

Historical Overview of Modal Logic

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- ▶ Modal logic treats reasoning that involves the concepts of 'necessity' or 'possibility', which are called 'modality'.
- ▶ Syntax: $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \Diamond\varphi$.

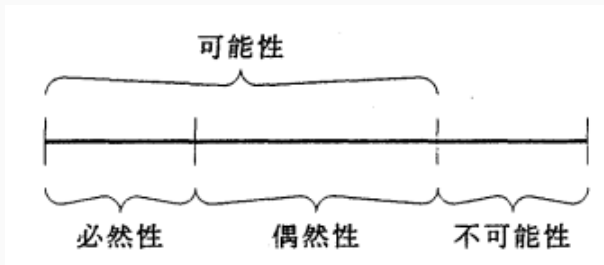
- ▶ Semantics: Kripke model $\langle W, R, V \rangle$, where
 - W is a non-empty set of possible worlds;
 - R is a binary relation on W ;
 - V is a valuation.
- ▶ Satisfaction relation: $M, w \models \Box\varphi :\Leftrightarrow$ for all v s.t. $(w, v) \in R$, $M, v \models \varphi$.
 $M, w \models \Diamond\varphi :\Leftrightarrow$ there exists v s.t. $(w, v) \in R$, $M, v \models \varphi$.
- ▶ Since S. A. Kripke developed, the semantics have been broadly used in the field.

Logic	Symbols	Expressions Symbolized
Modal Logic	\Box	It is necessary that ...
	\Diamond	It is possible that ...
Deontic Logic	O	It is obligatory that ...
	P	It is permitted that ...
	F	It is forbidden that ...
Temporal Logic	G	It will always be the case that ...
	F	It will be the case that ...
	H	It has always been the case that ...
	P	It was the case that ...
Doxastic Logic	Bx	x believes that ...
Epistemic Logic	Kx	x knows that ...

A List for a family of related logic quoted from [1].

Aristotle (384-322 B.C.E.)

- The study of alethic modality, such as necessity, possibility, contingency, and impossibility.



The relation between modalities quoted from [2]

- ▶ Dual: $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
- ▶ K: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- ▶ D: $\Box\varphi \rightarrow \Diamond\varphi$

- ▶ Arguing that facts that already happened have necessity.
- ▶ RN: $\vdash \varphi \Rightarrow \vdash \Box \varphi$

- ▶ Arguing that we live in the best of all possible worlds.
- ▶ Possible world: a being that is logically consistent world, including the actual world.

- ▶ Developing systems for strict implication S1-S5.
- ▶ Strict implication: $\neg\Diamond(p \wedge \neg q)$

- ▶ The study of the concept of 'iki' (粋).
- ▶ Translating 'modality' into '様相' in Japanese.

- ▶ Proposing **T** as the minimum system for alethic modality.

$$\mathbf{T} = \mathbf{K} + \Box\varphi \rightarrow \varphi.$$

K is a set of formulas that contains $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, and is closed under MP and RN.

- ▶ Deontic logic **D**: a modality is interpreted as 'obligatory.'

$$\mathbf{D} = \mathbf{K} + \Box\varphi \rightarrow \Diamond\varphi.$$

- ▶ Jónsson-Tarski theorem: every boolean algebra with an operator \mathfrak{A} is embeddable in the full complex algebra of its ultrafilter frame $(\mathfrak{A}_+)^+$.
- ▶ The full complex algebra \mathcal{F}^+ of \mathcal{F} is a tuple $\mathcal{F} = \langle \mathcal{P}(W), -, \cup, \emptyset, m_{R_\diamond} \rangle$, where $m_{R_\diamond} : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$ is defined by $m_{R_\diamond}(Y) = \{w \in W \mid \text{there exists } y \in Y \text{ such that } (w, y) \in R_\diamond\}$.

- ▶ Carnap's semantics: a tuple $\langle S, v \rangle$, where $S := \mathcal{P}(\mathcal{L})$. $s \in S$ is called *state description*.
- ▶ The semantics is sound and complete to **S5**.

- ▶ Kripke model.
- ▶ 'Naming and Necessity'

- Scott-Montague model: $\langle W, N, V \rangle$, where $N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$. This model can be a corresponding semantics to weaker modal logics than **K**.

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