Tableau Method for Modal Logic

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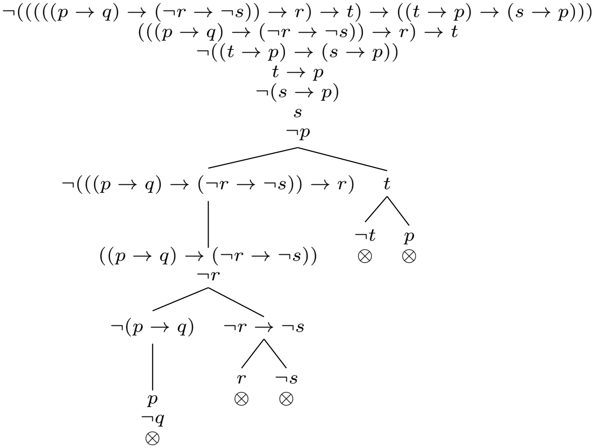
Tableau Method / System

- Tableau method is a type of proof procedure that defines proof.
- A tree (a connected acyclic directed graph) generated by a procedure is called a tableau.
- A closed tableau for $\neg A$ is a proof of A.



Prefixed Tableau Method

Definition 1

- A prefix σ is a finite sequence of positive integers.
- A prefixed formula is an expression of the form $\sigma\ \varphi$.

- A prefixed tableau method is a tableau method for prefixed formulas.
- We write prefixes using periods to separate integers.
- If σ is a prefix, and n is a positive integer, $\sigma.n$ is a concatenation of them (e.g. if $\sigma = 1.2.3$, and $n = 4$, $\sigma.n = 1.2.3.4$).

Definition of Branch

Definition 2

- A path of a tableau T is a finite sequence $(v_1, \cdots, v_n)$ of nodes (i.e. formulas) where there are edges $\langle v_i, v_{i+1} \rangle$ in T for $i = 1, \cdots, n-1$.
- A branch (maximal path) is a path satisfying the following conditions:
  (1) If a prefixed formula other than a possibility or necessity formula ($\Diamond A$ or $\neg\Box A$, $\Box A$ or $\neg\Diamond A$, respectively) appears on it, the applicable rules has been applied to it.
  (2) If a possibility formula appears on it, the possibility rule has been applied to it once.
  (3) If a necessity formula with σ or $\sigma.n$ appears on it, the applicable necessity rules have been applied to it once for each prefix σ or $\sigma.n$ that appears on it.

Closed Tableau and Open Tableau

Definition 3

- A tableau is saturated if there is a branch that includes the path for all paths.
- A branch is closed if it contains $\sigma\ A$ and $\sigma\ \neg A$ for some formulas A.
- A tableau is closed if all branchs are closed, and a tableau is open if it retains an open branch.
- A closed tableau for $1\ \neg A$ is a proof of A.

Branch Extention Rules for Connectives

Conjunctive Rules

$$
\begin{array}{c}
\sigma\ \ A \wedge B \\
\vdots \\
\sigma\ A \\
\sigma\ B
\end{array}
\qquad
\begin{array}{c}
\sigma\ \neg(A \vee B) \\
\vdots \\
\sigma\ \neg A \\
\sigma\ \neg B
\end{array}
\qquad
\begin{array}{c}
\sigma\ \neg(A \to B) \\
\vdots \\
\sigma\ A \\
\sigma\ \neg B
\end{array}
$$

Double Negation Rule

$$
\begin{array}{c}
\sigma\ \neg\neg A \\
\vdots \\
\sigma\ A
\end{array}
$$

Disjunctive Rules

$$
\begin{array}{c}
\sigma\ \neg(A \wedge B) \\
\vdots \\
\sigma\ \neg A \qquad \sigma\ \neg B
\end{array}
\qquad
\begin{array}{c}
\sigma\ A \vee B \\
\vdots \\
\sigma\ A \qquad \sigma\ B
\end{array}
\qquad
\begin{array}{c}
\sigma\ A \to B \\
\vdots \\
\sigma\ \neg A \qquad \sigma\ B
\end{array}
$$

Branch Extention Rules for Modal Operators

Possibility Rules : If the prefix $\sigma.n$ is new to the path,

$$\begin{array}{cc} \sigma\ \Diamond A & \sigma\ \neg\Box A \\ \vdots & \vdots \\ \sigma.n\ A & \sigma.n\ \neg A \end{array}$$

Basic Necessity Rules : If the prefix $\sigma.n$ is already occur in the path,

$$\begin{array}{cc} \sigma\ \Box A & \sigma\ \neg\Diamond A \\ \vdots & \vdots \\ \sigma.n\ A & \sigma.n\ \neg A \end{array}$$

Example

$$C : (\Box A \land \Box B) \to \Box(A \land B)$$

$$\begin{array}{c}
1\ \neg(\Box A \land \Box B) \to \Box(A \land B) \\
1\ \Box A \land \Box B \\
1\ \neg\Box(A \land B) \\
1\ \Box A \\
1\ \Box B \\
1.1\ \neg(A \land B)
\end{array}$$

$$\begin{array}{cc}
1.1\ \neg A & 1.1\ \neg B \\
1.1\ A & 1.1\ B \\
\otimes & \otimes
\end{array}$$

Soundness and Completeness

Definition 4

Suppose S is a set of prefixed formulas. We say S is satisfiable, if there are some models $M = \langle W, R, V \rangle$, and there is a function θ of assigning each prefix that appears in S to a possible world in W such that:

If σ and $\sigma.n$ appear in S, $(\theta(\sigma), \theta(\sigma.n)) \in R$;

If $\sigma\ \varphi$ appears in S, $M, \theta(\sigma) \vDash \varphi$.

Theorem 1

There is a closed tableau for $1\ \neg A$ ($\vdash A$) iff A is valid ($\vDash A$).

Strategies to Prove : Soundness

Soundness ($\vdash A \Rightarrow \vDash A$)

1. $\nvDash A \Rightarrow \nvdash A$.
2. $\{1\ \neg A\}$ is satisfiable, since there are some worlds v in some models M at which $M, v \vDash \neg A$.
3. Every tableau for $\{1\ \neg\varphi\}$ has at least one branch on which a set of prefixed formulas is satisfiable, since all rules preserve satisfiablity from a path before application to a path after application.
3. It is an open branch.
4. Every tableau for $1\ \neg A$ is not closed.

Strategies to Prove : Completeness

Completeness ($\vDash A \Rightarrow \vdash A$)

1. $\nvdash A \Rightarrow \nvDash A$.
2. There is a open saturated tableau, since every tableau for $\neg A$ is open.
3. Let $\mathcal{P}$ be a set of atomic propositions.
   We can create a model $M = \langle W, R, V \rangle$ where:
   - W is a collection of prefixes that appears on the open branch;
   - $(\sigma, \sigma.n) \in R :\Leftrightarrow$ both of σ and $\sigma.n$ are in W;
   - $V(p) :\Leftrightarrow \sigma\ p$ appears in the branch for each $p \in \mathcal{P}$.
4. The above model satisfies the property: For each formula A, if $\sigma\ A$ appears on the branch, then $M, \sigma \vDash A$.
5. $M, 1 \vDash \neg A$, since $1\ \neg A$ is a root.

More Tableau Methods

For prefixes σ and $\sigma.n$ already appears on the path,

Special Necessity Rule T :

$$\begin{array}{c} \sigma\ \Box A \\ \vdots \\ \sigma\ A \end{array} \qquad \begin{array}{c} \sigma\ \neg\Diamond A \\ \vdots \\ \sigma\ \neg A \end{array}$$

Special Necessity Rule D:

$$\begin{array}{c} \sigma\ \Box A \\ \vdots \\ \sigma\ \Diamond A \end{array} \qquad \begin{array}{c} \sigma\ \neg\Diamond A \\ \vdots \\ \sigma\ \neg\Box A \end{array}$$

Special Necessity Rule B:

$$\begin{array}{c} \sigma.n\ \Box A \\ \vdots \\ \sigma\ A \end{array} \qquad \begin{array}{c} \sigma.n\ \neg\Diamond A \\ \vdots \\ \sigma\ \neg A \end{array}$$

Special Necessity Rule 4:

$$\begin{array}{c} \sigma\ \Box A \\ \vdots \\ \sigma.n\ \Box A \end{array} \qquad \begin{array}{c} \sigma\ \neg\Diamond A \\ \vdots \\ \sigma.n\ \neg\Diamond A \end{array}$$

Special Necessity Rule $4r$:

$$\begin{array}{c} \sigma.n\ \Box A \\ \vdots \\ \sigma\ \Box A \end{array} \qquad \begin{array}{c} \sigma.n\ \neg\Diamond A \\ \vdots \\ \sigma\ \neg\Diamond A \end{array}$$

Soundness and Completeness to Extention

Definition 5

Suppose S is a set of prefixed formulas. We say S is $\mathbf{L} \in \{\mathbf{T}, \mathbf{D}, \mathbf{B}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$-satisfiable, if there are some $\mathbf{L} \in \{\mathbf{T}, \mathbf{D}, \mathbf{B}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$ -models $M = \langle W, R, V \rangle$, and there is a function θ of assigning each prefix that appears in S to a possible world in W such that:

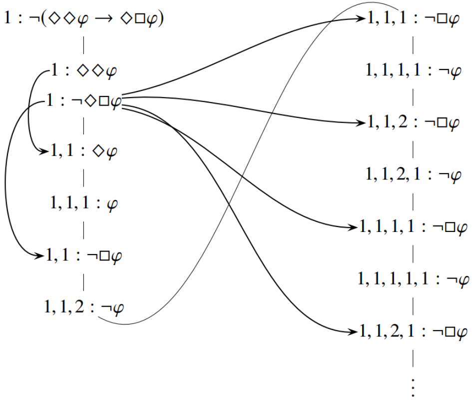
If σ and $\sigma.n$ appear in S, $(\theta(\sigma), \theta(\sigma.n)) \in R$;

If σ φ appears in S, $M, \theta(\sigma) \vDash \varphi$.

- Soundness : To prove to preserve the $\mathbf{L} \in \{\mathbf{T}, \mathbf{D}, \mathbf{B}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$-satisfiablity of the added rules.
- Completeness : To create a $\mathbf{L} \in \{\mathbf{T}, \mathbf{D}, \mathbf{B}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$-model satisfying 4's property.

Next Presentation

- As for $\mathbf{K4}$, a length of branches (i.e. the number of nodes on it) may be infinite.
- But there is a way to make it decidable.



A branch in $\mathbf{K4}$-tableaus (Takagi (2019))

[1] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. D. Reide1 Publisbing Company, 1983.

[2] M. Fitting and R.L. Mendelsohn. *First-order modal logic*. Kluwer Academic Publishers, 1998.

[3] T. Takagi. K4 タブローによる妥当性判定と濾過法. *Journal of Science and Philosophy*, 2(1), 4-23, 2019.