# Introduction to Word Problem

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## Outline



**Aim**

Solve chameleon puzzle by completion procedure



## Word Problem

**Definition (rewriting)**

Let $\mathcal{E}$ be set of equations on strings $\Sigma^*$. String $w_1 \in \Sigma^*$ rewrites to $w_2 \in \Sigma^*$ ($w_1 \to_{\mathcal{E}} w_2$) if there exists $l \approx r \in \mathcal{E}$ such that $w_1 = w_3 l w_4$ and $w_2 = w_3 r w_4$ for some strings $w_3, w_4$.

**Definition (word problem)**

Let $\mathcal{E}$ be set of equations. Following problem is called word problem on $\mathcal{E}$.

- Input: strings $w_1, w_2$
- Output: YES if $w_1 \leftrightarrow^*_{\mathcal{E}} w_2$ holds, NO otherwise

where $\leftrightarrow^*_{\mathcal{E}}$ denotes reflexive transitive symmetric closure of $\to_{\mathcal{E}}$.



## Example: Word Problem on Chameleons

**Example**

Consider word problem on following equations $\mathcal{E}$ (chameleon system):

| | | |
|---|---|---|
| RB ≈ GG | BG ≈ RR | RG ≈ BB |
| BR ≈ GG | GB ≈ RR | GR ≈ BB |

Questions:

1. GGBB $\leftrightarrow^*_{\mathcal{E}}$ RRRR ?
2. RRRR $\leftrightarrow^*_{\mathcal{E}}$ BBBB ?

Answers:

1. YES: GGBB $\to_{\mathcal{E}}$ GRRB $\leftarrow_{\mathcal{E}}$ GBGB $\to_{\mathcal{E}}$ RRGB $\to_{\mathcal{E}}$ RRRR
2. NO! (But why?)

## Idea

### Idea

- Restriction of $\leftrightarrow^*$ to $\to^* \cdot \leftarrow^*$
- Termination
- Church-Rosser property

**Example**

Questions:

1. $(1+2)+3 = 3+(1+2)$ ?
2. $1+(2+1) = 1+(2+3)$ ?

Answers:

1. YES: $(1+2)+3 \to 3+3 \to 6 \leftarrow 3+3 \leftarrow 3+(1+2)$.
2. NO: $1+(2+1) \to 1+3 \to 4 \neq 6 \leftarrow 1+5 \leftarrow 1+(2+3)$



## Termination

### Definition

When we restrict rewriting to one direction $\to$,

- set of equation is denoted by $\mathcal{R}$, and
- equation $l \approx r$ is denoted by $l \to r$.

### Definition (termination)

Let $\mathcal{R}$ be set of equations on strings. We say $\mathcal{R}$ is terminating if there is no infinite sequence $w_1 \to_{\mathcal{R}} w_2 \to_{\mathcal{R}} w_3 \to_{\mathcal{R}} \ldots$.

**Example**

Informally, addition $+$ on natural numbers is terminating.



## Termination of Chameleon System

**Example**

Consider chameleon system $\mathcal{R}$:

| | | |
|---|---|---|
| RB → GG | BG → RR | RG → BB |
| BR → GG | GB → RR | GR → BB |

Chameleon system $\mathcal{R}$ is not terminating since it admits loop:

$$\underline{\mathrm{BR}}\mathrm{RG} \to_{\mathcal{R}} \mathrm{G}\underline{\mathrm{GR}}\mathrm{G} \to_{\mathcal{R}} \underline{\mathrm{GB}}\mathrm{BG} \to_{\mathcal{R}} \mathrm{R}\underline{\mathrm{RB}}\mathrm{G} \to_{\mathcal{R}} \underline{\mathrm{RG}}\mathrm{GG} \to_{\mathcal{R}} \mathrm{B}\underline{\mathrm{BG}}\mathrm{G} \to_{\mathcal{R}} \mathrm{BRRG}$$



## Church-Rosser Property

### Definition (Church-Rosser property)

Let $\mathcal{R}$ be set of equations on strings. We say $\mathcal{R}$ has Church-Rosser property if $w_1 \to_{\mathcal{R}}^* \cdot \leftarrow_{\mathcal{R}}^* w_2$ whenever $w_1 \leftrightarrow_{\mathcal{R}}^* w_2$.

**Example**

- Chameleon system $\mathcal{R}$

  | | | |
  |---|---|---|
  | RB → GG | BG → RR | RG → BB |
  | BR → GG | GB → RR | GR → BB |

  is not confluent: GGG $\leftarrow_{\mathcal{R}}$ RBG $\to_{\mathcal{R}}$ RRR.
- Informally, addition on natural numbers is confluent.

## Complete Presentation

**Definition (complete presentation)**

Let $\mathcal{E}$ and $\mathcal{R}$ be sets of equations. $\mathcal{R}$ is complete presentation of $\mathcal{E}$ if

- $\leftrightarrow_{\mathcal{E}}^*$ and $\leftrightarrow_{\mathcal{R}}^*$ coincide, and
- $\mathcal{R}$ is terminating, and
- $\mathcal{R}$ has Church-Rosser property.

**Theorem**

If set of equations $\mathcal{E}$ admits complete presentation, word problem on $\mathcal{E}$ is decidable.



## Solve Word Problem by Complete Presentation

**Theorem**

If set of equations $\mathcal{E}$ admits complete presentation, word problem on $\mathcal{E}$ is decidable.

**Proof.**

Let $\mathcal{R}$ be complete presentation of $\mathcal{E}$, $w_1$ and $w_2$ strings. $\mathcal{R}$ yields following decision procedure:

1. Apply rewriting $\to_{\mathcal{R}}$ to $w_1$ and $w_2$ until no rules are applicable.
2. Compare $w_1$ and $w_2$.
   - If $w_1 = w_2$, output `YES` and terminate.
   - Otherwise, output `NO` and terminate.

□



## Completion Procedure

**Theorem (completion procedure)**

Let $\mathcal{E}$ be equations. There exists procedure $\psi$ such that $\psi(\mathcal{E})$ returns complete presentation of $\mathcal{E}$ if it terminates.

- Maxcomp is completion tool available at `https://www.jaist.ac.jp/project/maxcomp/`.
- mkbTT has web interface at `http://colo6-c703.uibk.ac.at/mkbtt/interface/index.php`.
- Mædmax is another completion tool available at `http://cl-informatik.uibk.ac.at/software/maedmax/`.



## Chameleon Problem Revisited

**Example**

Chameleon system $\mathcal{E}$

| | | |
|---|---|---|
| RB ≈ GG | BG ≈ RR | RG ≈ BB |
| BR ≈ GG | GB ≈ RR | GR ≈ BB |

admits complete presentation $\mathcal{R}$:

| | | |
|---|---|---|
| RB → GG | BG → GB | RG → GR |
| BR → GG | RR → GB | BB → GR |

RRRR $\leftrightarrow_{\mathcal{E}}^*$ BBBB does not hold since:

- RRRR $\to_{\mathcal{R}}$ GBRR $\to_{\mathcal{R}}$ GBGB $\to_{\mathcal{R}}$ GGBB $\to_{\mathcal{R}}$ GGGR
- BBBB $\to_{\mathcal{R}}$ GRBB $\to_{\mathcal{R}}$ GRGR $\to_{\mathcal{R}}$ GGRR $\to_{\mathcal{R}}$ GGGB

## Example: Group Theory

**Example**

Equational theory of group

$$x + 0 \approx x \qquad x + -x \approx 0 \qquad (x + y) + z \approx x + (y + z)$$

admits complete presentation:

$$\begin{array}{rrr}
x + 0 \to x & x + -x \to 0 & (x + y) + z \to x + (y + z) \\
0 + x \to x & -x + x \to 0 & -(x + y) \to -x + -y \\
-(-x) \to x & x + (-x + y) \to y & \\
-0 \to 0 & -x + (x + y) \to y &
\end{array}$$

Hence $x + y \approx y + x$ does not hold in general.



## Conclusion

Word problem can be solved by finding complete presentation.