Use of Convergence to Types

Michael P. Mersic

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1 Deriving the Gumbel Distribution

In the last paper [3] there were two missing steps to deriving the Gumbel Distribution. That is, how to prove Max-Stable Distributions are equivalent to Extreme Distributions and how to prove that the Limit Distribution of an Extreme Distriubtion is an Extreme Distribution.

Here is a quick reminder of what is meant by Extremal and Max-Stable Distributions:

Define: Extremal	Define: Max-Stable
$(F(c_n x + d_n))^n \to G(x)$	$(G(a_n x + b_n))^n \to G(x)$

In order to prove the intended statements we need to use the *Convergence to Types Theorem*. Before defining the *Convergence to Types Theorem* it is necessary to define what is meant by *Type*:

Definition 1. Two distribution functions $U(x)$ and $V(x)$ are of the same **type** if there exist constants $A \in \mathbb{R}^+$ and $B \in \mathbb{R}$ such that

$$V(x) = U(Ax + B)$$

The *Convergence to Types Theorem* comes from Aleksandr Khinchin work in probability theory. Khinchin was a student of Andrei Kolmogorov. One of Kolmogorov's other students, Boris Gnedenko gave the first rigorous derivation of the Extremal Distributions [1]. The *Convergence to Types Theorem* states:

Convergence to Types Theorem. *Suppose $U(x)$ and $V(x)$ are two non-degenerate distributions. Suppose for $n \geq 0$ that X_n are random variables with distribution function F_n and U, V are random variables with distribution functions U(x), V(x). We have constants $a_n \in \mathbb{R}^+$, $\alpha_n \in \mathbb{R}^+$, $b_n \in \mathbb{R}$, $\beta_n \in \mathbb{R}$.*

If

$$F_n(\alpha_n x + \beta_n) \xrightarrow{d} U(x), \qquad F_n(a_n x + b_n) \xrightarrow{d} V(x) \tag{1}$$

then there exist constants $A \in \mathbb{R}^+$, and $B \in \mathbb{R}$ such that,

$$\lim_{n\to\infty} \frac{a_n}{\alpha_n} \to A, \qquad \lim_{n\to\infty} \frac{b_n - \beta_n}{\alpha_n} \to B \tag{2}$$

and

$$V(x) = U(Ax + B) \tag{3}$$

In another section below I will introduce the proof to this theorem, however, I saved the full proof to the final paper. Now we are ready to prove that Max-Stable $\Leftrightarrow$ Extremal and that the Limit Distribution of an Extremal Distribution is also an Extremal Distribution.

2 Prove Max-Stable $\Leftrightarrow$ Extremal

It is important to know that Max-Stable distribution $\Leftrightarrow$ Extremal distribution because it tells us more about the Limit Distribution of F, specifically that it is Max-Stable. In the previous paper, we then used the fact that the Limit Distribution of F is Max-Stable to derive the Limit Distribution (the Extremal Distributions).

Theorem. *Extremal distributions and Max-Stable distributions are equivalent.*

Proof. We know Max-Stable distributions are Extremal automatically from the definitions of Max-Stable and Extremal. So we only need to show that Extremal distributions are Max-Stable.

So we have an Extremal distribution,

$$(F(c_n x + d_n))^n \to G(x) \tag{4}$$

then, for every $k \in \mathbb{N}$ we have each side raised to the k^{th} power.

In the first equation F and G are raised to the k^{th} power. In the second equation F is raised to the nk^{th} power and new constants c_{nk} and d_{nk} are chosen so that F^{nk} will still converge to G:

$$(F(c_n x + d_n))^{nk} \to G(x)^k = U(x) \quad and \quad (F(c_{nk}x + d_{nk}))^{nk} \to G(x) = V(x)$$

Now, let $U(x) = G(x)^k$ and $V(x) = G(x)$ like the $U(x)$ and $V(x)$ in the **Convergence to Types** thereom. Then, by the **Convergence to Types** thereom, there exist $c_k \in \mathbb{R}^+$ and $d_k \in \mathbb{R}$ such that,

$$\frac{c_{nk}}{c_n} = c_k \qquad and \qquad \frac{d_{nk} - d_n}{c_n} = d_k \qquad as \qquad n \to \infty \tag{5}$$

and,

$$V(x) = U(Ax + B), \qquad or \qquad G(c_k x + d_k)^k = G(x) \tag{6}$$

That is, Extremal distributions are Max-Stable. □

3 Limit Distributions of Extremal Distributions are also Extremal Distributions

The proof of $G^k(x) = G(Ax + B)$ follows almost precisely the same path as that of Max-Stable ⇔ Extremal. The difference is that we now choose $U(x)$ to be $G(x)$ and $V(x)$ to be $G^k(x)$.

Lemma. *If G is an extremal distribution, then $G^k(x) = G(Ax + B)$.*

Proof.

$$(F(c_{nk}x + d_{nk}))^{nk} \to G(x) = U(x) \quad (F(c_n x + d_n))^{nk} \to G^k(x) = V(x) \tag{7}$$

therefore by Convergence to Types Theorem there exist A and B such that

$$V(x) = U(Ax + B) \qquad and \qquad G^k(x) = G(Ax + B) \tag{8}$$

□

4 Introduction to the proof of the Convergence to Types Theorem

I will essentially be presenting Allan Gut's presentation of the proof of the **Convergence to Types Theorem** in [2].

Intro to the proof: Since the general case follows from the particular one by rescaling, it suffices to prove the latter. By general case, I mean the definition of Convergence to Types given above. By particular case, I mean: If

$$F_n(x) \xrightarrow{d} U(x), \qquad F_n(a_n x + b_n) \xrightarrow{d} V(x) \tag{9}$$

then there exist constants $A \in \mathbb{R}^+$, and $B \in \mathbb{R}$ such that,

$$\lim_{n\to\infty} a_n \to A, \qquad \lim_{n\to\infty} b_n \to B \tag{10}$$

and

$$V(x) = U(Ax + B) \tag{11}$$

The proof proceeds via the following steps:

1. $X_n \xrightarrow{d} U, a_n \to a > 0, b_n \to b \Rightarrow V_n \xrightarrow{d} \frac{X-b}{a}$ as $n \to \infty$
2. $X_n \xrightarrow{d} U, a_n \to +\infty \Rightarrow \frac{X_n}{a_n} \xrightarrow{p} 0$
3. $X_n \xrightarrow{d} U, \sup_n |b_n| = \infty \Rightarrow X_n - b_n \overset{d}{\nrightarrow}$ as $n \to \infty$.
4. $X_n \xrightarrow{d} U, V_n \xrightarrow{d} V \Rightarrow 0 < \inf_n a_n \leq \sup_n a_n < \infty, \sup_n |b_n| < \infty$
5. $U \overset{d}{=} \frac{U-b}{a} \Rightarrow a = 1, b = 0.$

1, 2, 3 are proved directly. 4 is proved using 1, 2, and 3. 5 is proved using 4. Once these five parts are proved, we can say that sequences of a_n and b_n for A and B exist, and we just need to prove they are unique. Using part 5 we can show there are not other sequences a_j, b_j for A^* and B^* where $A \neq A^*$ and/or $B \neq B^*$.

References

[1] B. Gnedenko. Sur la distribution limite du terme maximum d'une serie aleatoire. *Ann. Math.*, 44:423–453, 1943, Translated and reprinted in: Breakthroughs in Statistics, Vol. I, 1992, eds. S. Kotz and N.L. Johnson, Springer-Verlag, pp. 195-225.

[2] Allan Gut. *Probability: A Graduate Course*. Springer, Springer Science+Business Media, inc., 233 Spring Street, New York, NY 10013, USA, 2005.

[3] Michael Mersic. Deriving the gumbel distribution. University of Villanova Graduate Seminar, April 2007.