Introduction to Linear Algebra

Contents

To the Student

The main goal of this course is to increase your ability to read and understand new mathematics, connect concepts to justify new statements, and express your reasoning clearly both orally and in writing. The problems and theorems in these notes carefully build on each other, increasing in content and difficulty. Notice this text is very short compared to most math texts, it is important that you read it carefully and repeatedly. It is essential that you struggle with each new idea in turn, finding your own solution and not seeking out existing solutions from other texts, on-line sources, or written work done by other students. Talking to other students is encouraged, but only seek out other students once you have made as much progress on the exercises and theorem for yourself as you can.

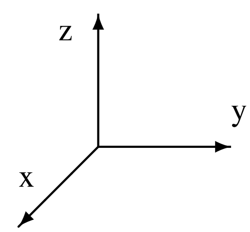
Chapter 1

Euclidean Spaces

1.1 Ordered Triples

Definition. An **ordered triple** of real numbers is written $\mathbf{v} = (v_1, v_2, v_3)$ where v_1, v_2, $v_3 \in \mathbb{R}$. Such a triple can be interpreted geometrically as either a point: indicating a position in 3 dimensional space, or as a vector: visualized as an arrow indicating direction which can occur at any position.

Unless otherwise indicated, the Cartesian coordinate system, also known as the rectangular coordinate system, will be used. In this system the x,y and z axis are perpendicular and indicate the direction for the first, second and third coordinates respectively.



As a **point** in the Cartesian coordinate system, $\mathbf{v} = (v_1, v_2, v_3)$, is the position reached by beginning where the axes meet and moving a distance of v_1 in the x direction, then v_2 in the y direction followed by v_3 in the z direction.

As a **vector** in the Cartesian coordinate system, $\mathbf{v} = (v_1, v_2, v_3)$, is visualized as an arrow which can start at any position and ends at the point obtained by moving a distance of v_1 in the x direction, then v_2 in the y direction followed by v_3 in the z direction.

Ordered triples can represent physical concepts such as position, velocity or force. They can also represent any triple of related data such as population of three related species: mice, smakes and hawks, or three related physical properties: temperature, pressure and volume.

Definition. The set of all ordered triples is called **Euclidean space** or **real 3-space**, and written: $\mathbb{R}^3$. Using set notation it is written:

$$\mathbb{R}^3 = \{(v_1, v_2, v_3) | v_1, v_2, v_3 \in \mathbb{R}\}$$

Definition. Let $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. The **norm**, also called **length** or the **magnitude** and denoted $\|\mathbf{v}\|$, is the distance of the point from the origin when $\mathbf{v}$ is interpreted as a position and the length of the arrow when $\mathbf{v}$ is interpreted as a vector.

Pythagorean Theorem. For any right triangle with sides of length a and b and hypotenuse c:

$$a^2 + b^2 = c^2$$

Exercise 1 (a). Give geometric representations of the points $\mathbf{v} = (3, -4, 0)$ and $\mathbf{w} = (3, -4, 2)$ using both point and vector interpretations.

Exercise 1 (b). For $\mathbf{v} = (3, -4, 0)$ and $\mathbf{w} = (3, -4, 2)$ find $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$. Show and justify each step.

Exercise 1 (c). Consider an ordered triple $\mathbf{v} = (v_1, v_2, v_3)$, with $\|\mathbf{v}\| = 10$. Does there exist such a $\mathbf{v}$? If so, is $\mathbf{v}$ unique? In other words is there exactly one such $\mathbf{v}$?

Theorem 1. *Let $\mathbf{v} = (v_1, v_2, v_3)$ where v_1, v_2, and v_3 are arbitrary real numbers. The norm of $\mathbf{v}$ is given by the formula:*

$$\|\mathbf{v}\| =$$

Challenge 1. In the case $\mathbf{v} = (v_1, v_2, v_3)$ is interpreted using spherical or cylindrical coordinates the norm of $\mathbf{v}$ is given by the formula:

i. $\|\mathbf{v}\|_{spherical} =$
ii. $\|\mathbf{v}\|_{cylindrical} =$

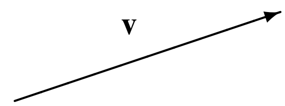
1.2 Vector Operations

Definition. Let $\mathbf{v} = (v_1, v_2, v_3)$ and $t \in \mathbb{R}$. The **scalar product** of t and $\mathbf{v}$ is defined by: $t\mathbf{v} = (tv_1, tv_2, tv_3)$

Definition. Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. The **sum** of $\mathbf{v}$ and $\mathbf{w}$ is defined by: $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$.

These are algebraic definition. It is also important to understand how these definitions affect vectors geometrically.

Exercise 2 (a). Use the vector $\mathbf{v}$ below and give geometric representations of the vectors described by the scalar products: $3\mathbf{v}$ and $-1\mathbf{v}$.



Exercise 2 (b). Given $\mathbf{v} = (1, -2, 2)$, find a vector with the same direction as $\mathbf{v}$, but with length 1, with length 10, and with length h.

Exercise 2 (c). Use the vectors $\mathbf{v}$ and $\mathbf{w}$ below to give a geometric representation of the vector which is the sum $\mathbf{v}+\mathbf{w}$. If this interpretation is applied to **w+v** is the result the same? If this interpretation is applied to $(\mathbf{v}+\mathbf{w})+\mathbf{u}$ and $\mathbf{v}+(\mathbf{w}+\mathbf{u})$ are those results the same?

Theorem 2. *For any ordered triple* $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ *there exists an ordered triple* $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ *such that when* $\mathbf{x}$ *is added to* $\mathbf{v}$ *the result is the **origin**,* $\mathbf{0} = (0,0,0)$. *For a given* $\mathbf{v}$ *such an ordered triple is unique. Geometrically the vector* $\mathbf{x}$ *can be visualized as _______.*
(Describe starting and ending positions, or direction and length.)

Also, for any ordered triples $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ *and* $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$ *there exists an ordered triple* $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ *such that when* $\mathbf{y}$ *is added to* $\mathbf{v}$ *the result is* $\mathbf{w} = (w_1, w_2, w_3)$. *For any given* $\mathbf{v}$ *and* $\mathbf{w}$ *such an ordered triple is unique. Geometrically the vector* $\mathbf{y}$ *can be visualized as _______.*
(Describe starting and ending positions, or direction and length.)

Theorem 2 provides justification for the following definitions.

Definition. Let $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. The **additive inverse** of $\mathbf{v}$ is the ordered triple, written $-\mathbf{v}$, that, when added to $\mathbf{v}$ results in the origin: $\mathbf{0} = (0,0,0)$. In symbols $-\mathbf{v}$ is the ordered triple such that $\mathbf{v} + -\mathbf{v} = 0$.

Definition. Let $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$. The **difference** of $\mathbf{w}$ from $\mathbf{v}$ is the ordered triple, written $\mathbf{w}-\mathbf{v}$, which when added to $\mathbf{v}$ results in $\mathbf{w}$. In symbols $\mathbf{w}-\mathbf{v}$ is the ordered triple such that $\mathbf{v}+(\mathbf{w}-\mathbf{v}) = \mathbf{w}$.

Discussion Question 2. Do $\mathbf{w}-\mathbf{v}$ and $\mathbf{w}+-\mathbf{v}$ describe the same ordered triple?

Challenge 2. Let $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $k \in \mathbb{R}$. Does $\|k\mathbf{v}\| = k\|\mathbf{v}\|$? Does $\|\mathbf{v}+\mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$? Include justification.

1.3 Sets of Ordered Triples

Exercise 3 (a). Give geometric representations for the set defined by

$$\mathbb{S} = \{\ t(-1,2,0) \mid t \ \in \ \mathbb{R}\ \}$$

both using point and using vector interpretations. (Unless otherwise indicated start vectors at the origin.)

Exercise 3 (b). Give geometric representations for the set defined by

$$\mathbb{T} = \{\ (3,1,0) + t(-1,2,0) \mid t \in \mathbb{R}\ \}$$

both using point and using vector interpretations.

Exercise 3 (c). Describe the set of points on the line in $\mathbb{R}^3$ that passes through points $\mathbf{v} = (1,2,3)$ and $\mathbf{w} = (2,0,-2)$.

Theorem 3. *For any two points* $\mathbf{v}$ *and* $\mathbf{w}$ *such that* $\mathbf{v} \neq \mathbf{w}$ *there exists a set of points describing a line passing through both* $\mathbf{v}$ *and* $\mathbf{w}$.

Challenge 3. For any three points $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$, such that $\mathbf{v} \neq \mathbf{w}$, and $\mathbf{u}$ is not on the line through $\mathbf{v}$ and $\mathbf{w}$, there exists a set of points describing a plane which passes through points $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$.

1.4 Systems of Linear Equations

Definition. A **linear equation** is an equation of the form:

$$a_1x + a_2y + a_3z = b$$

where the a_i represent specific real numbers called **constants**, b is also a real number and x, y and z are **variables** or **unknowns**. The **solution** to a linear equation is the set of ordered triples of real numbers, (x,y,z) for which the equation is true.

It is often important to be able to find solutions that work for multiple linear equations. To solve such a system of linear equations, transform it into simpler and simpler systems whose solutions are identical to the given system. Then use the simplified system to describe the values of x, y and z that make all equations in the system true, and give the result using set notation: $\mathbb{S} = \{(\quad,\quad,\quad) \mid \qquad\quad \}$. This set may be empty, contain one ordered triple or more than one ordered triple.

Definition. The three ways that can be used to transform a system of equations into a new system with exactly the same solutions are called **elementary row operations**. They are:

1. rearrange the order of the rows
2. multiply a row by a non-zero number
3. add a multiple of one row to another

When working with systems of equations it is more efficient to write only the constants, making sure to keep them in columns so their variables can be reattached once the matrix is fully simplified. The result is called an augmented matrix.

$$\begin{bmatrix} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3 \end{bmatrix} \quad or \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix}$$

The elementary row operations apply to either the system of equations or the associated matrix. Ideally, but not always, the most simplified version looks like:

$$\begin{bmatrix} x & & & = & s_1 \\ & y & & = & s_2 \\ & & z & = & s_3 \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & 0 & 0 & : & s_1 \\ 0 & 1 & 0 & : & s_2 \\ 0 & 0 & 1 & : & s_3 \end{bmatrix}$$

The solution in this case is a set with one ordered triple:

$$\mathbb{S} = \{(x,y,z) |\, x = s_1, y = s_2, z = s_3\} = \{(s_1, s_2, s_3)\}$$

Exercise 4 (a). Solve the following system of equations by transforming it into simpler and simpler systems of linear equations whose solution is identical to the given system.

$$\begin{bmatrix} 3x & + & 2y & - & z & = & 1 \\ x & + & y & & & = & 6 \\ & & -y & + & z & = & 0 \end{bmatrix} \quad or \quad \begin{bmatrix} 3 & 2 & -1 & : & 1 \\ 1 & 1 & 0 & : & 6 \\ 0 & -1 & 1 & : & 0 \end{bmatrix}$$

Exercise 4 (b). Solve the following system of equations. How many solutions are there?

$$\begin{bmatrix} 2x & + & 4y & + & 6z & = & 8 \\ x & + & 3y & + & 5z & = & 7 \\ 2x & + & 5y & + & 8z & = & 10 \end{bmatrix}$$

Exercise 4 (c). Solve the following system of equations. How many solutions are there?

$$\begin{bmatrix} 2x & + & 4y & + & 6z & = & 8 \\ x & + & 3y & + & 5z & = & 7 \\ 2x & + & 5y & + & 8z & = & 11 \end{bmatrix}$$

Write the solution in each of the following ways:

$$\mathbb{S} = \{\ (x,y,z) \mid x = \qquad , y = \qquad , z = \qquad \}$$
$$\mathbb{S} = \{\ (\qquad , \qquad , \qquad) \mid t \in \mathbb{R}\}$$
$$\mathbb{S} = \{\ (\quad , \quad , \quad) + t(\quad , \quad , \quad) \mid t \in \mathbb{R}\}$$

Theorem 4. *There exists a way to transform any system of linear equations in three unknowns into a well defined simplest form using only the three elementary row operations. This simplest form will allow any conditions on the three unknowns to be quickly identified.*
(Write an algorithm for simplifying any given system of equations in three unknowns. Describe how to identify when it is completely simplified and how to then identify the solution. Use the term "leading 1" for when 1 is the first non-zero number in a row.)

Discussion Question 4. Why are only the elementary row operations allowed when solving a system of equations? As a system of equations is simplified: What produces the possibility for no solutions? What produces the possibility for an infinite number of solutions? What might a fully reduced matrix look like in each case? Consider systems of two, three or four equations.

1.5 Spanning and Linear Independence

Definition. A vector $\mathbf{v}$ is a **linear combination of the vectors:** $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ if $\mathbf{v} = t_1\mathbf{v_1} + t_2\mathbf{v_2} + \cdots + t_n\mathbf{v_n}$ for some choice of real numbers t_i. The t_i are called coefficients.

Definition. The set of all linear combinations of the vectors from non-empty set $\mathbb{S} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is called the **span** of $\mathbb{S}$, and written:

$$span\mathbb{S} = \{t_1\mathbf{v_1} + t_2\mathbf{v_2} + \cdots + t_n\mathbf{v_n} \mid t_i \in \mathbb{R}\}$$

Exercise 5 (a). Is $(1,0,-1)$ in the span of $\mathbb{S} = \{(1,2,3),(4,5,6),(7,8,9)\}$? If so write $(1,0,-1)$ as a linear combination of the vectors in $\mathbb{S}$.

Exercise 5 (b). For each of the following sets $\mathbb{S}$, give a geometric description of $span\mathbb{S}$.

i. $\mathbb{S} = \{(0,1,0)\}$
ii. $\mathbb{S} = \{(5,-2,7)\}$
iii. $\mathbb{S} = \{(0,0,0)\}$
iv. $\mathbb{S} = \{(1,0,0),(0,1,1)\}$
v. $\mathbb{S} = \{(6,3,-9),(-4,-2,6)\}$
vi. $\mathbb{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$

vii. $\mathbb{S} = \{(1,0,0),(1,1,0),(1,1,1)\}$
viii. $\mathbb{S} = \{(2,0,2),(0,3,0),(1,1,1)\}$

When new vectors are made using linear combinations of vectors from a given set, $\mathbb{S}$, an important questions is: "Can all vectors in $\mathbb{R}^3$ be expressed as a linear combination of the vectors in $\mathbb{S}$?"

Definition. A set of vectors in $\mathbb{R}^3$ is said to **span** $\mathbb{R}^3$ if $span\mathbb{S} = \mathbb{R}^3$. This means that every vector in $\mathbb{R}^3$ is a linear combination of the vectors in $\mathbb{S}$. (Span has now been defined as a noun and as a verb.)

Exercise 5 (c). Determine algebraically if the following sets span $\mathbb{R}^3$. Showing a set spans $\mathbb{R}^3$ requires showing that any point $\mathbf{a} \in \mathbb{R}^3$ can be written as a linear combination of the vectors in the given set. Specifically, for $\mathbb{S} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, let $\mathbf{a} = (a,b,c) \in \mathbb{R}^3$ and find values for the real numbers $t_1, t_2, \ldots, t_n$ such that $\mathbf{a} = t_1\mathbf{v_1} + t_2\mathbf{v_2} + \cdots + t_n\mathbf{v_n}$. If it is not the case that any point in $\mathbb{R}^3$ can be written as a linear combination of the vectors in $\mathbb{S}$, describe the set of points that can. Where possible write $\mathbf{a} = (1,2,3)$ as a linear combination of the vectors in $\mathbb{S}$ in more than one way.

i. $\mathbb{S} = \{(1,2,0),(0,0,1),(2,4,6)\}$
ii. $\mathbb{S} = \{(1,0,0),(1,1,0),(1,1,1)\}$
iii. $\mathbb{S} = \{(1,0,0),(1,1,0),(1,1,1),(0,1,1)\}$
iv. $\mathbb{S} = \{(1,1,0),(1,1,1)\}$

Definition. A set $\mathbb{S} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is **linearly independent** if for any point in $span\mathbb{S}$ the choice of coefficients used to describe that point as a linear combination of the elements of $\mathbb{S}$ is unique.

Definition. A set that both spans $\mathbb{R}^3$ and is linearly independent is called a **basis** for $\mathbb{R}^3$.

The most familiar basis is $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$ where $(1,2,3) = 1(1,0,0)+2(0,1,0)+3(0,0,1)$. However sometimes it may be useful to use $\alpha = \{(1,1,1),(0,1,1),(0,0,1)\}$, making $(1,2,3) = 1(1,1,1)+1(0,1,1)+1(0,0,1)$. This is called making a change of basis and the list of coefficients written:

$$[(1,2,3)]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad [(1,2,3)]_\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Theorem 5. *Let σ and τ be two bases for $\mathbb{R}^3$. The process of taking $[\mathbf{v}]_\sigma$, the list of the coefficients needed to describe some point $\mathbf{v} \in \mathbb{R}^3$ using the basis σ to $[\mathbf{v}]_\tau$, a list of coefficients to describe that same point using the basis τ, is a well defined, 1-1 function that maps onto the set of all possible such lists.*

Chapter 2

Linear Spaces

2.1 Additive Inverses

The notes have been focusing on $\mathbb{R}^3$, but what about $\mathbb{R}^2$, can the same things be done there? or $\mathbb{R}^5$? What about any collection that behaves like $\mathbb{R}^3$ in that elements can be added together and multiplied by real numbers? Examples of such are 2×2 matrices and polynomials. It is a powerful tool to have a small set of elements that, by making linear combinations, can be used to represent the entire collection. This chapter generalizes the important ideas from Chapter 1.

Definition. A **linear space** is a set $\mathbb{L}$ containing objects called points where for any points $\mathbf{P}$ and $\mathbf{Q}$ in $\mathbb{L}$ and any real number t there exist unique points:

- $\mathbf{P}+\mathbf{Q}$ called the sum of $\mathbf{P}$ and $\mathbf{Q}$
- $t\mathbf{P}$ called the scalar product of t and $\mathbf{P}$

and where the following axioms are satisfied for any points $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R} \in \mathbb{L}$ and any $a, b \in \mathbb{R}$:

1. $\mathbf{P}+\mathbf{Q} = \mathbf{Q}+\mathbf{P}$
2. $\mathbf{P}+(\mathbf{Q}+\mathbf{R}) = (\mathbf{P}+\mathbf{Q})+\mathbf{R}$
3. $a(b\mathbf{P}) = (ab)\mathbf{P}$
4. $(a+b)\mathbf{P} = a\mathbf{P}+b\mathbf{P}$
5. $a(\mathbf{P}+\mathbf{Q}) = a\mathbf{P}+a\mathbf{Q}$
6. There exists a point $\mathbf{0}$, called the **additive identity**, such that $\mathbf{P}+\mathbf{0} = \mathbf{P}$
7. $a\mathbf{P} = \mathbf{0}$ if and only if $a = 0$ or $\mathbf{P} = \mathbf{0}$

From this definition, the goal is to discover as many consequences as possible. This often involves working with equations. When working with equations one side of an equation may be changed if the other is changed in exactly the same way. Also, an object may be substituted with another to which it is equal.

Exercise 6 (a). For points **P**, **Q**, and **R** in linear space $\mathbb{L}$, which of the following are also defined points in $\mathbb{L}$?

i. $1\mathbf{Q}$
ii. $1+\mathbf{Q}$
iii. $\mathbf{PQ}$
iv. $3\mathbf{P}+-1\mathbf{Q}$
v. $\mathbf{P}-\mathbf{Q}$
vi. $3\mathbf{P}+5$
vii. $\mathbf{P}+\mathbf{Q}+\mathbf{R}$

Exercise 6 (b). Does $\mathbf{0}=0$? Explain.

Exercise 6 (c). For point **P** in linear space $\mathbb{L}$, and $c \in \mathbb{R}$, does it follow that $0\mathbf{P}=\mathbf{0}$? Does it follow that $c\mathbf{0}=\mathbf{0}$? Is there a property that says $1\mathbf{P}=\mathbf{P}$?

Theorem 6. *For any point* $\boldsymbol{P}$ *in linear space* $\mathbb{L}$*, there exists a point* $\boldsymbol{X}$ *such that* $\boldsymbol{P}+\boldsymbol{X}=\boldsymbol{0}$*. In addition, for any given* $\boldsymbol{P}$ *this point is unique.*

Definition. Let $\mathbf{P} \in \mathbb{L}$. The **additive inverse** of the point **P** is the point, written $-\mathbf{P}$, that when added to **P** results in the additive identity element, $\mathbf{0} \in \mathbb{L}$. In symbols: $-\mathbf{Q}$ is the point such that $\mathbf{P}+-\mathbf{P}=\mathbf{0}$.

Discussion Questions 7. The symbol ”$-$” is now used to indicate the additive inverse of a real number and the additive inverse of a point which makes $-1\mathbf{P}$ ambiguous. How can parenthesis be used to clarify which meaning of ”$-$” is intended?

2.2 The Difference of Q from P

To define a specific Linear Space a set of points must be given and the actions of sum and scalar product described for these points.

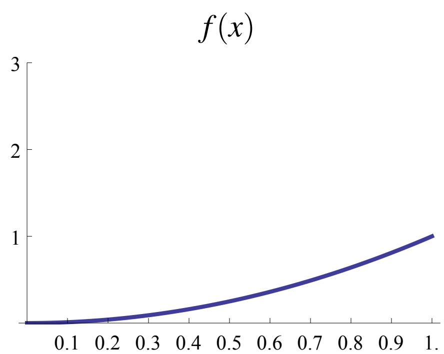
Exercise 7 (a). Does the set $\mathbb{R}^3$ with operations as defined in Chapter 1 satisfy all the conditions to be a linear space?

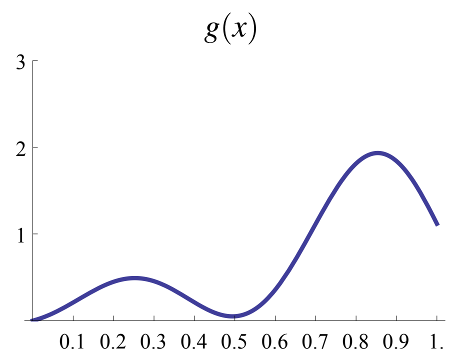
Exercise 7 (b). Which conditions of linear space does the set $\mathbb{R}^2$ satisfy when the sum and scalar product are defined in each of the following ways:

i. $(p_1,p_2)+(q_1,q_2)=(0,0)$ and $t(p_1,p_2)=(0,0)$
ii. $(p_1,p_2)+(q_1,q_2)=(p_1\cdot q_1,p_2\cdot q_2)$ and $t(p_1,p_2)=(p_1^t,p_2^t)$

Definition. The **set of all continuous functions defined on the interval [0,1]** is denoted: C[0,1]. Given functions f and g, and real number c, define the functions $f+g$ and cf by what they do to a given real number x, in this case: $(f+g)(x) := f(x)+g(x)$ and $(cf)(x) := c \cdot f(x)$. Recall, a function f is continuous at x if $\lim_{w \to x} f(w) = f(x)$.

Exercise 7 (c). For the graphs of continuous functions f and g given below, sketch graphs of the functions $f+g$ and $3g$. Are these functions continuous on [0,1]? Will the sum of continuous functions always be continuous? Will a scalar multiple of a continuous function always be continuous? A good justification will use the definition of continuous.





Theorem 7. *For any points $\boldsymbol{P}$ and $\boldsymbol{Q}$ in linear space $\mathbb{L}$, there is a unique point $\boldsymbol{Y}$ such that $\boldsymbol{P}+\boldsymbol{Y}=\boldsymbol{Q}$.*

Definition. Let $\mathbf{P}$ and $\mathbf{Q}$ be points in linear space $\mathbb{L}$. The **difference of Q from P** is the point, written $\mathbf{Q}-\mathbf{P}$, that when added to $\mathbf{P}$ results in $\mathbf{Q}$. In symbols: $\mathbf{Q}-\mathbf{P}$ is the vector such that $\mathbf{P}+(\mathbf{Q}-\mathbf{P})=\mathbf{Q}$.

2.3 Distributing over a Difference

Definition. The **set of all polynomials** is $\mathbb{P} = \{c_0 + c_1x + c_2x^2 + \cdots + c_kx^k \mid c_i \in \mathbb{R}, k \in \mathbb{Z}^+\}$. The **set of polynomials of degree less than n** is $\mathbb{P}_n = \{c_0 + c_1x + c_2x^2 + \cdots + c_kx^k \mid c_i \in \mathbb{R}, k \in \mathbb{Z}^+, k < n\}$. In both cases addition and scalar multiplication are done in the traditional way.

Exercise 8 (a). Is $\mathbb{P}$ a linear space?

Exercise 8 (b). Is the set of polynomials of degree exactly 3 a linear space? Is the set of polynomials of degree less than 3 a linear space? Why is $\mathbb{P}_n$ defined the way it is?

Exercise 8 (c). Write Theorem 8 as a sentence in English.

Theorem 8. *For any points $\boldsymbol{P}$ and $\boldsymbol{Q}$ in linear space $\mathbb{L}$, and any $c \in \mathbb{R}$:*

$$c(\boldsymbol{Q}-\boldsymbol{P}) = c\boldsymbol{Q} - c\boldsymbol{P}$$

Challenge 8. Is $\mathbf{0}$, the additive identity element in a linear space, unique?

2.4 When the Scalar Multiple is 1

Exercise 9 (a). What is the notation for the additive inverse of $-\mathbf{Q}$? Find a vector other than $\mathbf{Q}$ which satisfies the definition of additive inverse for $-\mathbf{Q}$.

Exercise 9 (b). Show: $-(-\mathbf{Q}) = 1\mathbf{Q}$.

Exercise 9 (c). Can a linear space have exactly one element? Include justification.

Theorem 9. *For any point $\boldsymbol{P}$ in linear space $\mathbb{L}$:*

$$1\boldsymbol{P} = \boldsymbol{P}$$

Challenge 9. Can a linear space have exactly two elements?

2.5 Subspaces

Definition. A subset $\mathbb{M}$ of a linear space $\mathbb{L}$ is a **subspace** of $\mathbb{L}$ if $\mathbb{M}$ is itself a linear space. This means:

$\mathbb{M} \subseteq \mathbb{L}$

- $\mathbb{M}$ is closed under addition: $\mathbf{P}, \mathbf{Q} \in \mathbb{M} \Rightarrow \mathbf{P} + \mathbf{Q} \in \mathbb{M}$
- $\mathbb{M}$ is closed under scalar multiplication: $c \in \mathbb{R}, \mathbf{P} \in \mathbb{M} \Rightarrow \mathbf{cP} \in \mathbb{M}$

$\mathbb{M}$ satisfies Axioms $1-7$ for linear spaces

Exercise 10 (a). Is $\mathbb{R}^3 \subseteq \mathbb{R}^3$? Is $\{\} \subseteq \mathbb{P}$? Is $\mathbb{R}^2 \subseteq \mathbb{R}^3$? Is $\mathbb{P} \subseteq C[0,1]$?

Exercise 10 (b). Write out several elements in the sets described using set notation below:

i. $\mathbb{S} = \{x \mid 3x \in \mathbb{Z}\}$
ii. $\mathbb{S} = \{3x \mid x \in \mathbb{Z}\}$

Exercise 10 (c). Which of the following sets $\mathbb{M}$ are subspaces of the given linear space? If a set is not a subspace which properties of subspace does it fail to satisfy?

i. $\mathbb{M} = \{(x,y,0) \mid x,y \in \mathbb{R}\} \subseteq \mathbb{R}^3$
ii. $\mathbb{M} = \{(x,y,z) \mid z = x+y,\ x,y \in \mathbb{R}\} = \{(x,y,x+y) \mid x,y \in \mathbb{R}\} \subseteq \mathbb{R}^3$
iii. $\mathbb{M} = \{(x,y,z) \mid x,y,z \in \mathbb{R}^+\} \subseteq \mathbb{R}^3$
iv. $\mathbb{M} = \{c_0 + c_1x + c_2x^2 \mid c_i \in \mathbb{Z}\} \subseteq \mathbb{P}$
v. $\mathbb{M} = \{f(x) \in C[0,1] \mid f(0) = \frac{1}{4}\} \subseteq C[0,1]$
vi. $\mathbb{M} = \{f(x) \in C[0,1] \mid f(\frac{1}{4}) = 0\} \subseteq C[0,1]$
vii. $\mathbb{M} = \{f(x) \in C[0,1] \mid f'(x) \in C[0,1]\} \subseteq C[0,1]$

Theorem 10. *Given a non-empty subset $\mathbb{M}$ of a linear space $\mathbb{L}$. $\mathbb{M}$ is a subspace of $\mathbb{L}$ if and only if $\mathbb{M}$ is closed under both addition and multiplication.*

Challenge 10. A non-empty subset $\mathbb{M}$ of a linear space $\mathbb{L}$ is a subspace of $\mathbb{L}$ if and only if for any $\mathbf{P}$ and $\mathbf{Q}$ in $\mathbb{M}$ and every pair of real numbers a and b, $a\mathbf{P} + b\mathbf{Q} \in \mathbb{M}$.

2.6 Span

Definition. A point $\mathbf{P}$ is said to be a **linear combination of the points**: $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ if $\mathbf{P} = t_1\mathbf{P}_1 + t_2\mathbf{P}_2 + \cdots + t_n\mathbf{P}_n$ for some list of real numbers t_i.

Definition. The set of all linear combinations using points from a non-empty set
$\mathbb{S} = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n\}$ is called the **span** of $\mathbb{S}$, and written:

$$span\mathbb{S} = \{\ t_1\mathbf{P}_1 + t_2\mathbf{P}_2 + \cdots + t_n\mathbf{P}_n \mid t_i \in \mathbb{R}\ \}$$

Definition. A set $\mathbb{S}$ of points in $\mathbb{L}$ is said to **span** $\mathbb{L}$ if $span\mathbb{S} = \mathbb{L}$. This means that every point in $\mathbb{L}$ is a linear combination of the points in $\mathbb{S}$. (Note: Span has again been defined as a noun and as a verb.)

Exercise 11 (a). Describe the span of $\mathbb{S} = \{(1,0,1),(0,1,1)\}$. Does $\mathbb{S}$ span $\mathbb{R}^3$?

Exercise 11 (b). Does $\mathbb{S} = \{x, x^2\}$ span $\mathbb{P}_3$? If not, find a set that does.

Exercise 11 (c). Find two different sets that span $\mathbb{R}^3$?

Theorem 11. *If $\mathbb{S}$ is a nonempty subset of a linear space $\mathbb{L}$, then $span\mathbb{S}$ is a subspace of $\mathbb{L}$. Moreover, $span\mathbb{S}$ is the smallest subspace of $\mathbb{L}$ containing $\mathbb{S}$, in other words, any subspace of $\mathbb{L}$ containing $\mathbb{S}$ must also contain $span\mathbb{S}$.*

Discussion Question 11. What definition for $span\{\}$ would make Theorem 11 true without the condition that $\mathbb{S}$ be non-empty?

2.7 Linear Independence

Definition. A subset $\mathbb{S} = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n\}$ of a linear space $\mathbb{L}$ is **linearly independent** if every point $\mathbf{Q}$ in $span\mathbb{S}$ can be written in only one way as a linear combination using elements of $\mathbb{S}$. More specifically, for any real numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$:

$$a_1\mathbf{P}_1 + a_2\mathbf{P}_2 + \cdots + a_n\mathbf{P}_n = b_1\mathbf{P}_1 + b_2\mathbf{P}_2 + \cdots + b_n\mathbf{P}_n \Longrightarrow a_1 = b_1, \dots a_n = b_n$$

A subset $\mathbb{S}$ is **linearly dependent** if $\mathbb{S}$ is not linearly independent.

Exercise 12 (a). Show how to write the point (a,b,c) as a linear combination of the points in $\mathbb{S} = \{(1,2,0),(1,0,0),(1,1,0)\}$. Is the set $\mathbb{S}$ linearly independent? If it is not, show how some point can be written in more than one way as a linear combination of the elements of $\mathbb{S}$.

Exercise 12 (b). What is the additive identity element in the linear space $\mathbb{R}^n$? ...in the linear space $\mathbb{P}$? ..in the linear space C[0,1]?

Exercise 12 (c). Describe the steps needed to prove:

$$(A \Longrightarrow B) \Longleftrightarrow (C \Longrightarrow D)$$

Theorem 12. *Let $\mathbb{S} = \{\boldsymbol{P}_1, \boldsymbol{P}_2, \dots, \boldsymbol{P}_n\}$ be a subset of linear space $\mathbb{L}$. Then the set $\mathbb{S}$ is linearly independent if and only if for all real numbers $c_1, c_2, \dots, c_n$, $c_1\boldsymbol{P}_1 + c_2\boldsymbol{P}_2 + \cdots + c_n\boldsymbol{P}_n = \boldsymbol{0}$ implies all of the coefficients c_i are zero.*

2.8 When a Set Contains 0

Exercise 13 (a). Negate the following:

i. It is raining $\Longrightarrow$ there are clouds in the sky
ii. For any math problem there is a solution.
iii. For all integers, prime $\Longrightarrow$ odd
iv. $x = 1,\ y = 2$, and $z = 3$

Discussion Question 13. Do the following mean the same thing?
"For any ..." "For all..." "For every..."

Exercise 13 (b). Negate the definition of $\mathbb{S}$ being linearly independent to get a definition for $\mathbb{S}$ being linearly dependent.

Exercise 13 (c). Show how to write the point (a,b,c) as a linear combination of the points in $\mathbb{S} = \{(1,2,0),(1,0,0),(1,1,1)\}$. Is the set $\mathbb{S}$ linearly independent?

Theorem 13. *If $\mathbb{S}$ is a subset of a linear space $\mathbb{L}$ and $\boldsymbol{0} \in \mathbb{S}$, then $\mathbb{S}$ is linearly dependent.*

2.9 Linear Dependence

Exercise 14 (a). Is the set $\mathbb{S} = \{x^2 - 1,\ x+1,\ x-1\}$ linearly independent in $\mathbb{P}_3$?

Exercise 14 (b). Is the set $\mathbb{S} = \{1,\ sin^2(x),\ cos^2(x)\}$ linearly independent in C[0,1]?

Exercise 14 (c). Use Theorem 12 to write another condition for $\mathbb{S}$ to be linearly dependent.

Theorem 14. *Let $\mathbb{S}$ be a subset of a linear space $\mathbb{L}$ containing more than one point. $\mathbb{S}$ is linearly dependent if and only if some point of $\mathbb{S}$ is a linear combination of the other points of $\mathbb{S}$.*

Discussion Question 14. The definition, Theorem 12, and Theorem 14 give three equivalent criteria for a set to be linearly independent. Which matches the meaning of "independent" most closely? Which seems easiest to use to prove set is linearly independent? Which makes the most sense paired with the definition of span? What do traditional Linear Algebra texts have for the definition of linearly independent?

2.10 Extending Linearly Independent Sets

Exercise 15 (a). Describe how to show a set $\mathbb{S}$ spans a linear space $\mathbb{L}$. Describe 3 ways to show a set $\mathbb{S}$ is linearly independent.

Exercise 15 (b). Show $\{(1,1,0,0),(0,1,1,0),(0,0,1,1)\}$ is linear independent in $\mathbb{R}^4$. Does is also span $\mathbb{R}^4$?

Exercise 15 (c). Given that $\mathbb{S} = \{(1,0,0),(0,1,0)\}$ is linearly independent in $\mathbb{R}^3$, find a point $\mathbf{P}$ such that $\mathbb{S} \cup \{\mathbf{P}\}$ is also linearly independent. Describe all possible such points.

Theorem 15. *If $\mathbb{S}$ is a linearly independent subset of $\mathbb{L}$ and $\boldsymbol{P}$ is a point of $\mathbb{L}$, not in $span(\mathbb{S})$, then $\mathbb{S} \cup \{\boldsymbol{P}\}$ is also linearly independent.*

Definition. A finite subset $\mathbb{B}$ of a linear space $\mathbb{L}$ is a **basis** if each point in $\mathbb{L}$ can be written in one and only one way as a linear combination of elements of $\mathbb{B}$. In other words, $\mathbb{B}$ is a basis for $\mathbb{L}$ if it spans $\mathbb{L}$ and is linearly independent.

Discussion Question 15. For infinite subsets $\mathbb{B}$ of $\mathbb{L}$ define $span(\mathbb{B})$ to be the set of all linear combinations involving a finite number of element of $\mathbb{B}$. With this addition the definitions of span and linearly independent can be extended to infinite subsets of $\mathbb{L}$. Thus a basis may contain an infinite number of elements; however, only a finite number of them may be used when making linear combinations. What problems could occur if infinite sums were allowed for linear combinations?

2.11 Maximal Linearly Independent Sets

Exercise 16 (a). Is $\mathbb{S} = \{(1,2,3),(1,2,0),(1,0,0)\}$ a basis for $\mathbb{R}^3$? Explain.

Exercise 16 (b). Is $\mathbb{S} = \{x^2+1, x+1, x-1\}$ a basis for $\mathbb{P}_3$? Explain.

Exercise 16 (c). State Theorem 15 in words.

Theorem 16. *$\mathbb{S}$ is a basis for $\mathbb{L}$ if and only if it is a maximal linearly independent subset of $\mathbb{L}$. In other words, $\mathbb{S}$ is a basis for $\mathbb{L}$ if and only if $\mathbb{S}$ is linearly independent and not a proper subset of any other linearly independent set.*

2.12 Reducing a Spanning Set

Exercise 17 (a). Show how to write the point $\mathbf{A} = (a,b,c)$ as a linear combination of the points in $\mathbb{S} = \{(1,0,-1),(1,2,3),(3,2,1),(0,1,0)\}$. Are all the points in $\mathbb{S}$ necessary?

Exercise 17 (b). Find a subset of $\mathbb{S} = \{(1,0,-1),(1,2,3),(3,2,1),(0,1,0)\}$ that spans $\mathbb{R}^3$.

Exercise 17 (c). Find a subset of $\mathbb{S} = \{(1,0,-1),(1,2,3),(3,2,1),(0,1,0)\}$ that does not span $\mathbb{R}^3$.

Theorem 17. *If $\mathbb{S}$ spans $\mathbb{L}$ and $\boldsymbol{P}$ is a point of $\mathbb{S}$ such that $\boldsymbol{P} \in span(\mathbb{S} - \{\boldsymbol{P}\})$ then $\mathbb{S} - \{\boldsymbol{P}\}$ also spans $\mathbb{L}$.*

2.13 Minimal Spanning Sets

Exercise 18 (a). Find a basis for $\mathbb{R}^n$. Now find a different basis for $\mathbb{R}^n$.

Exercise 18 (b). Find two different bases for $\mathbb{P}_n$. Find a basis for $\mathbb{P}$.

Exercise 18 (c). State theorem 17 in words.

Theorem 18. *$\mathbb{S}$ is a basis for $\mathbb{L}$ if and only if $\mathbb{S}$ is a minimal spanning set for $\mathbb{L}$. In other words $\mathbb{S}$ is a basis for $\mathbb{L}$ if and only if $\mathbb{S}$ spans $\mathbb{L}$ and no proper subset of $\mathbb{S}$ spans $\mathbb{L}$.*

2.14 The Replacement Lemma

Exercise 19 (a). Is the set $\{1, x, x^2, x^3, \ldots, x^n, \ldots\}$ a basis for C[0,1]?

Exercise 19 (b). Consider the set $\mathbb{S} = \{(1,0,1),(0,1,0)\}$ which spans some subspace $\mathbb{L} \subseteq \mathbb{R}^3$. Notice $(2,3,2)$ is a linear combination of the points in $\mathbb{S}$ since $(2,3,2) = 2(1,0,1)+3(0,1,0)$. Consider the set $\mathbb{S}' = \{(1,0,1),(2,3,2)\}$. Show how the point $(0,1,0)$ can be written as a linear combination of the points in $\mathbb{S}'$? Does $\mathbb{S}'$ also span $\mathbb{L}$? Explain.

Exercise 19 (c). If $\mathbf{Q} = 2\mathbf{P}_1 + 3\mathbf{P}_2 + 0\mathbf{P}_3 + 4\mathbf{P}_4 + -1\mathbf{P}_5 + 0\mathbf{P}_6$ which **P**'s can be written as linear combinations **Q** and the other **P**'s?

Theorem 19 (Replacement Lemma)**.** *Suppose that $\mathbb{S}$ spans $\mathbb{L}$, $\boldsymbol{Q} \in \mathbb{L}$ and $\boldsymbol{P}$ is a point of $\mathbb{S}$ such that when $\boldsymbol{Q}$ is written as a linear combination of points of $\mathbb{S}$, the coefficient of $\boldsymbol{P}$ is not zero. If $\mathbb{S}'$ is the set obtained from $\mathbb{S}$ by replacing $\boldsymbol{P}$ with $\boldsymbol{Q}$, then $\mathbb{S}'$ also spans $\mathbb{L}$.*

2.15 Preparing to Define Dimension

Discussion Question 20. What problems might occur if the Dimension of a Vector Space were defined to be the number of elements in a basis? How is this similar to other situations where the term 'well defined' is used?

Exercise 20 (a). Does Theorem 16 say there is no linearly independent set with more elements than a basis? Explain.

Exercise 20 (b). Does Theorem 18 say there is no spanning set with fewer elements than a basis?

Exercise 20 (c). Find another way to state Theorem 20.

Theorem 20. *Considering only finite subsets of $\mathbb{L}$, no linearly independent set has more points than a spanning set.*

2.16 Properties of Linearly Independent Sets

Exercise 21 (a). Is there an infinite subset in C[0,1] which is linearly independent? To demonstrate such a set one option is to use pictures and establish a pattern.

Exercise 21 (b). Describe how to extend a linearly independent set to get a basis. Will this always work?

Exercise 21 (c). Use negation to rewrite Theorem 16.

Theorem 21. *If $\mathbb{L}$ has a basis $\mathbb{B} = \{\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n\}$ with a finite number of points, then the following hold:*

- *i. No linearly independent set contains more than n points.*
- *ii. Every linearly independent set with n points is a basis.*
- *iii. Every linearly independent set is contained in a basis.*

2.17 Properties of Spanning Sets

Exercise 22 (a). Show that no finite set spans C[0,1].

Exercise 22 (b). Describe how to reduce a spanning set to get a basis. Will this always work?

Exercise 22 (c). Use negation to rewrite Theorem 18.

Theorem 22. *If $\mathbb{L}$ has a basis $\mathbb{B} = \{\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n\}$ with a finite number of points, then the following hold:*

i. No spanning set contains fewer than n points.
ii. Every spanning set with n points is a basis.
iii. Every finite spanning set containes a basis.

Theorems 21 and 22 together give the result that if a linear space has a basis with *n* points, then every basis must have *n* points. This observation leads to the following important definition.

Definition. If a linear space $\mathbb{L}$ has a basis with *n* elements $\mathbb{L}$ is called an **n dimensional vector space**. If there is no finite set that forms a basis for $\mathbb{L}$, $\mathbb{L}$ is said to be **infinite dimensional**.

2.18 When the Vectors are the Rows

Until now our ordered n-tuples have appeared in matrices as columns. They can also be placed as rows. The following Theorem shows what can be learned by row-reducing such a matrix.

Exercise 23 (a). Put the points of $\mathbb{S} = \{(3,0,-3),(1,2,3)\}$ into a matrix as rows then row reduce the resulting matrix. Use $\mathbb{S}'$ to denote the rows of the resulting matrix. What does Theorem 23 say about *span*$\mathbb{S}'$?

Exercise 23 (b). Find a "nicer" set that spas the same space as the set spanned by $\mathbb{S} = \{(3,0,-3),(1,2,3),(0,1,0)\}$. Is this new set linearly independent?

Exercise 23 (c). Find a "nicer" set that spans the same space as the set spanned by $\mathbb{S} = \{(3,0,-3),(1,2,3),(3,2,1),(0,1,0)\}$. Is this new set linearly independent?

Theorem 23. *Let $\mathbb{S}$ be a subset of a linear space $\mathbb{R}^n$ and let $\boldsymbol{P}$ be in $\mathbb{S}$. Assume that $\boldsymbol{Q}$ is obtained from $\boldsymbol{P}$ either by*

i. multiplying $\boldsymbol{P}$ by a non-zero number ***or***
ii. adding to $\boldsymbol{P}$ a scalar multiple of another element of $\mathbb{S}$

Let $\mathbb{S}'$ be obtained from $\mathbb{S}$ by replacing $\boldsymbol{P}$ with $\boldsymbol{Q}$. Then $\mathbb{S}'$ will span the same linear space as $\mathbb{S}$. In other words: $span\mathbb{S}' = span\mathbb{S}$.

In addition, if $\mathbb{S}''$ consists of the non-zero rows of a fully simplified matrix, then $\mathbb{S}''$ is linearly independent.

Chapter 3

Linear Transformations

Functions play a central role in mathematics and linear combinations play a central role in Linear Algebra. We want a type of function that respects the organization of points into linear combinations. Throughout $\mathbb{L}_1, \mathbb{L}_2$ and $\mathbb{L}_3$ will be general linear spaces.

3.1 Properties of Linear Transformations

Definition. A **linear transformation** is a function $f : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$, such that for any $\mathbf{P}$, $\mathbf{Q} \in \mathbb{L}_1$, and $c \in \mathbb{R}$:

1. $f(\mathbf{P}+\mathbf{Q}) = f(\mathbf{P}) + f(\mathbf{Q})$
2. $f(c\mathbf{P}) = cf(\mathbf{P})$

Exercise 24 (a). If $T : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ is a linear transformation explain why:

i. $T(\mathbf{0}) = \mathbf{0}$
ii. $T(-\mathbf{A}) = -T(\mathbf{A})$ for any $\mathbf{A} \in \mathbb{L}_1$
iii. $T(\mathbf{A}-\mathbf{B}) = T(\mathbf{A}) - T(\mathbf{B})$ for any $\mathbf{A}$, $\mathbf{B} \in \mathbb{L}_1$

Exercise 24 (b). Which of the following are functions?

i. $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(\frac{a}{b}) = ab$
ii. $g : \mathbb{Q} \to \mathbb{Q}$ defined by $g(\frac{a}{b}) = \frac{b}{a}$
iii. $h : \mathbb{Q}^2 \to \mathbb{Q}$ defined by $h(\frac{a}{b}, \frac{c}{d}) = \frac{ac}{bd}$
iv. $j : \mathbb{Q} \to \mathbb{Q}^2$ defined by $j(\frac{a}{b}) = (a,b)$
v. $k : \mathbb{Q} \to \mathbb{Q}$ defined by $k(\frac{a}{b}) = \frac{a^2}{b^2}$

Which of the following functions are linear transformations?

i. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$
ii. $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 3x+1$

iii. $h : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $h(x_1,x_2,x_3) = (x_1,x_2)$
iv. $j : \mathbb{P} \to \mathbb{R}$ defined by $k(p(x)) = p(1)$
v. $k : C[0,1] \to C[0,1]$ defined by $k(x) = k'(x)$

Which of the following functions are $1-1$?

i. $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1,x_2) = (x_2,x_1)$
ii. $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x_1,x_2) = (x_1,0)$
iii. $h : C[0,1] \to C[0,1]$ defined by $h(f) = \int_0^x f$
iv. $j : C[0,1] \to \mathbb{R}$ defined by $h(f) = \int_0^1 f$
v. $k : \mathbb{P} \to \mathbb{P}$ defined by $k(p(x)) = x \cdot p(x)$

Which of the following functions are onto?

i. $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1,x_2) = (x_2,x_1)$
ii. $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x_1,x_2) = (x_1,0)$
iii. $h : C[0,1] \to C[0,1]$ defined by $h(f) = \int_0^x f$
iv. $j : C[0,1] \to \mathbb{R}$ defined by $h(f) = \int_0^1 f$
v. $k : \mathbb{P} \to \mathbb{P}$ defined by $k(p(x)) = x \cdot p(x)$

Discussion Question 24. For a function to be $1-1$ it must have the property that if you put in two different inputs, $x \neq y$, then their outputs are different, $f(x) \neq f(y)$. Recall that the contrapositive of an implication is equivalent to that implication. Use the contrapositive to rewrite what it means for a function for be $1-1$. What does it mean for a function to be onto?

Exercise 24 (c). Show

i. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $f(x_1,x_2) = (x_1, x_1+x_2, x_1-x_2)$ is $1-1$
ii. $g : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ defined by $g(x_1,x_2,x_3,x_4) = (x_1 \cdot x_2, x_3 \cdot x_4)$ is onto

Theorem 24. *Let $\mathbb{L}_1$,$\mathbb{L}_2$ and $\mathbb{L}_3$ be linear spaces.*

i. If $T : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ is a linear transformation that is $1-1$ and maps onto $\mathbb{L}_2$ then the function $T^{-1} : \mathbb{L}_2 \longrightarrow \mathbb{L}_1$ exists and is also a linear transformation.

ii. If $T_1 : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ and $T_2 : \mathbb{L}_2 \longrightarrow \mathbb{L}_3$ are $1-1$, onto linear transformations then $T_2 \circ T_1$ is a 1-1, onto linear transformation.

3.2 The Kernel

Definition. Given a linear transformation $T : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ the set of points in $\mathbb{L}_1$ that T sends to $\mathbf{0} \in \mathbb{L}_2$ is called the **kernel** of T and written:

$$ker(T) = \{\mathbf{P} \in \mathbb{L}_1 \mid T(\mathbf{P}) = \mathbf{0}\}$$

.

Exercise 25 (a). Describe the kernel of $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ if T is the linear transformation that

i. rotates all points around the z-axis by 90°
ii. projects all points perpendicularly onto the xy-plane
iii. projects all points horizontally onto the z-axis
iv. reflects all points across the xy-plane
v. moves all points straight out to twice their distance from origin

Exercise 25 (b). What is the kernel of $T : C[0,1] \longrightarrow C[0,1]$ if T is the linear transformation defined by $T(f) = f'$.

Exercise 25 (c). For $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ find $ker(T)$ and $dim(ker(T))$ if T is the linear transformation defined by

i. $T(x,y) = (0,0,0)$
ii. $T(x,y) = (0,0,x+y)$
iii. $T(x,y) = (x,y,0)$
iv. $T(x,y) = (x,y,x+y)$

Theorem 25. *For any linear transformation* $T : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$, $ker(T) = \{\mathbf{0}\}$ *if and only if* T *is* $1-1$.

3.3 The Range

Definition. Given a linear transformation $T : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ the set of points in $\mathbb{L}_2$ that are the result of T applied to at least one element of $\mathbb{L}_1$ is called the **range** of T and written:

$$T(\mathbb{L}_1) = \{T(\mathbf{P}) \mid \mathbf{P} \in \mathbb{L}_1\}$$

.

Exercise 26 (a). Describe the range of $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ if T is the linear transformation that

i. rotates all points around the z-axis by 90°
ii. projects all points perpendicularly onto the xy-plane
iii. projects all points horizontally onto the z-axis
iv. reflects all points across the xy-plane
v. moves all points straight out to twice their distance from origin

Exercise 26 (b). What is the range of $T : \mathbb{P} \longrightarrow \mathbb{P}$ if T is the linear transformation defined by $T(p(x)) = x \cdot p(x)$.

Exercise 26 (c). For $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ find $T(\mathbb{R}^2)$ and $dim(T(\mathbb{R}^2))$ if T is the linear transformation defined by

i. $T(x,y) = (0,0,0)$
ii. $T(x,y) = (0,0,x+y)$
iii. $T(x,y) = (x,y,0)$
iv. $T(x,y) = (x,y,x+y)$

Theorem 26. *For any linear transformation $T : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$*

i. the kernel of T, $ker(T)$ is a subspace of $\mathbb{L}_1$
ii. the range of T, $T(\mathbb{L}_1)$ is a subspace of $\mathbb{L}_2$

Challenge 26. Prove for any linear transformation $T : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ where $\mathbb{L}_1$ is a finite dimensional linear space: $dim(ker(T)) + dim(T(\mathbb{L}_1)) = dim(\mathbb{L}_1)$

3.4 Isomorphisms

Definition. A function $F : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ is an **Isomorphism** if F is a linear transformation that is $1-1$ and maps onto $\mathbb{L}_2$.

Exercise 27 (a). Given that $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a linear transformation such that $T(1,1,0) = (1,2)$, $T(0,0,1) = (0,-1)$ and $T(0,1,0) = (2,0)$. Determine $T(3,3,0)$, $T(1,2,3)$, and $T(x,y,z)$.

Discussion Question 27. Describe the problems that could occur if the set on which the linear transformation is defined is:

i. not a spanning set for $\mathbb{R}^3$
ii. not linearly independent

Exercise 27 (b). Use a system of equations to

i. write (5,4,3) as a linear combination of (1,1,1),(0,1,1) and (0,0,1).
ii. write $5+4x+3x^2$ as a linear combination of $1+x+x^2$, $x+x^2$ and x^2.

Exercise 27 (c). Define a linear transformation from $\mathbb{P}_3$ to $\mathbb{R}^3$. Is it 1-1? Is it onto? What is its inverse transformation?

Theorem 27. *Any n-dimensional linear space is isomorphic to* $\mathbb{R}^n$

Challenge 27. Prove any two n-dimensional linear spaces are isomorphic.

3.5 Matrices

From now on assume all linear spaces are finite dimensional and a basis, β, has been chosen. Each point **P** in the linear space can then be uniquely identified using the list of coefficients used to write **P** as a linear combination of the points in β. This list is called its coordinate vector and is written

$$[\mathbf{P}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

For example if $\beta = \{1, x, x^2\}$ and $\alpha = \{1+x+x^2, x+x^2, x^2\}$

$$[5+4x+3x^2]_\beta = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} \text{ and } [5+4x+3x^2]_\alpha = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

Lists of coordinate vectors can be organized into what are called coordinate matrices.

Definition. A **Matrix** is a rectangular array of numbers, written

$$\mathtt{A}_{r\times c} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix}$$

Definition. For matrices $\mathtt{A}_{r\times c} = [a_{ij}]$ and $\mathtt{B}_{m\times n} = [b_{ij}]$

i. The **sum** of $\mathtt{A}$ and $\mathtt{B}$ is possible when $r = m$ and $c = n$ and is defined by
   $\mathtt{A}+\mathtt{B} = [s_{ij}]$ where $s_{ij} = a_{ij} + b_{ij}$

ii. The **scalar product** of $t \in \mathbb{R}$ and $\mathtt{A}$ is defined by
   $t\mathtt{A} = [q_{ij}]$ where $q_{ij} = ta_{ij}$

iii. The **product** of $\mathtt{A}$ and $\mathtt{B}$ is possible when $c = m$ and is defined by
   $\mathtt{AB} = [p_{ij}]$ where $p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

Exercise 28 (a). Perform, if possible, the indicated matrix operations.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 10 & 100 \\ 0 & -1 & 0 \end{bmatrix}$$

i. $3A =$
ii. $A+B =$
iii. $AB =$
iv. $AC =$
v. $3+A =$
vi. $A+C =$
vii. $BA =$
viii. $CA =$

Definition. An **additive identity** for the set of $n \times n$ matrices is an $n \times n$ matrix $\mathtt{O}$ such that for all $n \times n$ matrices $\mathtt{M}$, $\mathtt{M}+\mathtt{O} = \mathtt{M}$. $\mathtt{O}$ is also called the zero matrix. A **multiplicative identity** for the set of $n \times n$ matrices is an $n \times n$ matrix $\mathtt{I}$ such that for all $n \times n$ matrices $\mathtt{M}$, $\mathtt{MI} = \mathtt{M}$ and $\mathtt{IM} = \mathtt{M}$.

Exercise 28 (b). Show that for the set of 3×3 matrices there exists a unique additive identity and a unique multiplicative identity.

Definition. A **multiplicative inverse** of a matrix $\mathtt{A}$ is a matrix $\mathtt{A}^{-1}$ such that $\mathtt{AA}^{-1} = \mathtt{I}$ and $\mathtt{A}^{-1}\mathtt{A} = \mathtt{I}$.

Exercise 28 (c). Does a multiplicative inverse exist for all matrices? Is the multiplicative inverse of a given matrix unique?

Discussion Question 28: Where else is the notation $(\quad)^{-1}$ used? Is this notation used consistently?

Theorem 28. *For any* 2×2 *matrices:*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

i. $k(AB) = (kA)B = A(kB)$
ii. $(AB)C = A(BC)$
iii. $A+B = B+A$
iv. $A(B+C) = AB+AC$
v. $A^{-1} = \frac{1}{a_{11}a_{22}-a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ *if* $a_{11}a_{22} - a_{12}a_{21} \neq 0$

**i - iv. will hold for any matrices for which the required operations are defined.*

Challenge 28. Prove or disprove:

i. $(\mathtt{A} \neq \mathtt{O}$ and $\mathtt{AB} = \mathtt{AC}) \Longrightarrow \mathtt{B} = \mathtt{C}$
ii. $\mathtt{AB} = \mathtt{O} \Longrightarrow (\mathtt{A} = \mathtt{O}$ or $\mathtt{B} = \mathtt{O})$

3.6 Linear Transformations as Matrix Multiplication

Exercise 29 (a). Given that for $T:\mathbb{R}^2 \to \mathbb{R}^2$, $T(1,0) = (2,3)$ and $T(0,1) = (4,5)$, find a formula for $T(x_1,x_2)$.

Exercise 29 (b). Let $f:\mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x_1,x_2) = (x_1, x_1 \cdot x_2, x_2)$. Is f a linear transformation? If possible find a matrix $\mathtt{A}$ such that

$$f(x_1,x_2) = \mathtt{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Exercise 29 (c). Let $g:\mathbb{R}^2 \to \mathbb{R}^3$ be defined by $g(x_1,x_2) = (x_1, x_1 + x_2, x_2)$. Is g a linear transformation? If possible find a matrix $\mathtt{A}$ such that

$$g(x_1,x_2) = \mathtt{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Theorem 29. *A function $f:\mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a linear transformation if and only if it can be defined as multiplication by a matrix: $f(\mathbf{v}) = \mathtt{A}\mathbf{v}$. The size of the matrix will be $\_ \times \_$.*

To indicate the linear transformation that results from multiplication by the matrix $\mathtt{A}$, write $T_{\mathtt{A}}:\mathbb{R}^m \longrightarrow \mathbb{R}^n$.

Challenge 29. What must be true about the columns of $\mathtt{A}$ for the linear transformation $T_{\mathtt{A}}$ to be onto? What must be true about the columns of $\mathtt{A}$ for $T_{\mathtt{A}}$ to be $1-1$?

To indicate the matrix for a given linear transformation $T:\mathbb{L}_1 \longrightarrow \mathbb{L}_2$ uses α as the basis for $\mathbb{L}_1$ and β as the basis for $\mathbb{L}_2$ the matrix is written $[T]_{\beta,\alpha}$. This means $[T(P)]_\beta = [T]_{\beta,\alpha}[P]_\alpha$ and $[T_2 \circ T_1]_{\gamma,\alpha}$ and $[T_2]_{\gamma,\beta}[T_1]_{\beta,\alpha}$.

3.7 The Determinant

Definition. There is a useful process for assigning a real number, called the **determinant**, to any square matrix. For $n \times n$ matrix $\mathtt{A}$ this number is written det($\mathtt{A}$) or $|\mathtt{A}|$.

The determinant for 2×2 matrix $\mathtt{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the real number:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant for a 3×3 matrix is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

Discussion Question 29. There are many ways this sum can be factored, each giving a different way of understanding the determinant of a 3×3 matrix, one example is:

$$a(ei - fh) - b(di - fg) + c(dh - eg) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Another example is:

$$-b(di - fg) + e(ei - cg) - h(af - cd) = -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

Find another example. What pattern do all of these examples have in common. This pattern can be used to extend the definition of determinant to larger square matrices.

Definition. An **elementary matrix** is a matrix that when multiplied on the left of a given matrix performs one of the 3 elementary row operations for reducing a matrix.

Exercise 30 (a). Find 3×3 elementary matrices $\mathtt{E}$ that do the following actions. For each, also find det($\mathtt{E}$) and $\mathtt{E}^{-1}$.

i. switch rows 1 and 3
ii. multiply row 3 by the number 5
iii. add 10 times row 2 to row 3

Exercise 30 (b). Compute determinants for the following matrices. Also find the elementary matrices indicated and their determinant. Choose carefully how to compute each determinant.

i. $\mathtt{A} = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

ii. $\mathtt{E}_1\mathtt{A} = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$ where $\mathtt{E}_1 =$

iii. $\mathtt{E}_2\mathtt{A} = \begin{bmatrix} -10 & 20 & 30 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ where $\mathtt{E}_2 =$

iv. $\mathtt{B} = \begin{bmatrix} -1 & 2 & 3 \\ -1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

v. $\mathtt{E}_3\mathtt{A} = \begin{bmatrix} -1 & 2 & 3 \\ 4+-1 & 5+2 & 6+3 \\ 7 & 8 & 9 \end{bmatrix}$ where $\mathtt{E}_3 =$

vi. $\mathtt{C} = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

vii. $\mathtt{A}^T = \begin{bmatrix} -1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ (This is called the transpose of A)

Exercise 30 (c). (Elementary Matrix Lemma) For any $n \times n$ matrix M and any $n \times n$ elementary matrix E, find a relationship between det(EM), det(E) and det(M) .

Theorem 30. *For any two $n \times n$ matrices A and B:*

$$det(AB) = det(A)det(B)$$

3.8 The Big Equivalences Theorem

Recall that the point $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ can be written as the list of coefficients using the standard basis β which gives the $n \times 1$ matrix

$$\mathbf{v} = [(v_1, v_2, .., v_n)]_\beta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

This allows multiplication on the left by an $n \times n$ matrix M to result in a new vector $\mathbf{b} \in \mathbb{R}^n$ written: $\mathtt{M}\mathbf{v} = \mathbf{b}$.

Exercise 31 (a). For any $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ and $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, with $c_i \in \mathbb{R}$, write the equation:
$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{a}$ as:

i. a linear combination: $\_\_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} + \_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} + \_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$

ii. an augmented matrix: $\begin{bmatrix} \_\_ & \_\_ & \_\_ & \vdots & \_\_ \\ \_\_ & \_\_ & \_\_ & \vdots & \_\_ \\ \_\_ & \_\_ & \_\_ & \vdots & \_\_ \end{bmatrix}$

iii. a matrix equation: $\begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$

Exercise 31 (b). Write the augmented matrix obtained by setting $\mathbf{a} = (a_1, a_2, a_3)$ equal to a linear combination of the vectors $\{(1,4,2),(0,2,1),(-1,0,0),(1,2,3)\}$. Determine if these vectors are linearly independent and span $\mathbb{R}^3$.

Exercise 31 (c). For a given $r \times c$ matrix $\mathtt{A}$, sort the following statements into two groups such that the statements in each group are equivalent to each other:

The columns of $\mathtt{A}$ are linearly independent.
The columns of $\mathtt{A}$ span $\mathbb{R}^r$.
There is a leading 1 in each row when $\mathtt{A}$ is row reduced.
There is a leading 1 in each column when $\mathtt{A}$ is row reduced.
Any system of equations with coefficients from $\mathtt{A}$ will have ≤ 1 solution.
Any system of equations with coefficients from $\mathtt{A}$ will have ≥ 1 solution.
The matrix equation $\mathtt{A}\mathbf{x} = \mathbf{b}$ will have ≤ 1 solution.
The matrix equation $\mathtt{A}\mathbf{x} = \mathbf{b}$ will have ≥ 1 solution.

Theorem 31 (The Big Theorem)**.** *For A an $n \times n$ matrix, the following are equivalent:*

i. $det(A) \neq 0$
ii. *The columns of A span $\mathbb{R}^n$*
iii. *The columns of A are linearly independent*
iv. $A\mathbf{x} = \mathbf{b}$ *has a unique solution, $x \in \mathbb{R}^n$, for each* $\mathbf{b}$ *in* $\mathbb{R}^n$
v. *A has a multiplicative inverse*

Challenge 31. The following can be added to the above list

vi. *The rows of A form a basis for $\mathbb{R}^n$*

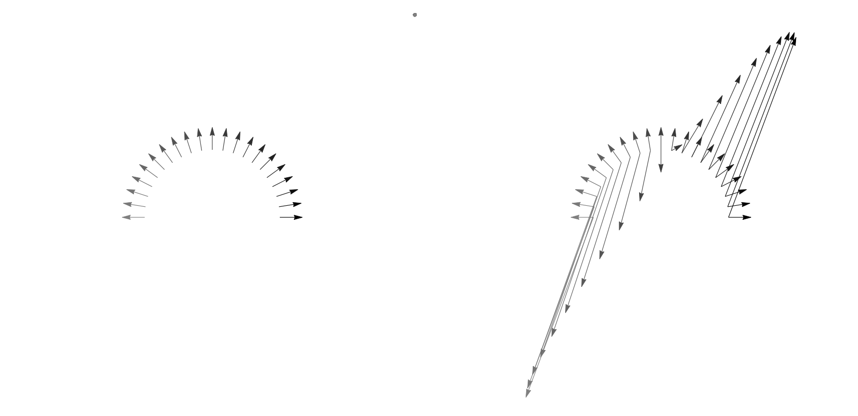
Discussion Question 31. For an $n \times n$ matrix $\mathtt{A}$ with $det(\mathtt{A}) \neq 0$, how can elementary operations be used to find $\mathtt{A}^{-1}$?

3.9 Eigenspaces

Exercise 32 (a). Given $\mathtt{A} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ and $\mathbf{v} = (v_1, v_2)$. Compute $\mathtt{A}\mathbf{v} - 3\mathbf{v}$ and $(\mathtt{A} - 3\mathtt{I})\mathbf{v}$. What do you notice? What does it mean when $(\mathtt{A} - 3\mathtt{I})\mathbf{v} = \mathbf{0}$? Find at least two vectors $\mathbf{v}$ such that $\mathtt{A}\mathbf{v} = 3\mathbf{v}$.

Definition. Given a square matrix $\mathtt{A}$, when there exists a non-zero vector $\mathbf{v}$ such that $\mathtt{A}\mathbf{v} = \lambda\mathbf{v}$ for some real number λ that vector is called an **eigenvector** for $\mathtt{A}$, and λ its corresponding **eigenvalue**.

Example. The effect of multiplication by a matrix on a representative set of vectors is illustrated below. Shown is a set of vectors and then those same vectors with the result of multiplying each of them by $\mathtt{A} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.
The vectors $(0,1)$ and $(1,2)$ are eigenvectors. Any scalar multiple of these vectors is also an eigenvector. The eigenvalue corresponding to $(0,1)$ is $\lambda = -1$ and the eigenvalue corresponding to $(1,2)$ is $\lambda = 3$.



Exercise 32 (b). Given $\mathtt{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, find all values for λ such that the matrix equation

$$(\mathtt{A} - \lambda\mathtt{I})\mathbf{v} = \mathbf{0}$$

has a solution other that other than $\mathbf{v} = \mathbf{0}$. These are the eigenvalues for $\mathtt{A}$.

Exercise 32 (c). Given $\mathtt{A} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ has eigenvalues $\lambda_{\mathbf{v}} = 2$ and $\lambda_{\mathbf{w}} = -1$.

i. Describe the set of vectors $\mathbf{v}$ such that $\mathtt{A}\mathbf{v}$ = 2$\mathbf{v}$
ii. Describe the set of vectors $\mathbf{w}$ such that $\mathtt{A}\mathbf{w}$ = -1$\mathbf{w}$

Discussion Question 32. If an eigenvector could be zero what would the corresponding eigenvalue be?

Theorem 32. *For any $n \times n$ matrix $\mathtt{A}$, the set of eigenvectors corresponding to a given eigenvalue λ becomes a subspace of $\mathbb{R}^n$ if you include $\mathbf{0}$.*

Challenge 32. Given matrix $\mathtt{A}$ has eigenvector $\mathbf{v}$ with eigenvalue λ and $k \in \mathbb{R}$. What can you say about the eigenvectors and eigenvalues of the matrix $k\mathtt{A}$? What can you say about the eigenvectors and eigenvalues of the matrix $\mathtt{A}^k$? Does it matter what k is?

3.10 When an Eigenvalue is 0

Exercise 33 (a). Recall that if there are elementary matrices $\mathtt{E}_1$, $\mathtt{E}_2$,...$\mathtt{E}_k$ such that $\mathtt{E}_k \cdots \mathtt{E}_2\mathtt{E}_1\mathtt{A} = \mathtt{I}$ then $\mathtt{A}^{-1} = \mathtt{E}_k \cdots \mathtt{E}_2\mathtt{E}_1$. For $\mathtt{A} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ find $\mathtt{A}^{-1}$ by starting with $[\mathtt{A} : \mathtt{I}]$ then using the same elementary row operation on both sides to get $[\mathtt{I} : \mathtt{A}^{-1}]$. Check your answer by computing $\mathtt{A}^{-1}\mathtt{A}$.

Discussion Question 33. Can eigenvalues be zero? What would it mean if a vector's eigenvalue were zero?

Exercise 33 (b). Compute the eigenvalues and their corresponding eigenspaces for $\mathtt{A} = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$.

Exercise 33 (c). Compute the eigenvalues and their corresponding eigenspaces for $\mathtt{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Theorem 33. *An $n \times n$ matrices $\mathtt{A}$ is invertible if and only if none of its eigenvalues are zero.*

Challenge 33. Given an invertible square matrix $\mathtt{A}$ with eigenvector $\mathbf{v}$ and eigenvalue λ, show $\mathtt{A}^{-1}$ also has eigenvector $\mathbf{v}$, but with eigenvalue $\frac{1}{\lambda}$.

3.11 When Eigenvalues are Distinct

Exercise 34 (a). Compute the eigenvalues and a representative eigenvector for $\mathtt{A} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ and for $2\mathtt{A} = \begin{bmatrix} 2 & 4 \\ 6 & -8 \end{bmatrix}$. What do you notice?

Discussion Question 34. For a given eigenvalue can there be just one corresponding eigenvector? Usually books talk about eigenvectors as if there were just one, why doesn't this cause problems?

Exercise 34 (b). Compute the eigenvalues and eigenvectors for $\mathtt{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$
.

Exercise 34 (c). Compute the eigenvalues and eigenvectors for $\mathtt{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

Theorem 34. *Suppose* $\mathtt{A}$ *is a square matrix with eigenvectors* $\mathbf{v}$ *and* $\mathbf{w}$*, and corresponding eigenvalues* $\lambda_{\mathbf{v}}$ *and* $\lambda_{\mathbf{w}}$*. If* $\lambda_{\mathbf{v}} \neq \lambda_{\mathbf{w}}$ *then* $\mathbf{v}$ *and* $\mathbf{w}$ *are linearly independent.*

Challenge 34. Given a square matrix $\mathtt{A}$ with three eigenvectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, having distinct corresponding eigenvalues $\lambda_{\mathbf{u}}$, $\lambda_{\mathbf{v}}$ and $\lambda_{\mathbf{w}}$, show $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

3.12 Diagonalization

Exercise 35 (a). Compute the following matrix products.

i. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 1000 \end{bmatrix} =$

ii. $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 1000 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} =$

Exercise 35 (b). Use that $\mathbf{v_1}$ =(3,0,1), $\mathbf{v_2}$ =(0,2,1) and $\mathbf{v_3}$ =(1,1,1) are the eigenvectors corresponding to eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = 1$ for the matrix $\mathtt{A} = \begin{bmatrix} -2 & -3 & 6 \\ 2 & 5 & -6 \\ 0 & 1 & 0 \end{bmatrix}$ to fill in the blanks below.

$$\begin{bmatrix} -2 & -3 & 6 \\ 2 & 5 & -6 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & -3 & 6 \\ 2 & 5 & -6 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} \_\_ & 0 & 0 \\ 0 & \_\_ & 0 \\ 0 & 0 & \_\_ \end{bmatrix} \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

Exercise 35 (c). Given $\mathtt{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$, find invertible matrix $\mathtt{P}$ and diagonal matrix $\mathtt{D}$ such that $\mathtt{A} = \mathtt{PDP}^{-1}$

Theorem 35. *If an $n \times n$ matrix has n linearly independent eigenvectors then there exists an invertible matrix P and a diagonal matrix D such that:*

$$A = P\ D P^{-1}$$

Challenge 35. If an $n \times n$ matrix $\mathtt{A}$ has n linearly independent eigenvectors then $\det(\mathtt{A})$ is the product of the n corresponding eigenvalues.

Discussion Question 35. How would Theorem 35 help us efficiently compute $\mathtt{A}^{100}$?