第三次作业

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1.考虑单天线点对点通信平坦瑞利衰落信道下采用 QPSK 调制的数字通信系统，每个符号间隔内的接收信号为：$y = hx + w$，其中：$h \sim \mathcal{CN}(0,1)$，$w \sim \mathcal{CN}(0,N_0)$。

1.a 请给出 AWGN 信道下 QPSK 的符号错误概率。通过计算机仿真给出 AWGN 信道下 QPSK 的符号错误概率曲线，横轴为E_b/N_0；

$$P_e = 2Q\left(\frac{d_{min}}{2\sigma}\right) - Q^2\left(\frac{d_{min}}{2\,\sigma}\right),$$

where $\sigma = \sqrt{\frac{N_0}{2}}$, $d_{min} = 2\sqrt{E} = 2\sqrt{E_b}$ and $Q(\cdot)$ is the Gaussian Q function. Thus, we have

$$P_e = 2Q\left(\sqrt{\frac{2E}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E}{N_0}}\right).$$

符号错误概率曲线见 Fig.1

1.b 通过计算机仿真给出该系统的符号错误概率曲线，横轴为E_b/N_0；对比平坦瑞利衰落和 AWGN 信道下的 QPSK 符号错误概率曲线。

符号错误概率曲线见 Fig.2

1.c 假设系统采用重复编码方案获取时间分集，分集支路为 2，接收机采用最大比合并算法。通过计算机仿真给出此时系统符号错误概率曲线。

符号错误概率曲线见 Fig.3

1.d 包含 L 条分集支路的重复编码的最优相干接收机为最大比合并器，出于实现

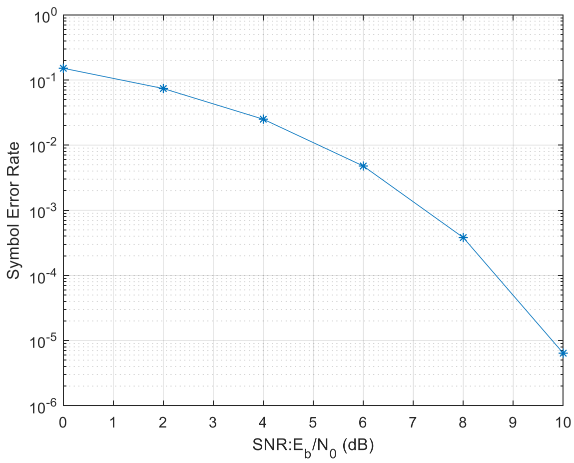

Fig.1 SER of QPSK in AWGN channel

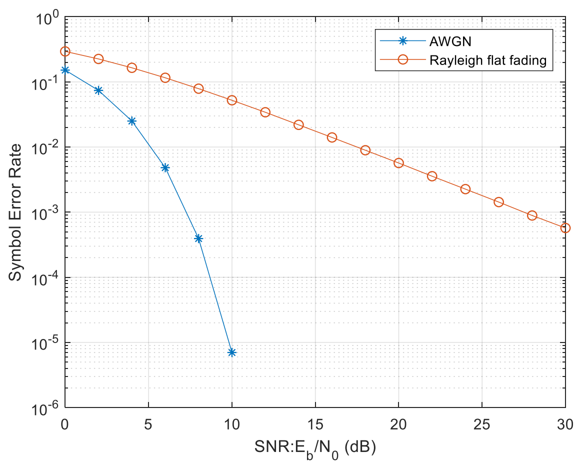

Fig.2 Comparison of SERs of QPSK in AWGN and Rayleigh flat fading

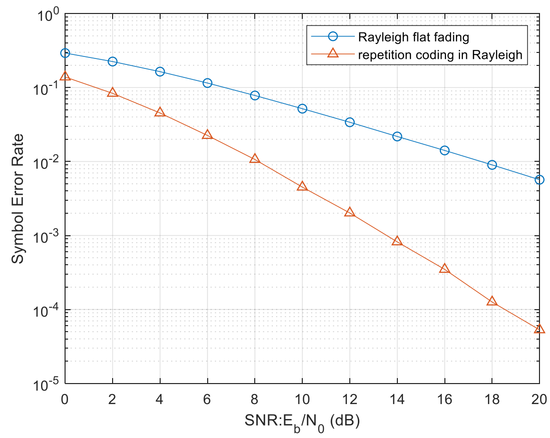

Fig.3 Comparison of SERs of QPSK in Rayleigh flat fading and that with repetition coding

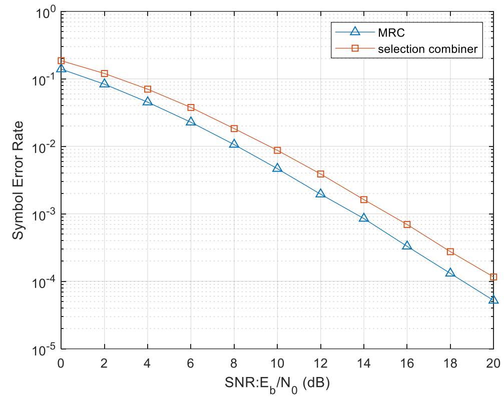

Fig.4 Comparison of SERs of QPSK in Rayleigh flat fading with repetition coding by different combining method

的原因，实际中也有可能采用更为简单的接收机结构，其中比较典型的一种就是选择合并器（selection combiner），它仅根据增益最强的支路上的接收信号进行检测，并忽略其他信号。在接收机采用选择合并器的情况下重新完成(b)中的仿真，并比较选择合并与最大比合并之间的性能差异。

概率曲线见 Fig.4。选择合并器相较于 MRC 大概有 1.5dB 的损失。

1.e 尝试从理论上证明以上最大比合并器和选择合并器的分集阶数。

In the case where there are $L=2$ branches, first we analyze MRC as follows. The probability of error is given by

$$P_{e|\boldsymbol{h}} = Q\left(\sqrt{2\left|\left|\boldsymbol{h}\right|\right|^2 SNR}\right).$$

Let $x = 2\left|\left|\boldsymbol{h}\right|\right|^2 = 2|h_1|^2 + 2|h_2|^2$, thus $X \sim \chi^2(4)$, which means we have PDF of x

$$\begin{aligned} f_X(x) &= \frac{1}{4\Gamma(2)} x e^{-\frac{x}{2}} \\ &= \frac{1}{4} x e^{-\frac{x}{2}}, x > 0 \end{aligned}$$

As $P_e = \int P_{e|\boldsymbol{h}} f_X(x)dx$, we get

$$\begin{aligned} P_e &= \int_0^\infty Q(\sqrt{xSNR}\frac{1}{4}xe^{-\frac{x}{2}}dx \\ &= \int_0^\infty Q\left(\sqrt{xSNR}\right)\frac{1}{4}xe^{-\frac{x}{2}}dx \\ &= -\frac{1}{2}\left[xe^{-\frac{x}{2}}Q\left(\sqrt{xSNR}\right)|_0^\infty - \int_0^\infty e^{-\frac{x}{2}}dxQ\left(\sqrt{xSNR}\right)\right] \\ &= \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}}dxQ\left(\sqrt{xSNR}\right) \\ &= \frac{1}{2}\left[\int_0^\infty e^{-\frac{x}{2}}Q\left(\sqrt{xSNR}\right)dx + \int_0^\infty xe^{-\frac{x}{2}}dQ\left(\sqrt{xSNR}\right)\right]. \end{aligned}$$

We calculate the first term of the above formula:

$$\int_0^\infty e^{-\frac{x}{2}}Q(\sqrt{xSNR})dx = -2\left[e^{-\frac{x}{2}}Q(\sqrt{xSNR})|_0^\infty - \int_0^\infty e^{-\frac{x}{2}}dQ(\sqrt{xSNR})\right].$$

As $dQ(\sqrt{xSNR}) = -\frac{SNR}{2\sqrt{2\pi xSNR}}e^{-\frac{SNR}{2}x}dx$, we derive

$$\int_0^\infty e^{-\frac{x}{2}}Q(\sqrt{xSNR})dx = 1 - \frac{\sqrt{SNR}}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{x}}e^{\left(-\frac{1+SNR}{2}\right)x}dx.$$

$$= 1 - \left(\frac{1+SNR}{2}\right)^{-1}\sqrt{\frac{SNR}{2\pi}}\Gamma\left(\frac{1}{2}\right)$$

$$= 1 - \sqrt{\frac{SNR}{1+SNR}}$$

$$= 1 - \left(1 + \frac{1}{SNR}\right)^{-\frac{1}{2}}$$

And the second term of P_e being

$$\int_0^\infty xe^{-\frac{x}{2}}dQ(\sqrt{xSNR}) = -\frac{\sqrt{SNR}}{2\sqrt{2\pi}}\int_0^\infty \sqrt{x}e^{-\frac{1+SNR}{2}}dx$$

$$= -\frac{1}{2}\frac{\sqrt{SNR}}{\sqrt{SNR+1}^3}$$

$$= -\frac{1}{2}\frac{1}{SNR}\left(1 + \frac{1}{SNR}\right)^{-\frac{3}{2}}.$$

Hence we present P_e by

$$P_e = \frac{1}{2}\left[1 - \left(1 + \frac{1}{SNR}\right)^{-\frac{1}{2}} - \frac{1}{2}\frac{1}{SNR}\left(1 + \frac{1}{SNR}\right)^{-\frac{3}{2}}\right].$$

By using $li\,m_{x\to 0}(1+x)^b \approx 1 + bx$, we get approximation of P_e when $SNR \to \infty$

$$P_e \approx \frac{1}{2}[1 - \left(1 - \frac{1}{2SNR}\right) - \frac{1}{2}\frac{1}{SNR}(1 - \frac{3}{2SNR})$$

$$= \frac{3}{8}\frac{1}{SNR^2}$$

indicating the diversity gain being 2.

As for the selection combiner,

令 $X_i = |h_i|^2$, i.i.d

则 $X_i \sim exp(1)$

即 $f_{X_i}(x) = e^{-x}$, $x \geq 0$

令 $Y = \max\limits_i X_i$

$$
\begin{aligned}
F_Y(y) &= P\{Y \le y\} \\
&= P\{X_1 \le y\} P\{X_2 \le y\} \\
&= \int_0^y e^{-x} dx \int_0^y e^{-x} dx \\
&= \left(\int_0^y e^{-x} dx\right)^2 \\
&= \left(-e^{-x}\Big|_0^y\right)^2 \\
&= (1-e^{-y})^2, \quad y \ge 0
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= F_Y'(y) \\
&= 2(1-e^{-y}) \cdot e^{-y} \\
&= 2e^{-y} - 2e^{-2y}, \quad y \ge 0
\end{aligned}
$$

$$
\begin{aligned}
P_{e|h} &= Q\left(\sqrt{2\, SNR \cdot \max(x_i)}\right) \\
&= Q\left(\sqrt{2 SNR \cdot y}\right)
\end{aligned}
$$

$$
\begin{aligned}
P_e &= \int_0^\infty Q(\sqrt{2SNR\, y}) \cdot f_Y(y) dy \\
&= 2\int_0^\infty Q(\sqrt{2SNR\, y}) \cdot e^{-y} dy - 2\int_0^\infty Q(\sqrt{2SNR\, y}) e^{-2y} dy \\
&= -2\int_0^\infty Q(\sqrt{2SNR\, y}) de^{-y} + \int_0^\infty Q(\sqrt{2SNR\, y}) de^{-2y} \\
&= -2\left[Q(\sqrt{2SNR\, y}) e^{-y}\Big|_0^\infty - \int_0^\infty e^{-y} dQ(\sqrt{2SNR\, y})\right] \\
&\quad + \left[Q(\sqrt{2SNR\, y}) e^{-2y}\Big|_0^\infty - \int_0^\infty e^{-2y} dQ(\sqrt{2SNR\, y})\right] \\
&= -2 \times \left[0 - \frac{1}{2} - \int_0^\infty e^{-y} dQ(\sqrt{\ })\right] \\
&\quad + \left[0 - \frac{1}{2} - \int_0^\infty e^{-2y} dQ(\sqrt{\ })\right] \\
&= \frac{1}{2} + 2\underbrace{\int_0^\infty e^{-y} dQ(\sqrt{\ })} - \int_0^\infty e^{-2y} dQ(\sqrt{\ })
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty e^{-y} dQ(\sqrt{\ }) &= \int_0^\infty e^{-y} (-1) \cdot \frac{\sqrt{SNR}}{2\sqrt{\pi y}} e^{-SNR\, y} dy \\
&= -\frac{\sqrt{SNR}}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{y}} e^{-(1+SNR)y} dy \\
&= -\frac{\sqrt{SNR}}{2\sqrt{\pi}} \cdot \frac{\sqrt{1+SNR}}{1+SNR} \cdot \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} dt \\
&= -\frac{1}{2} \frac{\sqrt{SNR}}{\sqrt{1+SNR}} \cdot \frac{1}{\sqrt{\pi}} \cdot \Gamma\left(\frac{1}{2}\right) \\
&= -\frac{1}{2}\left(1 + \frac{1}{SNR}\right)^{-\frac{1}{2}}
\end{aligned}
$$

$$\int_0^\infty e^{-2y} dQ(\sqrt{\ }) = \int_0^\infty e^{-2y} \cdot (-1) \cdot \frac{\sqrt{SNR}}{2\sqrt{\pi}\sqrt{y}} e^{-SNRy} dy$$

$$= -\frac{\sqrt{SNR}}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{y}} e^{-(SNR+2)y} dy$$

$$= -\frac{1}{2} \cdot \frac{\sqrt{SNR}}{\sqrt{2+SNR}} \cdot \frac{1}{\sqrt{\pi}} \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= -\frac{1}{2}\left(1+\frac{2}{SNR}\right)^{-\frac{1}{2}}$$

$$\Rightarrow P_e = \frac{1}{2} - \left(1+\frac{1}{SNR}\right)^{-\frac{1}{2}} + \frac{1}{2}\left(1+\frac{2}{SNR}\right)^{-\frac{1}{2}}$$

由 $x \to 0$ 时 $(1+x)^b \approx 1 + bx + \frac{b(b-1)}{2} \cdot x^2$

得 $P_e = \frac{1}{2} - \left(1 - \frac{1}{2} \cdot \frac{1}{SNR} + \frac{\frac{1}{2}\times\frac{3}{2}}{2} \frac{1}{SNR^2}\right) + \frac{1}{2}\times\left(1 - \frac{1}{SNR} + \frac{\frac{1}{2}\times\frac{3}{2}}{2} \frac{4}{SNR^2}\right)$

$$= \left(\frac{3}{8} + \frac{3}{4}\right) \frac{1}{SNR^2}$$

$$= \frac{3}{8} \frac{1}{SNR^2}$$

我感觉我哪里解错了，感觉$P_e^{selection}$和P_e^{MRC}中的常数因子应该不一样。但既然 SNR 的指数是对的，也就不细究了。

2.若想在瑞利衰落信道中获得 L 阶分集增益，请问有哪些办法？分别利用了什么资源？请分别举例说明。

a) 重复编码，重复次数 L，或交织编码，块长度 L。利用了时间资源。

b) 使用 L 个频率发送同样的内容。利用了频率资源。

c) 使用 L 个天线接收并用 MRC 或选择合并。利用了接收端空间资源。

d) 发射端多天线时的空时编码。比如 OSTBC 码，即双天线时的 Alamouti 码。利用了发射端的空间资源。