Homework 9

MATH2534 CRN:15708

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1. Prove by set identities (or by element method) or disprove the following:

   (a) If $B \subseteq C$, then $(A - B) \subseteq (A - C)$
   $A = \{1, 2, 3, 4\}, B = \{2, 3\}, C = \{2, 3, 4\}$
   $A - B = \{1, 4\}$
   $A - C = \{1\}$
   $\{1, 4\} \nsubseteq \{1\}$

   (b) $(A \cup B) - B = A$
   $A = \{1, 2, 3, 4\}, B = \{4, 5, 6\}$
   $A \cup B = \{1, 2, 3, 4, 5, 6\}$
   $(A \cup B) - B = \{1, 2, 3\}$
   $(A \cup B) - B \neq A$

   (c) $A \cup (B - C) = (A \cup B) - (A \cup C)$
   $A = \{1, 2, 3\}, B = \{2, 3, 4\}, C = \{2, 3\}$
   $B - C = \{4\}$
   $A \cup (B - C) = \{1, 2, 3, 4\}$
   $A \cup B = \{1, 2, 3, 4\}$
   $A \cup C = \{1, 2, 3\}$
   $(A \cup B) - (A \cup C) = \{4\}$
   $A \cup (B - C) \neq (A \cup B) - (A \cup C)$

2. Simplify $(A \cap (B \cup C)) \cap (A - B) \cap (B \cup C^{\mathcal{C}})$
   $(A \cap (B \cup C)) \cap (A - B) \cap (B \cup C^{\mathcal{C}}) = A \cap (B \cup C) \cap A \cap B^{\mathcal{C}} \cap (B \cup C^{\mathcal{C}})$
   By set difference definition and associative laws
   $A \cap (B \cup C) \cap A \cap B^{\mathcal{C}} \cap (B \cup C^{\mathcal{C}}) = (B \cup C) \cap A \cap B^{\mathcal{C}} \cap (B \cup C^{\mathcal{C}})$
   By idempotent laws
   $(B \cup C) \cap A \cap B^{\mathcal{C}} \cap (B \cup C^{\mathcal{C}}) = A \cap B^{\mathcal{C}} \cap (B \cup (C \cap C^{\mathcal{C}}))$
   by distributive laws
   $A \cap B^{\mathcal{C}} \cap (B \cup (C \cap C^{\mathcal{C}})) = A \cap B^{\mathcal{C}} \cap B$
   by complement laws and universal bounds laws
   $A \cap B^{\mathcal{C}} \cap B = A \cap \emptyset$
   by complement laws
   $A \cap \emptyset = \emptyset$
   by universal bounds laws
   $(A \cap (B \cup C)) \cap (A - B) \cap (B \cup C^{\mathcal{C}}) = \emptyset$

3. Prove that for all sets A and B, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

   Suppose that $\forall x$ if $x \in \mathcal{P}(A \cap B)$, then $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$

   $x \subseteq A \cap B$ by definition of a power set

   $x \subseteq A, x \subseteq B$ by definition of an intersection

   $x \in \mathcal{P}(A), x \in \mathcal{P}(B)$ by definition of a power set

   $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$ by definition of an interesection

   Suppose that $\forall x$ if $\mathcal{P}(A) \cap \mathcal{P}(B)$, then $x \in \mathcal{P}(A \cap B)$

   $x \in \mathcal{P}(A), x \in \mathcal{P}(B)$ by definition of an intersection

   $x \subseteq A, x \subseteq B$ by definition of a power set

   $x \subseteq A \cap B$ by definition of an intersection

   $x \in \mathcal{P}(A \cap B)$ by definition of a power set

   Therefore as $x \in \mathcal{P}(A \cap B)$ and $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

4. Let A, B, and C be sets. Prove that $(A - B) - (B - C) = A - B$ with double set inclusion.

   Let $\exists x \in U$

   Suppose that $\forall x$ if $x \in (A - B) - (B - C)$, then $x \in A - B$.

   $x \in A, x \notin B, x \not in(B - C)$ by set difference definition

   $x \in A, x \notin B$ by absorption rule

   $x \in (A - B)$ by set difference definition

   Suppose that $\forall x$ if $x \in A - B$, then $x \in (A - B) - (B - C)$.

   $x \in A, x \notin B$ by set difference definition

   $x \in A, x \notin B, x \notin (B - C)$ by absorption rule

   $x \in (A - B), x \notin (B - C)$ by set difference definition

   $x \in (A - B) - (B - C)$ by set difference definition

   Therefore as $x \in (A - B)$ and $x \in (A - B) - (B - C)$, $(A - B) - (B - C) = A - B$.

5. Consider the set A of all divisors of 70, $A = \{1, 2, 5, 7, 10, 14, 35, 70\}$, with the operations defined as follows: $a + b = \text{GCD}(a, b)$, $a \cdot b = \text{LCM}(a, b)$ and the complement is defined to be $a = 70/a$. Assume that A is a Boolean algebra with operations $+$ and $\cdot$.

   (a) Evaluate $14 \cdot 35$, $5 + \overline{5}$, and $14 \cdot \overline{14}$

   $14 \cdot 35 = \text{LCM}(14, 35) = 70$

   $5 + \overline{5} = 5 + 70/5 = 5 + 14 = \text{GCD}(5, 14) = 1$

   $14 \cdot \overline{14} = 14 \cdot 70/14 = 14 \cdot 5 = \text{LCM}(14, 5) = 70$

   (b) Show $2 + (7 \cdot 14) = (2 + 7) \cdot (2 + 14)$

   $2 + (7 \cdot 14) = 2 + \text{LCM}(7, 14) = 2 + 14 = \text{GCD}(2, 14) = 2$

   $(2 + 7) \cdot (2 + 14) = \text{GCD}(2, 7) \cdot \text{GCD}(2, 14) = \text{LCM}(1, 2) = 2$

   $2 = 2$

   $2 + (7 \cdot 14) = (2 + 7) \cdot (2 + 14)$

   (c) Show $\overline{10 + 35} = \overline{10} \cdot \overline{35}$

   $\overline{10 + 35} = 70/\text{GCD}(10, 35) = 70/5 = 14$

   $\overline{10} \cdot \overline{35} = 70/10 \cdot 70/35 = 7 \cdot 2 = \text{LCM}(7, 2) = 14$

   $14 = 14$

   $\overline{10 + 35} = \overline{10} \cdot \overline{35}$

   (d) Find the identities 0 and 1 for operations $+$ and $\cdot$, respectively. Justify your answer.

   $a + 0 = \text{GCD}(a, 0) = a$ 0 is divisible by all integers

   $a + 1 = \text{GCD}(a, 1) = 1$ 1 is only divisible by 1

   $a \cdot 0 = \text{LCM}(a, 0) = 0$ 0 multiplied by anything is 0

   $a \cdot 1 = \text{LCM}(a, 1) = a$ The GCD of any number, a and one is a.

   (e) Verify $a + \overline{a} = 1$ and $a \cdot \overline{a} = 0$ for all a in A.

   $a + \overline{a} = a + 70/a = \text{GCD}(a, 70/a) = 1$ $70/a$ will never be divisible by a and either a or $\overline{a}$ will be prime.

   $a \cdot \overline{a} = \text{LCM}(a, 70/a) = 70$ The LCM of a and $70/a$ will always be 70 as the $\frac{1}{a}$ must always be canceled out first.

6. Assume that B is a Boolean algebra with operations $+$ and $\cdot$. For all a and b in B, prove the absorption law $(a + b) \cdot a = a$.

   $(a + b) \cdot a = (a + T) \cdot (a + b)$ by identity law

   $(a + T) \cdot (a + b) = a \cdot (1 + b)$ by distributive law

   $a \cdot (1 + b) = a \cdot 1$ by universal bounds law

   $a \cdot 1 = a$ by identity law

   $(a + b) \cdot a = a$