信号与系统：连续信号的正交分解

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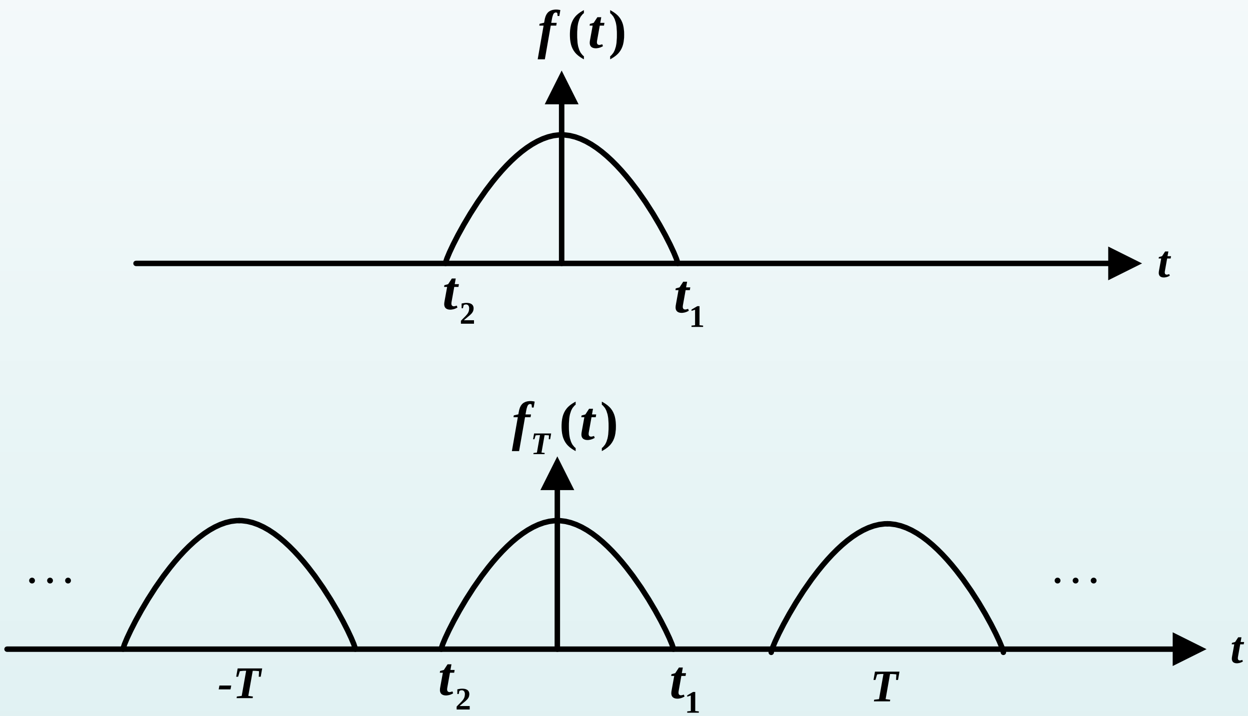
本章内容：

- 分析周期信号（利用傅里叶级数）
  ——谐波分析法
- 分析非周期信号（$T \to \infty$）
  ——傅里叶变换

延拓目的：

- 分析系统的I/O特性，并用频率方法求$r_{zi}(t)$

一、傅里叶变换(Fourier Transform, 记作FT)导出



$$f(t)=\lim_{T\to\infty} f_T(t)$$

非周期信号的傅里叶变换

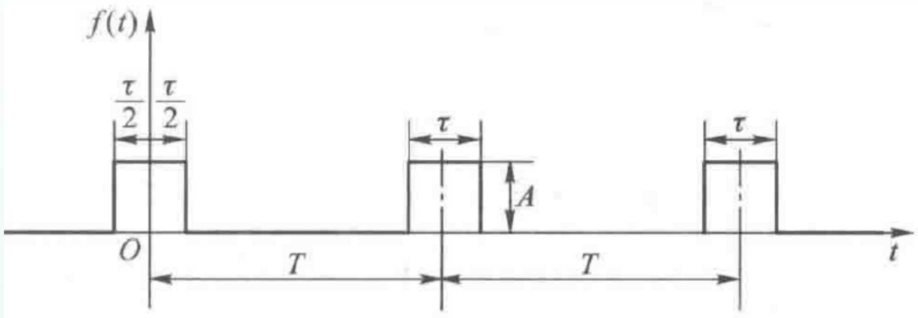


图 3-11 周期性矩形脉冲信号

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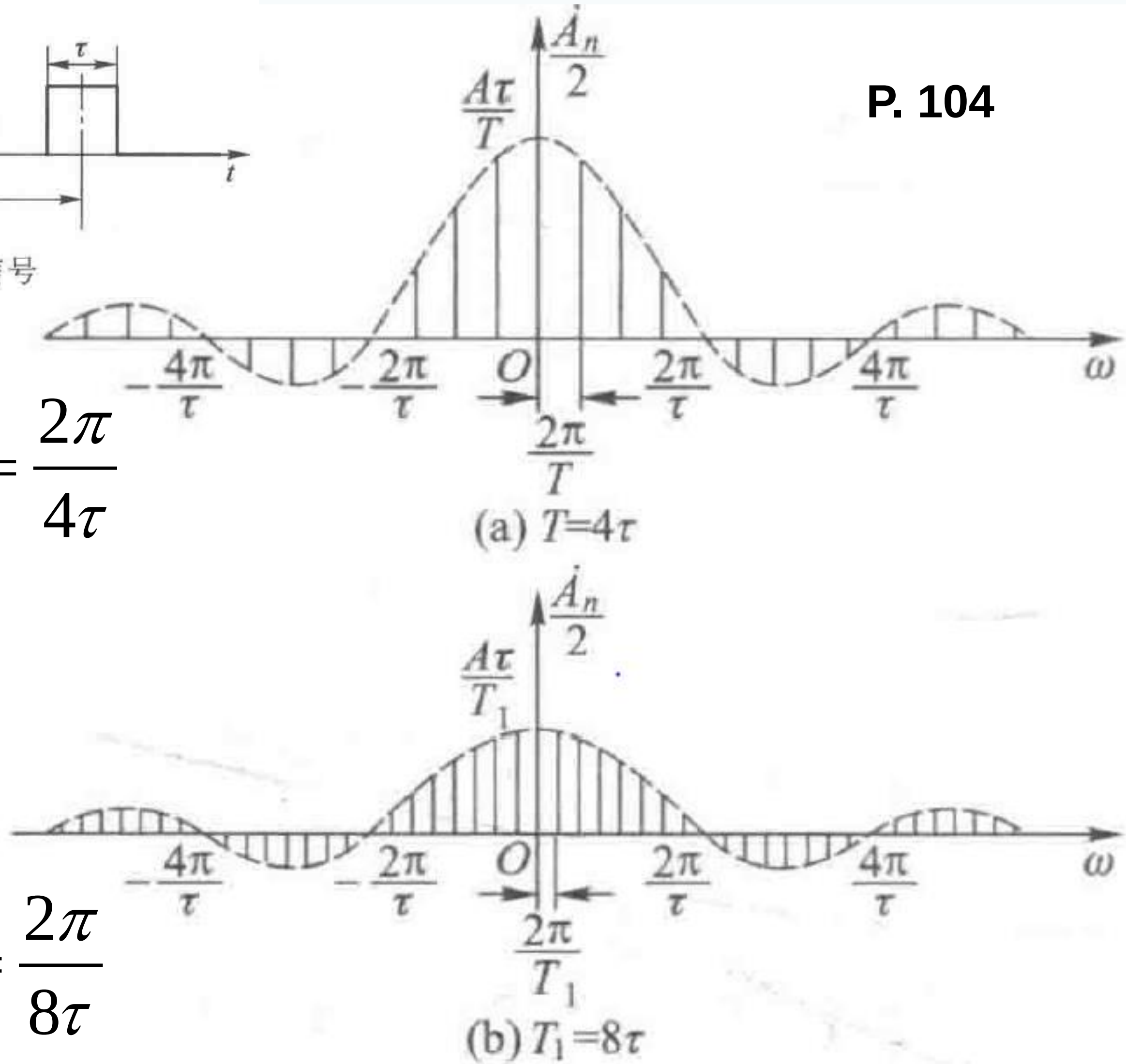


$$\Omega = \frac{2\pi}{T}$$

$$\Omega = \frac{2\pi}{4\tau}$$

定义谱线间隔！

$$\Omega = \frac{2\pi}{8\tau}$$

非周期信号的傅里叶变换

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\Omega t}$$

$$F_n=\frac{1}{T}\int_{t_0}^{t_0+T}f(t)e^{-jn\Omega t}dt$$

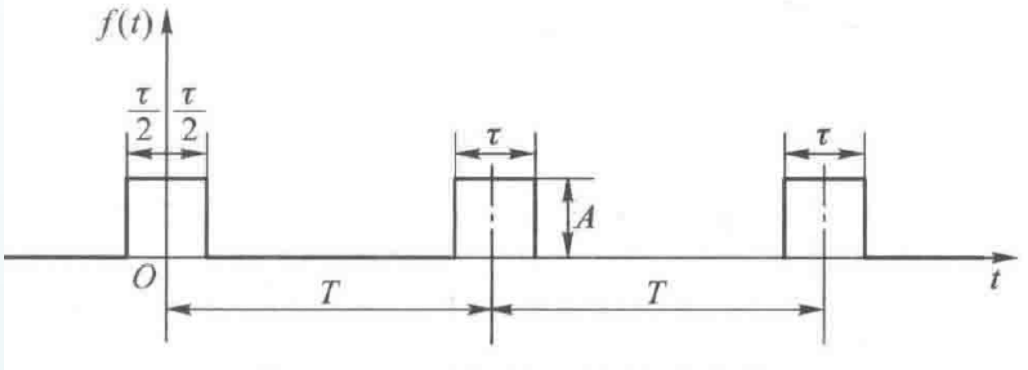
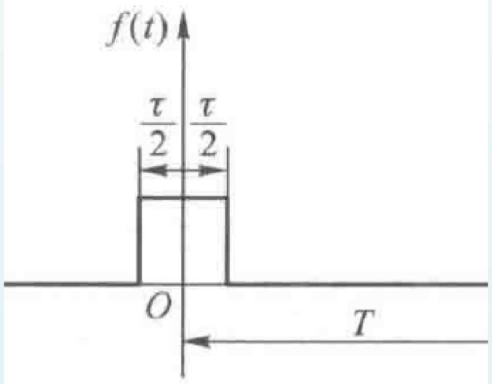


图 3-11 周期性矩形脉冲信号

$$TF_n=\int_{t_0}^{t_0+T}f(t)e^{-jn\Omega t}dt$$

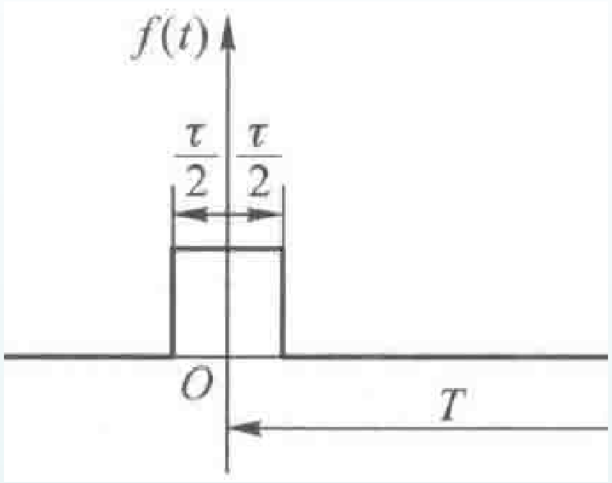


$$T\to\infty \Rightarrow \Omega\to 0, n\Omega\to w$$

$$TF_n=\int_{-\infty}^{\infty}f(t)e^{-jwt}dt=F(jw)$$

傅里叶变换!!

非周期信号的傅里叶变换



$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\Omega t}$$

$$f(t)=\sum_{n=-\infty}^{\infty}TF_n e^{jn\Omega t}\frac{1}{T}$$

$$T\to\infty \Rightarrow TF_n=F(jw),\frac{1}{T}=\frac{\Omega}{2\pi},\Omega\to dw$$

$$f(t)=\sum_{n=-\infty}^{\infty}F\left(jw\right)e^{jwt}\frac{dw}{2\pi}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}F\left(jw\right)e^{jwt}dw$$

傅里叶逆变换!!

非周期信号的傅里叶变换

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt \overset{\Delta}{=} \mathscr{F}[f(t)]$$ ——$f(t)$的FT

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega \overset{\Delta}{=} \mathscr{F}^{-1}[F(j\omega)]$$ ——$F(j\omega)$的IFT

$$f(t)\leftrightarrow F(j\omega)$$ ——傅里叶变换对

意义：

1.由$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega$ ——非周期信号$f(t)$的频域分解式

2. $$F(j\omega)=\lim_{T\to\infty}TF_n=\lim_{\Omega\to 0}\frac{2\pi}{\Omega}F_n=\lim_{\Delta f\to 0}\frac{F_n}{\Delta f}$$

——$F(j\omega)$称为频谱密度，简称频谱

$F(j\omega)$与F_n（周期信号频谱）的区别：

F_n是谐波离散的，代表各分量绝对大小

$F(j\omega)$是连续的，代表各分量相对大小

非周期信号的傅里叶变换

$$F(j\omega)=|F(j\omega)|e^{j\varphi(\omega)}$$

$|F(j\omega)|\sim\omega$——幅度频谱

反映各分量相对大小的关系

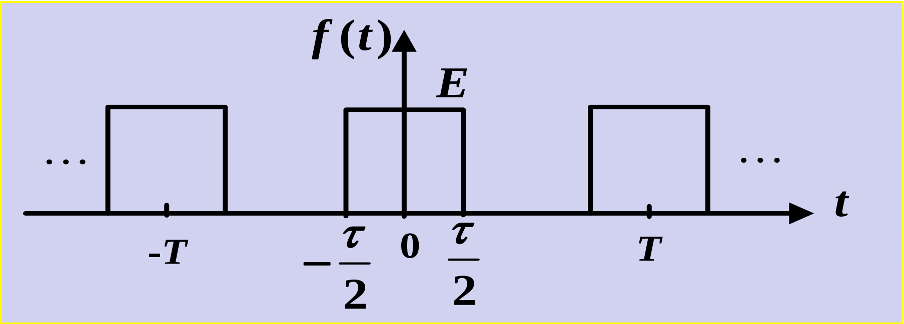
$\varphi(\omega)\sim\omega$ ——相位频谱反映各分量的初始相位

$|F(j\omega)|$和$\varphi(\omega)$随ω的变化规律分别称为信号的幅频特性和相频特性

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$

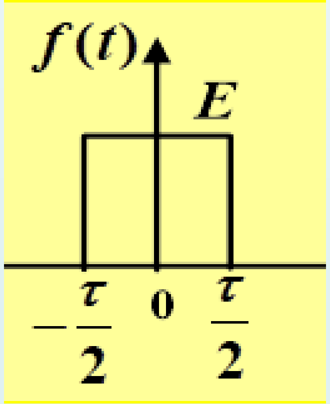
$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega$$

典型信号的傅里叶变换: 矩形窗



$$f(t)=\sum_{n=-\infty}^{\infty}\frac{E\tau}{T}Sa(\frac{n\Omega\tau}{2})e^{jn\Omega t}$$

$$F_n=\frac{E\tau}{T}Sa(\frac{n\Omega\tau}{2})$$

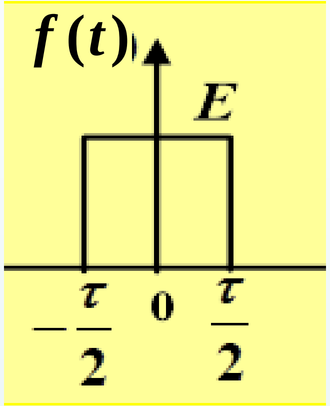


$$TF_n=E\tau Sa(\frac{n\Omega\tau}{2})$$

$$T\to\infty => \Omega\to 0, n\Omega\to w$$

$$TF_n=F(jw)=E\tau Sa(\frac{w\tau}{2})$$

典型信号的傅里叶变换: 矩形窗



$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

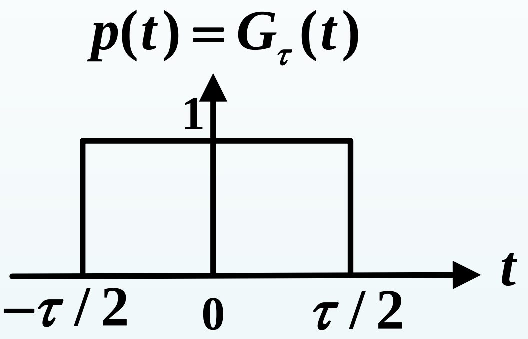
$$F(j\omega)=E\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)e^{-j\omega t}dt \Longrightarrow F(j\omega)=E\frac{e^{-j\omega\frac{\tau}{2}}-e^{j\omega\frac{\tau}{2}}}{-j\omega}$$

$$F(j\omega)=E\frac{2\sin\left(\omega\frac{\tau}{2}\right)}{\omega} \Longrightarrow F(j\omega)=E\tau Sa\left(\frac{\omega\tau}{2}\right)$$

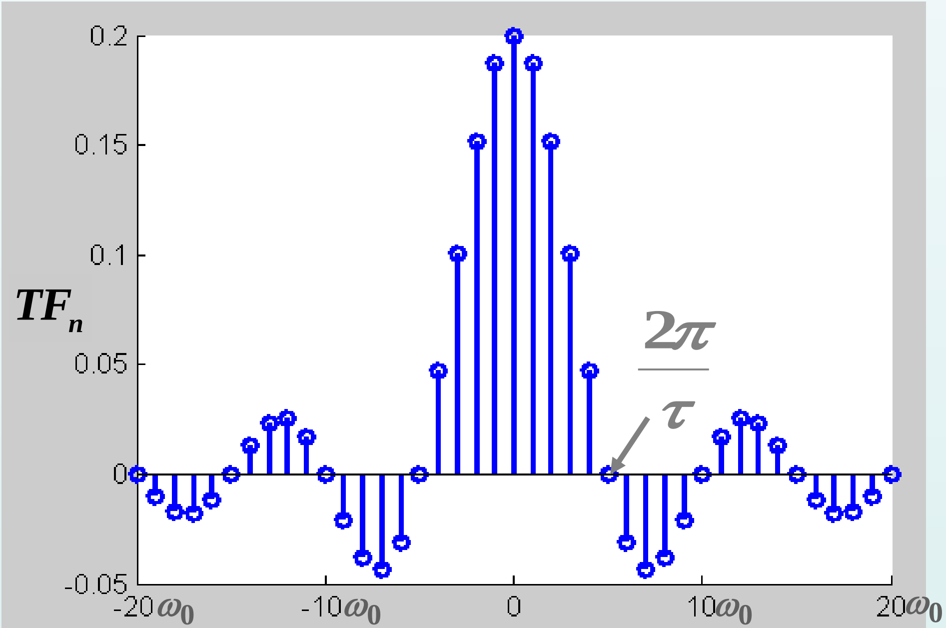
其频域分解式： $$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} E\tau Sa(\frac{\omega\tau}{2})e^{j\omega t}dw$$

典型信号的傅里叶变换：矩形窗

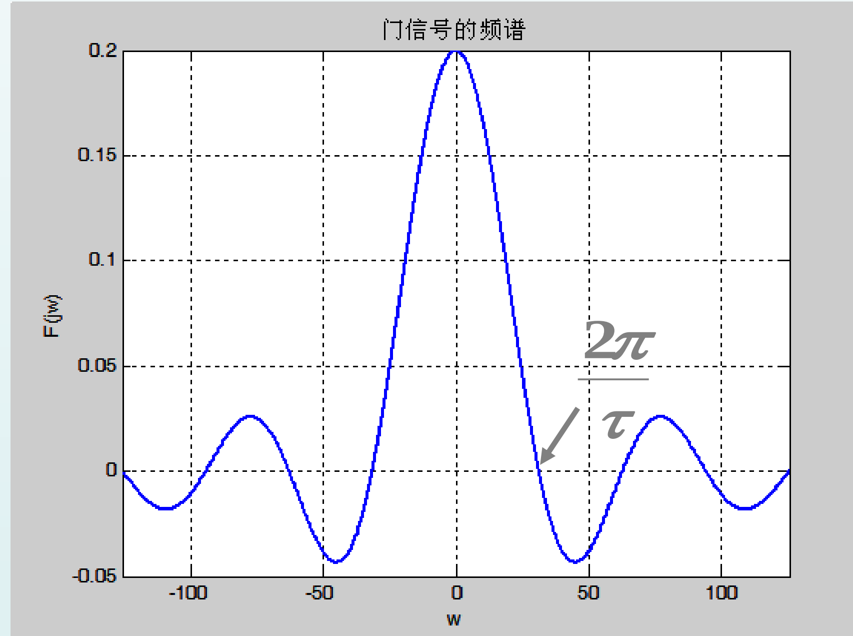
矩形脉冲信号的频谱： $G_\tau(t) \leftrightarrow \tau \cdot Sa\left(\frac{\omega\tau}{2}\right)$



周期矩形脉冲的傅里叶级数： $F_n = \frac{\tau}{T} Sa\left(\frac{n\Omega\tau}{2}\right)$



周期矩形脉冲的傅里叶级数



非周期门信号的傅立叶变换

周期信号的频谱是对应的非周期信号频谱的离散抽样；
而非周期信号的频谱是对应的周期信号频谱的包络。

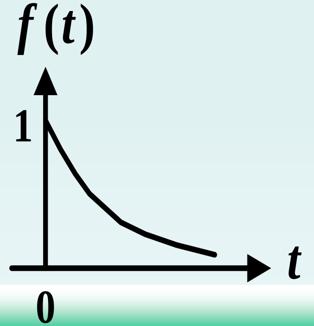
单边指数信号的傅里叶变换

$$f(t)=e^{-at}\varepsilon(t) \qquad (a>0) \qquad Give\ F(j\omega)$$

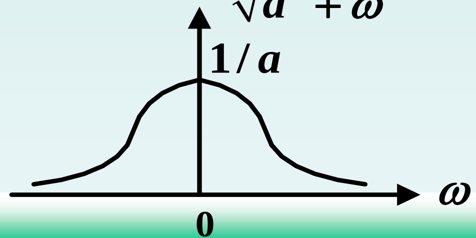
$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \Rightarrow F(j\omega)=\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$F(j\omega)=\frac{1}{-a-j\omega}\left[e^{(-a-j\omega)\infty}-e^{(-a-j\omega)0}\right] \Rightarrow F(j\omega)=\frac{1}{a+j\omega}$$

$$F(jw)=\frac{a-j\omega}{a^2+\omega^2}=\frac{|a-j\omega|e^{j\varphi(w)}}{a^2+\omega^2}=\frac{\sqrt{a^2+\omega^2}e^{j\varphi(w)}}{a^2+\omega^2}=\frac{1}{\sqrt{a^2+\omega^2}}e^{j\varphi(w)}$$



$$|F(j\omega)|=\frac{1}{\sqrt{a^2+\omega^2}}$$

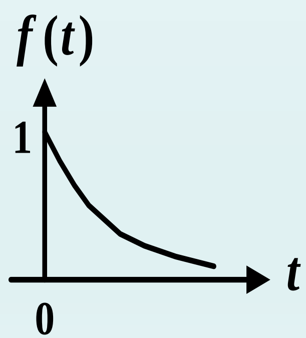


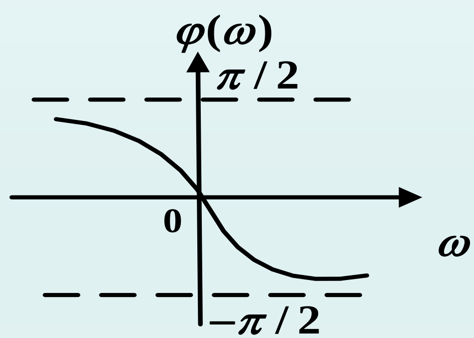
单边指数信号的傅里叶变换

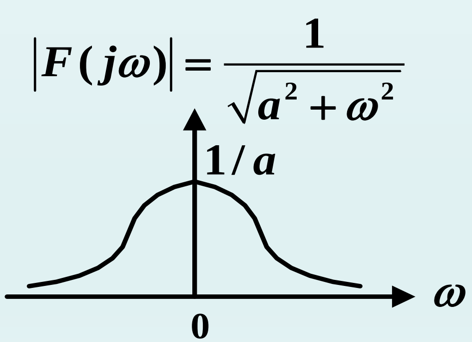
$$F(jw)=\frac{a-j\omega}{a^2+\omega^2}=\frac{|a-j\omega|e^{j\varphi(w)}}{a^2+\omega^2}=\frac{\sqrt{a^2+\omega^2}\,e^{j\varphi(w)}}{a^2+\omega^2}$$

$$\left.\begin{aligned} a &= \sqrt{a^2+\omega^2}\cos\left(\varphi\left(w\right)\right) \\ -w &= \sqrt{a^2+\omega^2}\sin\left(\varphi\left(w\right)\right)\end{aligned}\right\}\begin{cases}\varphi\left(w\right)\text{位于第四象限, w>0}\\ \varphi\left(w\right)\text{位于第一象限, w<0}\end{cases}$$

$$\tan\left(\varphi\left(w\right)\right)=\frac{-w}{a}$$







单位冲激信号的频谱

求单位冲激信号$\delta(t)$的频谱密度函数，并写出它的频域分解式

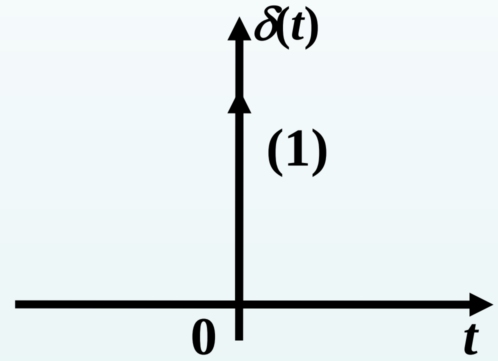
$$F(jw)=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt$$

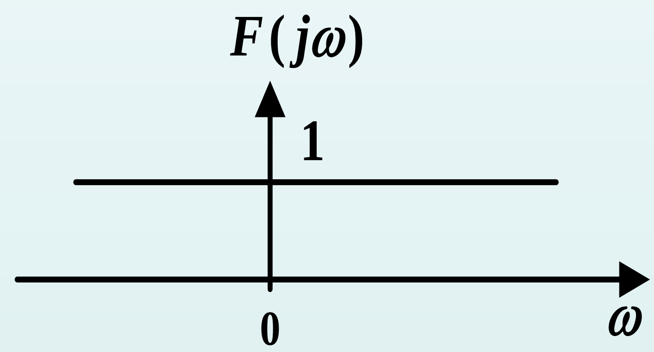
$$\Downarrow$$

$$F(jw)=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega 0}dt$$

$$\Downarrow$$

$$F(jw)=\int_{-\infty}^{\infty}\delta(t)dt=1$$





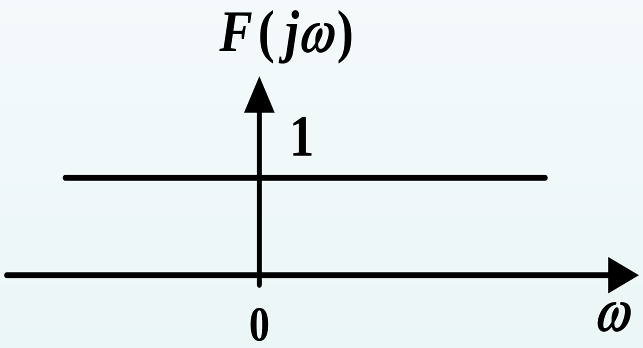
$$\delta(t)\leftrightarrow 1$$

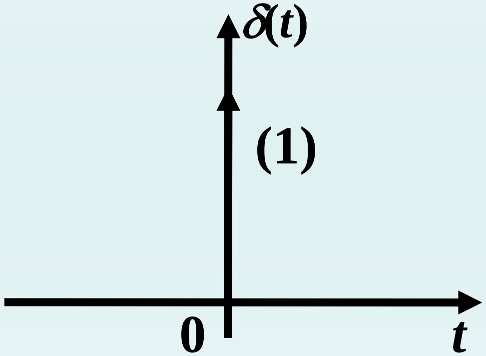
$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}d\omega$$

$\delta(t)$的频谱包含了所有频率分量，且各个频率分量的幅度、相位完全相同。故称为白色谱。

单位冲激信号的频谱

联想白色的光！

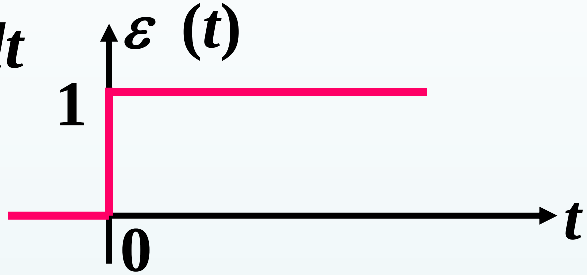




白色的光在另一个域或者空间，
会不会只是一个瞬间尼？

单位阶跃信号的频谱

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt \qquad F(j\omega)=\int_{0}^{\infty}1\times e^{-j\omega t}dt$$



积分不满足绝对可积条件：$\int_{-\infty}^{\infty}|f(t)|dt<\infty$

借助已有函数的频谱的近似。

$$e^{-at}\varepsilon(t)\leftrightarrow\frac{1}{a+j\omega}=\frac{a}{a^2+\omega^2}-j\frac{\omega}{a^2+\omega^2}$$

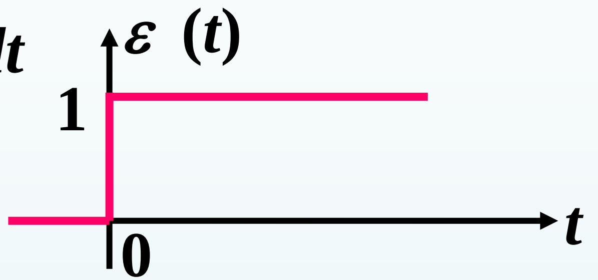
$$a\to 0, e^{-at}\varepsilon(t)\to\varepsilon(t),$$

$$\frac{\omega}{a^2+\omega^2}\xrightarrow{a=0}\frac{1}{\omega},$$

$$\frac{a}{a^2+\omega^2}\xrightarrow{a=0}\begin{cases}0, \omega\neq 0\\ \infty, \omega=0\end{cases}$$

单位阶跃信号的频谱

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad F(j\omega)=\int_{0}^{\infty} 1\times e^{-j\omega t}dt$$



积分不满足绝对可积条件：$\int_{-\infty}^{\infty}\left|f(t)\right|dt<\infty$

$$\int_{-\infty}^{\infty}\frac{a}{a^2+\omega^2}d\omega=\int_{-\infty}^{\infty}\frac{1}{1+\left(\frac{\omega}{a}\right)^2}d\left(\frac{\omega}{a}\right)$$

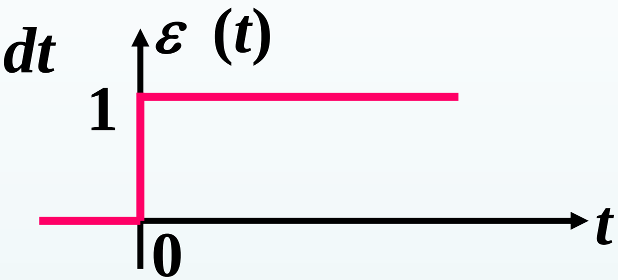
$$=\int_{-\infty}^{\infty}d\left(\arctan\left(\frac{\omega}{a}\right)\right)$$

$$=\arctan\left(\frac{\omega}{a}\right)\Big|_{-\infty}^{\infty}=\pi$$

$$\boxed{\int\frac{1}{1+x^2}dx=\arctan x+c}$$

单位阶跃信号的频谱

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad F(j\omega)=\int_{0}^{\infty} 1\times e^{-j\omega t}dt$$



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$$e^{-at}\varepsilon(t)\leftrightarrow\frac{1}{a+j\omega}=\frac{a}{a^2+\omega^2}-j\frac{\omega}{a^2+\omega^2}$$

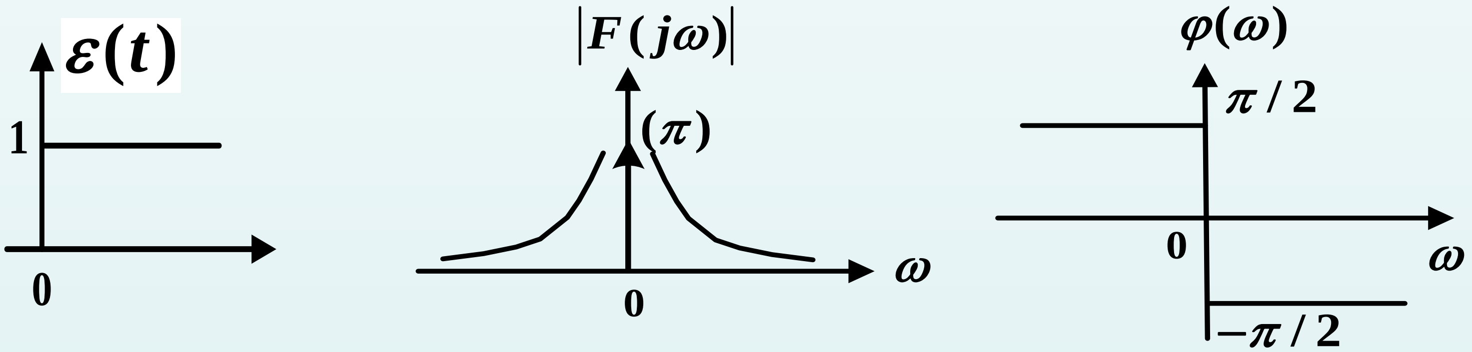
$$a\to 0, e^{-at}\varepsilon(t)\to\varepsilon(t),$$

$$\boxed{\frac{a}{a^2+\omega^2}\xrightarrow{a=0}\pi\delta(\omega)}, \boxed{\frac{\omega}{a^2+\omega^2}\xrightarrow{a=0}\frac{1}{\omega}}$$

$$\boxed{\varepsilon(t)\leftrightarrow\pi\delta(\omega)+\frac{1}{j\omega}}$$

单位阶跃信号的频谱

$$\varepsilon(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

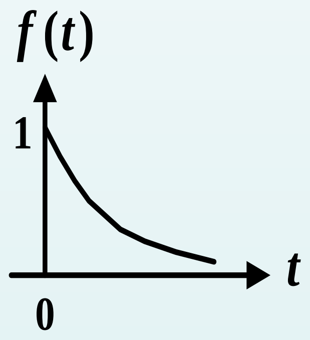


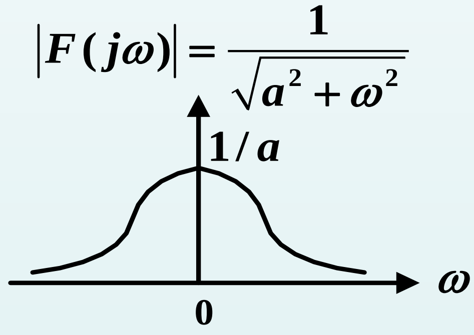
$$j = e^{j\frac{\pi}{2}}$$

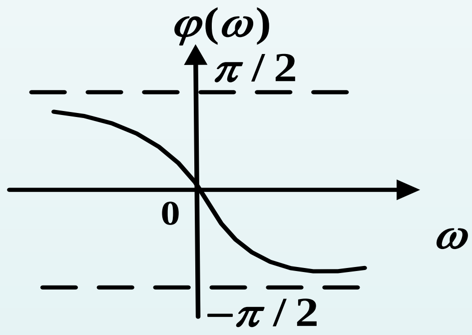
$$\frac{1}{j} = -j = e^{-j\frac{\pi}{2}}$$

典型信号的傅里叶变换对

单边指数函数 $$e^{-at}\varepsilon(t) \leftrightarrow \frac{1}{a+j\omega},\quad a>0$$



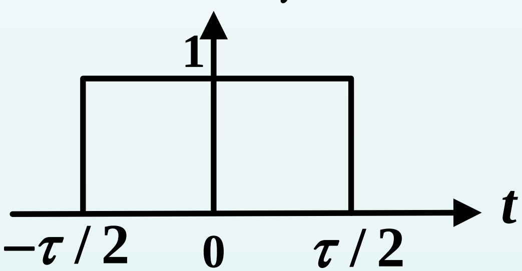




典型信号的傅里叶变换对

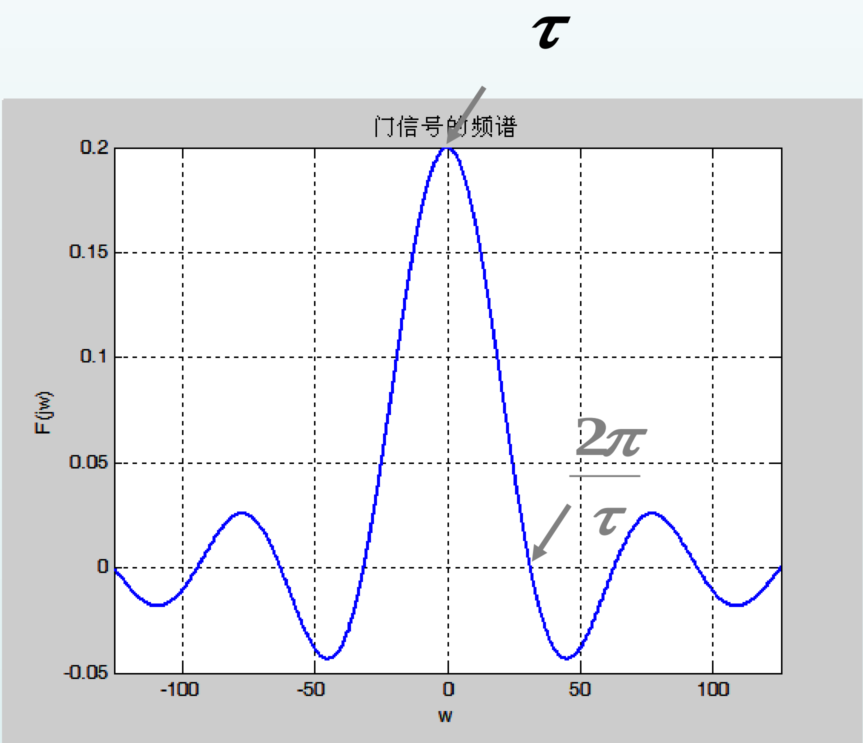
单位矩形窗/门函数 $$G_\tau(t) \leftrightarrow \tau Sa\left(\frac{\omega\tau}{2}\right)$$

$$p(t) = G_\tau(t)$$



$$\tau = 0.2$$

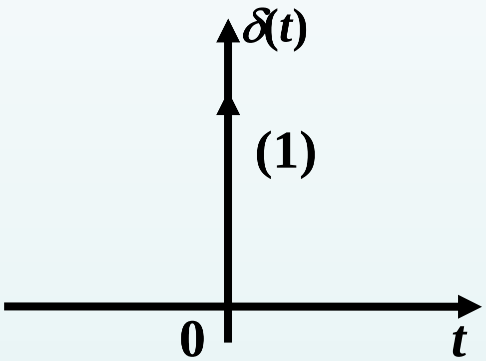
$$\frac{2\pi}{\tau} = 10\pi$$

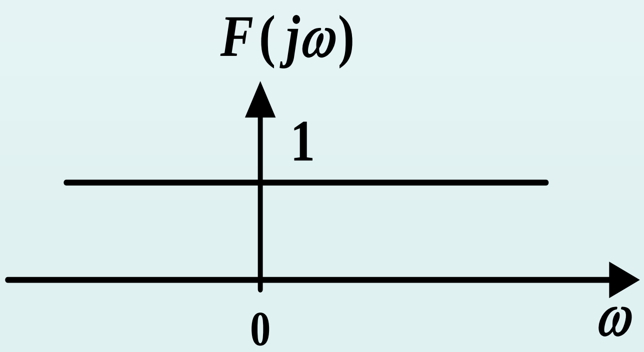


非周期门信号的傅立叶变换

典型信号的傅里叶变换对

冲激函数 $\delta(t) \leftrightarrow 1$

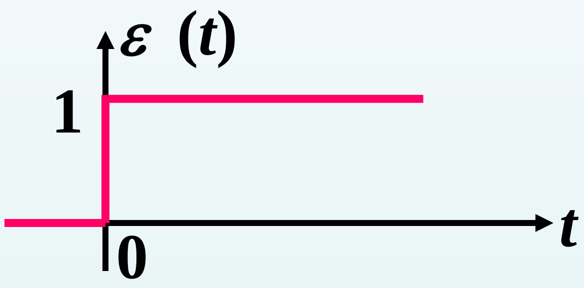


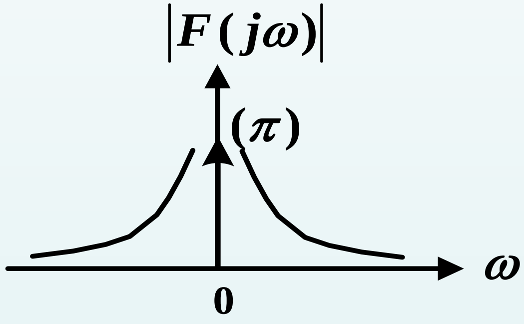


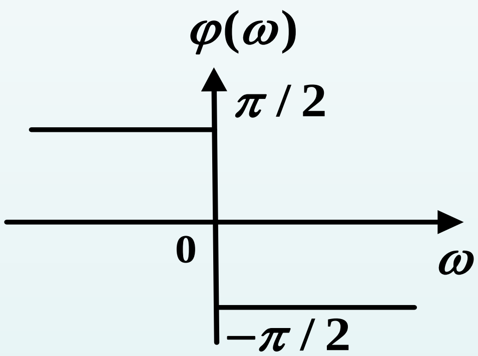
典型信号的傅里叶变换对

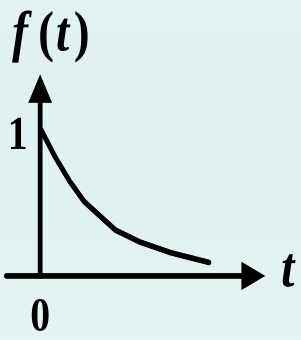
阶跃函数

$$\varepsilon(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$









$$|F(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$