PALの完全性について

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What is Dynamic Epistemic Logic?

- 動的認識論理（Dynamic Epistemic Logic）はモデルに対する動的な操作可能とする認識論理の一種.
- 動的論理（Dynamic Logic）とは別物.
  - PAL (Public Announcement Logic)
  - Action Model
  - Belief Revision (AGM-approach)

Basic Epistemic Logic : Syntax

- A language of Epistemic Logic (EL) is the set of formulas generated by the following grammar:

$$\varphi \in \mathcal{L}^{\mathrm{EL}}_{\mathrm{AP}} ::= \mathbf{p} \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi,$$

where $\mathbf{p} \in \mathrm{AP}$. Other connectives $\vee$, $\to$, and $\leftrightarrow$ are defined as usual.

Basic Epistemic Logic : Semantics 1/2

- A kripke Model is a tuple $M = \langle W, R, V \rangle$, where:
  - W is a non-empty set of possible worlds;
  - R is a binary relation on W;
  - V is a valuation that is $V : \mathrm{AP} \to \mathcal{P}(W)$.

Basic Epistemic Logic : Semantics 2/2

- For each kripke model M and $w \in W$, the satisfaction relation $\models$ is defined as follows:

$$\begin{aligned} M, w \models \mathbf{p} &:\Leftrightarrow w \in V(\mathbf{p}); \\ M, w \models \neg\varphi &:\Leftrightarrow M, w \not\models \varphi; \\ M, w \models \varphi \wedge \psi &:\Leftrightarrow M, w \models \varphi \text{ and } M, w \models \psi; \\ M, w \models \Box\varphi &:\Leftrightarrow \text{ For all } s \text{ s.t. } \langle w, s \rangle \in R, M, s \models \varphi. \end{aligned}$$

Public Announcement Logic（PAL）

- The language of *Public Announcement Logic* (PAL) is the set of formulas generated by the following grammar:
$$\varphi \in \mathcal{L}^{\mathrm{PAL}}_{\mathrm{AP}} ::= \mathbf{p} \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid [\varphi!]\varphi,$$
where $\mathbf{p} \in \mathrm{AP}$. Other connectives $\vee$, $\to$, and $\leftrightarrow$ are defined as usual.

Semantics 1/2

- For each PAL model M and $w \in W$, the satisfaction relation $\models$ is defined as follows:

$$
\begin{aligned}
M, w \models \mathbf{p} &:\Leftrightarrow w \in V(\mathbf{p}); \\
M, w \models \neg\varphi &:\Leftrightarrow M, w \not\models \varphi; \\
M, w \models \varphi \wedge \psi &:\Leftrightarrow M, w \models \varphi \text{ and } M, w \models \psi; \\
M, w \models K\varphi &:\Leftrightarrow \text{ for all s s.t. } \langle w, s\rangle \in R, M, s \models \varphi; \\
M, w \models [\varphi!]\psi &:\Leftrightarrow (M, w \models \varphi \Rightarrow M^{\varphi}, w \models \psi).
\end{aligned}
$$

The Relativization of a Model

- $M^{\varphi} = \langle W', R', V' \rangle$
  - $W' := W \cap \{w \mid M, w \models \varphi\}$;
  - $R' := R \cap (W' \times W')$;
  - $V'(\psi) = V(\psi) \cap W'$ for all $\psi \in \mathcal{L}^{\mathrm{PAL}}$.

Axiomatization PA

Prop	The set of propositional tautologies
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
5	$\neg K\varphi \to K\neg K\varphi$
AP	$[\varphi!]p \leftrightarrow (\varphi \to p)$
AN	$[\varphi!]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi!]\psi)$
AC	$[\varphi!](\psi \wedge \chi) \leftrightarrow ([\varphi!]\psi \wedge [\varphi!]\chi)$
AK	$[\varphi!]K\psi \leftrightarrow (\varphi \to K[\varphi!]\psi)$
AC	$[\varphi!][\psi!]\chi \leftrightarrow [\varphi \wedge [\varphi!]\psi]\chi$
MP	$(\vdash \varphi$ and $\vdash \varphi \to \psi) \Rightarrow \psi$
NR	$\vdash \varphi \Rightarrow K\varphi$.

Soundness

Theorem 2.1

$\vdash_{\mathrm{PA}} \varphi \Rightarrow$ for all M in the class $\mathrm{C(E)}$ of reflexive euclidean models, for all $w \in W$, $M, w \models \varphi$.

- PA によって生成される全ての論理式が妥当であることを示す.
- 公理の妥当性と推論規則の妥当性の保存を証明すればよい.

Proof 2.2

$[\varphi!]p \leftrightarrow \varphi \to p$

$\Rightarrow$ (1) 仮定と定義より, 任意の M と w について $M, w \models \varphi \Rightarrow M^{\varphi}, w \models p$ と $M, w \models \varphi$.
(2) (1) より $M^{\varphi}, w \models p$.
(3) (2) と valuation の定義より, $M, w \models p$.

$\Leftarrow$ (1) 同様にして, 任意の M と w について $M, w \models \varphi \Rightarrow M, w \models p$ と $M, w \models \varphi$.
(2) (1) より $M, w \models p$.
(3) (2) と valuation の定義より, $M^{\varphi}, w \models p$.

Completeness 1/3

Theorem 2.3

For all M in the class $\mathrm{C}(\mathrm{E})$ of reflexive euclidean models, for all $w \in W$, $M, w \models \varphi \Rightarrow \vdash_{\mathrm{PA}} \varphi$.

- Translation $t : \mathcal{L}^{\mathrm{PAL}} \to \mathcal{L}^{\mathrm{EL}}$
  - $t(p) = p$;
  - $t(\neg\varphi) = \neg t(\varphi)$;

    $\vdots$
  - $t([\varphi!]p) = t(\varphi \to p)$;

    $\vdots$
  - $t([\varphi!][\psi!]\chi = t([\varphi! \wedge [\varphi!]\psi]\chi))$.
- Complexity $c : \mathcal{L}^{\mathrm{PAL}} \to \mathbb{N}$
  - $c(p) = 1$;
  - $c(\neg\varphi) = 1 + c(\varphi)$;
  - $c(\varphi \wedge \psi) = 1 + \max(c(\varphi), c(\psi))$;
  - $c(K\varphi) = 1 + c(\varphi)$;
  - $c([\varphi!]\psi) = (4 + c(\varphi)) \times c(\psi)$;

Completeness 2/3

(1) Complexity に関する帰納法を使い, 全ての $\varphi \in \mathcal{L}^{\mathrm{PAL}}$ について, $\vdash_{\mathrm{PA}} \varphi \leftrightarrow t(\varphi)$ が成立することを示す.

1. Base case : $\vdash_{\mathrm{PA}} p \leftrightarrow t(p)$.
2. complexity の小さい論理式に関する $\vdash_{\mathrm{PA}} \varphi \leftrightarrow t(\varphi)$ を I.H. として全ての論理式に対して証明する.

Proof 2.4

φ が $[\psi!]\chi$ のとき

$c([\psi!]\chi) > c(\psi \to \chi)$ のため, I.H. より $\vdash_{\mathrm{PA}} [\psi!]\chi \leftrightarrow t(\psi \to \chi)$. よって,
$\vdash_{\mathrm{PA}} [\psi!]\chi \leftrightarrow t([\psi!]\chi)$.

Completeness 3/3

(2) 健全性より $\models \varphi \leftrightarrow t(\varphi)$.

(3) 仮定 $\models \varphi$ より $\models t(\varphi)$.

(4) $t(\varphi) \in \mathcal{L}^{\mathrm{EL}}$ のため, $\vdash_{\mathrm{S5}} t(\varphi)$.

(5) $S5 \subseteq \mathrm{PA}$ のため, $\vdash_{\mathrm{PA}} t(\varphi)$.

(6) (1) と (5) より, $\vdash_{\mathrm{PA}} \varphi$.

Appendix : Completeness of S5

(1) For all M in the class $\mathrm{C(E)}$ of reflexive euclidean models, for all $w \in W$, $M, w \models \varphi \Rightarrow$ for the canonical model for $\mathrm{S5}$ M^c, for all $w \in W^c$, $M^c \models \varphi$, since $C(M^{c_K}) \subseteq C(M)$.

(2) For the canonical model for $\mathrm{S5}$ M^c, for all $w \in W^c$, $M^c \models \varphi \Rightarrow \vdash_{\mathrm{S5}} \varphi$, since for every $\mathrm{Max}_K \Delta, \varphi \in \Delta \Leftrightarrow \vdash_K \varphi$.

まとめ

- モデルの変更を伴うオペレータを含んだ論理体系の完全性定理の証明に使える汎用性の高いテクニックなので，よく使われる．

- Hans Van Ditmarsch, Wiebe van Der Hoek, and Barteld Kooi. *Dynamic epistemic logic*, volume 337. Springer Science & Business Media, 2007.
- Hans Van Ditmarsch, Wiebe van Der Hoek, Joseph Y Halpern, and Barteld Kooi. *Handbook of epistemic logic*, College Publications, 2015.
- Brian F Chellas. *Modal logic: an introduction*. Cambridge university press, 1980.