The Art of Linear Algebra

– Graphic Notes on “Linear Algebra for Everyone” –

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with the kindest help of Gilbert Strang †

September 1, 2021/updated October 15, 2022

Abstract

I tried intuitive visualizations of important concepts introduced in “Linear Algebra for Everyone”.[1] This is aimed at promoting understanding of vector/matrix calculations and algorithms from the perspectives of matrix factorizations. They include Column-Row ($\boldsymbol{CR}$), Gaussian Elimination ($\boldsymbol{LU}$), Gram-Schmidt Orthogonalization ($\boldsymbol{QR}$), Eigenvalues and Diagonalization ($\boldsymbol{Q\Lambda Q^{\mathrm{T}}}$), and Singular Value Decomposition ($\boldsymbol{U\Sigma V^{\mathrm{T}}}$).

Foreword

I am happy to see Kenji Hiranabe’s pictures of matrix operations in linear algebra ! The pictures are an excellent way to show the algebra. We can think of matrix multiplications by row · column dot products, but that is not all – it is “linear combinations” and “rank 1 matrices” that complete the algebra and the art. I am very grateful to see the books in Japanese translation and the ideas in Kenji’s pictures.

– Gilbert Strang
Professor of Mathematics at MIT

Contents

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[1] “Linear Algebra for Everyone”: `http://math.mit.edu/everyone/` with Japanese translation started by Kindai Kagaku.

1 Viewing a Matrix – 4 Ways

A matrix ($m \times n$) can be seen as 1 matrix, mn numbers, n columns and m rows.

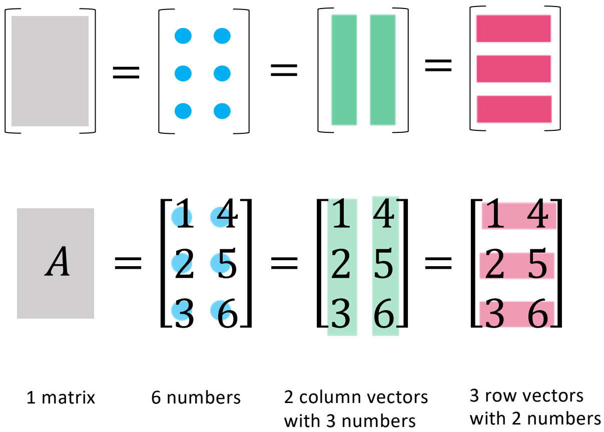


Figure 1: Viewing a Matrix in 4 Ways

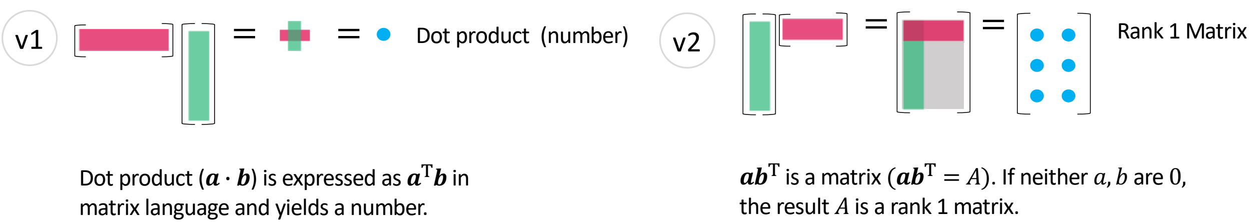
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} | & | \\ \boldsymbol{a_1} & \boldsymbol{a_2} \\ | & | \end{bmatrix} = \begin{bmatrix} -\boldsymbol{a}_1^*- \\ -\boldsymbol{a}_2^*- \\ -\boldsymbol{a}_3^*- \end{bmatrix}$$

Here, the column vectors are in bold as $\boldsymbol{a_1}$. Row vectors include $*$ as in $\boldsymbol{a}_1^*$. Transposed vectors and matrices are indicated by T as in $\boldsymbol{a}^{\mathrm{T}}$ and A^{T}.

2 Vector times Vector – 2 Ways

Hereafter I point to specific sections of "Linear Algebra for Everyone" and present graphics which illustrate the concepts with short names in colored circles.

- Sec. 1.1 (p.2) Linear combination and dot products
- Sec. 1.3 (p.25) Matrix of Rank One
- Sec. 1.4 (p.29) Row way and column way



$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 3x_3$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x & y \\ 2x & 2y \\ 3x & 3y \end{bmatrix}$$

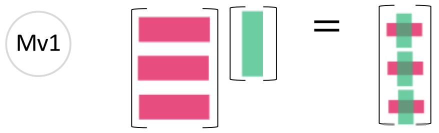
Figure 2: Vector times Vector - (v1), (v2)

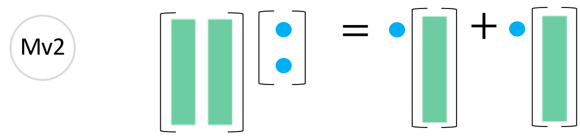
(v1) is a elementary operation of two vectors, but (v2) multiplies the column to the row and produce a rank 1 matrix. Knowing this outer product (v2) is the key for the later sections.

3 Matrix times Vector – 2 Ways

A matrix times a vector creates a vector of three dot products (Mv1) as well as a linear combination (Mv2) of the column vectors of A.

- Sec. 1.1 (p.3) Linear combinations
- Sec. 1.3 (p.21) Matrices and Column Spaces





The row vectors of A are multiplied by a vector $\boldsymbol{x}$ and become the three dot-product elements of $A\boldsymbol{x}$.

The product $A\boldsymbol{x}$ is a linear combination of the column vectors of A.

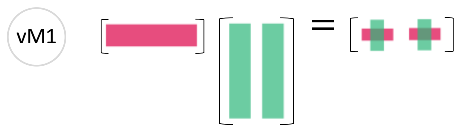
$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (x_1+2x_2) \\ (3x_1+4x_2) \\ (5x_1+6x_2) \end{bmatrix}$$

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Figure 3: Matrix times Vector - (Mv1), (Mv2)

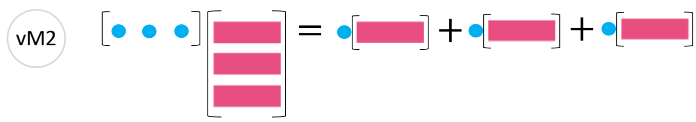
At first, you learn (Mv1). But when you get used to viewing it as (Mv2), you can understand $A\boldsymbol{x}$ as a linear combination of the columns of A. Those products fill the column space of A denoted as $\mathbf{C}(A)$. The solution space of $A\boldsymbol{x} = \mathbf{0}$ is the nullspace of A denoted as $\mathbf{N}(A)$.

Also, (vM1) and (vM2) shows the same patterns for a row vector times a matrix.



$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = [(y_1+3y_2+5y_3) \quad (2y_1+4y_2+6y_3)]$$

A row vector $\boldsymbol{y}$ is multiplied by the two column vectors of A and become the two dot-product elements of $\boldsymbol{y}A$.



$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = y_1[1 \quad 2] + y_2[3 \quad 4] + y_3[5 \quad 6]$$

The product $\boldsymbol{y}A$ is a linear combination of the row vectors of A.

Figure 4: Vector times Matrix - (vM1), (vM2)

The products fill the row space of A denoted as $\mathbf{C}(A^{\mathrm{T}})$. The solution space of $yA = 0$ is the left-nullspace of A denoted as $\mathbf{N}(A^{\mathrm{T}})$.

The four subspaces consists of $\mathbf{N}(A) + \mathbf{C}(A^{\mathrm{T}})$ (which are perpendicular to each other) in $\mathbb{R}^n$ and $\mathbf{N}(A^{\mathrm{T}})$ + $\mathbf{C}(A)$ in $\mathbb{R}^m$ (which are perpendicular to each other).

- Sec. 3.5 (p.124) Dimensions of the Four Subspaces

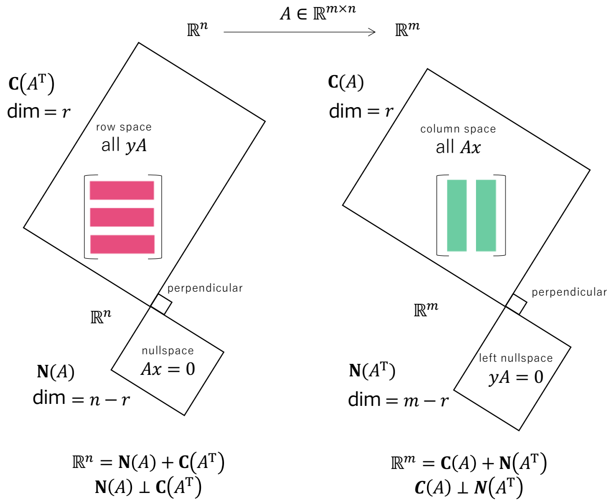


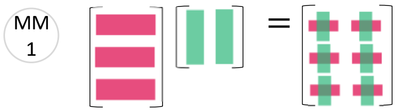
Figure 5: The Four Subspaces

See $A = CR$ (Sec 6.1) for the rank r.

4 Matrix times Matrix – 4 Ways

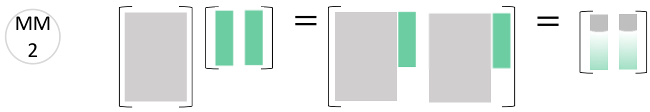
"Matrix times Vector" naturally extends to "Matrix times Matrix".

- Sec. 1.4 (p.35) Four Ways to Multiply $\boldsymbol{AB} = \boldsymbol{C}$
- Also see the back cover of the book



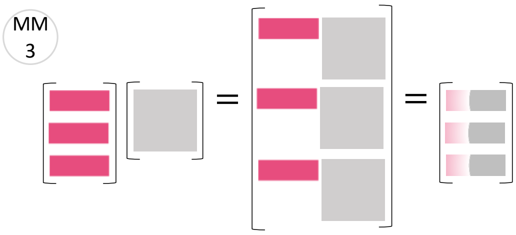
Every element becomes a dot product of row vector and column vector.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix} = \begin{bmatrix}(x_1+2x_2) & (y_1+2y_2)\\(3x_1+4x_2) & (3y_1+4y_2)\\(5x_1+6x_2) & (5y_1+6y_2)\end{bmatrix}$$



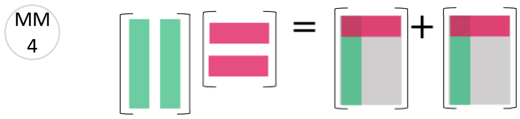
$A\boldsymbol{x}$ and $A\boldsymbol{y}$ are linear combinations of columns of A.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix} = A[\boldsymbol{x} \quad \boldsymbol{y}] = [A\boldsymbol{x} \quad A\boldsymbol{y}]$$



The produced rows are linear combinations of rows.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{a}_1^*\\\boldsymbol{a}_2^*\\\boldsymbol{a}_3^*\end{bmatrix}X = \begin{bmatrix}\boldsymbol{a}_1^*X\\\boldsymbol{a}_2^*X\\\boldsymbol{a}_3^*X\end{bmatrix}$$



Multiplication AB is broken down to a sum of rank 1 matrices.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{bmatrix} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2]\begin{bmatrix}\boldsymbol{b}_1^*\\\boldsymbol{b}_2^*\end{bmatrix} = \boldsymbol{a}_1\boldsymbol{b}_1^* + \boldsymbol{a}_2\boldsymbol{b}_2^*$$

$$= \begin{bmatrix}1\\3\\5\end{bmatrix}[b_{11} \quad b_{12}] + \begin{bmatrix}2\\4\\6\end{bmatrix}[b_{21} \quad b_{22}] = \begin{bmatrix}b_{11} & b_{12}\\3b_{11} & 3b_{12}\\5b_{11} & 5b_{12}\end{bmatrix} + \begin{bmatrix}2b_{21} & 2b_{22}\\4b_{21} & 4b_{22}\\6b_{21} & 6b_{22}\end{bmatrix}$$

Figure 6: Matrix times Matrix - (MM1), (MM2), (MM3), (MM4)

5 Practical Patterns

Here, I show some practical patterns which allow you to capture the coming factorizations more intuitively.

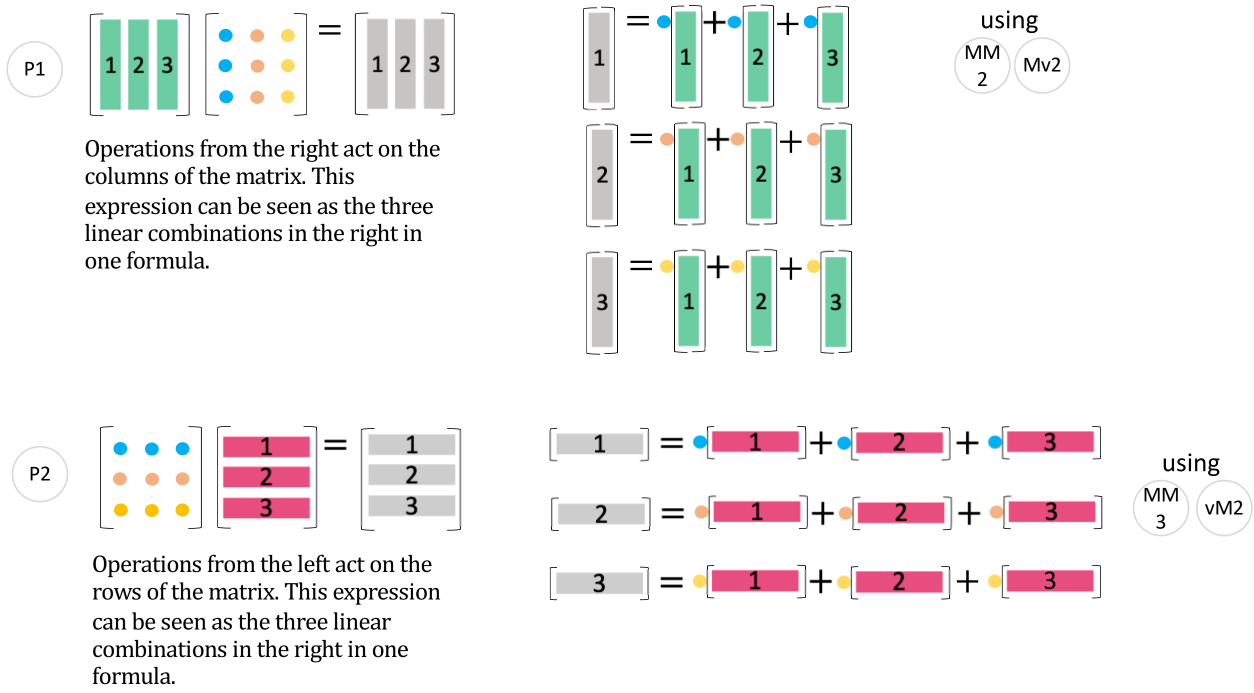
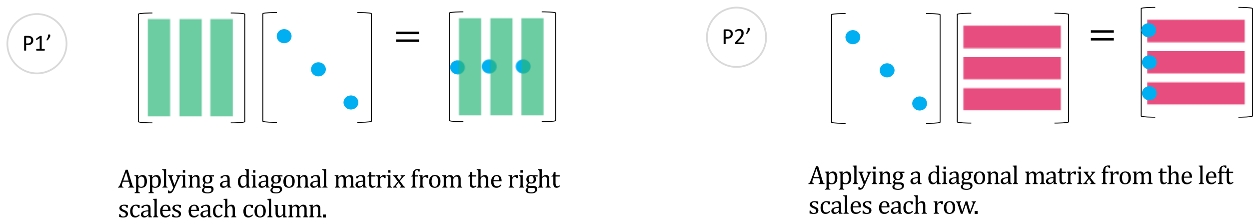


Figure 7: Pattern 1, 2 - (P1), (P1)

Pattern 1 is a combination of (MM2) and (Mv2). Pattern 2 is an extention of (MM3). Note that Pattern 1 is a column operation (multiplying a matrix from right), whereas Pattern 2 is a row operation (multiplying a matrix from left).

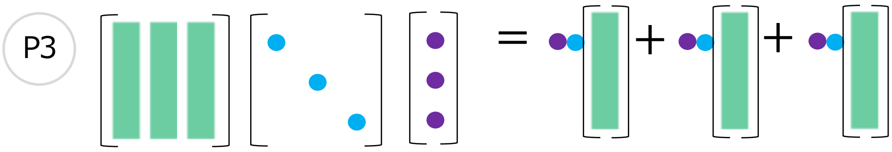


$$AD = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3]\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} = [d_1\boldsymbol{a}_1 \quad d_2\boldsymbol{a}_2 \quad d_3\boldsymbol{a}_3]$$

$$DB = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} \boldsymbol{b}_1^* \\ \boldsymbol{b}_2^* \\ \boldsymbol{b}_3^* \end{bmatrix} = \begin{bmatrix} d_1\boldsymbol{b}_1^* \\ d_2\boldsymbol{b}_2^* \\ d_3\boldsymbol{b}_3^* \end{bmatrix}$$

Figure 8: Pattern 1′, 2′ - (P1′), (P2′)

(P1′) multipies the diagonal numbers to the columns of the matrix, whereas (P2′) multipies the diagonal numbers to the row of the matrx. Both are variants of (P1) and (P2).



This pattern makes another combination of columns.
You will encounter this in differential/recurrence equations.

$$XD\boldsymbol{c} = [\boldsymbol{x_1} \quad \boldsymbol{x_2} \quad \boldsymbol{x_3}]\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 d_1 \boldsymbol{x_1} + c_2 d_2 \boldsymbol{x_2} + c_3 d_3 \boldsymbol{x_3}$$

Figure 9: Pattern 3 - (P3)

This pattern appears when you solve differential equations and recurrence equations:

- Sec. 6 (p.201) Eigenvalues and Eigenvectors
- Sec. 6.4 (p.243) Systems of Differential Equations

$$\begin{aligned} \frac{d\boldsymbol{u}(t)}{dt} &= A\boldsymbol{u}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \\ \boldsymbol{u}_{n+1} &= A\boldsymbol{u}_n, \quad \boldsymbol{u_0} = \boldsymbol{u}_0 \end{aligned}$$

In both cases, the solutions are expressed with eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, eigenvectors $X = \begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3\end{bmatrix}$ of A, and the coefficients $c = \begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}^{\mathrm{T}}$ which are the coordinates of the initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in terms of the eigenvectors X.

$$\begin{aligned} \boldsymbol{u}_0 &= c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 \\ \boldsymbol{c} &= \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = X^{-1}\boldsymbol{u}_0 \end{aligned}$$

and the general solution of the two equations are:

$$\begin{aligned} \boldsymbol{u}(t) &= e^{At}\boldsymbol{u}_0 = Xe^{\Lambda t}X^{-1}\boldsymbol{u_0} && = Xe^{\Lambda t}\boldsymbol{c} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2 + c_3 e^{\lambda_3 t}\boldsymbol{x}_3 \\ \boldsymbol{u}_n &= A^n\boldsymbol{u}_0 = X\Lambda^n X^{-1}\boldsymbol{u_0} && = X\Lambda^n\boldsymbol{c} = c_1\lambda_1^n\boldsymbol{x}_1 + c_2\lambda_2^n\boldsymbol{x}_2 + c_3\lambda_3^n\boldsymbol{x}_3 \end{aligned}$$

See Figure 8: Pattern 3 (P3) above again for XDc.

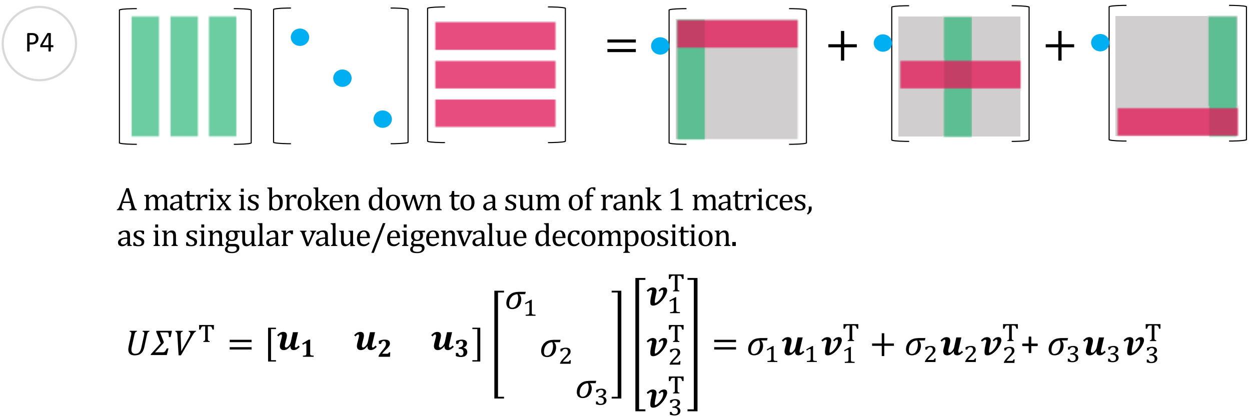


Figure 10: Pattern 4 - (P4)

This pattern (P4) works in both eigenvalue decomposition and singular value decomposition. Both decompositions are expressed as a product of three matrices with a diagonal matrix in the middle, and also a sum of rank 1 matrices with the eigenvalue/singular value coefficients.

More details are discussed in the next section.

6 The Five Factorizations of a Matrix

- Preface p.vii, The Plan for the Book.

$A = CR$, $A = LU$, $A = QR$, $A = Q\Lambda Q^{\mathrm{T}}$, $A = U\Sigma V^{\mathrm{T}}$ are illustrated one by one.

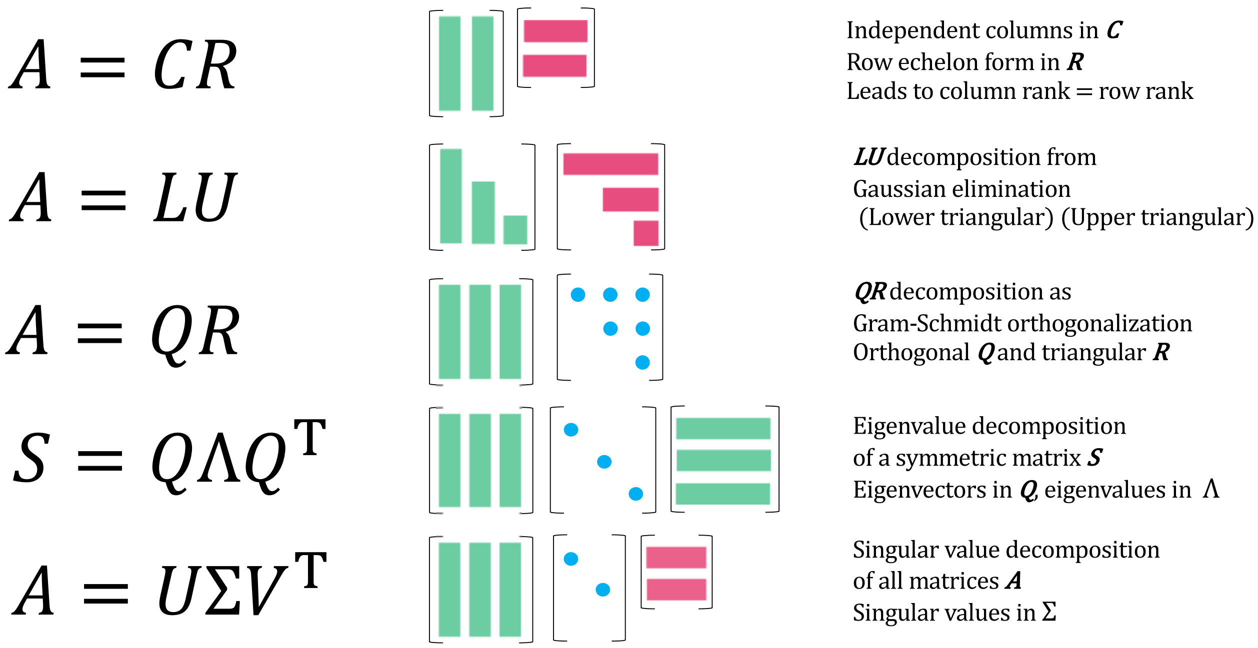


Figure 11: The Five Factorizations

6.1 $\boldsymbol{A = CR}$

- Sec.1.4 Matrix Multiplication and $\boldsymbol{A = CR}$ (p.29)

All general rectangular matrices A have the same row rank as the column rank. This factorization is the most intuitive way to understand this theorem. C consists of independent columns of A, and R is the row reduced echelon form of A. $A = CR$ reduces to r independent columns in C times r independent rows in R.

$$A = CR$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Procedure: Look at the columns of A from left to right. Keep independent ones, discard dependent ones which can be created by the former columns. The column 1 and the column 2 survive, and the column 3 is discarded because it is expressed as a sum of the former two columns. To rebuild A by the independent columns 1, 2, you find a row echelon form R appearing in the right.

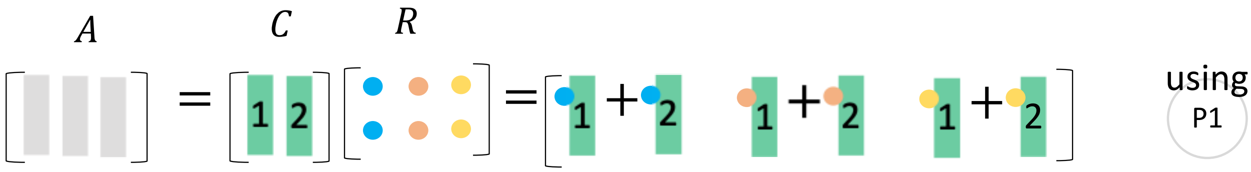


Figure 12: Column Rank in CR

Now you see the column rank is two because there are only two independent columns in C and all the columns of A are linear combinations of the two columns of C.

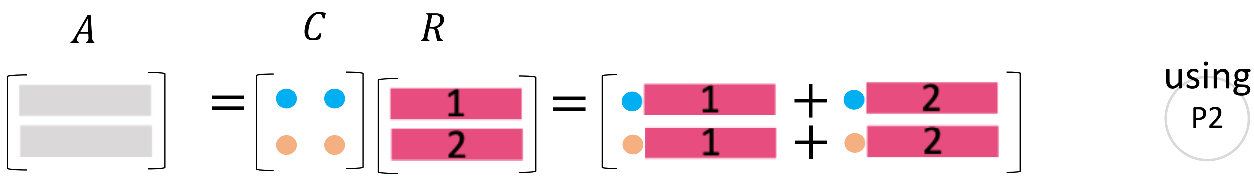


Figure 13: Row Rank in CR

And you see the row rank is two because there are only two independent rows in R and all the rows of A are linear combinations of the two rows of R.

6.2 $\boldsymbol{A = LU}$

Solving $A\boldsymbol{x} = \boldsymbol{b}$ via Gaussian elimination can be expressed as an LU factorization. Usually, you apply elementary row operation matrices (E) from left of A to make upper trianglar U.

$$\begin{aligned} EA &= U \\ A &= E^{-1}U \\ \text{let } L = E^{-1}, \quad A &= LU \end{aligned}$$

Now solve $A\boldsymbol{x} = \boldsymbol{b}$ in 2 steps: (1) forward $L\boldsymbol{c} = \boldsymbol{b}$ and (2) back $U\boldsymbol{x} = \boldsymbol{c}$.

- Sec.2.3 (p.57) Matrix Computations and $\boldsymbol{A = LU}$

Here, we directly calculate L and U from A.

$$A = \begin{bmatrix} | \\ \boldsymbol{l}_1 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{u}_1^* - \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & & A_2 \end{bmatrix} = \begin{bmatrix} | \\ \boldsymbol{l}_1 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{u}_1^* - \end{bmatrix} + \begin{bmatrix} | \\ \boldsymbol{l}_2 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{u}_2^* - \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3 \end{bmatrix} = LU$$

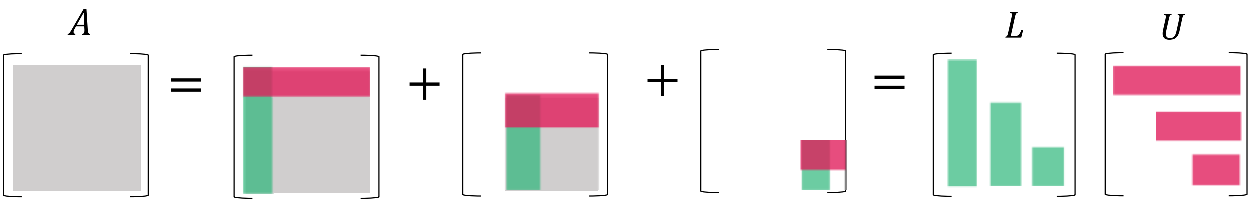


Figure 14: Recursive Rank 1 Matrix Peeling from A

To find L and U, peel off the rank 1 matrix made of the first row and the first column of A. This leaves A_2. Do this recursively and decompose A into the sum of rank 1 matrices.

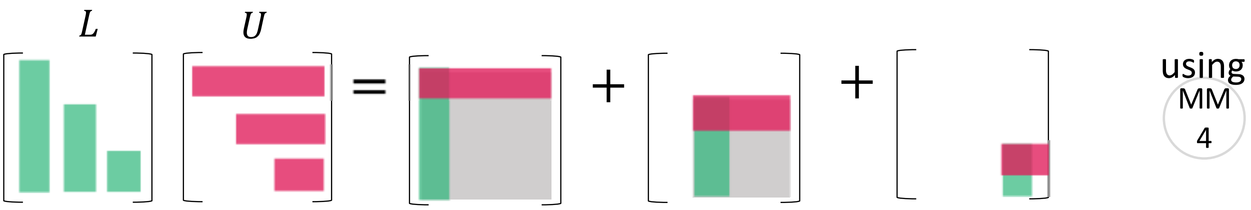


Figure 15: LU rebuilds A

To rebuild A from L times U is easy.

6.3 $\boldsymbol{A = QR}$

$A = QR$ changes the columns of A into perpendicular columns of Q, keeping $\boldsymbol{C}(A) = \boldsymbol{C}(Q)$.

- Sec.4.4 Orthogonal matrices and Gram-Schmidt (p.165)

In Gram-Schmidt, the normalized $\boldsymbol{a}_1$ is picked up as $\boldsymbol{q}_1$ first and then $\boldsymbol{a}_2$ is adjusted to be perpendicular to $\boldsymbol{q}_1$ to create $\boldsymbol{q}_2$, and this procedure goes on.

$$\begin{aligned}
&\boldsymbol{q}_1 = \boldsymbol{a}_1/||\boldsymbol{a}_1|| \\
&\boldsymbol{q}_2 = \boldsymbol{a}_2 - (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_2)\boldsymbol{q}_1, \quad \boldsymbol{q}_2 = \boldsymbol{q}_2/||\boldsymbol{q}_2|| \\
&\boldsymbol{q}_3 = \boldsymbol{a}_3 - (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_3)\boldsymbol{q}_1 - (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{a}_3)\boldsymbol{q}_2, \quad \boldsymbol{q}_3 = \boldsymbol{q}_3/||\boldsymbol{q}_3||
\end{aligned}$$

or you can write with $r_{ij} = \boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{a}_j$:

$$\begin{aligned}
\boldsymbol{a}_1 &= r_{11}\boldsymbol{q}_1 \\
\boldsymbol{a}_2 &= r_{12}\boldsymbol{q}_1 + r_{22}\boldsymbol{q}_2 \\
\boldsymbol{a}_3 &= r_{13}\boldsymbol{q}_1 + r_{23}\boldsymbol{q}_2 + r_{33}\boldsymbol{q}_3
\end{aligned}$$

The original A becomes QR: orthogonal times triangular.

$$A = \begin{bmatrix} | & | & | \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix} = QR$$

$$QQ^{\mathrm{T}} = Q^{\mathrm{T}}Q = I$$

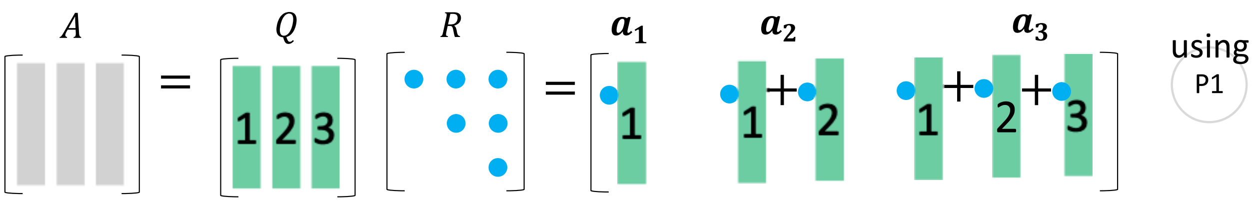


Figure 16: $A = QR$

The column vectors of A can be adjusted into an orthonormal set: the column vectors of Q. Each column vector of A can be rebuilt from Q and the upper triangular matrix R .

See Pattern 1 (P1) again for the graphic interpretation.

6.4 $S = Q\Lambda Q^{\mathrm{T}}$

All symmetric matrices S must have real eigenvalues and orthogonal eigenvectors. The eigenvalues are the diagonal elements of Λ and the eigenvectors are in Q.

- Sec.6.3 (p.227) Symmetric Positive Definite Matrices

$$S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} | & | & | \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} -\boldsymbol{q}_1^{\mathrm{T}}- \\ -\boldsymbol{q}_2^{\mathrm{T}}- \\ -\boldsymbol{q}_3^{\mathrm{T}}- \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} | \\ \boldsymbol{q}_1 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{q}_1^{\mathrm{T}}- \end{bmatrix} + \lambda_2 \begin{bmatrix} | \\ \boldsymbol{q}_2 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{q}_2^{\mathrm{T}}- \end{bmatrix} + \lambda_3 \begin{bmatrix} | \\ \boldsymbol{q}_3 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{q}_3^{\mathrm{T}}- \end{bmatrix}$$

$$= \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$$

$$P_1 = \boldsymbol{q}_1\boldsymbol{q}_1^{\mathrm{T}}, \quad P_2 = \boldsymbol{q}_2\boldsymbol{q}_2^{\mathrm{T}}, \quad P_3 = \boldsymbol{q}_3\boldsymbol{q}_3^{\mathrm{T}}$$

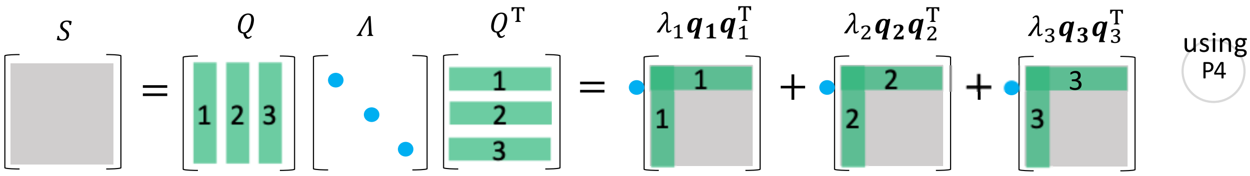


Figure 17: $S = Q\Lambda Q^{\mathrm{T}}$

A symmetric matrix S is diagonalized into Λ by an orthogonal matrix Q and its transpose. And it is broken down into a combination of rank 1 projection matrices $P = qq^{\mathrm{T}}$. This is the spectral theorem.

Note that Pattern 4 (P4) is working for the decomposition.

$$S = S^{\mathrm{T}} = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$$
$$QQ^{\mathrm{T}} = P_1 + P_2 + P_3 = I$$
$$P_1P_2 = P_2P_3 = P_3P_1 = O$$
$$P_1^2 = P_1 = P_1^{\mathrm{T}}, \quad P_2^2 = P_2 = P_2^{\mathrm{T}}, \quad P_3^2 = P_3 = P_3^{\mathrm{T}}$$

6.5 $\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$

- Sec.7.1 (p.259) Singular Values and Singular Vecrtors

All matrices including rectangular ones have a singular value decomposition (SVD). $A = U\Sigma V^{\mathrm{T}}$ has the singular vectors of A in U and V. And its singular values line up in Λ's diagonal elements. The following illustrates the 'reduced' SVD.

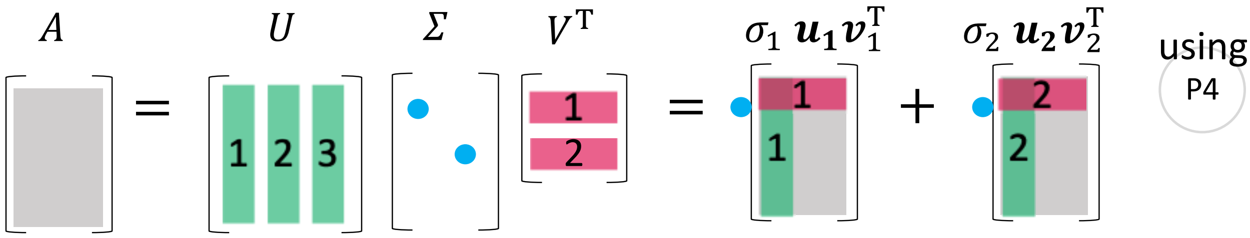


Figure 18: $A = U\Sigma V^{\mathrm{T}}$

You can find V as an orthonormal basis of $\mathbb{R}^n$ (eigenvectors of $A^{\mathrm{T}}A$), and U as an orthonormal basis of $\mathbb{R}^m$ (eigenvectors of AA^{T}). Together they diagonalize A into Σ. This is also expressed as a combination of rank 1 matrices.

$$A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} | & | & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \\ & \end{bmatrix} \begin{bmatrix} -\boldsymbol{v}_1^{\mathrm{T}}- \\ -\boldsymbol{v}_2^{\mathrm{T}}- \end{bmatrix} = \sigma_1 \begin{bmatrix} | \\ \boldsymbol{u}_1 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{v}_1^{\mathrm{T}}- \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ \boldsymbol{u}_2 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{v}_2^{\mathrm{T}}- \end{bmatrix}$$
$$= \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}}$$

Note that:

$$UU^{\mathrm{T}} = I_m$$
$$VV^{\mathrm{T}} = I_n$$

See Pattern 4 (P4) for the graphic notation.

Conclusion and Acknowledgements

I presented systematic visualizations of matrix/vector multiplication and their application to the Five Matrix Factorizations. I hope you enjoyed them and will use them in your understanding of Linear Algebra.

Ashley Fernandes helped me with beautifying this paper in typesetting and made it much more consistent and professional.

To conclude this paper, I'd like to thank Prof. Gilbert Strang for publishing "Linear Algebra for Everyone". It guides us through a new vision to these beautiful landscapes in Linear Algebra. Everyone can reach a fundamental understanding of its underlying ideas in a practical manner that introduces us to contemporary and also traditional data science and machine learning. An important part of the matrix world.

References and Related Works

1. Gilbert Strang(2020),*Linear Algebra for Everyone*, Wellesley Cambridge Press., http://math.mit.edu/everyone

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3. Kenji Hiranabe(2021), *Map of Eigenvalues*, An Agile Way(blog),
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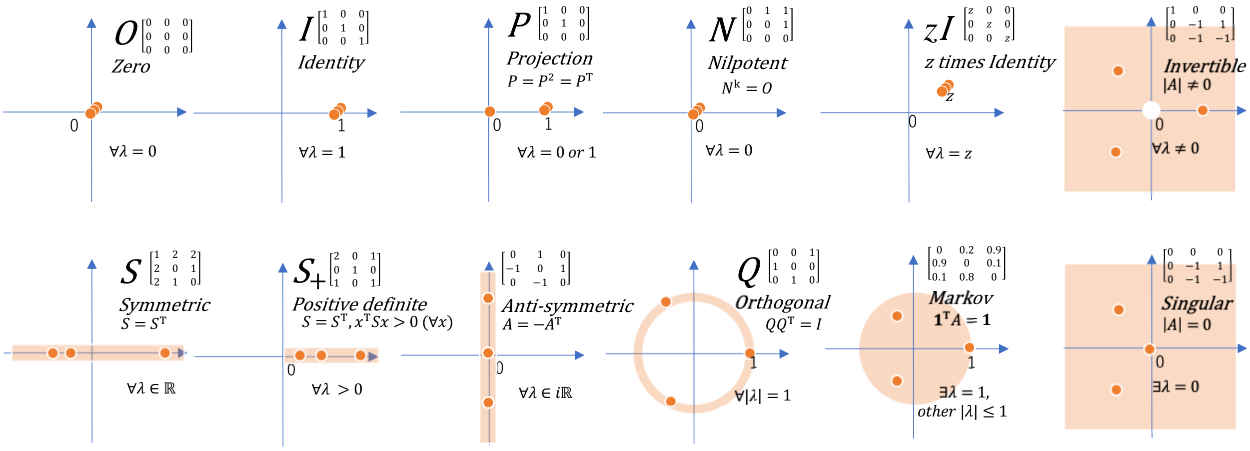

Figure 19: Map of Eigenvalues

4. Kenji Hiranabe(2020), *Matrix World*, An Agile Way(blog),
https://anagileway.com/2020/09/29/matrix-world-in-linear-algebra-for-everyone/

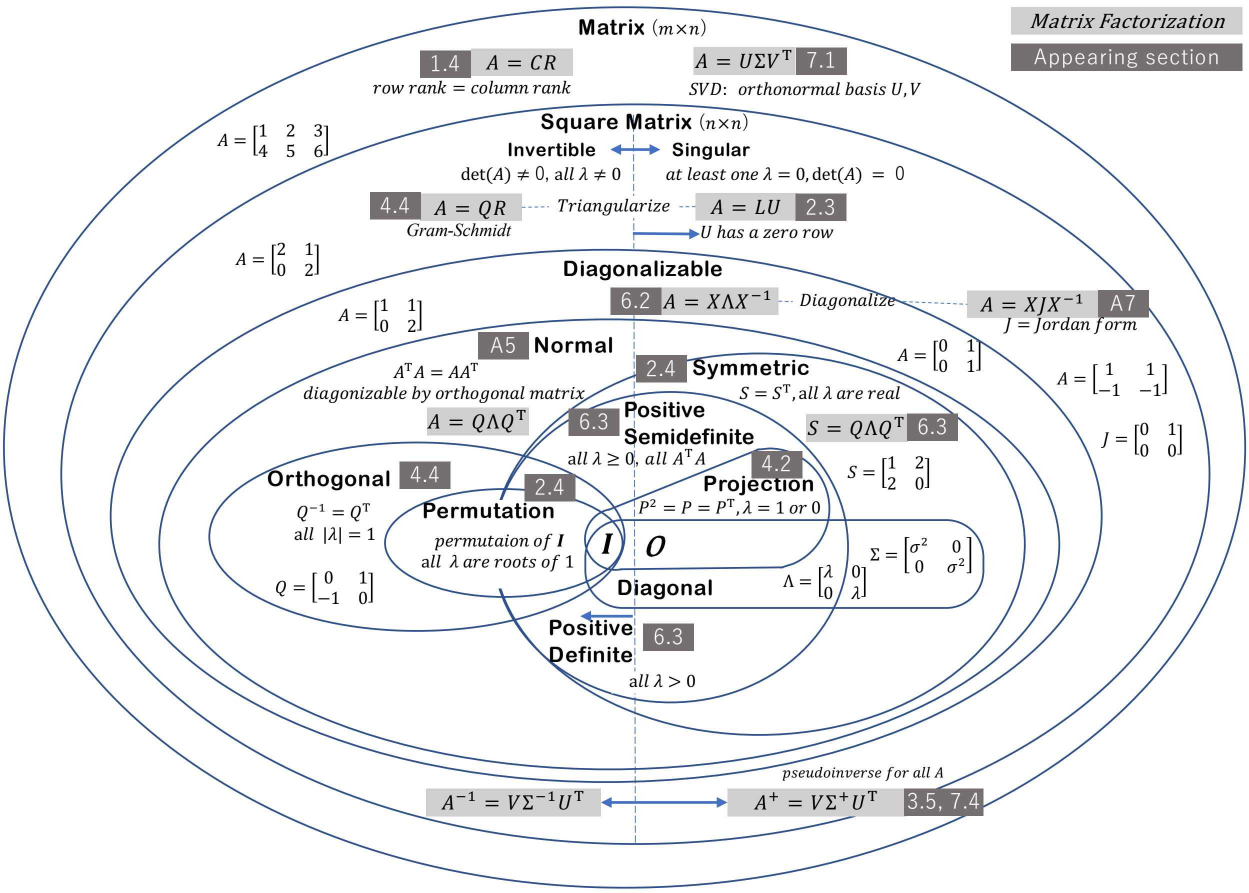

Figure 20: Matrix World