Use of Convergence to Types

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May 9, 2007

1 Introduction

In this paper I will restate what the Convergence to Types Theorem (from here on abbreviated CtoTT) is used for, in the context of Extreme Value Theory. I will introduce two new concepts that are used in the proof of the CtoTT: Convergence in Distribution, and the Continuous Mapping Theorem (abbreviated CMT). Finally, I will present a proof of the CtoTT.

2 Uses of Convergence to Types Theorem

In the previous paper the Convergence to Types Theorem was used to prove two results [4]. These two results were important parts of the derivation of the Extremal Distribution where $\gamma = 0$ in the second paper [3].

1. Extremal $\Rightarrow$ Max-Stable

   Proof.

   $(F(c_n x + d_n))^{nk} \to G^k(x) = U(x) \quad (F(c_{nk} x + d_{nk}))^{nk} \to G(x) = V(x)$

   therefore by Convergence to Types Theorem there exist A and B such that
   $$G(x) = G^k(Ax + B)$$
   $\square$

2. $G^k(x) = G(Ax + B)$

   Proof.

   $(F(c_{nk}x + d_{nk}))^{nk} \to G(x) = U(x) \quad (F(c_n x + d_n))^{nk} \to G^k(x) = V(x)$

   therefore by Convergence to Types Theorem there exist A and B such that

   $$G^k(x) = G(Ax + B)$$

   □

Before getting into the proof of the CtoTT, I need to introduce Convergence in Distribution and the Continuous Mapping Theorem.

3 Convergence in Distribution

Suppose that $F_1, F_2, ...$ is a sequence of cumulative distriubtion functions corresponding to the random variables $X_1, X_2, ...$, and that F is a distribution function corresponding to a random variable X. We say that the sequence X_n converges towards X in distribution, if

$$\lim_{n\to\infty} F_n(a) = F(a)$$

for every real number a at which F is continuous. In this paper, Convergence in Distribution is notated $X_n \xrightarrow{d} X$. Convergence in Distribution is also called Weak Convergence, oddly enough, because it is the weakest form of convergence [2]. Now I will immediately use Convergence in Distribution to state the Continuous Mapping Theorem.

4 Continuous Mapping Theorem

The Continuous Mapping Theorem is used in the first two steps of the proof of the Convergence to Types Theorem. The two steps are summarized here for easy reference:

1. $X_n \xrightarrow{d} U, a_n \to a > 0, b_n \to b \Rightarrow V_n \xrightarrow{d} \frac{U-b}{a}$ as $n \to \infty$

2. $X_n \xrightarrow{d} U, a_n \to +\infty \Rightarrow \frac{X_n}{a_n} \xrightarrow{p} 0$

Continuous Mapping Theorem. *Let X, X_1, X_2, ... be random variables, and suppose that*

$$X_n \xrightarrow{d} X \qquad as \qquad n \to \infty$$

If g is continuous, then

$$g(X_n) \xrightarrow{d} g(X) \qquad as \qquad n \to \infty.$$

Therefore if X_1, X_2, ... is a sequence of random variables, such that $X_n \xrightarrow{d} X$ as $n \to \infty$ for some random variable X, then, for $a_n \in \mathbb{R}^+$ and $b_n \in \mathbb{R}$, such that $a_n \to a$, and $b_n \to b$ as $n \to \infty$, we have:

$$a_n X_n + b_n \xrightarrow{d} aX + b.$$

This statement of CMT is from [1]. Now we have the tools to prove the Convergence in Types Theorem.

5 Convergence To Types Theorem

Like most of the material is this paper, this presentation of the Convergence to Types Theorem is from Allan Gut's, Probability: A Graduate Course [1]. However, it was originally proved by Aleksandr Khinchin. I hope to add value by combining the separate topics in a step by step manner to prove the CtoTT and show how it ties into the derivation of the Extremal Distribution.

Convergence to Types Theorem. *Let X_1, X_2, ... be a sequence of random variables, a_n, $\alpha_n \in \mathbb{R}^+$, b_n, $\beta_n \in \mathbb{R}$, $n \geq 1$, be such that,*

$$\frac{X_n - \beta_n}{\alpha_n} \xrightarrow{d} U, \qquad \frac{X_n - b_n}{a_n} \xrightarrow{d} V \tag{1}$$

where U and V are non-degenerate random variables. Then there exist $A \in \mathbb{R}^+$ and $B \in \mathbb{R}$, such that

$$\lim_{n\to\infty} \frac{a_n}{\alpha_n} \to A, \qquad \lim_{n\to\infty} \frac{b_n - \beta_n}{\alpha_n} \to B \tag{2}$$

In particular, if

$$X_n \xrightarrow{d} U \qquad \text{and} \qquad V_n = \frac{X_n - b_n}{a_n} \xrightarrow{d} V \qquad \text{as} \qquad n \to \infty, \tag{3}$$

then there exists $A \in \mathbb{R}^+$ and $B \in \mathbb{R}$, such that

$$a_n \to A \qquad \text{and} \qquad b_n \to B \qquad \text{as} \qquad n \to \infty \tag{4}$$

and,

$$V \overset{d}{=} \frac{U - B}{A} \qquad \text{or, equivalently,} \qquad U \overset{d}{=} AV + B. \tag{5}$$

Since the general case follows from the particular one by rescaling, it suffices to prove the latter.

The proof proceeds via the following steps:

(a) $X_n \xrightarrow{d} U, a_n \to a > 0, b_n \to b \Rightarrow V_n \xrightarrow{d} \frac{U-b}{a}$ as $n \to \infty$

(b) $X_n \xrightarrow{d} U, a_n \to +\infty \Rightarrow \frac{X_n}{a_n} \xrightarrow{p} 0$

(c) $X_n \xrightarrow{d} U, \sup_n |b_n| = \infty \Rightarrow X_n - b_n \overset{d}{\nrightarrow}$ as $n \to \infty$.

(d) $X_n \xrightarrow{d} U, V_n \xrightarrow{d} V \Rightarrow 0 < \inf_n a_n \leq \sup_n a_n < \infty, \sup_n |b_n| < \infty$

(e) $U \overset{d}{=} \frac{U-b}{a} \Rightarrow a = 1, b = 0.$

The first three steps, (a), (b), and (c). are proved directly. The proof of (d) uses (a), (b), and (c). Then, (e) is proved using (d). Once these five parts are proved, we can say that sequences of a_n and b_n for A and B exist, and we just need to prove they are unique. Using (e) we can show there are not other sequences a_j, b_j for A^* and B^* where $A \neq A^*$ and/or $B \neq B^*$.

6 Proving Convergence to Types

6.1

(a) $X_n \xrightarrow{d} U, a_n \to a > 0, b_n \to b \Rightarrow V_n \xrightarrow{d} \frac{U-b}{a}$ as $n \to \infty$

Proof. Since $X_n \xrightarrow{d} U, a_n \to a > 0, b_n \to b$ then,

$$V_n = \frac{X_n - b_n}{a_n} \qquad \text{is the definition of } V_n \text{ and} \tag{6}$$

$$\frac{X_n - b_n}{a_n} \xrightarrow{d} \frac{U-b}{a} \qquad \text{by the continuous mapping theorem, so} \tag{7}$$

$$V_n \xrightarrow{d} \frac{U-b}{a} \tag{8}$$

□

6.2

(b) $X_n \xrightarrow{d} U, a_n \to +\infty \Rightarrow \frac{X_n}{a_n} \xrightarrow{p} 0$

Proof. Since $X_n \xrightarrow{d} U, a_n \to +\infty$ then,

$$\frac{X_n}{a_n} \xrightarrow{d} \frac{U}{a} \qquad \text{by the continuous mapping theorem, and} \tag{9}$$

$$\frac{U}{a} = \frac{U}{+\infty} \xrightarrow{p} 0 \qquad \text{since } U \text{ is not degenerate.} \tag{10}$$

□

6.3

(c) $X_n \xrightarrow{d} U, \sup_n |b_n| = \infty \Rightarrow X_n - b_n \overset{d}{\nrightarrow}$ as $n \to \infty$

Proof. Since $X_n \xrightarrow{d} U, \sup_n |b_n| = \infty$ then there exist subsequences:

$$b'_n \to +\infty \text{ and/or } b''_n \to -\infty \text{ as } n', n'' \to \infty \text{ that implies} \tag{11}$$

$$\begin{aligned} P(X'_n - b'_n \le x) &= P(X'_n \le x + \infty) \to 1 && \text{as } n' \to \infty \\ P(X''_n - b''_n \le x) &= P(X''_n \le x + -\infty) \to 0 && \text{as } n'' \to \infty \end{aligned} \tag{12}$$

□

6.4

(d) $X_n \xrightarrow{d} U, V_n \xrightarrow{d} V \Rightarrow 0 < \inf_n a_n \leq \sup_n a_n < \infty, \sup_n |b_n| < \infty$

Proof. This proof is by contradiction. Suppose that $\sup_n a_n = +\infty$ then there exists a subsequence $\{n_k, k \geq 1\}$, such that $a_{nk} \to \infty$ as $k \to \infty$, so that, by (b), $\frac{X_{nk}}{a_{nk}} \xrightarrow{p} 0$ as $k \to \infty$. Since $V_{nk} \xrightarrow{d} V$ as $k \to \infty$, (c) tells us that $\sup_k \frac{|b_{nk}|}{a_{nk}} < +\infty$. That implies there exists a further subsequence n_{kj}, $j \geq 1$, such that $\frac{|b_{nkj}|}{a_{nkj}} \to c$ for some $c \in \mathbb{R}$ as $j \to \infty$. But,

$$V_n \xrightarrow{d} \frac{U-b}{a} \qquad \text{by (a), and } V_n \text{ is} \tag{13}$$

$$\frac{X_{nk} - b_{nk}}{a_{nk}} = \frac{X_{nk}}{a_{nk}} - \frac{b_{nk}}{a_{nk}} \qquad \text{therefore} \tag{14}$$

$$V_{nkj} = \frac{X_{nkj}}{a_{nkj}} - c = \frac{X_{nkj}}{+\infty} - c = -c \qquad \text{so} \tag{15}$$

$$V_{nkj} \xrightarrow{p} -c \qquad \Rightarrow\Leftarrow \tag{16}$$

this contradicts the fact that V is not degenerate, therefore $\sup_n a_n < \infty$. By the same argument, reversing the roles of X_n and V_n shows that $\inf_n a_n > 0$.

To show that $\sup_n |b_n| < \infty$, assume $\sup_n |b_n| \to \infty$, then there exists a subsequence $\{n_k, k \geq 1\}$, such that $a_{nk} \to a$ and $b_{nk} \to \infty$ as $k \to \infty$. Then $\frac{X_{nk}}{a_{nk}}$ converges in distribution, but V_{nk} by (c), does not:

$$V_{nk} = \frac{X_{nk}}{a_{nk}} - \frac{b_{nk}}{a_{nk}} = -\infty \qquad \Rightarrow\Leftarrow \tag{17}$$

By the same argument, if $b_{nk} \to -\infty$, then

$$V_{nk} = \frac{X_{nk}}{a_{nk}} - \frac{b_{nk}}{a_{nk}} = \infty \qquad \Rightarrow\Leftarrow \tag{18}$$

therefore $\sup_n |b_n| < \infty$. □

6.5

(e) $U \overset{d}{=} \frac{U-b}{a} \Rightarrow a = 1, b = 0$

Proof. Apply the assumption to itself:

$$U \overset{d}{=} \frac{\frac{U-b}{a} - b}{a} = \frac{U}{a^2} - \frac{b}{a^2} - \frac{b}{a} \overset{d}{=} ... \overset{d}{=} \frac{U}{a^n} - b\sum_{k=1}^{n} \frac{1}{a^k} = \frac{U - a^n b \sum_{k=0}^{n-1} a^k}{a^n}. \tag{19}$$

Now, (d) tells us:

$$0 < \inf_n a_n \leq \sup_n a_n \leq \infty \text{ and } \sup_n a^n |b| \sum_{k=0}^{n-1} a^k < \infty \tag{20}$$

therefore $a = 1$ and $b = 0$. □

6.6

Proof. a

Finally, by the assumptions of the theorem and the boundedness implied by (d) we can select a subsequence $\{n_k, k \geq 1\}$, such that $a_{nk} \to A > 0$ and $b_{nk} \to B \in \mathbb{R}$ as $k \to \infty$, which, implies that

$$V_{nk} \xrightarrow{d} \frac{U - B}{A} \text{ as } k \to \infty, \tag{21}$$

so that

$$V \overset{d}{=} \frac{U - B}{A}. \tag{22}$$

Now if along another subsequence, $\{m_j, j \geq 1\}$, we have $a_{mj} \to A^* > 0$ and $b_{mj} \to B^* \in \mathbb{R}$ as $j \to \infty$, and

$$V_{mj} \xrightarrow{d} \frac{U - B^*}{A^*} \text{ as } j \to \infty, \tag{23}$$

so that

$$V \overset{d}{=} \frac{U - B^*}{A^*}. \tag{24}$$

it follows by (e), that $B = B^*$ and $A = A^*$. This shows that every convergent pair of normalizations yields the same limit, and, hence, that the whole sequence does too. □

7 Conclusion

I would like to take a moment to try and explain, in plainer english, what we have proven in this paper, and in the previous two papers [3][4]. We first took a non-degenerate, distribution F and defined it's Extremal Distribution. Then, using the CtoTT we showed that the Extremal Distribution is Max-Stable, that is, the Extremal Distribution of an Extremal Distribution is also an Extremal Distribution. Using that critical fact we were able to derive the Gumbel Distribution. Finally, the reason we care about the Extremal Distribution is that it exists (there is a non-degenerate distribution G such that G is the Extremal Distribution of F) for many common distributions.

References

[1] Allan Gut. *Probability: A Graduate Course*. Springer, Springer Science+Business Media, inc., 233 Spring Street, New York, NY 10013, USA, 2005.

[2] Convergence in Distribution. en.wikipedia.org.

[3] Michael Mersic. Deriving the gumbel distribution. University of Villanova Graduate Seminar, April 2007.

[4] Michael Mersic. Use of convergence to types. Presented in Graduate Seminar, April 2007., April 2007.