Inverse Kinematics of 2 and 3 link finger

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1 Inverse Kinematics of 1 finger

As 2 link manipulator
For end effector position $[a, b, c]^T$

$$a = L_1 c_1 + L_2 c_{12}$$

$$b = L_1 s_1 + L_2 s_{12}$$

$$c = 0$$

1. Finding θ_2

To find θ_2 first the geometry of the right angled triangle with sides a and b and a hypotenuse of r_1:

$$a^2 + b^2 = r_1^2$$

$$r_1^2 = L_1^2 + L_2^2 + 2L_1 L_2 (c_{12} + s_{12})$$

since $c_{12} + s_{12} = c_2$

$$c_2 = \frac{r_1^2 - L_1^2 - L_2^2}{2L_1 L_2}$$

It therefore follows that

$$\theta_2 = cos^{-1}(\frac{r_1^2 - L_1^2 - L_2^2}{2L_1 L_2})$$

Case 1

$$\frac{r_1^2 - L_1^2 - L_2^2}{2L_1 L_2} > 1,\ 0\ solutions\ for\ \theta_2$$

Case 2

$$\frac{r_1^2 - L_1^2 - L_2^2}{2L_1 L_2} = 0,\ cos(\theta_2) = 1$$

Therefore

$$a = (L_1 + L_2)c_1,\ b = (L_1 + L_2)s_1$$

Rearranging:

$$c_1 = \frac{a}{L_1 + L_2} s_1 = \frac{b}{L_1 + L_2}$$

$$tan(\theta_1) = \frac{s_1}{c_1}$$

$$\theta_1 = Atan2(b, a)$$

Case 3
When

$$-1 < \frac{r_1^2 - L_1^2 - L_2^2}{2L_1L_2} < 1,$$

$$\theta_2 = cos^{-1}(\frac{r_1^2 - L_1^2 - L_2^2}{2L_1L_2}) \text{ and } \theta_2 = 2\pi - cos^{-1}(\frac{r_1^2 - L_1^2 - L_2^2}{2L_1L_2})$$

Case 4
When

$$\frac{r_1^2 - L_1^2 - L_2^2}{2L_1L_2} = -1$$

So

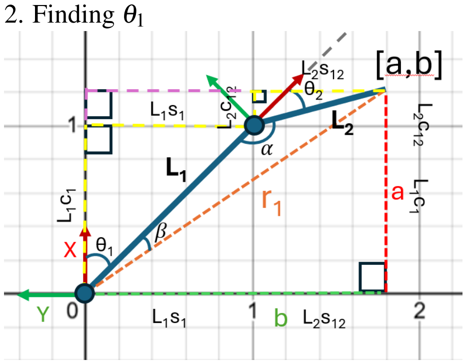
$$\theta_2 = \pi$$

Case 5
When

$$\frac{r_1^2 - L_1^2 - L_2^2}{2L_1L_2} < -1$$

So 0 solutions for when ¡-1.
The solution we require is when elbow is up only because the finger doesn't bend backwards, and also only in half of the workspace because there is a palm, but i just put all the different cases here. Need to apply constraints

2. Finding θ_1



The line r_1 makes an angle $\theta_1 + \beta$ with the x-axis. This makes a right angled triangle with sides a and b, which gives:

$$tan(\theta_1 + \beta) = \frac{b}{a}, \text{ where } b = L_1s_1 + L_2s_{12} \text{ and } a = L_1c_1 + L_2c_{12}.$$

$$\theta_1 = tan^{-1}\left(\frac{b}{a}\right) - \beta \tag{1}$$

And using the law of cosines

$$cos(\beta) = \frac{r_1^2 + L_1^2 - L_2^2}{2r_1L_2}$$

$$\beta = cos^{-1}\left(\frac{r_1^2 + L_1^2 - L_2^2}{2r_1L_2}\right) \tag{2}$$

Meaning:

$$\theta_1 = tan^{-1}\left(\frac{b}{a}\right) - cos^{-1}\left(\frac{r_1^2 + L_1^2 - L_2^2}{2r_1L_2}\right) \tag{3}$$

As 3 link manipulator

For end effector position $[a, b, \phi]^T$, where a and b are the x and y coordinates of the end effector and ϕ is the relative angle of the end effector from the base frame(or the orientation.

$$a = L_1c_1 + L_2c_{12} + L_3c_{123} \tag{4}$$

$$b = L_1s_1 + L_2s_{12} + L_3s_{123} \tag{5}$$

and

$$\phi = \theta_1 + \theta_2 + \theta_3 \tag{6}$$

Firstly substituting ϕ into equations (4) and (5) yields:

$$a = L_1c_1 + L_2c_{12} + L_3c_\phi$$

$$b = L_1s_1 + L_2s_{12} + L_3s_\phi$$

eliminating θ_3, and rearranging so the unknowns θ_1 and θ_2 are on the right hand side:

$$a - L_3c_\phi = L_1c_1 + L_2c_{12}$$

$$b - L_3s_\phi = L_1s_1 + L_2s_{12}$$

the left hand side expression for each can be renamed as e and f:

$$e = L_1c_1 + L_2c_{12} \tag{7}$$

$$f = L_1s_1 + L_2s_{12} \tag{8}$$

Rearranging:

$$e - L_1 c_1 = L_2 c_{12}$$

$$f - L_1 s_1 = L_2 s_{12}$$

squaring the two equations and adding them together yields:

$$e^2 + f^2 + L_1^2 c_1^2 + L_1^2 s_1^2 - 2eL_1 c_1 - 2fL_1 s_1 - L_2^2 c_{12}^2 - L_2^2 s_{12}^2 = 0$$

Simplifying using the trigonometric identity $sin^2\theta + cos^2\theta = 1$:

$$e^2 + f^2 + L_1^2 - L_2^2 - 2eL_1 c_1 - 2fL_1 s_1 = 0$$

This equation is now in the following form:

$$Pcos\theta + Qsin\theta + R = 0$$

with $R = -e^2 - f^2 - L_1^2 + L_2^2$, $P = 2eL_1$ and $Q = 2fL_1$ This form of trigonometric equation can be solved for θ_1 in the following way:

$$\theta_1 = \gamma \pm cos^{-1}\left(\frac{-R}{\sqrt{P^2 + Q^2}}\right) \tag{9}$$

Where

$$\gamma = atan2\left(\frac{Q}{\sqrt{P^2 + Q^2}}, \frac{P}{\sqrt{P^2 + Q^2}}\right) \tag{10}$$

Now that the two solutions (positive and negative) for θ_1 are known, to find θ_2 we simply rearrange equations (7) and (8) for $\sin(\theta_1 + \theta_2)$ and $\cos(\theta_1 + \theta_2)$ and dividing:

$$sin(\theta_1 + \theta_2) = \frac{e - L_1 c_1}{L_2} \tag{11}$$

$$cos(\theta_1 + \theta_2) = \frac{f - L_1 s_1}{L_2} \tag{12}$$

and using the trigonometric identity $\tan\theta$ = $\frac{sin\theta}{cos\theta}$

$$tan(\theta_1 + \theta_2) = \frac{f - L_1 s_1 \frac{1}{L_2}}{e - L_1 c_1 \frac{1}{L_2}} \tag{13}$$

And finally rearranging for θ_2

$$\theta_2 = atan2(\frac{f - L_1 s_1}{L_2}, \frac{e - L_1 c_1}{L_2}) - \theta_1 \tag{14}$$

Therefore giving θ_2 a unique solution for each solution for θ_1. θ_3 It follows that θ_3 is found by:

$$\theta_3 = \phi - \theta_1 - \theta_2 \tag{15}$$

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References

[1] “Introduction to Robot Geometry and Kinematics.” Accessed: Jan. 10, 2025. [Online]. Available: http://fricke.co.uk/Teaching/CS591_Swarm_Robotics_2017fall/Readings/IntroRobotKinematics5.pdf