设 X 和 Y 是两个相互独立的零均值高斯随机变量，方差同为 σ^2，证明：$Z=\sqrt{X^2+Y^2}$ 服从瑞利分布，Z^2 服从指数分布。

$$X, Y \sim N(0, \sigma^2)$$

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

Let $Z = \sqrt{X^2+Y^2}$ and thus, CDF of Z is

$$\begin{aligned}
F_Z(z) &= P\{R \le r\} \\
&= P\left\{\sqrt{X^2+Y^2} \le z\right\} \\
&= \iint_{\sqrt{X^2+Y^2}\le z} f_X(x) f_Y(y)\,dxdy \\
&= \iint_{\sqrt{X^2+Y^2}\le z} \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}\,dxdy \\
&= \begin{cases} \int_0^{2\pi}\int_0^{z} \frac{1}{2\pi\sigma^2} e^{-\frac{\rho^2}{2\sigma^2}} \rho d\rho d\varphi, & z \ge 0 \\ 0, & z < 0 \end{cases} \\
&= \begin{cases} 1 - e^{-\frac{z^2}{2\sigma^2}}, & z \ge 0 \\ 0, & z < 0 \end{cases}.
\end{aligned}$$

PDF of Z is

$$\begin{aligned}
f_Z(z) &= F_Z'(z) \\
&= \begin{cases} \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}}, & z \ge 0 \\ 0, & z < 0 \end{cases},
\end{aligned}$$

which shows the variable Z obeys Rayleigh distribution.

Then, let $R = X^2 + Y^2$. Following the same procedure, we get CDF of R

$$\begin{aligned}
F_R(r) &= \iint_{X^2+Y^2\le r} f_X(x) f_Y(y)\,dxdy \\
&= \begin{cases} \int_0^{2\pi}\int_0^{\sqrt{r}} \frac{1}{2\pi\sigma^2} e^{-\frac{\rho^2}{2\sigma^2}} \rho d\rho d\varphi, & r \ge 0 \\ 0, & r < 0 \end{cases} \\
&= \begin{cases} 1 - e^{-\frac{r}{2\sigma^2}}, & r \ge 0 \\ 0, & r < 0 \end{cases}
\end{aligned}$$

And PDF of R

$$\begin{aligned}
f_R(r) &= F_R'(r) \\
&= \begin{cases} \frac{1}{2\sigma^2} e^{-\frac{r}{2\sigma^2}}, & r \ge 0 \\ 0, & r < 0 \end{cases},
\end{aligned}$$

which shows the variable $R = X^2 + Y^2$ obeys Exponential distribution.