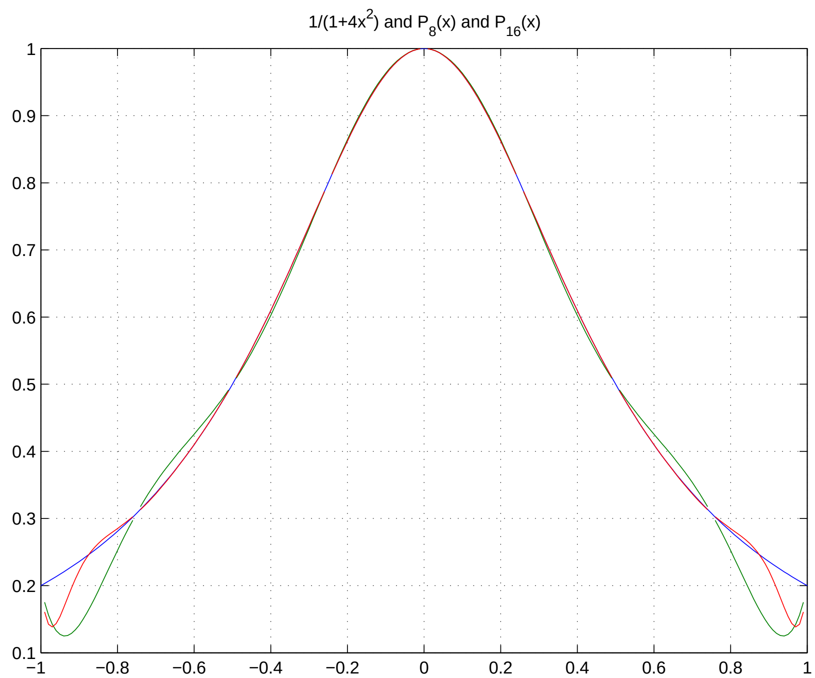
SPLINE INTERPOLATION

Spline Background

- Problem: high degree interpolating polynomials often have extra oscillations.
  Example: "Runge" function $f(x) = \frac{1}{1+4x^2}$, $x \in [-1, 1]$.



- **Piecewise Polynomials** provide alternative to high degree polynomials: approximation interval $[a, b]$ is subdivided into pieces $[x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$, with $a = x_1 < x_2 < \cdots < x_n = b$, and a low degree polynomial is used to approximate $f(x)$ on each subinterval.
  Example: piecewise linear approximation $S(x)$

$$S(x) = f(x_j) + (x - x_j)\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}, \text{ if } x \in [x_j, x_{j+1}]$$

- **Splines** are piecewise polynomial approximations, connected at x_j's with various continuity conditions.

CUBIC SPLINE INTERPOLATION

Cubic Interpolating Splines for $a = x_1 < \cdots < x_n = b$ with given data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

- Properties of **Cubic Interpolating Spline** $S(x)$,

  a) $S(x)$ is composed of cubic polynomial pieces $S_j(x)$

  $$S(x) = S_j(x) \text{ if } x \in [x_j, x_{j+1}],\ j = 1, 2, \ldots, n-1.$$

  b) $S(x_j) = y_j$, $j = 1, \ldots, n$. (interpolation)

  c) $S_{j-1}(x_j) = S_j(x_j)$, $j = 2, \ldots, n-1$ ($S \in C[a,b]$).

  d) $S'_{j-1}(x_j) = S'_j(x_j)$, $j = 2, \ldots, n-1$ ($S \in C^1[a,b]$).

  e) $S''_{j-1}(x_j) = S''_j(x_j)$, $j = 2, \ldots, n-1$ ($S \in C^2[a,b]$).

  f) two end conditions: examples

    i) $S''(x_1) = S''(x_n) = 0$ (**natural or free spline**);

    ii) $S'(x_1) = f'(x_1), S'(x_n) = f'(x_n)$
    (**complete or clamped spline**);

    iii) $S_1''' = S_{n-1}''' = 0$ (**parabolically terminated**);

    iv) $S_1'''(x_2) = S_2'''(x_2),\ S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$
    (**not-a-knot**).

  Note: if

  $$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3,$$

  condition a) provides $4(n-1)$ free parameters;
  b)-f) give $n + 3(n-2) + 2 = 4(n-1)$ constraints.

CUBIC SPLINE INTERPOLATION

- Example: $n = 3$, natural, data (1,2), (2,3), (3,5).

$$S_1(x) = 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3,$$
$$S_2(x) = 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3,$$

CUBIC SPLINE INTERPOLATION

Cubic Interpolating Spline Construction

- Spline Linear System: let $h_j = x_{j+1} - x_j$; start with

$$\begin{aligned} S_j(x) &= a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \\ &= y_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3. \end{aligned}$$

c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$, implies

$$y_{j+1} = y_j + b_j h_j + c_j h_j^2 + d_j h_j^3;\ \frac{\Delta y_j}{h_j} = b_j + c_j h_j + d_j h_j^2,$$

where $\Delta y_j = y_{j+1} - y_j$.
Notice $S''_j(x) = 2c_j + 6d_j(x - x_j)$, so
e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$, implies

$$2c_{j+1} = 2c_j + 6d_j h_j;\ d_j h_j = (c_{j+1} - c_j)/3,$$

with extra unknown $c_n = S''_{n-1}(x_n)/2$ added. Then

$$\frac{\Delta y_j}{h_j} = b_j + c_j h_j + \frac{(c_{j+1} - c_j)h_j}{3} = b_j + \frac{(c_{j+1} + 2c_j)h_j}{3};$$

$$\frac{3\Delta y_{j+1}}{h_{j+1}} - \frac{3\Delta y_j}{h_j} = 3\Delta b_j + (c_{j+2} + 2c_{j+1})h_{j+1} - (c_{j+1} + 2c_j)h_j.$$

Also $S'_j(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$, so
d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$; $b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2$;

$$\Delta b_j = 2c_j h_j + (c_{j+1} - c_j)h_j = (c_{j+1} + c_j)h_j$$

$$\frac{3\Delta y_{j+1}}{h_{j+1}} - \frac{3\Delta y_j}{h_j} = c_j h_j + 2c_{j+1}(h_j + h_{j+1}) + c_{j+2}h_{j+1}$$

CUBIC SPLINE INTERPOLATION

- Natural Splines: $S''(x_1) = S''(x_n) = 0$, so $c_1 = c_n = 0$
  Linear system equations are a "tridiagonal" system

$$
\begin{aligned}
c_1 &= 0 \\
c_1h_1 + 2c_2(h_1{+}h_2) + c_3h_2 &= \frac{3\Delta y_2}{h_2} - \frac{3\Delta y_1}{h_1} \\
c_2h_2 + 2c_3(h_2{+}h_3) + c_4h_3 &= \frac{3\Delta y_3}{h_3} - \frac{3\Delta y_2}{h_2} \\
&\vdots \qquad\qquad \vdots \\
c_{n-3}h_{n-3} + 2c_{n-2}(h_{n-3}{+}h_{n-2}) + c_{n-1}h_{n-2} &= \frac{3\Delta y_{n-2}}{h_{n-2}} - \frac{3\Delta y_{3-2}}{h_{n-3}} \\
c_{n-2}h_{n-2} + 2c_{n-1}(h_{n-2}{+}h_{n-1}) + c_nh_{n-1} &= \frac{3\Delta y_{n-1}}{h_{n-1}} - \frac{3\Delta y_{n-2}}{h_{n-2}} \\
c_n &= 0.
\end{aligned}
$$

  which can be solved uniquely for c_j's with $O(n)$ work;
  $d_j = (c_{j+1} - c_j)/(3h_j)$, $b_j = \Delta y_j/h_j - c_jh_j - d_jh_j^2$
  can be used to find remaining coefficients for $S_j(x)$'s.
  Note: if all $h_j = h$, a simpler tridiagonal system.

CUBIC SPLINE INTERPOLATION

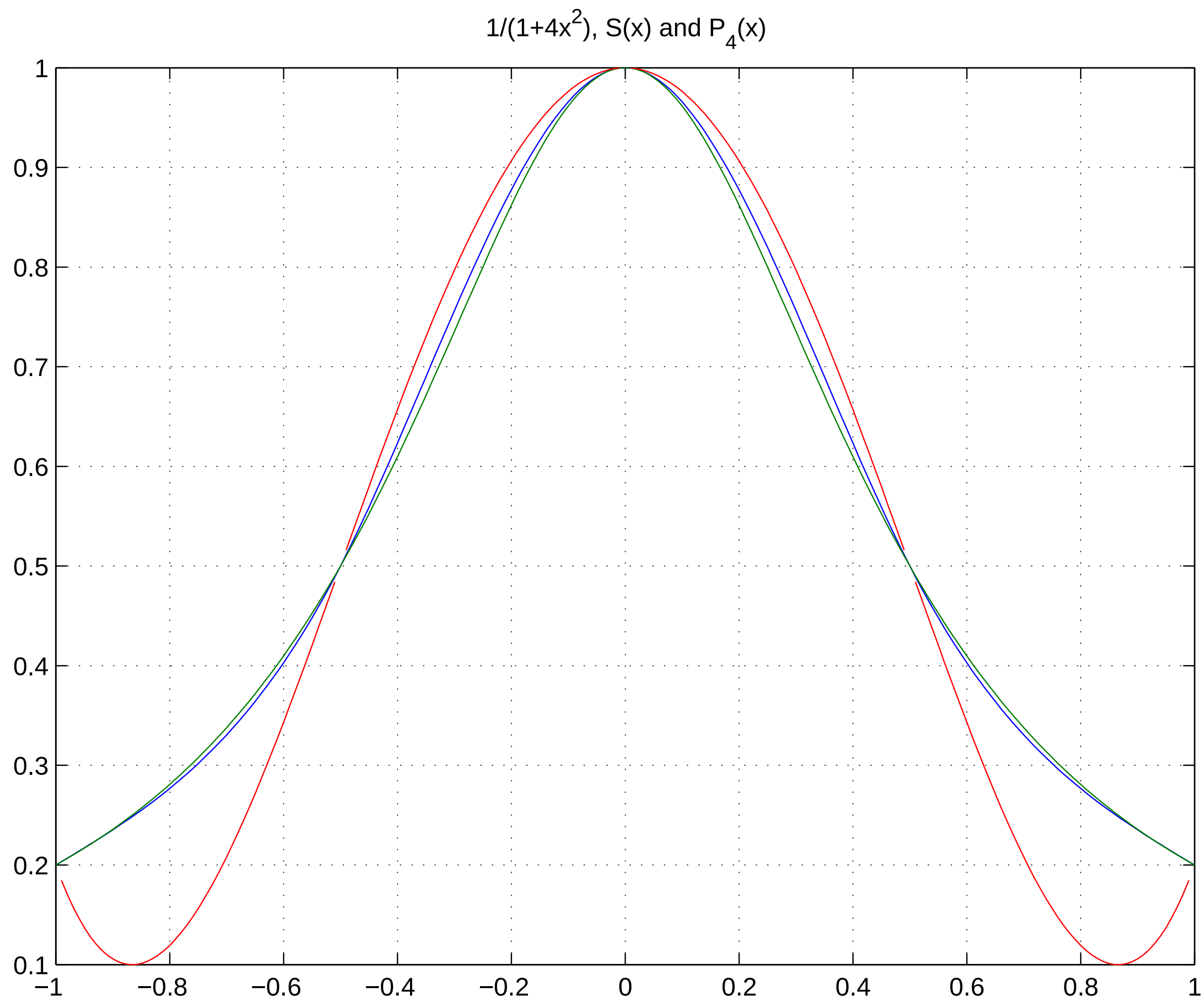
- Example: $n = 3$, natural, with data (1,2), (2,3), (3,5), so $h_1 = h_2 = 1$, $\Delta y_1 = 1$, $\Delta y_2 = 2$.

  Only one equation $2c_2(1+1) = 3(2-1)$, so $c_2 = 3/4$, $d_1 = 1/4$, $d_2 = -1/4$;
  $b_1 = 1 - 0 - 1/4 = 3/4$, $b_2 = 2 - 3/4 + 1/4 = 3/2$.

$$S_1(x) = 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3,$$

$$S_2(x) = 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3.$$

- Example: "Runge" function $f(x) = \frac{1}{1+4x^2}$, $x \in [-1, 1]$.



CUBIC SPLINE INTERPOLATION

- Clamped Splines: let $S'(x_1) = y'_1,\ S''(x_n) = y'_n$,
  so $y'_1 = b_1$, $y'_n = b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h^2_{n-1}$
  Using $\frac{\Delta y_j}{h_j} = b_j + \frac{(c_{j+1}+2c_j)h_j}{3}$, $3h_jd_j = (c_{j+1} - c_j)$,
  "1^{st}" and "n^{th}" equations become
  $2c_1h_1 + c_2h_1 = \frac{3\Delta y_1}{h_1} - 3y'_1$, and
  $c_{n-1}h_{n-1} + 2c_nh_{n-1} = 3y'_n - \frac{3\Delta y_{n-1}}{h_{n-1}}$.
  Linear system equations are a "tridiagonal" system

$$
\begin{aligned}
2c_1h_1 + c_2h_1 \;&=\; \frac{3\Delta y_1}{h_1} - 3y'_1 \\
c_1h_1 + 2c_2(h_1{+}h_2) + c_3h_2 \;&=\; \frac{3\Delta y_2}{h_2} - \frac{3\Delta y_1}{h_1} \\
&\;\;\vdots \qquad\qquad \vdots \\
c_{n-2}h_{n-2} + 2c_{n-1}(h_{n-2}{+}h_{n-1}) + c_nh_{n-1} \;&=\; \frac{3\Delta y_{n-1}}{h_{n-1}} - \frac{3\Delta y_{n-2}}{h_{n-2}} \\
c_{n-1}h_{n-1} + 2c_nh_{n-1} \;&=\; 3y'_n - \frac{3\Delta y_{n-1}}{h_{n-1}}.
\end{aligned}
$$

  which can be solved (uniquely) for c_j's with $O(n)$ work;
  $d_j = (c_{j+1} - c_j)/(3h_j)$, $b_j = \Delta y_j/h_j - c_jh_j - d_jh_j^2$
  can be used to find remaining coefficients for $S_j(x)$'s.

CUBIC SPLINE INTERPOLATION

- Example: $n = 3$, clamped, with data (1,2), (2,3), (3,5), and $y_1' = 1$, $y_3' = 2$. Three equations:

  $2c_1 + c_2 = 3(2-1) - 3 = 0,$

  $c_1 + 4c_2 + c_3 = 3,$

  $c_2 + 2c_3 = 6 - 3(2) = 0,$

  so $c_1 = c_3 = -1/2, c_2 = 1;$

  $d_1 = 1/2,\ d_2 = -1/2; b_1 = 1,\ b_2 = 2{-}1{+}1/2 = 3/2.$

  $S_1(x) = 2 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3,$

  $S_2(x) = 3 + \frac{3}{2}(x-2) + (x-2)^2 - \frac{1}{2}(x-2)^3,$

- Parabolically Terminated Splines: $S_1''' = S_{n-1}''' = 0$, so $d_1 = d_{n-1} = 0,\ c_1 = c_2,\ c_{n-1} = c_n$.

  Linear system equations are a "tridiagonal" system

$$
\begin{aligned}
c_1 - c_2 &= 0 \\
c_1h_1 + 2c_2(h_1{+}h_2) + c_3h_2 &= \frac{3\Delta y_2}{h_2} - \frac{3\Delta y_1}{h_1} \\
&\ \,\vdots \qquad\qquad \vdots \\
c_{n-2}h_{n-2} + 2c_{n-1}(h_{n-2}{+}h_{n-1}) + c_nh_{n-1} &= \frac{3\Delta y_{n-1}}{h_{n-1}} - \frac{3\Delta y_{n-2}}{h_{n-2}} \\
c_{n-1} - c_n &= 0.
\end{aligned}
$$

  which can be solved (uniquely) for c_j's with $O(n)$ work;

  $d_j = (c_{j+1} - c_j)/(3h_j),\ b_j = \Delta y_j/h_j - c_jh_j - d_jh_j^2$

  can be used to find remaining coefficients for $S_j(x)$'s.

CUBIC SPLINE INTERPOLATION

- Not-a-Knot Splines:
  $S_1'''(x_2) = S_2'''(x_2),\ S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$, so
  $d_1 = d_2$, $d_{n-2} = d_{n-1}$, and $S_1 = S_2,\ S_{n-2} = S_{n-1}$.
  Then $(c_2 - c_1)/h_1 = (c_3 - c_2)/h_2$,
  $(c_{n-1} - c_{n-2})/h_{n-2} = (c_n - c_{n-1})/h_{n-1}$, so
  "1^{st}" and "n^{th}" equations become
  $c_1 h_2 - c_2(h_1 + h_2) + c_3 h_1 = 0$
  $c_{n-2} h_{n-1} - c_{n-1}(h_{n-2} + h_{n-1}) + c_n h_{n-2} = 0.$
  Linear system equations are a "tridiagonal" system

$$\begin{aligned}
c_1 h_2 - c_2(h_1 + h_2) + c_3 h_1 &= 0 \\
c_1 h_1 + 2c_2(h_1 + h_2) + c_3 h_2 &= \frac{3\Delta y_2}{h_2} - \frac{3\Delta y_1}{h_1} \\
&\ \ \vdots \qquad\qquad \vdots \\
c_{n-2} h_{n-2} + 2c_{n-1}(h_{n-2} + h_{n-1}) + c_n h_{n-1} &= \frac{3\Delta y_{n-1}}{h_{n-1}} - \frac{3\Delta y_{n-2}}{h_{n-2}} \\
c_{n-2} h_{n-1} - c_{n-1}(h_{n-2} + h_{n-1}) + c_n h_{n-2} &= 0.
\end{aligned}$$

  which can be solved (uniquely) for c_j's with $O(n)$ work;
  $d_j = (c_{j+1} - c_j)/(3h_j)$, $b_j = \Delta y_j / h_j - c_j h_j - d_j h_j^2$
  can be used to find remaining coefficients for $S_j(x)$'s.

CUBIC SPLINE INTERPOLATION

Efficient Spline Evaluation

- Setup: solve linear system for c_j's in $O(n)$ time
- For each evaluation point x, find interval $x \in [x_j, x_{j+1}]$ in $O(\log(n))$ time and evaluate $S_j(x)$ in $O(1)$ time. Alternate formula for $S_j(x)$ (without b_j's and d_j's):

$$S_j(x) = \frac{c_j}{3h_j}(x_{j+1} - x)^3 + \frac{c_{j+1}}{3h_{j+1}}(x - x_j)^3 + \left(\frac{y_j}{h_j} - \frac{c_j h_j}{3}\right)(x_{j+1} - x) + \left(\frac{y_{j+1}}{h_{j+1}} - \frac{c_{j+1} h_{j+1}}{3}\right)(x - x_j).$$

- Compare with polynomial interpolation, where setup time is $O(n^2)$ and evauation time is $O(n)$.

Error Theorem :
if $f \in C^4[a, b]$, with $\max_{x \in [a,b]} |f^{(4)}(x)| = M$, and $S(x)$ is the unique clamped spline for $f(x)$ with nodes $a = x_1 < x_2 \cdots < x_n = b$, then

$$|f(x) - S(x)| \le \frac{5M}{384} \max_{1 \le j < n} h_j^4.$$