Expressivity of Epistemic Logics I

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December 21, 2024

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Issues of Expressivity

- Expressive power ($\mathcal{L}_1 \preceq \mathcal{L}_2$): given two languages, which one is more expressive than the other?
- The ability to distinguish non-identical models ($((M, w) \underline{\leftrightarrow} (M', w') \Rightarrow (M, w) \equiv_{\mathcal{L}} (M', w'))$: does a language have the ability (enough expressive) to distinguish non-identical models?
  - $(M, w) \equiv_{\mathcal{L}} (M', w')$ iff, $M, w \vDash \varphi$ iff $M', w' \vDash \varphi$ for every $\varphi \in \mathcal{L}$.

Expressive Power

- $\varphi \equiv \psi$ iff, $M, w \vDash \varphi$ iff $M, w \vDash \psi$ for all models and worlds.
- $\mathcal{L}_1 \preceq \mathcal{L}_2$ iff for every $\varphi_1 \in \mathcal{L}_1$, there is $\varphi_2 \in \mathcal{L}_2$ s.t. $\varphi_1 \equiv \varphi_2$.
  $\mathcal{L}_1 \prec \mathcal{L}_2$ iff $\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_2 \npreceq \mathcal{L}_1$.
  $\mathcal{L}_1 \simeq \mathcal{L}_2$ iff $\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_2 \preceq \mathcal{L}_1$.

Expressive Power of Propositional Logics

- p is an element of a set of atomic propositions.
- $\mathcal{L}_{PL} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi$.
- $\mathcal{L}_{even} \ni \varphi ::= p \mid \varphi \nabla \varphi \mid \varphi \leftrightarrow \varphi$.

φ	ψ	$\varphi \nabla \psi$	$\varphi \leftrightarrow \psi$
1	1	0	1
1	0	1	0
0	1	1	0
0	0	0	1

- $\neg\varphi \equiv \varphi \nabla (\varphi \leftrightarrow \varphi)$

Theorem 1

$\mathcal{L}_{even} \prec \mathcal{L}_{PL}$

Proof.

($\mathcal{L}_{even} \preceq \mathcal{L}_{PL}$) It is sufficient to show $t : \mathcal{L}_{even} \to \mathcal{L}_{PL}$ such that $\varphi \equiv t(\varphi)$ for every $\varphi \in \mathcal{L}_{even}$. This is proven by induction on a formula. The only non-trivial case is ∇, and the translation for this connective is provided by
$t(\varphi \nabla \psi) = \neg((\varphi \wedge \neg\psi) \wedge (\neg\varphi \wedge \psi))$.
($\mathcal{L}_{PL} \not\preceq \mathcal{L}_{even}$) It is sufficient to find a formula $\varphi \in \mathcal{L}_{PL}$ such that there is no formula $\psi \in \mathcal{L}_{even}$ such that $\varphi \equiv \psi$. Every formula in $\mathcal{L}_{even}$ shares a property that some formula in $\mathcal{L}_{even}$ does not have. The property is that in the truth table for a formula generated by at least two atomic propositions, there are an even number of truth values in every column. This is proven by induction on a formula.

Proof(Continued).

We show the case of $\varphi \nabla \psi$ here. Let

- x be the number of rows where φ is true and ψ is true,
- y be the number of rows where φ is true and ψ is false, and
- z be the number of rows where φ is false and ψ is true.

It follows from the induction hypothesis that $x + y$ and $x + z$ are even, which leads to $y + z$ is also even. Thus, the number of truth values in the column of $\varphi \nabla \psi$ is even. Consider a formula $p \wedge q$. The column in the truth table for this formula has three false values. Thus, there is no formula $\varphi \in \mathcal{L}_{PL}$ such that $p \wedge q \equiv \varphi$ and $\mathcal{L}_{PL} \not\preceq \mathcal{L}_{even}$. $\square$

Epistemic Logics

- $\mathcal{L}_K \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi$
  - $M, w \vDash K_i\varphi$ iff for all v s.t. $(w, v) \in \sim_i$, $M, v \vDash \varphi$.
- $\mathcal{L}_{K+C} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid C_{\mathcal{G}}\varphi$
  - $M, w \vDash C_{\mathcal{G}}\varphi$ iff for all v s.t. $(w, v) \in (\bigcup_{i \in \mathcal{G}} \sim_i)^*$, $M, v \vDash \varphi$.
- $\mathcal{L}_{K+PA} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid [\varphi]\varphi$
  - $M, w \vDash [\varphi]\psi$ iff $M, w \vDash \varphi$ implies $M|\varphi, w \vDash \psi$, where $M|\varphi = \langle W', \sim', V' \rangle$ is defined as follows:
    - $W' = W \cap \{w \mid M, w \models \varphi\}$,
    - $\sim' = \sim \cap (W' \times W')$,
    - $V'(\psi) = V(\psi) \cap W'$.

Expressive Power of Epistemic Logics

- $\circledcirc$ $\mathcal{L}_K \prec \mathcal{L}_{K+C}$
- $\mathcal{L}_K \simeq \mathcal{L}_{K+PA}$
- $\mathcal{L}_{K+C} \prec \mathcal{L}_{K+C+PA}$

Theorem 2

$\mathcal{L}_K \prec \mathcal{L}_{K+C}$

- We have to show that $\mathcal{L}_K \preceq \mathcal{L}_{K+C}$ and $\mathcal{L}_{K+C} \not\preceq \mathcal{L}_K$. The former is clear since $\mathcal{L}_K$ is a sublanguage of $\mathcal{L}_{K+C}$.

Spine Models

- Let n, m, k be natural numbers.

Definition 1

$Spine(n)$ is a tuple $\langle W, \sim_a, \sim_b, V \rangle$, where

- $W = \{m \mid m \leq (n+1)\}$,
- $\sim_a = \{(w, w) \mid w \in W\} \cup \{(m, k) \mid min(m, k) \bmod 2 = 0$ *and* $|m - k| = 1\}$,
- $\sim_b = \{(w, w) \mid w \in W\} \cup \{(m, k) \mid min(m, k) \bmod 2 = 1$ *and* $|m - k| = 1\}$,
- $V(p) = \{n + 1\}$ *for every* p.

$$0 \overset{a}{\rule[0.5ex]{2em}{0.4pt}} 1 \overset{b}{\rule[0.5ex]{2em}{0.4pt}} 2 \overset{a}{\rule[0.5ex]{2em}{0.4pt}} 3 \overset{b}{\rule[0.5ex]{2em}{0.4pt}} 4 \overset{a}{\rule[0.5ex]{2em}{0.4pt}} \underset{p}{5}$$

Definition 2

$Spine(\omega)$ is a tuple $\langle W, \sim_a, \sim_b, V \rangle$, where

- $W = \mathbb{N}$,
- $\sim_a = \{(w, w) \mid w \in W\} \cup \{(m, k) \mid min(m, k) \bmod 2 = 0$ *and* $|m - k| = 1\}$,
- $\sim_b = \{(w, w) \mid w \in W\} \cup \{(m, k) \mid min(m, k) \bmod 2 = 1$ *and* $|m - k| = 1\}$,
- $V(p) = \emptyset$ *for every* p.

$$0 \overset{a}{\rule{2em}{0.4pt}} 1 \overset{b}{\rule{2em}{0.4pt}} 2 \overset{a}{\rule{2em}{0.4pt}} 3 \overset{b}{\rule{2em}{0.4pt}} 4 \overset{a}{\rule{2em}{0.4pt}} 5 \cdots$$

Theorem 2

$\mathcal{L}_K \prec \mathcal{L}_{K+C}$

- ($\mathcal{L}_K \preceq \mathcal{L}_{K+C}$) Since $\mathcal{L}_K$ is a sublanguage of $\mathcal{L}_{K+C}$, it is clear that $\mathcal{L}_K \preceq \mathcal{L}_{K+C}$.
- ($\mathcal{L}_{K+C} \not\preceq \mathcal{L}_K$) It is sufficient to find a formula $\varphi \in \mathcal{L}_{K+C}$ such that there is no formula $\psi \in \mathcal{L}_K$ such that $\varphi \equiv \psi$. Consider a formula $C_{\{a,b\}}\neg p$. This formula distinguishes two spine models $(spine(n), 0)$ and $(spine(\omega), 0)$ for every n: $(spine(n), 0) \not\equiv_{\{C_{\{a,b\}}\neg p\}} (spine(\omega), 0)$. However, there is no formula in $\mathcal{L}_K$ that is false in $(spine(n), 0)$ but true in $(spine(\omega), 0)$: $(spine(n), 0) \equiv_{\mathcal{L}_K} (spine(\omega), 0)$. Thus, we cannot provide a translation to fulfill the condition $C_{\{a,b\}}\neg p \equiv t(C_{\{a,b\}}\neg p)$.

REFERENCES

- H. van Ditmarsch, W. van Der Hoek, and B. Kooi. (2007). *Dynamic Epistemic Logic*. Springer Science & Business Media.