# A Proof of Arrow's Impossibility Theorem

Yudai Kubono

April 26, 2025

Graduate School of Science and Technology, Shizuoka University

# What is Arrow's (Impossibility) Theorem?

アローの不完全性定理について教えてください。

もちろんです！

**アローの不完全性定理（Arrow's Impossibility Theorem）**は、経済学者ケネス・アロー（Kenneth Arrow）が1951年に著書『Social Choice and Individual Values（社会的選択と個人的価値）』の中で示した、とても有名な定理です。簡単にいうと、

> 「個人の好みを集約して社会全体の好みを決めるとき、いくつか自然な条件を満たす方法は、**独裁制**以外には存在しない」

というものです。

Source: Text generated by ChatGPT.

**アローの不可能性定理とは？**

多数決(投票)に参加する人が2人以上で、選択肢が3つ以上ある時は、**公正な投票制度が存在しない**という定理。

Source:『アローの不可能性定理を分かりやすく】どうすれば選挙は上手く機能するのか. https://kitaguni-economics.com/arrows-impossibilitytheorem/.

# Arrow's Theorem

**Theorem 1 (Arrow's Theorem)**

*Suppose that there are more than two alternatives and finite individuals. Then, any social welfare function that respects transitivity, completeness, independence of irrelevant alternatives, and unanimity is a dictatorship.*

- Individuals: $I = \{1, \ldots, n\}$
- Alternatives: $A = \{a, b, \ldots, n\}$
- Individual preference $\preceq_i$: $i$'s preference ordering over $A$ ($a \preceq_i b \prec_i c$).
- Social welfare function $S$: a function that maps $n$-tuple of individual preferences to a social preference $\preceq$.

## Conditions

- Transitivity and Completeness: Individual and social preferences are transitive and complete relation (weak ordering).
- Independence of irrelevant alternatives (IIA): the social preference of any two alternatives depends only on individuals' preferences of them.
  - Let $\preceq_i| \{a, b\}$ denote the part of $\preceq_i$ concerning alternatives $a$ and $b$. For any $a, b \in A$, if for all $i \in I$, $\preceq_i| \{a, b\} = \preceq_i'| \{a, b\}$, then $\preceq| \{a, b\} = \preceq'| \{a, b\}$.
- Unanimity (U): For any $a, b$, if for all $i$, $a \prec_i b$, then $a \prec b$ (weak Pareto).
- Dictatorship: there is $i$ such that for any $a, b$, if $a \prec_i b$, then $a \prec b$.

# Extremal Lemma

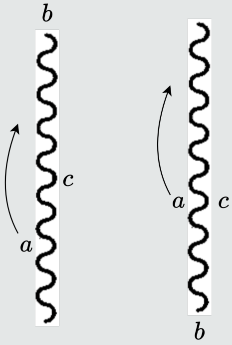
**Lemma 1 (Extremal Lemma)**

*Let alternative $b$ be chosen arbitrarily. If all individuals put $b$ at the very top or bottom of their preference, then the social preference must as well.*

**Proof.**

Suppose to the contrary that for such individual preferences and some $a, c \in A$, the social preference put $a \preceq b$ and $b \preceq c$. If every $i$ moves $a$ above $c$, the relations continue to hold due to IIA.

## Proof (continued).



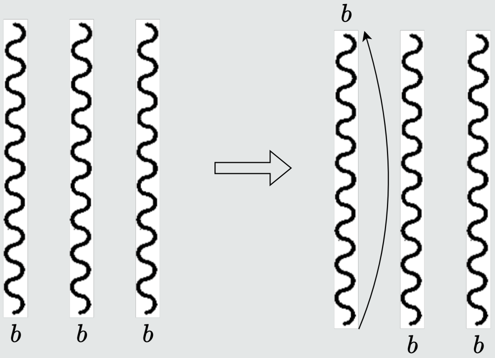

Thus, by transitivity, it continues to put $a \preceq c$, but by U, it also puts $c \preceq a$. This is a contradiction. $\square$

**Theorem 1 (Arrow's Theorem)**

*Suppose that there are more than two alternatives and finite individuals. Then, any social welfare function that respects transitivity, completeness, independence of irrelevant alternatives, and unanimity is a dictatorship.*

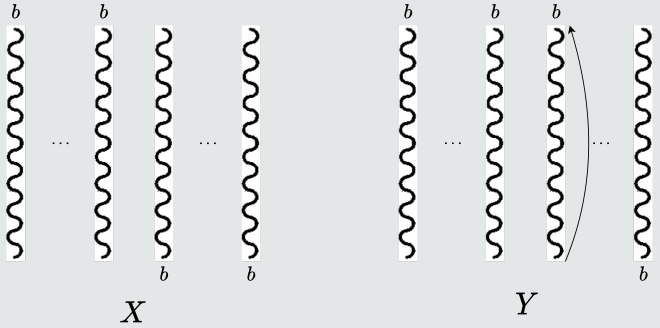
**Proof.**

1. Let alternative $b$ be chosen arbitrarily and every individual put $b$ at the very bottom of their preferences. Then, let individuals $\{1, \ldots, n\}$ successively move $b$ from the very bottom to the very top of their preferences while keeping the other relative orderings unchanged.



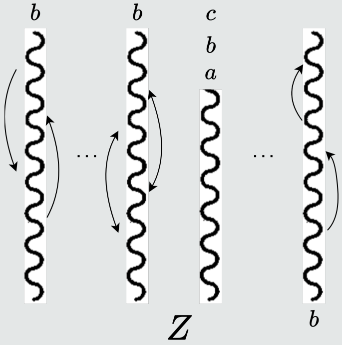
## Proof (continued).

It follows from the Extremal Lemma that there exists $i \in I$ such that by moving $b$ to the very top of his/her preference, she can move $b$ from the very bottom of the social preference to the very top, who is denoted by $i(b)$. We denote by tuple $X$ the list of all individual preferences just before $i$'s moving and $Y$ the list just after his/her moving.

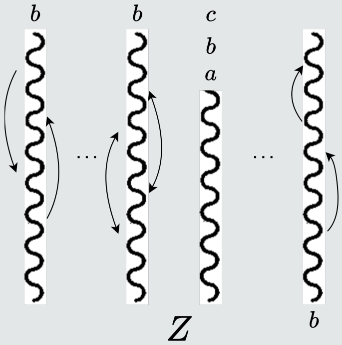


## Proof (continued).

2. We argue that $i(b)$ is a dictator over any alternative pair $a, c$ not involving $b$. To prove this, we construct tuple $Z$ from $Y$ by letting $i(b)$ put $a \prec_i b \prec_i c$ and all the other individuals arbitrarily rearrange their orderings of $a$ and $c$ while leaving $b$ in its extreme position.



## Proof (continued).




By IIA, the social preferences corresponding to $Z$ put $b \prec c$ as in $X$ and $a \prec b$ as in $Y$. By transitivity and IIA, $a \prec c$, which agree with $i(b)$'s preference ordering.

**Proof (continued).**

3. If we take another alternative $d$ different from $b$, there must be an individual $i(d)$, who is a dictator over any alternative pairs not involving $d$. This means that $i(d)$ dominates the social preference of any pair, including $a$ and $c$. This dictator must be $i(b)$. Thus, there exists only one dictator over every pair of alternatives.

Since this argument can be applied to any $n$-tuple of individual preferences, if a social structure function satisfies the conditions, then there exists a dictator for any $n$-tuple of individual preferences. □

# REFERENCES


- Geanakoplos, J. (2005). Three brief proofs of Arrow's impossibility theorem. *Economic Theory*, 26(1), 211-215.
- Morreau, M. (2019). Arrow's Theorem. https://plato.stanford.edu/entries/arrows-theorem/ (accessed April 24, 2025).
- OpenAI. (2025). ChatGPT 4o (March 27 version). [Large language model]. https://chat.openai.com.
- 『アローの不可能性定理を分かりやすく】どうすれば選挙は上手く機能するのか. (2019). https://kitaguni-economics.com/arrows-impossibilitytheorem/ (accessed April 26, 2025).