Homework 7

MATH2534 CRN:15708

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March 26, 2018

Prove the following statements using strong mathematical induction.

1. Let $a \neq b$ be two distinct integers, and let $n \geq 1$ be any integer. Prove that $a^n - b^n$ is divisible by $a - b$.

   Let $\forall a, b, n \in \mathbb{Z}, a \neq b, n \geq 1$.

   $P(1): \ a^1 - b^1 = a - b \implies (a-b)|(a^1 - b^1)$

   $P(2): \ a^2 - b^2 = (a-b)(a+b) \implies (a-b)|(a^2 - b^2)$

   $P(3): \ a^3 - b^3 = (a-b)(a+b)(a^2 - b^2) \implies (a-b)|(a^3 - b^3)$

   Suppose this holds true for $P(n): \ (a-b)|(a^n - b^n)$.

   $P(n+1): \ (a-b)|(a^{n+1} - b^{n+1})$

   $a^{n+1} - b^{n+1} = a \times a^n - b \times b^n = aa^n - ab^n + ab^n - bb^n = (a^n - b^n)a + (a-b)b^n$

   $a^n - b^n$ and $a - b$ are both divisible by $a - b$. This means that $a^{n+1} - b^{n+1}$ is divisible by $a - b$.

   Therefore by induction, $a^n - b^n$ is divisible by $a - b$ for all $n \geq 1$.

2. Let $a_1, a_2, a_3, ...$ be a sequence such that $a_1 = 1, a_2 = 1$ and $a_k = 2a_{k-1} + 3a_{k-2}$ for $k \geq 3$. Prove that $a_n \leq 2 \times 3^{n-2}$ for all $n \geq 3$.

   Let $\forall k, n, \exists a \in \mathbb{Z}, n \geq 3, k \geq 3, a_k = 2a_{k-1} + 3a_{k-2}$.

   $P(1): \ a_1 = 1 \leq 2 \times 3^{1-2} = \frac{2}{3}$

   $P(2): \ a_2 = 1 \leq 2 \times 3^{2-2} = 2$

   $P(3): \ a_3 = 2a_{3-1} + 3a_{3-2} = 2a_2 + 3a_1 = 2 + 3 = 5 \leq 2 \times 3^{3-2} = 6 \implies 5 \leq 6$

   Suppose this holds true for $P(n): \ a_n \leq 2 \times 3^{n-2}$.

   $P(n+1): \ a_{n+1} \leq 2 \times 3^{n+1-2}$

   $a_{n+1} = 2a_{n+1-1} + 3a_{n+1-2} = 2a_n + 3a_{n-1}$

   $2 \times 3^{n+1-2} = 2 \times 3^{n-1}$

   $2a_n + 3a_{n-1} \leq 2 \times 3^{n-1}$

   As $a_n \leq 2 \times 3^{n-2}$, $a_{n-1} \leq 2 \times 3^{n-3}$

   $2a_n + 3a_{n-1} \leq 2 \times 2 \times 3^{n-2} + 3 \times 2 \times 3^{n-3} = 4 \times 3^{n-2} + 2 \times 3^{n-2} = 6 \times 3^{n-2} = 2 \times 3^{n-1}$

   $2 \times 3^{n-1} \leq 2 \times 3^{n-1}$

   Therefore by induction, $a_n \leq 2 \times 3^{n-2}$ for all $n \geq 3$.

3. Let $a_1, a_2, a_3, ...$ be a sequence such that $a_1 = 1, a_2 = 8$ and $a_k = a_{k-1} + 2a_{k-2}$ for $k \geq 3$. Prove that $a_n = 3 \times 2^{n-1} + 2(-1)^n$ for all $n \geq 1$.

   Let $\forall k, n, \exists a \in \mathbb{Z}, n \geq 1, k \geq 3, a_k = a_{k-1} + 2a_{k-2}$.

   $P(1):\ a_1 = 1 = 3 \times 2^{1-1} + 2(-1)^1 = 3 - 2 = 1$

   $P(2):\ a_2 = 8 = 3 \times 2^{2-1} + 2(-1)^2 = 6 + 2 = 8$

   $P(3):\ a_3 = a_{3-1} + 2a_{3-2} = a_2 + 2a_1 = 8 + 2 = 10 = 3 \times 2^{3-1} + 2(-1)^3 = 12 - 2 = 10$

   Suppose this holds true for $P(n):\ a_n = 3 \times 2^{n-1} + 2(-1)^n$.

   $P(n+1):\ a_{n+1} = 3 \times 2^{n+1-1} + 2(-1)^{n-1} = 3 \times 2^n - 2(-1)^n$

   $a_{n+1} = a_{n+1-1} + 2a_{n+1-2} = a_n + 2a_{n-1}$

   As $a_n = 3 \times 2^{n-1} + 2(-1)^n$, $a_{n-1} = 3 \times 2^{n-2} + 2(-1)^{n-1}$

   $a_{n+1} = (3 \times 2^{n-1} + 2(-1)^n) + 2(3 \times 2^{n-2} + 2(-1)^{n-1}) = 3 \times 2^n - 2(-1)^n$

   $a_{n+1} = 3 \times 2^n - 2(-1)^n = 3 \times 2^n - 2(-1)^n$

   Therefore by induction, $P(n):\ a_n = 3 \times 2^{n-1} + 2(-1)^n$ for all $n \geq 1$

4. Let $a_1, a_2, a_3, ...$ be a sequence such that $a_1 = 1, a_2 = 1, a_3 = 3$ and $a_k = a_{k-1} + a_{k-2} + a_{k-3}$ for $k \geq 4$. Prove that $a_n < 2^n$ for all $n \geq 1$.

   Let $\forall k, n, \exists a \in \mathbb{Z}, n \geq 1, k \geq 4, a_k = a_{k-1} + a_{k-2} + a_{k-3}$.

   $P(1):\ a_1 = 1 < 2^1 = 2$

   $P(2):\ a_2 = 1 < 2^2 = 4$

   $P(3):\ a_3 = 3 < 2^3 = 8$

   $P(4):\ a_4 = a_{4-1} + a_{4-2} + a_{4-3} = 8 + 4 + 2 = 14 < 2^4 = 16$

   Suppose this holds true for $P(n):\ a_n < 2^n$.

   $P(n+1):\ a_{n+1} < 2^{n+1}$

   $a_{n+1} = a_{n+1-1} + a_{n+1-2} + a_{n+1-3} = a_n + a_{n-1} + a_{n-2}$

   As $a_n < 2^n$, $a_{n-1} < 2^{n-1}$ and $a_{n-2} < 2^{n-2}$

   $a_{n+1} < 2^n + 2^{n-1} + 2^{n-2} = \frac{7}{4}2^n = 7 \times 2^{n-2} < 2^{n+1}$

   $7 \times 2^{n-2} < 2^{n+1}$

   Therefore by induction, $P(n):\ a_n < 2^n$ for all $n \geq 1$.

5. Prove that every amount of postage that is at least 44¢ can be made from 5¢ and 12¢ stamps.

   Let $\forall n, \exists x, y \in \mathbb{Z}, n \geq 44$.

   $P(44):\ 44 = 5 \times 4 + 12 \times 2$

   $P(45):\ 45 = 5 \times 9 + 12 \times 0$

   $P(46):\ 46 = 5 \times 2 + 12 \times 3$

   $P(47):\ 47 = 5 \times 7 + 12 \times 1$

   $P(48):\ 48 = 5 \times 0 + 12 \times 4$

   Suppose this holds true for $P(n):\ n = 5x + 12y$.

   $P(n+1):\ n + 1 = 5x + 12y$

   Because we can have shown that 48¢ postage can be made from 5¢ and 12¢ stamps, we can assume that $n + 1 \geq 49$

   $44 \leq (n+1) - 5 \leq n \implies 44 \leq n - 4 \implies 15 \leq n$

   $(n+1) - 5 = 5x + 12y \implies n + 1 = 5(x+1) + 12y$

   $n + 1 = 5(x+1) + 12y, x + 1 > 0, y \geq 0$

   Therefore by induction, every amount of postage that is at least 44¢ can be made from 5¢ and 12¢ stamps.