x^2 - 4\sin x$
- 求:$y = e^x(\sin x + \cos x) \rightarrow y' = 2e^x\cos x$
- 求:$y = \tan x \rightarrow y' = \sec^2 x$
- 求:$y = \sec x \rightarrow y' = \sec x \tan x$
- 求:$(\cot x)' = -\csc^2 x, (\csc x)' = -\csc x \cot x$
- 求:$y = \arcsin x \rightarrow y' = \frac{1}{\sqrt{(1-x^2)}}$
- 求:$y = \arccos x \rightarrow y' = -\frac{1}{\sqrt{(1-x^2)}}$
- 求:$y = \arctan x \rightarrow y' = \frac{1}{1+x^2}$
- 求:$y = arccotx \rightarrow y' = -\frac{1}{1+x^2}$
- 求:$y = e^{x^3} \rightarrow \frac{dy}{dx} = 3x^2e^{x^3}$
- 求:$y = \sin\frac{2x}{1+x^2} \rightarrow \frac{dy}{dx} = \frac{2(1-x^2)}{(1+x^2)^2} \cdot \cos\frac{2x}{1+x^2}$
- 求:$y = \ln\sin x \rightarrow \frac{dy}{dx} = \cot x$
- 求:$y = \sqrt[3]{1-2x^2} \rightarrow \frac{dy}{dx} = \frac{-4x}{3\sqrt[3]{(1-2x^2)^2}}$
- 求:$y = \ln\cos(e^x) \rightarrow \frac{dy}{dx} = -e^x\tan(e^x)$
- 求:$y = e^{\sin\frac{1}{x}} \rightarrow \frac{dy}{dx} = -\frac{1}{x^2}e^{\sin\frac{1}{x}} \cdot \cos\frac{1}{x}.$
- 求:$y = \sin nx \cdot \sin^n x \rightarrow \frac{dy}{dx} = n\sin^{n-1}x \cdot \sin(n+1)x$
- 求:$(shx)' = chx, (chx)' = shx, (thx)' = \frac{1}{ch^2x}$
- 求:$(arshx)' = \frac{1}{\sqrt{1+x^2}}, (archx)' = \frac{1}{\sqrt{x^2-1}}, (arthx)' = \frac{1}{1-x^2}$
- 求:$y^5 + 2y - x - 3x^7 = 0, \rightarrow \frac{dy}{dx} = \frac{1+21x^6}{5y^4+2}$
- 求:$\frac{x^2}{16} + \frac{y^2}{9} = 1, \rightarrow \frac{dy}{dx} = -\frac{9x}{16y}$
- 求:$x - y + \frac{1}{2}\sin y = 0, \rightarrow \frac{dy}{dx} = \frac{2}{2-\cos y}, \rightarrow \frac{d^2y}{dx^2} = \frac{-4\sin y}{(2-\cos y)^3}$