Algebraic Semantics for Modal Logic

Yudai Kubono

May 27, 2023

Shizuoka University, Graduate School of Science and Technology

Basic Notions

- An algebraic similarity type is a pair $\mathcal{T} = \langle F, \rho \rangle$, where F is a non-empty set of function symbols, and ρ is a function $F \to \mathbb{N}$ assigning a finite rank (arity) to each function symbol.
- Algebras of type $\mathcal{T}$ is a tuple $\mathfrak{A} = \langle A, I \rangle$, where A is a non-empty carrier, and I is an interpretation, that is a function assigning n-ary operations on A to each function symbol f of rank n. Notationally, we write $\langle A, f_{\mathfrak{A}} \rangle_{f \in F}$ for such an algebra.
- Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras of the same similarity type. A function $\eta : A \to B$ is a homomorphism if for all $f_{\mathfrak{A}}, a_1, \cdots, a_n$, $\eta f_{\mathfrak{A}}(a_i, \cdots, a_n) = f_{\mathfrak{B}}(\eta(a_i), \cdots, \eta(a_n))$, where n is the rank of $f_{\mathfrak{A}}$. Besides, we call $\mathfrak{B}$ the homomorphic image of $\mathfrak{A}$ if η is a surjective homomorphism.

- Let $\langle A, f_{\mathfrak{A}}\rangle_{f\in F}$ be an algebra. If $B \subseteq A$ is closed under every operations $f_{\mathfrak{A}}$, $\mathfrak{B} = \langle A, f_{\mathfrak{A}} \restriction_B\rangle_{f\in F}$ is a subalgebra of the algebra.
- $\mathfrak{A} = \langle A, f_{\mathfrak{A}}\rangle$ is a (direct) product of $\{\mathfrak{A}_i\}_{i\in\mathcal{I}}$ where $A = \prod_{i\in\mathcal{I}} A_i$, and $f_{\mathfrak{A}}$ is defined componentwisely: for each $i \in \mathcal{I}$, $f_{\mathfrak{A}}(a_1, \cdots, a_n)(i) = f_{\mathfrak{A}_i}(a_1(i), \cdots, a_n(i))$, where $a_n(i) := \pi_i(a)$, and $\pi_i : A \to A_i$ is a projection.
- A class is called a variety if the class is closed under homomorphic images, subalgebras and direct products. $\mathbb{V}(C)$ denotes the smallest variety containing C.

- Let X be a set of variables, and $\mathcal{T}$ be a similarity type. $\mathrm{Term}_{\mathcal{T}}(X)$ is the smallest set of $\mathcal{T}$-terms over X, which are recursively composed of X and function symbols in F.
- A function $\theta : X \to A$ is called an assignment. the extention $\tilde{\theta} : \mathrm{Term}_F(X) \to A$ called a meaning is defined as follow: $\tilde{\theta}(p) = \theta(p)$; $\tilde{\theta}(c) = c_{\mathfrak{A}}$; $\tilde{\theta}(f(t_1, \cdots, t_n)) = f(\tilde{\theta}(t_1), \cdots, \tilde{\theta}(t_n))$, where $p \in X$.
- An equation is a pair of $\mathrm{Term}_{\mathcal{T}}(X)$. An equation $t \approx t'$ is true in an algebra if $\tilde{\theta}(t) = \tilde{\theta}(t')$ for all θ on $\mathfrak{A}$. It is denoted by $\mathfrak{A} \vDash t \approx t'$. The algebra is called a model for $t \approx t'$.

- A class of algebras is equationally definable if there is a set of equations E such that the class precisely contains models for E.
- Birkhoff's theorem: a class of algebras is equationally definable if and only if it is a variety.
- A term algebra of $\mathcal{T}$ over X is a tuple $\mathfrak{Term}_{\mathcal{F}}(X) = \langle \mathrm{Term}_{\mathcal{T}}(X), I \rangle$, where $I(f)(t_1, \cdots, t_n) \coloneqq f(t_1, \cdots, t_n)$.

Equational Logic

- Let E be a set of equations. A derivation of E is a list of equations such that every element is axioms (elements in E), has the form of $t \approx t$, or is obtained from earlier elements using symmetry, transitivity, replacement (congruence), or substitution rules.
- An equation (formula) is derivable from E, which is denoted by $E \vdash t \approx u$, if the equation appears at the end of a derivation of E.
- An equation is a semantic consequence of E, which is denoted by $E \vDash t \approx u$, if for all $\mathfrak{A}$, $\mathfrak{A} \vDash E \Rightarrow \mathfrak{A} \vDash t \approx u$.
- Completeness theorem: for all equations $t \approx u$, $E \vDash t \approx u \Leftrightarrow E \vdash t \approx u$.

Algebraizing Propositional Logic

- The type Bool $\mathcal{B}$ is a tuple $\langle\{\neg,\vee,\bot\},\rho\rangle$ such that $\rho(\neg)=1,\rho(\vee)=2$, and $\rho(\bot)=0$.
- The propositional formula algebra over X is the term algebra of $\mathcal{B}$ over X, that is $\mathfrak{Form}_{\mathcal{B}}(X)=\langle \mathrm{Form}_{\mathcal{B}}(X),-,+,\bot\rangle$, where $-\varphi := \neg\varphi$, and $\varphi+\psi := \varphi\vee\psi$.
- The algebra of truth value $\mathbf{2}$ is a tuple $\langle\{1,0\},-,+,0\rangle$, where $-a := 1-a$, and $a+b := \max(a,b)$.
- Given an assignment θ, we obtain a meaning $\tilde{\theta}:\mathrm{Form}_{\mathcal{B}}(X)\to\{1,0\}$, that is a homomorphism.

Theorem 1

$\vDash_{\mathrm{CL}} \varphi \Leftrightarrow \mathbf{2}\vDash\varphi\approx\top$.

Proof.

We prove it immediately from the definition of a valuation and an assignment. $\square$

- A set algebra $\mathfrak{S}$ of A is a subalgebra of a power set algebra $\mathfrak{P}$ of A, which is a tuple $\mathfrak{P} = \langle \mathcal{P}(A), ^{-}, \cup, \emptyset \rangle$.
- A $\mathcal{I}$ power of $\mathtt{2}$ is the (direct) product of $(\mathtt{2}_i)_{i \in \mathcal{I}}$.

Lemma 1

Every power set algebra is isomorphic to a power of $\mathtt{2}$.

Proof.

Let A be an arbitrary set and B and C be subsets of A. we consider the characteristic function ζ of B, that is $\zeta : \mathcal{P}(A) \to \prod_{i \in A}\{1, 0\}_i$. For all operations in a power set algebra, $\zeta(\bar{B}) = -\zeta(B)$, $\zeta(B \cup C) = \zeta(B) + \zeta(C)$ and $\zeta(\emptyset) = 0$. Thus, the function ζ is a homomorphism between a power set algebra $\mathfrak{P}$ of A and a A power of $\mathtt{2}$. Moreover, if $B \neq C$, then $\zeta(B) \neq \zeta(C)$. For each element t of $\prod_{i \in A}\{1, 0\}_i$, there exists B such that $\zeta(B) = t$. Therefore, ζ is an isomorphism. $\square$

Lemma 2

If $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, then the validity of equations is preserved.

Proof.

Let ι be an isomorphism between A and B, and $\varphi \approx \psi$ be an equation. For an arbitrary meaning $\tilde{\theta}_{\mathfrak{B}}$ in $\mathfrak{B}$, there is a corresponding meaning $\tilde{\theta}_{\mathfrak{A}}$ such that $\tilde{\theta}_{\mathfrak{B}}(\varphi) = \iota(\tilde{\theta}_{\mathfrak{A}}(\varphi))$. By the assumption $\tilde{\theta}_{\mathfrak{A}}(\varphi) = \tilde{\theta}_{\mathfrak{A}}(\psi)$. Thus, $\tilde{\theta}_{\mathfrak{B}}(\varphi) = \tilde{\theta}_{\mathfrak{B}}(\psi)$. $\square$

Lemma 3

The validity of equations is preserved under taking a direct product and a subalgebra.

Proof.

As for a direct product, it is easy to prove by contraposition. The other one is trivial. $\square$

Theorem 2

$\models_{\mathrm{CL}} \varphi \Leftrightarrow \mathfrak{S} \models \varphi \approx \top$.

Proof.

We prove it immediately from theorem 1, lemma 1, 2, and 3. $\square$

- If $\vdash \varphi \leftrightarrow \psi$, it is called that two formulas φ and ψ are provably equivalent. $\equiv$ denotes the relation. The relation is a congruence on the propositional formula algebra.
- Lindenbaum-Tarski algebra is the quotient algebra of the propositional formula algebra over X, that is $\mathfrak{L}_{\equiv}(X) = \langle \mathrm{Form}_{\mathcal{B}}(X)/\equiv, -, +, 0\rangle$, where $-[\varphi] := [\neg\varphi]$, $[\varphi] + [\psi] := [\varphi \vee \psi]$, and $0 := [\bot]$.

Theorem 3

$\vdash_{\mathrm{CL}} \varphi \Leftrightarrow \mathfrak{L}_{\equiv}(X) \vDash \varphi \approx \top$.

Proof.

$(\Rightarrow)$ Let θ be an arbitrary assignment, and σ be a substitution, that is $\theta(p) = [\sigma(p)]$. It follows from induction of the complexicity of formulas that $\tilde{\theta}(\psi) = [\sigma(\psi)]$. By the assumption, $\sigma(\varphi)$ is a theorem. Thus, $[\sigma(\varphi)] = [\top]$.

$(\Leftarrow)$ We prove it by contraposition. Suppose that φ is not the theorem, then $[\varphi] \neq [\top]$. We consider an assignment such that $\tilde{\theta}(\psi) = [\psi]$. This assignment is an example that $\tilde{\theta}(\varphi) \neq \tilde{\theta}(\top)$. $\square$

- An algebra of type Bool is called a boolean algebra iff it satisfies commutativity, associativity, distributivity, identity, and complementation. BA denotes the class of boolean algebras.

Lemma 4

Both $\mathfrak{S}$ and $\mathfrak{L}_{\equiv}(X)$ are boolean algebras.

Theorem 4 (Algebraic Completeness)

$\vdash_{\mathrm{CL}} \varphi \Leftrightarrow \mathrm{BA} \vDash \varphi \approx \top$.

Proof.

($\Rightarrow$) We prove it by induction on the construction of logic.

($\Leftarrow$) This direction follows immediately from lemma 4. $\square$

- A ultrafilter of $\mathfrak{A} = \langle A, +, -, 0, f_\Diamond \rangle$ is a subset $L \subseteq A$ satisfying four conditions: $1 \in L$; L is closed under taking meets; L is upward closed; $0 \notin$ Ł; for every $a \in A$, $a \in L$ or $-a \in L$. A class $U(\mathfrak{A})$ denotes the collection of ultrafilters of $\mathfrak{A}$.

Theorem 5 (Stone's Representation Theorem)

Any boolean algebra is isomorphic to a set algebra.

Proof.

Let $\mathfrak{A}$ be a boolean algebra. We consider a map $\iota : A \to \mathcal{P}(U(\mathfrak{A}))$ such that $\iota(a) = \{u \in U(\mathfrak{A}) \mid a \in u\}$. This map is a homomorphism, since it is established that $\iota(f_{\mathfrak{A}}(a_1, \cdots, a_n)) = f_{\mathfrak{U}(\mathfrak{A})}(\iota(a_1), \cdots, \iota(a_n))$ (e.g. $\iota(a + b) = \iota(a) \cup \iota(b)$). Moreover, if $a \neq b$, $\iota(a) \neq \iota(b)$. Therefore, any boolean algebra is enbeddable in a power set algebra of $U(\mathfrak{A})$. $\square$

Theorem 6 (Completeness Theorem)

$\vdash_{\mathrm{CL}} \varphi \Leftrightarrow \vDash_{\mathrm{CL}} \varphi$.

Proof.

($\Rightarrow$) We prove it by induction on the construction of logic.
($\Leftarrow$) This direction follows immediately from lemma 4 and theorem 5, that is the fact, the Lindenbaum-Tarski algebra is isomorphic to a set algebra of ultrafilters of the algebra. $\square$

Algebraizing Modal Logic

- Let the concatenation of the boolean similarity type $\mathcal{B}$ and the modal similarity type τ be $\mathcal{B}_\tau$. The boolean similarity type is $\langle\{\neg, \vee, \bot\}, \rho\rangle$ such that $\rho(\neg) = 1, \rho(\vee) = 2$, and $\rho(\bot) = 0$. The modal similarity type is $\langle O, \rho\rangle$. In basic modal logic, $O = \{\Diamond\}$ and $\rho(\Diamond) = 1$.
- The modal formula algebra over X is the term algebra of $\mathcal{B}_\tau$ over X, that is $\mathfrak{Form}_{\mathcal{B}_\tau}(X) = \langle \mathrm{Form}_{\mathcal{B}_\tau}(X), -, +, \bot, f_\Diamond\rangle$.

- A τ-frame is a tuple $\mathcal{F} = \langle W, R_\Diamond \rangle$. A function in a frame $m_{R_\Diamond} : \mathcal{P}(W) \to \mathcal{P}(W)$ is defined by $m_{R_\Diamond}(Y) = \{w \in W \mid \text{there exists } y \in Y \text{ such that } (w, y) \in R_\Diamond\}$. It means that $V(\Diamond\varphi) = m_{R_\Diamond}(V(\varphi))$.
- A complex algebra of $\mathcal{F}$ is a subalgebra of the full complex algebra $\mathcal{F}^+$ of $\mathcal{F}$, that is a tuple $\mathcal{F} = \langle \mathcal{P}(W), ^-, \cup, \emptyset, m_{R_\Diamond} \rangle$. If K is a class of τ-frame, CmK denotes the class of full complex algebras of frames in K.

Lemma 5

$(\mathcal{F}, \theta), w \vDash_K \varphi \Leftrightarrow w \in \tilde{\theta}(\varphi)$.

Lemma 6

$\mathcal{F} \vDash_K \varphi \Leftrightarrow \mathcal{F}^+ \vDash \varphi \approx \top$.

Proof.

By lemma 5, for an arbitary meaning $\tilde{\theta}$ and $w \in W$ in $\mathcal{F}$, $w \in \tilde{\theta}(\varphi)$. Since $w \in \tilde{\theta}(\top)$, $\tilde{\theta}(\varphi) = \tilde{\theta}(\top)$. $\square$

Theorem 7

$\vDash_{\mathrm{K}} \varphi \Leftrightarrow \mathrm{CmK} \vDash \varphi \approx \top$.

- A relation of provable equivalence $\equiv_\Lambda$ ($\varphi \equiv_\Lambda \psi :\Leftrightarrow \vdash_\Lambda \varphi \leftrightarrow \psi$) is a congruence on the modal formula algebra $\mathfrak{Form}_{\mathcal{B}_\tau}(X)$, if Λ is normal modal τ-logic.
- Lindenbaum-Tarski algebra of Λ over X is the quotient algebra of the formula algebra over $\equiv_\Lambda$, that is $\mathfrak{L}_\Lambda(X) = \langle \mathrm{Form}_{\mathcal{B}_\tau}(X)/\equiv_\Lambda, -, +, 0, f_\Diamond \rangle$.

Theorem 8

$\vdash_\Lambda \varphi \Leftrightarrow \mathfrak{L}_\Lambda(X) \models \varphi \approx \top$.

Proof.

This proof is analogous to theorem 3. As for the right direction, we additionally prove a $\Diamond$ case of $\tilde{\theta}(\psi) = [\sigma(\psi)]$, that is $\tilde{\theta}(\Diamond\chi) = [\sigma(\Diamond\chi)]$. This can be proven by the induction hypothesis and the definition of the algebra. $\square$

- A boolean algebra with τ-operators is an algebra $\langle A, +, -, 0, f_\Diamond\rangle$ such that $\langle A, +, -, 0\rangle$ is a boolean algebra, and $f_\Diamond$ is an operation satisfying normality and additivity. BAO denotes the class of boolean algebras with τ-operators, if τ is known from the context.
- All of $\mathcal{F}^+$ and $\mathfrak{L}_\Lambda(X)$ are examples of a boolean algebra with an operator.

Theorem 9 (Algebraic Completeness)

$\vdash_\Lambda \varphi \Leftrightarrow \mathrm{BAO} \vDash \varphi \approx \top$.

Proof.

The proof is completely analogous to theorem 4. $\square$

- The ultrafilter frame of $\mathfrak{A} = \langle A, +, -, 0, f_\Diamond \rangle$ is a tuple $\langle U(\mathfrak{A}), Q_{f_\Diamond} \rangle$, where $Q_{f_\Diamond}$ is a binary relation on $U(\mathfrak{A})$ to be defined by $(u, u') \in Q_{f_\Diamond} \Leftrightarrow \{f_\Diamond(a) \mid a \in u'\} \subseteq u$. It is denoted as $\mathfrak{A}_+$.

Theorem 10 (Jónsson-Tarski Theorem)

Every boolean algebra with an operator $\mathfrak{A}$ is embeddable in the full complex algebra of its ultrafilter frame $(\mathfrak{A}_+)^+$.

Proof.

We consider a map $\iota : A \to \mathcal{P}(U(\mathfrak{A}))$ such that $\iota(a) = \{u \in U(\mathfrak{A}) \mid a \in u\}$. For boolean operations, it is an embedding by the stone's representation theorem (theorem 5). Thus, It suffices to prove that $\iota(f_\Diamond(a)) = m_{Q_{f_\Diamond}}(\iota(a))$. Moreover, for every $a, b \in A$, if $a \neq b$ then $\iota(a) \neq \iota(b)$. Therefore, any boolean algebra with an operator is enbeddable in the full complex algebra of its ultrafilter frame. $\square$

Theorem 11 (Completeness Theorem)

$\vdash_\Lambda \varphi \Leftrightarrow \vDash_\Lambda \varphi$.

Proof.

The proof is completely analogous to the stone's representation theorem (theorem 5). $\square$

Reference

[1] Blackburn, P., Rijke, M., and Venema, Y. (2001). *Modal Logic*. Cambridge: Cambridge University Press. doi:10.1017/CBO9781107050884