Homework 4

MATH2534 CRN:15708

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February 28, 2018

Prove the following using definitions given in class.

1. The square of any rational number subtracted from 5 is a rational number.

   Let $\forall n \in \mathbb{Q}, \exists a, b \in \mathbb{Z}, b \neq 0$.

   As n is rational it is the quotient of some a and b.

$$n = \frac{a}{b} \tag{1}$$

   n^2 is a composite of n and n and therefore is a composite of $\frac{a}{b}$ and $\frac{a}{b}$

$$n^2 = n \times n = \frac{a}{b} \times \frac{a}{b} \tag{2}$$

   By simplifying, it can be shown that n^2 is still a rational number.

$$n^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2} \tag{3}$$

   As 5 is an integer and therefore a rational number and $\frac{b^2}{b^2}$ is equivalent to 1, $5 \times \frac{b^2}{b^2}$ is still rational. It is now shown that the fully simplified form is solely composed of a sum of integers divided by integers.

$$5 - n^2 = 5 - \frac{a^2}{b^2} = \frac{b^2}{b^2} \times 5 - \frac{a^2}{b^2} = \frac{5b^2 - a^2}{b^2} \tag{4}$$

   Therefore by the definition of a rational number, $n^2 - 5$ is rational.

2. If x is any nonzero rational number and y is any rational number, then $\frac{5y}{x}$ is rational.

   Let $\forall x, y \in \mathbb{Q}, \exists a, b, c, d \in \mathbb{Z}, x \neq 0, a \neq 0, b \neq 0, d \neq 0$

   As x and y are rational, they are equivalent to $\frac{a}{b}$ and $\frac{c}{d}$ respectively where a, b, and d are not 0.

$$x = \frac{a}{b}, y = \frac{c}{d} \tag{5}$$

   As $y \times 5$ is equivalent to y added to itself 5 times, $5y$ is still rational.

$$5y = y + y + y + y + y \tag{6}$$

   As $5bc$ and ad are composites of integers, they are still integers. Additionally ad is non zero due to the zero product property.

$$\frac{5y}{x} = \frac{5\frac{c}{d}}{\frac{a}{b}} = \frac{5bc}{ad} \tag{7}$$

   Therefore $\frac{5y}{x}$ is rational.

3. For all integers a, b, and c, if $a|b$ and $a|c$, then $a|(5b-7c)$.

   Let $\forall a, b, c \in \mathbb{Z}, \exists x, y \in \mathbb{Z}$

   As $a|b$, b is a composite of a and some other integer x. Similarly $a|c$, c is a composite of a and some other integer y.

$$5b - 7c = 5ax - 7ay \tag{8}$$

   By factoring the expression, we can show that $5b - 7c$ is a composite of a and $5x - 7y$.

$$5b - 7c = 5ax - 7ay = (a)(5x - 7y) \tag{9}$$

   As $5b - 7c$ is a composite containing a, it is divisible by a.

$$\frac{5b - 7c}{a} = \frac{(a)(5x - 7y)}{a} = 5x - 7y \tag{10}$$

   Therefore $a|(5b - 7c)$.