Dynamics considerations

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1 Calculating the Jacobian matrix and consideration of joint velocities

Considering the forward kinematics of the manipulator:

$$
{}^{0}T_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & L_1c_1 + L_2c_{12} + L_3c_{123} \\ s_{123} & c_{123} & 0 & L_1s_1 + L_2s_{12} + L_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}
$$

The position and orientation of the end effector [a, b, ϕ]^T is given by:

$$
a = L_1c_1 + L_2c_{12} + L_3c_{123} \tag{2}
$$

$$
b = L_1s_1 + L_2s_{12} + L_3s_{123} \tag{3}
$$

$$
\phi = \theta_1 + \theta_2 + \theta_3 \tag{4}
$$

In order to find the manipulator's dynamics, the following relationship between Cartesian and joint space coordinates can be considered[1]:

$$
\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}} \tag{5}
$$

Where $\dot{\mathbf{x}}$ represents the vector of the Cartesian (end effector) velocities, $\mathbf{J} \in \mathbb{R}^{mxn}$(where m is the number of Cartesian coordinates and n is the degrees of freedom, or number of joints, of the manipulator, so in this case m=3 and n=3) is the Jacobian matrix, and $\dot{\mathbf{q}}$ is the vector of joint-space velocities.
To find the Cartesian space velocities the time derivatives of a, b, and ϕ are first taken:

$$
\dot{\mathbf{x}} = \begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{\phi} \end{bmatrix} \tag{6}
$$

$$
\dot{a} = -\dot{\theta}_1L_1s_1 - (\dot{\theta}_1 + \dot{\theta}_2)L_2s_{12} - (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)L_3s_{123} \tag{7}
$$

$$
\dot{b} = \dot{\theta}_1L_1c_1 + (\dot{\theta}_1 + \dot{\theta}_2)L_2c_{12} + (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)L_3c_{123} \tag{8}
$$

$$
\dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \tag{9}
$$

$\dot{\mathbf{q}}$. The Jacobian matrix of the manipulator can be found by rearranging the time derivatives of each state element so that $\dot{\mathbf{q}}$ is factored out:

$$\begin{bmatrix}\dot{a}\\ \dot{b}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}-L_1s_1-L_2s_{12}-L_3s_{123} & -L_2s_{12}-L_3s_{123} & -L_3s_{123}\\ L_1c_1+L_2c_{12}+L_3c_{123} & L_2c_{12}+L_3c_{123} & L_3c_{123}\\ 1 & 1 & 1\end{bmatrix} \cdot \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix} \tag{10}$$

The Jacobian matrix, given by the following formula:

$$\mathbf{J} = \begin{bmatrix}\frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_1}{\partial \theta_2} & \frac{\partial x_1}{\partial \theta_3}\\ \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_2} & \frac{\partial x_2}{\partial \theta_3}\\ \frac{\partial \phi}{\partial \theta_1} & \frac{\partial \phi}{\partial \theta_2} & \frac{\partial \phi}{\partial \theta_3}\end{bmatrix} \tag{11}$$

2 Manipulator equation

The general manipulator equation for a 2 link manipulator would be given by[2]:

$$\mathbf{M}(q)\ddot{q} + \mathbf{C}(q,\dot{q})\dot{q} - \tau_g(q) = \tau \tag{12}$$

Where $\mathbf{M}$ represents the Inertia Matrix:

$$\mathbf{M} = \begin{bmatrix}(m_1+m_2)L_1^2 + m_2L_2^2 + 2m_2L_1L_2cos(q_2) & m_2L_2^2 + m_2L_1L_2cos(q_2)\\ m_2L_2^2 + m_2L_1L_2cos(q_2) & m_2L_2^2\end{bmatrix} \tag{13}$$

$\mathbf{C}$ represents the Coriolis or centrifugal Matrix:

$$\mathbf{C} = \begin{bmatrix}-2L_1L_2sin(q_2)\dot{q_2} & -L_1L_2m_2sin(q_2)\dot{q_2}\\ L_1L_2m_2sin(q_2)\dot{q_1} & 0\end{bmatrix} \tag{14}$$

and τ_g represents the gravitational torque vector:

$$\tau_g = \begin{bmatrix}(m_1 + m_2gL_1sin(q_1) + m_2gL_2sin(q_1+q_2)\\ m_2gL_2sin(q_1+q_2)\end{bmatrix} \tag{15}$$

References

[1] "Introduction to Robot Geometry and Kinematics." Accessed: Jan. 10, 2025. [Online]. Available: `http://fricke.co.uk/Teaching/CS591_Swarm_Robotics_2017fall/Readings/IntroRobotKinematics5.pdf`

[2] K. M. Lynch and F. C. Park, Modern robotics : mechanics, planning, and control. Cambridge: University Press, 2017.