

Lecture Note 1 on Gradient Descent

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Abstract—In this lecture note, we give a gradient descent example for its applications in Neural Networks (NN), e.g., the concept of the negative gradient $-\nabla f$ follows the direction of steepest descent.

I. INTRODUCTION

In this lecture note, we give a gradient descent example for Neural Networks (NN) applications. In particular, the basic concept of the negative gradient $-\nabla f$ follows the direction of steepest descent of a given function f which can be an error function.

II. PARTIAL DERIVATIVE VS. GRADIENT

Given a scalar-valued multivariable functions, e.g. the function with a multidimensional input x_1, x_2, \dots, x_n , and a one-dimensional output as $y = f(x_1, x_2, \dots, x_n)$, where $f : R^n \rightarrow R$. The partial derivative of $f(x_1, x_2, \dots, x_n)$ with respect to x_i for $i = 1, 2, \dots, n$:

$$\frac{\partial f}{\partial x_i} = \lim_{\delta x \rightarrow 0} \frac{f(x_1, \dots, x_i + \delta x_i, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{\delta x_i} \quad (1)$$

The gradient of the function ∇f , is the collection of all its partial derivatives into a vector form, e.g., vector valued function [1].

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \dots \\ \frac{\partial f}{\partial x_i} \\ \dots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \quad (2)$$

Suppose standing on a surface of $f(x_1, x_2, \dots, x_n)$ at a point (x_1, x_2, \dots, x_n) , ∇f tells you which direction to travel to increase the value of f most rapidly. Hence, we make the following claim.

Claim 1. The negative gradient $-\nabla f$ follows the direction of steepest descent [2].

III. GRADIENT EXAMPLE

Example 1. Given

$$y = f(x_1, x_2) = x_1^2 + x_1 x_2 \quad (3)$$

Find its gradient as follows [1]:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 \end{pmatrix} \quad (4)$$

IV. GRADIENT STEEPEST DESCENT FOR MINIMIZATION

Now, let's define f as an error function, and we would like to minimize it. So we can try to change its inputs (x_1, x_2) by iteration steps:

$$(x_1^{k+1}, x_2^{k+1}) = (x_1^k, x_2^k) + [-\eta(\nabla f)^t] \quad (5)$$

which will reduce the function value f .

To verify this, write f in terms of Taylor expansion as follows

$$f(x_1, x_2) \simeq f(a, b) + \frac{\partial f}{\partial x_1}(x_1 - a) + \frac{\partial f}{\partial x_2}(x_2 - b) \quad (6)$$

use simplified notation for the partial derivative f_{x_1} and f_{x_2} , we have

$$f(x_1, x_2) \simeq f(a, b) + f_{x_1}(a, b) * (x_1 - a) + f_{x_2}(a, b) * (x_2 - b) \quad (7)$$

we want to update (x_1^k, x_2^k) to (x_1^{k+1}, x_2^{k+1}) such that $f(x_1^{k+1}, x_2^{k+1}) < f(x_1^k, x_2^k)$. From equation (6), replace (x_1, x_2) by (x_1^{k+1}, x_2^{k+1}) , and let $(x_1^k, x_2^k) = (a, b)$, so we have

$$f(x_1, x_2) - f(a, b) \simeq f_{x_1}(a, b) * (x_1 - a) + f_{x_2}(a, b) * (x_2 - b) \quad (8)$$

Or,

$$f(x_1, x_2) - f(a, b) = (x_1 - a, x_2 - b) \begin{pmatrix} f_{x_1}(a, b) \\ f_{x_2}(a, b) \end{pmatrix} \quad (9)$$

Which can be written as

$$f(x_1, x_2) - f(a, b) = (x_1 - a, x_2 - b) \nabla f \quad (10)$$

Based on the notion in Claim 1, let

$$\Delta x_1 = x_1 - a = -f_{x_1}, \Delta x_2 = x_2 - b = -f_{x_2} \quad (11)$$

hence, we have

$$f(x_1, x_2) - f(a, b) = (\Delta x_1, \Delta x_2) \nabla f = -(f_{x_1}^2 + f_{x_2}^2) \quad (12)$$

So apparently,

$$f(x_1, x_2) - f(a, b) = -(f_{x_1}^2 + f_{x_2}^2) < 0 \quad (13)$$

which shows the error function satisfies

$$f(x_1, x_2) < f(a, b) \quad (14)$$

e.g.,

$$f(x_1^{k+1}, x_2^{k+1}) < f(x_1^k, x_2^k). \quad (15)$$

V. CONCLUSION

In this lecture note, we describe the basic concept of gradient and we have noted that The negative gradient $-\nabla f$ follows the direction of steepest descent.

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