Computer Graphics and AG Homework 3 (F2021)

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***Abstract*—Homework 1 for Introduction to Computer Graphics and Augmented Reality.** [1]

I. The Theory

Given two points, a starting point $P_i(x_i, y_i)$ and an ending point $P_{i+1}(x_{i+1}, y_{i+1})$ as illustrated in the following figure, we can connect them to form a line,

$$P(x,y) = P_i(x_i, y_i) + \lambda(P_{i+1}(x_{i+1}, y_{i+1}) - P_i(x_i, y_i)), \quad (1)$$

where

$$-\infty < \lambda < +\infty \quad (2)$$

and the directional vector is denoted as $d(x, y)$ defined as

$$d(x,y) = P_{i+1}(x_{i+1}, y_{i+1}) - P_i(x_i, y_i) \quad (3)$$

Note, just reminder, the above equation can be written in a short hand notation as we will often do in the future

$$P = P_i + \lambda(P_{i+1} - P_i), \quad (4)$$

or equivalently in explicity x-y components form as

$$(x, y) = (x_i, y_i) + \lambda((x_{i+1}, y_{i+1}) - (x_i, y_i)), \quad (5)$$

so we can easily to write the equation in C/C++ or python code as

$$x = x_i + \lambda(x_{i+1} - x_i), \quad (6)$$

and

$$y = y_i + \lambda(y_{i+1} - y_i), \quad (7)$$

Once we write the vector form equation 1 in its x-component, y-component format, as in 6 and 7 we can code it easily in C/C++ or python.

This is illustrated below I.

```
// C-code
x_tmp = x[i] + lambda *  (x_[i+1] - x[i]);
y_tmp = y[i] + lambda *  (y_[i+1] - y[i]);
```

Now, let's take a look at the physical meaning of λ.

1 when $\lambda = 0$, from 1, we have

$$P(x,y) = P_i(x_i, y_i) \quad (8)$$

e.g., the starting point of the line segment.

2 when $\lambda = 1$, we have

$$P(x,y) = P_{i+1}(x_{i+1}, y_{i+1}) \quad (9)$$

e.g., the ending point of the line segment.

2 when $0 < \lambda < 1$, we have

$$P_i(x_i, y_i) < P(x,y) < P_{i+1}(x_{i+1}, y_{i+1}) \quad (10)$$

So, P_i is any arbitrary point between the starting point P_i and ending point P_{i+1}.

e.g., any arbitrary point between the startin and ending point.

II. 2D Transformations

Two dimensional transformations form the foundation for manipulation of 2D graphics. Here we give 3 most useful transformations.

1 Translation;
2 Rotation;
3 Scaling.

A. 2D Translations

First about notation, we define a point, or vertex $P_i(x_i, y_i) = (x_i, y_i)$, we write this point as follows with the understanding that we have simplified the vector transpose symoble t,

$$(x_i, y_i) = \begin{bmatrix} x_i' \\ y_i' \end{bmatrix}$$

Let a given vertex be (x_i, y_i). After a translation of this point is denoted as (x_i', y_i'). Hence, we have the following mathematic formulation to descrbe this translation. Note we are using the concept of "before" and "after" this graphics manipulation to connect the vertex (x_i, y_i) (before) to its new location (x_i', y_i') (after). Note we have added the 3rd "dummy" dimension for the purpose of unified mathematic formulation for translations and rotations. This will become clear once we discuss the composition of 2D transforms.

$$\begin{bmatrix} x_i' \\ y_i' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix}$$

The derivation of the above mathematic formula can be done using constructive approach for each x and y dimension respectively. Consider the x dimension of a translation, we want to establish a mathematic equation

$$x_i' \stackrel{?}{=} x_i, \quad (11)$$

which is basically the following question to answer

$$x_i' \stackrel{?}{=} \alpha x_i + \beta, \quad (12)$$

[1]This homework assignment is for CMPE 163.

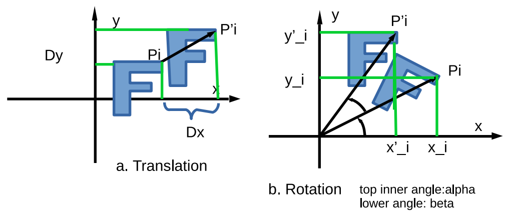

Fig. 1. Two dimensional transformations such as rotation and translaton form the foundation for manipulation of 2D graphics.

how do we find α and β to balance the equation before and after. Inspection of translation in Fig 1, we know

$$\alpha = 1 \tag{13}$$

and

$$\beta = \Delta x. \tag{14}$$

hence,

$$x_i' = x_i + \Delta x, \tag{15}$$

Similarly, for the y,

$$y_i' = y_i + \Delta y. \tag{16}$$

B. 2D Rotations

Consider a 2D rotation with the following conditions:

1 A rotation of 2D graphics pattern is defined as a positive rotation α with reference to the positve x-axis in the counter clockwise direction.

2 A rotation angle α is defined with respect to the origin of x-y coordinate system. Note this definition, especially when designing rotations in your application.

Based on the concept of "before" and "after" this graphics manipulation to connect the vertex (x_i, y_i) (before) to its new location (x_i', y_i') (after).

$$\begin{bmatrix} x_i' \\ y_i' \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix}$$

The derivation of the above mathematic formula can be carried out by referencing back to the illustration of the rotation in Figure 1 based on the inspection of before rotation and after rotation of a vertex P_i, we have

$$x_i' = r\cos(\alpha+\beta) = r\cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{17}$$

It is easy to observer from Figure 1, we have

$$x_i = r\cos\beta, \tag{18}$$

$$y_i = r\sin\beta. \tag{19}$$

Substitute them into the x_i' equation, we have

$$x_i' = \cos\alpha x_i - \sin\alpha y_i = \tag{20}$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \end{bmatrix} \begin{bmatrix} x_i, \\ y_i, \\ 0 \end{bmatrix}$$

Using the same approach, we can verify

$$y_i' = \sin\alpha x_i + \cos\alpha y_i = \tag{21}$$

$$\begin{bmatrix} \sin\alpha & \cos\alpha & 0 \end{bmatrix} \begin{bmatrix} x_i, \\ y_i, \\ 0 \end{bmatrix}$$

III. The Homework

Question 1. Given two points, a starting point $P_i(x_i, y_i) = (-11.3, -5.0)$ and an ending point $P_{i+1}(x_{i+1}, y_{i+1}) = (5.6, 1.0)$,

$P_{i+1}(5.6, 1) - P_i(-11.3, -5) =$

1.1 Find the directional vector $d(x_d, y_d) =$? $(5.6+11.3, 1+5) = (16.9, 6)$

1.2 Find the λ to give a point on the line at 80% point from the starting point? $\lambda = 0.8$

1.3 Find the λ to give a point on the line extended from the ending point by 25%? $\lambda = 1+0.25 = 1.25$

Please show your work step by step in details.

Question 2. Continued from the question above, now we would like to perform a rotation of the ending point P_{i+1} by angle α equal 30 degree, the rotation is clockwise, and the rotation is defined with respect to the starting point P_i (Note, not with respect to the ~~origi~~ (0,0)). Answer the following questions:

$\Delta X = -(-11.3) = 11.3,$ $\Delta Y = 5$

2.1 Find the preprocessing matrix $T_{pr} =$?

2.2 Find the rotation materix $R_{3x3} =$? $\alpha = -30$

2.3 Find the post processing matrix $T_{pp} =$? $-\Delta X, -\Delta Y$

2.4 Find the compostion matrix $T_\Sigma =$? $T^{-1}RT$

2.5 Find the new point after this rotation $P_{i+1}' =$?

(END)

References

[1] Harry Li, Lecture Notes, https://github.com/hualili/opencv/blob/master /ComputerGraphics_AR /F2018/2021F-2-lecture-note-2021-09-09.pdf