Algebraizing Propositional Logic

Yudai Kubono

March 25, 2023

Japan Advanced Institute of Science and Technology

Basic Notions

- (Algebraic) Similarity type is a pair $\mathcal{F} = \langle F, \rho \rangle$, where F is a non-empty set of function symbols, and ρ is a function $F \to \mathbb{N}$ assigning a finite rank (arity) to each function symbol.
- Algebras of type $\mathcal{F}$ is a tuple $\mathfrak{A} = \langle A, I \rangle$, where A is a non-empty carrier, and I is an interpretation, that is a function assigning n-ary operations on A to each function symbol f of rank n. Notationally, we write $\langle A, f_{\mathfrak{A}} \rangle_{f \in F}$ for such an algebra.
- Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras of the same similarity type. A function $\eta : A \to B$ is a homomorphism if for all $f_{\mathfrak{A}}, a_1 \cdots a_n$, $\eta f_{\mathfrak{A}}(a_i, \cdots, a_n) = f_{\mathfrak{B}}(\eta(a_i), \cdots, \eta(a_n))$, where n is the rank of $f_{\mathfrak{A}}$. Besides, we call $\mathfrak{B}$ the homomorphic image of $\mathfrak{A}$ if η is a surjective homomorphism.

- Let $\langle A, f_{\mathfrak{A}}\rangle_{f\in F}$ be an algebra. If $B \subseteq A$ is closed under every operations $f_{\mathfrak{A}}$, $\mathfrak{B} = \langle A, f_{\mathfrak{A}} \restriction_B\rangle_{f\in F}$ is a subalgebra of the algebra.
- $\mathfrak{A} = \langle A, f_{\mathfrak{A}}\rangle$ is a (direct) product of $\{\mathfrak{A}_i\}_{i\in\mathcal{I}}$ where $A = \prod_{i\in\mathcal{I}} A_i$, and $f_{\mathfrak{A}}$ is defined componentwisely: for each $i \in \mathcal{I}$, $f_{\mathfrak{A}}(a_1, \cdots, a_n)(i) = f_{\mathfrak{A}_i}(a_1(i), \cdots, a_n(i))$, where $a_n(i) := \pi_i(a)$ where $\pi_i : A \to A_i$ is a projection.
- A class is called a variety if the class is closed under homomorphic images, subalgebras and direct products. $\mathbb{V}(C)$ denotes the smallest variety containing C.

- Let X be a set of variables, and $\mathcal{F}$ be a similarity type. $\mathrm{Term}_{\mathcal{F}}(X)$ is the smallest set of $\mathcal{F}$-terms over X, which are recursively composed of X and function symbols in F.
- An function $\theta : X \to A$ is called an assignment. the extention $\tilde{\theta} : \mathrm{Term}_F(X) \to A$ called a meaning is defined as follow: $\tilde{\theta}(p) = \theta(p)$; $\tilde{\theta}(c) = c_{\mathfrak{A}}$; $\tilde{\theta}(f(t_1, \cdots, t_n)) = f(\tilde{\theta}(t_1), \cdots, \tilde{\theta}(t_n))$, where $p \in X$.
- An equation is a pair of $\mathrm{Term}_{\mathcal{F}}(X)$. An equation $t \approx t'$ is true in an algebra if $\tilde{\theta}(t) = \tilde{\theta}(t')$ for all θ on $\mathfrak{A}$. It is denoted by $\mathfrak{A} \vDash t \approx t'$. The algebra is called a model for $t \approx t'$.

- A class of algebras is equationally definable if there is a set of equations E such that the class precisely contains models for E.
- Birkhoff's theorem: a class of algebras is equationally definable if and only if it is a variety.

Equational Logic

- Let E be a set of equations. A derivation of E is a list of equations such that every element is axioms (elements in E), has the form of $t \approx t$, or is obtained from earlier elements using symmetry, transitivity, replacement (congruence), or substitution rules.
- An equation (formula) is derivable from E, which is denoted by $E \vdash t \approx u$ if the equation appears at the end of a derivation of E.
- Completeness theorem: for every equations $t \approx u$, $E \vDash t \approx u \Leftrightarrow E \vdash t \approx u$, where $E \vDash t \approx u$ is defined by $\mathfrak{A} \vDash E \Rightarrow \mathfrak{A} \vDash t \approx u$.

Algebraizing Propositional Logic

- A term algebra of $\mathcal{F}$ over X is a tuple $\mathfrak{Term}_{\mathfrak{F}}(X) = \langle \mathrm{Term}_{\mathcal{F}}(X), I \rangle$, where $I(f)(t_1, \cdots, t_n) := f(t_1, \cdots, t_n)$.
- The propositional formula algebra over X is the term algebra of type Bool $\mathcal{B}$ over X, that is $\mathfrak{Form}_{\mathcal{B}}(X) = \langle \mathrm{Form}_{\mathcal{B}}(X), -, +, \bot \rangle$, where $-\varphi := \neg\varphi$, and $\varphi + \psi := \varphi \vee \psi$ (type Bool: $\langle \{\neg, \vee, \bot\}, \rho \rangle$ such that $\rho(\neg) = 1, \rho(\vee) = 2$, and $\rho(\bot) = 0$).
- The algebra of truth value $\mathbf{2}$ is a tuple $\langle \{1, 0\}, -, +, 0 \rangle$, where $-a := 1 - a$, and $a + b := \max(a, b)$.
- Given an assignment θ, we obtain a meaning $\tilde{\theta}$:$\mathrm{Form}_{\mathcal{F}}(X) \to \{1, 0\}$, that is a homomorphism.

Theorem 1

$\vDash_{\mathrm{CL}} \varphi \Leftrightarrow \mathbf{2} \vDash \varphi \approx \top$.

- A set algebra $\mathfrak{S}$ of A is a subalgebra of a power set algebra $\mathfrak{P}$ of A, which is a tuple $\mathfrak{P} = \langle \mathcal{P}(A), {}^{-}, \cup, \emptyset \rangle$.
- A power of 2 is the (direct) product of $(2_i)_{i \in \mathcal{I}}$.

Lemma 1

Every power set algebra is isomorphic to a power of 2.

Lemma 2

The validity of equations preserves under taking a direct product and a subalgebra.

Theorem 2

$\models_{\mathrm{CL}} \varphi \Leftrightarrow \mathfrak{S} \models \varphi \approx \top$.

- A relation $\equiv$ of provable equivalence is congruence on the propositional formula algebra.
- Lindenbaum-Tarski algebra is a tuple $\mathfrak{L}_{\equiv}(X) = \langle \mathrm{Form}_{\mathcal{B}}(X)/\equiv, -, +, 0 \rangle$, where $-[\varphi] := [\neg\varphi]$, $[\varphi] + [\psi] := [\varphi \vee \psi]$, and $0 := [\bot]$.

Theorem 3

$\vdash \varphi \Leftrightarrow \mathfrak{L}_{\equiv}(X) \models \varphi \approx \top$

- An algebra of type Bool is called a boolean algebra iff it satisfies commutativity, associativity, distributivity, identity, and complementation. BA denotes the class of boolean algebras.
- All of $\mathit{2}$, $\mathfrak{S}$, and $\mathfrak{L}_{\equiv}(X)$ are the examples of boolean algebras.

Theorem 4

$\vdash \varphi \Leftrightarrow \mathrm{BA} \models \varphi \approx \top$

Theorem 5 (Stone's Representation Theorem)

Any boolean algebra is isomorphic to a set algebra.

Theorem 6 (Completeness Theorem)

$\vdash \varphi \Leftrightarrow \models \varphi$

Reference

[1] Blackburn, P., Rijke, M., and Venema, Y. (2001). *Modal Logic*. Cambridge: Cambridge University Press. doi:10.1017/CBO9781107050884