Common Knowledge の形式化

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Example : Muddy Children 1/2

Imagine n children playing together. The mother of these children has told them that if they get dirty there will be severe consequences. So, of course, each child wants to keep clean, but each would love to see the others get dirty. Now it happens during their play that some of the children, say k of them, get mud on their foreheads. Each can see the mud on others but not on his own forehead. So, of course, no one says a thing. Along comes the father, who says, "At least one of you has mud on your forehead", thus expressing a fact known to each of them before he spoke (if $k > 1$). The father then asks the following question, over and over: "Does any of you know whether you have mud on your own forehead?" Assuming that all the children are perceptive, intelligent, truthful, and that they answer simultaneously, what will happen? [1]

Example : Muddy Children 2/2

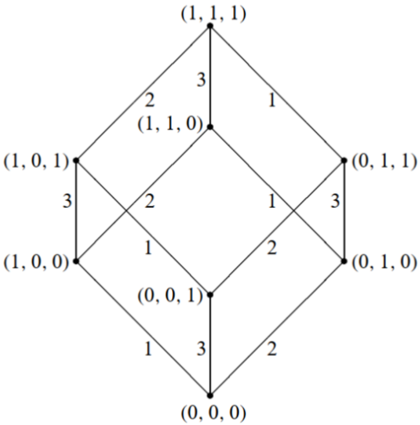

Figure: The Kripke structure for the muddy children puzzle with $n = 3$ [1]

Syntax

- Let AP be a countable set of atomic propositions, and AG be a finite set of agent-symbols. A language of multi-agent epistemic Logic with common knowledge (ELC) is the set of formulas generated by the following grammar:

$$\mathcal{L}_{\mathrm{AP}}^{\mathrm{ELC}} \ni \varphi ::= \mathbf{p} \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid C_G\varphi,$$

  where $\mathbf{p} \in \mathrm{AP}$, $i \in \mathrm{AG}$, and $G \subseteq \mathrm{AG}$. Other connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as usual.
- $E_G\varphi \coloneqq \bigwedge_{i\in G} K_i\varphi$.

$C_G\varphi = \bigwedge_{k=1}^{\infty} E_G^k\varphi$, but this definition is not a well-formed formula of $\mathcal{L}_{\mathrm{AP}}^{\mathrm{EL}}$.

Semantics 1/2

- A epistemic model is a tuple $M = \langle W, R^{\mathrm{AG}}, V \rangle$, where:
  - W is a non-empty set of possible worlds;
  - For every $i \in \mathrm{AG}$, R_i is a binary relation on W;
  - V is a valuation that is $V : \mathrm{AP} \to \mathcal{P}(W)$.

Semantics 2/2

- For each kripke model M and $w \in W$, the satisfaction relation $\models$ is defined as follows:

$$
\begin{aligned}
M, w &\models \mathbf{p} \text{ iff } w \in V(\mathbf{p}); \\
M, w &\models \neg\varphi \text{ iff } M, w \not\models \varphi; \\
M, w &\models \varphi \wedge \psi \text{ iff } M, w \models \varphi \text{ and } M, w \models \psi; \\
M, w &\models K_i\varphi \text{ iff } \text{ for all } t \text{ s.t. } \langle w, t\rangle \in R_i, M, t \models \varphi; \\
M, w &\models C_G\varphi \text{ iff } \text{ for all } t \text{ s.t. } \langle w, t\rangle \in (\bigcup_{i \in G} R_i)^*, M, t \models \varphi.
\end{aligned}
$$

- R^* is the smallest set satisfying the following conditions:
  - $R \subseteq R^*$;
  - For all x, y, z, if $\langle x, y\rangle \in R^*$ and $\langle y, z\rangle \in R^*$, then $\langle x, z\rangle \in R^*$;
  - For all x, $\langle x, x\rangle \in R^*$.

Axiomatization $\mathrm{S5C}$

- $\mathrm{S5C}$ consists of all the axioms and rules of $\mathrm{S5}$, plus the axioms and rules of the following table.

DC	$C_G(\varphi \to \psi) \to (C_G\varphi \to C_G\psi)$
MIX	$C_G\varphi \to (\varphi \wedge E_G C_G\varphi)$
IC	$C_G(\varphi \to E_B\varphi) \to (\varphi \to C_G\varphi)$
NC	From φ, infer $C_G\varphi$

Aumann Structure

- An aumann structure is a tuple $M = \langle S, \mathcal{P}_1, \cdots, \mathcal{P}_n \rangle$, where:
  - S is a set of possible worlds;
  - $\mathcal{P}_i$ is a patition of S for every agent i.
- We denote by $\mathcal{P}_i(s)$ the part (cell) of the partition where s appears.

Knowledge Operators

- Knowledge operator $K_i : 2^S \to 2^S$ for each i.

  $K_i(e) \coloneqq \{s \in S \mid \mathcal{P}_i(s) \subseteq e\}$.
  $E_G(e) \coloneqq \bigcap_{i \in \mathrm{AG}} K_i(e)$.
  $C_G(e) \coloneqq \bigcap_{k=1}^{\infty} E_G^k(e)$.

Aumann Structure

- Given an epistemic model, there is the aumann structure such that a truth set of φ matches the corresponding event for every formulas.

Conclusions

- 全員が知っていることと、Common knowledge（共有知識）という概念の差。
- （少なくとも）2 種類の形式化ができる。
- 前提条件の記述に使われる（ゲーム理論：common knowledge of rationality）。

[1] Ronald Fagin, Joseph Y Halpern, Yoram Moses, and Moshe Vardi. *Reasoning about knowledge*. MIT press, 2004.

[2] Peter Vanderschraaf, Giacomo Sillari. “Common Knowledge.” Stanford Encyclopedia of Philosophy. accessed 17 june 2022. https://plato.stanford.edu/entries/common-knowledge.