Quaternion Kinematic and Dynamic Differential Equations

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***Abstract*—Using quaternions to describe finite rotations brings attention to their capacity to specify arbitrary rotations in space without degeneration to singularity and to their usefulness in extending the vector algebra to encompass multiplication and division for both scalars and spatial vectors. A quaternion contains four parameters, and they have been proved to be a minimal set for defining a nonsingular mapping between the parameters and their corresponding rotational transformation matrix. Many useful identities pertaining to quaternion multiplications are generalized in this paper. Among them multiplicative commutativity is the most powerful. Since quaternion space includes the three-dimensional vector space, the physical quantities related to rotations, such as angular displacement, velocity, acceleration, and momentum, are shown to be vector quaternions, and their expressions in quaternion space are derived. These kinematic and dynamic differential equations are further shown to be invertible due to the fact that they are written in quaternion space, and the highest order term of the rotation parameters can be expressed explicitly in closed form.**

I. Introduction

In 1843, the Irish mathematician W. R. Hamilton invented quaternions in order to extend three-dimensional vector algebra for inclusion of multiplications and divisions. Although it has been found that ordinary vector algebra provides a better mathematical apparatus for investigating physical problems, quaternion algebra, nevertheless, provides us with a simple and elegant representation for describing finite rotations in space.

Finite rotations in space are described by 3×3 rotational transformation matrices, and these matrices can be specified by a set of three to nine parameters. Commonly used parameter triples for finite rotations include Euler angles, Cardan angles (or Bryant angles), and Rodrigues parameters. Four parameters are Euler axis and angle, Euler parameters, and Cayley–Klein parameters. Directly using direction cosines for describing rotations requires a total of nine parameters with six constraints. A thorough comparison of most of these rotation parameters was given by Spring in 1986 [1]. Among all the representations for finite rotations, only those of four parameters behave well for arbitrary rotations because a nonsingular mapping between parameters and their corresponding rotational transformation matrix requires a set of four parameters at least; this fact was addressed by Stuelpnagel in 1964 [2].

The study of quaternions can be traced back to the work by Hamilton (1853 and 1969) [3], [4], Tait (1867) [5], Kelland and Tait (1882) [6], Hardy (1881) [7], Cayley (1885 and 1889) [8]–[10], and Klein (1925) [11]. It did not become popular until the recent applications to spatial kinematic analysis by Yang and Freudenstein in 1964 [12] and Nikravesh in 1984 [13]; spatial kinematic synthesis by Sandor in 1968 [14]; rigid-body dynamics by Robinson (1958) [15], Yang and Freudenstein (1964) [12], Nikravesh and Chung (1982) [16], Koshlyakov (1983) [17], Chelnokov (1984) [18], Nikravesh (1984) [13], and Chou *et al.* (1986) [19]–[21]; robot trajectory planning by Taylor (1979) [22]; robot dynamics by Huston and Kelly (1982) [23] and Chou *et al.* (1987) [24]; spacecraft control by Ickes (1970) [25], Meyer (1971) [26], Hendley (1971) [27], Junkins and Turner (1980) [28], Werz (1980) [29], Dwyer (1984) [30], and Wen and Kreutz (1988) [31]; camera calibration by Chou and Kamel (1988 and 1991) [32], [33] and Chou (1991) [34], [35]; and photogrammetry by Horn (1987) [36].

Euler parameters are unit quaternions. They are a set of four parameters required to satisfy the normality condition that the norm is unity. Using Euler-parameter representations to describe the orientation of a coordinate system has several advantages over the conventional usage of direction cosines and Euler angles. First, a mechanical system that involves rotation between various coordinate systems does not degenerate for any angular orientation [37]. Second, the computational cost of using Euler parameters is less than using direction cosines or Euler angles when quaternion multiplication is applied [15], [22], [25]. Furthermore, physical quantities pertaining to the motion of rotation such as angular displacement, velocity, acceleration, and momentum are derived in terms of Euler parameters in a simple manner. Manipulating equations is much easier when using quaternion algebra.

II. Quaternions

A quaternion $\boldsymbol{\alpha}$ is defined as a complex number

$$\boldsymbol{\alpha} = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \tag{1}$$

formed from four different units (1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$) by means of the real parameters $a_i (i = 0, 1, 2, 3)$ [3], [11], [38]–[42], where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are three orthogonal unit spatial vectors. With ideas from both vector and matrix algebra, the quaternion $\boldsymbol{\alpha}$ may be viewed as a linear combination of a scalar a_0 and a spatial vector $\vec{a}$:

$$\boldsymbol{\alpha} = a_0 + \vec{a}. \tag{2}$$

If $a_0 = 0$, $\boldsymbol{\alpha}$ is a purely imaginary number and is called a *vector quaternion*; when $\vec{a} = \vec{0}$, $\boldsymbol{\alpha}$ is a real number and is

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called a *scalar quaternion*. As we can observe, scalars and spatial vectors are quaternions, and they are in the subspace of quaternions.

The *conjugate* of a quaternion $\boldsymbol{\alpha}$, denoted by $\boldsymbol{\alpha}^*$, is defined by negating its vector part (or imaginary part); that is

$$\boldsymbol{\alpha}^* = a_0 - \vec{a}. \tag{3}$$

It is convenient to represent quaternions and their algebra in matrix form to simplify equation manipulations. The matrix (column vector) representation of an arbitrary quaternion $\boldsymbol{\alpha}$ with respect to the basis (1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$) is merely the collection of its parameters:

$$\boldsymbol{\alpha} = [a_0, a_1, a_2, a_3]^T = [a_0, \boldsymbol{a}^T]^T \tag{4}$$

where superscript T indicates the transpose of a matrix.

III. Quaternion Algebra

Since scalars and spatial vectors are in the subspace of quaternions, the rules in scalar and vector algebra also apply to quaternions. Let us consider the following three quaternions: $\boldsymbol{\alpha} = a_0+\vec{a}$, $\boldsymbol{\beta} = b_0+\vec{b}$, and $\boldsymbol{\gamma} = c_0+\vec{c}$, or in matrix form: $\boldsymbol{\alpha} = [a_0, a_1, a_2, a_3]^T = [a_0, \boldsymbol{a}^T]^T$, $\boldsymbol{\beta} = [b_0, b_1, b_2, b_3]^T = [b_0, \boldsymbol{b}^T]^T$, and $\boldsymbol{\gamma} = [c_0, c_1, c_2, c_3]^T = [c_0, \boldsymbol{c}^T]^T$.

A. Addition and Subtraction

Addition and *subtraction*, $\pm$, of two quaternions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined as

$$\boldsymbol{\alpha} \pm \boldsymbol{\beta} = (a_0 \pm b_0) + (\vec{a} \pm \vec{b}) \tag{5}$$

or in matrix form

$$\boldsymbol{\alpha} \pm \boldsymbol{\beta} = [a_0 \pm b_0, a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3]^T. \tag{6}$$

The quaternion addition and subtraction obey associative and commutative laws.

B. Multiplication

Quaternion multiplication, designated by $\otimes$, is defined as

$$\begin{aligned}\boldsymbol{\alpha} \otimes \boldsymbol{\beta} &= (a_0 + \vec{a}) \otimes (b_0 + \vec{b}) \\ &= a_0 \otimes b_0 + a_0 \otimes \vec{b} + b_0 \otimes \vec{a} + \vec{a} \otimes \vec{b}\end{aligned} \tag{7}$$

where the *scalar-scalar* and *scalar-vector quaternion products* are defined, respectively, the same way as scalars and spatial vectors; thus, $a_0 \otimes b_0 = a_0 b_0$, $a_0 \otimes \vec{b} = a_0\vec{b}$, and $b_0 \otimes \vec{a} = b_0\vec{a}$. The *vector-vector quaternion product* is defined as[1]

$$\vec{a} \otimes \vec{b} = -\vec{a} \cdot \vec{b} + \vec{a} \times \vec{b} \tag{8}$$

where the operations "$\cdot$" and "$\times$" define the dot product and the cross product in the space of spatial vectors. In terms of matrices, we separate the scalar and the vector parts in (7) and rewrite it as

$$\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \begin{bmatrix} a_0 b_0 - \boldsymbol{a}^T\boldsymbol{b} \\ \boldsymbol{a} b_0 + (a_0 \boldsymbol{U} + \tilde{\boldsymbol{a}})\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} b_0 a_0 - \boldsymbol{b}^T\boldsymbol{a} \\ \boldsymbol{b} a_0 + (b_0 \boldsymbol{U} - \tilde{\boldsymbol{b}})\boldsymbol{a} \end{bmatrix} \tag{9}$$

[1] This equation can be obtained by multiplying $(a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})$ and $(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$ directly and using the rules: $1\boldsymbol{i} = \boldsymbol{i}1 = \boldsymbol{i}$, $1\boldsymbol{j} = \boldsymbol{j}1 = \boldsymbol{j}$, $1\boldsymbol{k} = \boldsymbol{k}1 = \boldsymbol{k}$, $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$, $\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}$, and $\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}$, which were defined by Hamilton.

where $\boldsymbol{U}$ is a 3×3 unit matrix (or identity matrix), and

$$\tilde{\boldsymbol{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \text{ and } \tilde{\boldsymbol{b}} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}.$$

Letting $\boldsymbol{\gamma} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ and factoring (9) into a product of two matrices pertaining to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we get

$$\begin{bmatrix} c_0 \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} a_0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & a_0\boldsymbol{U} + \tilde{\boldsymbol{a}} \end{bmatrix} \begin{bmatrix} b_0 \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} b_0 & -\boldsymbol{b}^T \\ \boldsymbol{b} & b_0\boldsymbol{U} - \tilde{\boldsymbol{b}} \end{bmatrix} \begin{bmatrix} a_0 \\ \boldsymbol{a} \end{bmatrix}. \tag{10}$$

The quaternion multiplication is associative and distributive: $(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \otimes \boldsymbol{\gamma} = \boldsymbol{\alpha} \otimes (\boldsymbol{\beta} \otimes \boldsymbol{\gamma})$ and $\boldsymbol{\alpha} \otimes (\boldsymbol{\beta} \pm \boldsymbol{\gamma}) = \boldsymbol{\alpha} \otimes \boldsymbol{\beta} \pm \boldsymbol{\alpha} \otimes \boldsymbol{\gamma}$, respectively, but it is not commutative: $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} \neq \boldsymbol{\beta} \otimes \boldsymbol{\alpha}$.

C. Multiplicative Commutativity

Quaternion multiplication is associative and distributive with respect to addition and subtraction, but the commutative law does *not* hold in general. However, from (10) we can observe that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can commute simply with a sign change. This property is very useful; therefore, two compact notations, first introduced by Wehage (1984) [42], are designed for the leading matrices:

$$\overset{+}{\boldsymbol{\alpha}} = \begin{bmatrix} a_0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & a_0\boldsymbol{U} + \tilde{\boldsymbol{a}} \end{bmatrix} \text{ and } \overset{-}{\boldsymbol{\beta}} = \begin{bmatrix} b_0 & -\boldsymbol{b}^T \\ \boldsymbol{b} & b_0\boldsymbol{U} - \tilde{\boldsymbol{b}} \end{bmatrix} \tag{11}$$

where the hats "+" and "−" used in $\overset{+}{\boldsymbol{\alpha}}$ and $\overset{-}{\boldsymbol{\beta}}$ correspond to the "+" and "−" signs attached to the matrices $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$, respectively. The hats "+" and "−" may also be considered as mathematical *operators* that transform a four-vector $\boldsymbol{\alpha}$ into a 4 × 4 matrix in the form of $\overset{+}{\boldsymbol{\alpha}}$ and $\overset{-}{\boldsymbol{\alpha}}$, respectively. With the notation given in (11), the commutative property of multiplication demonstrated in (10) can be expressed in a compact form as

$$\boxed{\boldsymbol{\gamma} = \overset{+}{\boldsymbol{\alpha}}\,\boldsymbol{\beta} = \overset{-}{\boldsymbol{\beta}}\,\boldsymbol{\alpha}.} \tag{12}$$

Furthermore, consider a direct application of (12) to the multiplication of three quaternions in matrix form. The triple multiplication, $\boldsymbol{\alpha} \otimes \boldsymbol{\gamma} \otimes \boldsymbol{\beta}$, can be written as

$$\overset{+}{\boldsymbol{\alpha}}\overset{+}{\boldsymbol{\gamma}}\,\boldsymbol{\beta} = \overset{+}{\boldsymbol{\alpha}}\overset{-}{\boldsymbol{\beta}}\,\boldsymbol{\gamma} \text{ or } \overset{+}{\boldsymbol{\alpha}}\overset{+}{\boldsymbol{\gamma}}\,\boldsymbol{\beta} = \overset{-}{\boldsymbol{\beta}}\,(\overset{+}{\boldsymbol{\alpha}}\,\boldsymbol{\gamma}).$$

Thus, we get

$$\boxed{\overset{+}{\boldsymbol{\alpha}}\overset{-}{\boldsymbol{\beta}} = \overset{-}{\boldsymbol{\beta}}\overset{+}{\boldsymbol{\alpha}}\,.} \tag{13}$$

Equation (12) can be further generalized as

$$\boxed{\begin{aligned}\overset{+}{\boldsymbol{\alpha}}_1\overset{+}{\boldsymbol{\alpha}}_2 \cdots \overset{+}{\boldsymbol{\alpha}}_{n-1}\boldsymbol{\alpha}_n &= \overset{-}{\boldsymbol{\alpha}}_n\overset{-}{\boldsymbol{\alpha}}_{n-1} \cdots \overset{-}{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_1 \\ &= (\overset{+}{\boldsymbol{\alpha}}_1\overset{+}{\boldsymbol{\alpha}}_2 \cdots \overset{+}{\boldsymbol{\alpha}}_{i-1})(\overset{-}{\boldsymbol{\alpha}}_n\overset{-}{\boldsymbol{\alpha}}_{n-1} \cdots \overset{-}{\boldsymbol{\alpha}}_{i+1})\boldsymbol{\alpha}_i\end{aligned}} \tag{14}$$

and (13) can be generalized as

$$\boxed{(\overset{+}{\boldsymbol{\alpha}}_1\overset{+}{\boldsymbol{\alpha}}_2\cdots\overset{+}{\boldsymbol{\alpha}}_n)(\overset{-}{\boldsymbol{\beta}}_1\overset{-}{\boldsymbol{\beta}}_2\cdots\overset{-}{\boldsymbol{\beta}}_m) = (\overset{-}{\boldsymbol{\beta}}_1\overset{-}{\boldsymbol{\beta}}_2\cdots\overset{-}{\boldsymbol{\beta}}_m)(\overset{+}{\boldsymbol{\alpha}}_1\overset{+}{\boldsymbol{\alpha}}_2\cdots\overset{+}{\boldsymbol{\alpha}}_n).} \tag{15}$$

Using (12) and (13), we can prove (14) and (15) very easily.

If we want to use "+" and "−" as operators to transform a quaternion that is the resultant of a sequence of quaternion multiplications into its 4×4 matrix, the following equations will be useful:

$$\boxed{\overset{+}{(\overset{+}{\boldsymbol{\alpha}}_1\overset{+}{\boldsymbol{\alpha}}_2\cdots\overset{+}{\boldsymbol{\alpha}}_{n-1}\boldsymbol{\alpha}_n)} = \overset{+}{\boldsymbol{\alpha}}_1\overset{+}{\boldsymbol{\alpha}}_2\cdots\overset{+}{\boldsymbol{\alpha}}_{n-1}\overset{+}{\boldsymbol{\alpha}}_n} \tag{16}$$

$$\boxed{\overset{-}{(\overset{+}{\boldsymbol{\alpha}}_1\overset{+}{\boldsymbol{\alpha}}_2\cdots\overset{+}{\boldsymbol{\alpha}}_{n-1}\boldsymbol{\alpha}_n)} = \overset{-}{\boldsymbol{\alpha}}_n\overset{-}{\boldsymbol{\alpha}}_{n-1}\cdots\overset{-}{\boldsymbol{\alpha}}_2\overset{-}{\boldsymbol{\alpha}}_1} \tag{17}$$

$$\boxed{\overset{+}{(\overset{-}{\boldsymbol{\alpha}}_1\overset{-}{\boldsymbol{\alpha}}_2\cdots\overset{-}{\boldsymbol{\alpha}}_{n-1}\boldsymbol{\alpha}_n)} = \overset{+}{\boldsymbol{\alpha}}_n\overset{+}{\boldsymbol{\alpha}}_{n-1}\cdots\overset{+}{\boldsymbol{\alpha}}_2\overset{+}{\boldsymbol{\alpha}}_1} \tag{18}$$

$$\boxed{\overset{-}{(\overset{-}{\boldsymbol{\alpha}}_1\overset{-}{\boldsymbol{\alpha}}_2\cdots\overset{-}{\boldsymbol{\alpha}}_{n-1}\boldsymbol{\alpha}_n)} = \overset{-}{\boldsymbol{\alpha}}_1\overset{-}{\boldsymbol{\alpha}}_2\cdots\overset{-}{\boldsymbol{\alpha}}_{n-1}\overset{-}{\boldsymbol{\alpha}}_n.} \tag{19}$$

The proof of these four equations can be developed using (12) and (13).

D. Conjugate Multiplications

If the multiplication involves the conjugates of quaternions, the commutative properties in (12) to (19) still hold. For instance, the product of two quaternion conjugates, $\boldsymbol{\gamma} = \boldsymbol{\alpha}^* \otimes \boldsymbol{\beta}^*$, can be written in matrix form as $\boldsymbol{\gamma} = \overset{+}{\boldsymbol{\alpha}^*}\boldsymbol{\beta}^* = \overset{-}{\boldsymbol{\beta}^*}\boldsymbol{\alpha}^* = (\overset{+}{\boldsymbol{\alpha}})^T\boldsymbol{\beta}^* = (\overset{-}{\boldsymbol{\beta}})^T\boldsymbol{\alpha}^*$. Here, we also find that

$$\overset{+}{\boldsymbol{\alpha}^*} = (\overset{+}{\boldsymbol{\alpha}})^T \text{ and } \overset{-}{\boldsymbol{\beta}^*} = (\overset{-}{\boldsymbol{\beta}})^T. \tag{20}$$

If the multiplication involves the conjugate of a sequence of quaternions, the following identity demonstrates the multiplicative commutativity of conjugate products:

$$\boxed{(\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2 \otimes \cdots \otimes \boldsymbol{\alpha}_n)^* = \boldsymbol{\alpha}_n^* \otimes \cdots \otimes \boldsymbol{\alpha}_2^* \otimes \boldsymbol{\alpha}_1^*} \tag{21}$$

or in matrix form

$$(\overset{+}{\boldsymbol{\alpha}}_1\overset{+}{\boldsymbol{\alpha}}_2\cdots\overset{+}{\boldsymbol{\alpha}}_{n-1}\,\boldsymbol{\alpha}_n)^* = \overset{+}{\boldsymbol{\alpha}}_n^T\cdots\overset{+}{\boldsymbol{\alpha}}_2^T\,\boldsymbol{\alpha}_1^*.$$

E. Norm, Division, and Inverse

The *norm* of a quaternion $\boldsymbol{\alpha}$, denoted by $N(\boldsymbol{\alpha})$, is a scalar quaternion and is defined as $N(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^* \otimes \boldsymbol{\alpha} = \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}^* = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Apparently, it is identical to the square of the Euclidean norm (l_2 norm) of a general vector: $\|\boldsymbol{\alpha}\|_2^2 = \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = \boldsymbol{\alpha}^T\boldsymbol{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2$. We conclude that the norm of $\boldsymbol{\alpha}$ can be written as

$$\boxed{N(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha}^* \otimes \boldsymbol{\alpha} = \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}^* = \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = \|\boldsymbol{\alpha}\|_2^2.} \tag{22}$$

Unlike spatial vectors, the set of quaternions forms a *division algebra* [38]–[40], since for each nonzero quaternion $\boldsymbol{\alpha}$ there is an inverse $\boldsymbol{\alpha}^{-1}$ such that $\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}^{-1} = \boldsymbol{\alpha}^{-1} \otimes \boldsymbol{\alpha} = 1$. Consider two nonzero quaternions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = \frac{\boldsymbol{\alpha}^*}{N(\boldsymbol{\alpha})}$. Since $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \frac{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}^*}{N(\boldsymbol{\alpha})} = 1$, we find the *inverse* of $\boldsymbol{\alpha}$ to be

$$\boxed{\boldsymbol{\alpha}^{-1} = \frac{\boldsymbol{\alpha}^*}{N(\boldsymbol{\alpha})}.} \tag{23}$$

If $N(\boldsymbol{\alpha}) = 1$, $\boldsymbol{\alpha}$ is normalized and is called a *unit quaternion*; in this case, the inverse of $\boldsymbol{\alpha}$ is $\boldsymbol{\alpha}^*$. Since the inverse of $\boldsymbol{\alpha}$ is equal to its conjugate scaled by its norm, the commutative property of conjugate products still holds:

$$\boxed{(\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2 \otimes \cdots \otimes \boldsymbol{\alpha}_n)^{-1} = \boldsymbol{\alpha}_n^{-1} \otimes \cdots \otimes \boldsymbol{\alpha}_2^{-1} \otimes \boldsymbol{\alpha}_1^{-1}.} \tag{24}$$

F. Some Interesting Matrices

Furthermore, defining two 3×4 matrices

$$\overset{-}{\boldsymbol{E}}{}^{\alpha} = \begin{bmatrix} -a_1 & a_0 & a_3 & -a_2 \\ -a_2 & -a_3 & a_0 & a_1 \\ -a_3 & a_2 & -a_1 & a_0 \end{bmatrix} = [-\boldsymbol{a} \quad (a_0\boldsymbol{U} - \tilde{\boldsymbol{a}})] \tag{25}$$

$$\overset{+}{\boldsymbol{E}}{}^{\beta} = \begin{bmatrix} -b_1 & b_0 & -b_3 & b_2 \\ -b_2 & b_3 & b_0 & -b_1 \\ -b_3 & -b_2 & b_1 & b_0 \end{bmatrix} = [-\boldsymbol{b} \quad (b_0\boldsymbol{U} + \tilde{\boldsymbol{b}})] \tag{26}$$

we can rewrite $\overset{+}{\boldsymbol{\alpha}}$ and $\overset{-}{\boldsymbol{\beta}}$ as

$$\overset{+}{\boldsymbol{\alpha}} = \left[\boldsymbol{\alpha} \quad (\overset{-}{\boldsymbol{E}}{}^{\alpha})^T\right] = \begin{bmatrix} (\boldsymbol{\alpha}^*)^T \\ \overset{-}{\boldsymbol{E}}{}^{\alpha^*} \end{bmatrix} \tag{27}$$

and

$$\overset{-}{\boldsymbol{\beta}} = \left[\boldsymbol{\beta} \quad (\overset{+}{\boldsymbol{E}}{}^{\beta})^T\right] = \begin{bmatrix} (\boldsymbol{\beta}^*)^T \\ \overset{+}{\boldsymbol{E}}{}^{\beta^*} \end{bmatrix}. \tag{28}$$

The matrices $\overset{-}{\boldsymbol{E}}{}^{\alpha}$, $\overset{+}{\boldsymbol{E}}{}^{\beta}$, $\overset{-}{\boldsymbol{E}}{}^{\alpha^*}$, and $\overset{+}{\boldsymbol{E}}{}^{\beta^*}$ represent some matrices of particular structures; the superscripts α, β, α^*, and β^* indicate that the entries of these matrices are made by the quaternions $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\alpha}^*$, and $\boldsymbol{\beta}^*$, respectively. Therefore, we may also have $\overset{+}{\boldsymbol{E}}{}^{\alpha}$ and $\overset{-}{\boldsymbol{E}}{}^{\beta}$, or other $\overset{+}{\boldsymbol{E}}$ and $\overset{-}{\boldsymbol{E}}$ associated with any given quaternion.

IV. Euler Parameters and Finite Rotations

Euler parameters, denoted by $\boldsymbol{p} = [e_0, e_1, e_2, e_3]^T = [e_0, \boldsymbol{e}^T]^T$, are unit quaternions. They can be expressed in the form [38]

$$\boldsymbol{p} = \cos\frac{\theta}{2} + (\sin\frac{\theta}{2})\boldsymbol{u}, \quad 0 \le \theta \le 2\pi \tag{29}$$

where $\cos(\theta/2) = e_0$, $\sin(\theta/2) = \pm\sqrt{\boldsymbol{e}^T\boldsymbol{e}}$, and $\boldsymbol{u} = \pm\frac{\boldsymbol{e}}{\sqrt{\boldsymbol{e}^T\boldsymbol{e}}}$. The vector $\boldsymbol{u}$ is a unit vector when $\sqrt{\boldsymbol{e}^T\boldsymbol{e}}$ is not zero. The Euler parameters are required to satisfy the normality constraint

$$\boldsymbol{p}^T\boldsymbol{p} = 1. \tag{30}$$

Let $\boldsymbol{p}$ be a unit quaternion and $\boldsymbol{\alpha}$ be an arbitrary quaternion. The operation $\boldsymbol{p}\otimes\boldsymbol{\alpha}\otimes\boldsymbol{p}^*$ transforms $\boldsymbol{\alpha}$ into another quaternion $\boldsymbol{\alpha}'$ without changing its norm. Expressing $\boldsymbol{\alpha}' = \boldsymbol{p}\otimes\boldsymbol{\alpha}\otimes\boldsymbol{p}^*$ in matrix form, we have $\boldsymbol{\alpha}' = (\overset{+}{\boldsymbol{p}}\overset{-}{\boldsymbol{p}}^T)\boldsymbol{\alpha} = (\overset{-}{\boldsymbol{p}}^T\overset{+}{\boldsymbol{p}})\boldsymbol{\alpha}$ or

$$\boldsymbol{\alpha}' = \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \overset{+}{\boldsymbol{E}^p}\overset{-}{\boldsymbol{E}^p}^T \end{bmatrix} \begin{bmatrix} a_0 \\ \boldsymbol{a} \end{bmatrix} = \boldsymbol{A}\boldsymbol{\alpha} \tag{31}$$

where $\boldsymbol{p}^T\boldsymbol{p} = 1$ and $\overset{+}{\boldsymbol{E}^p}\boldsymbol{p} = \overset{-}{\boldsymbol{E}^p}\boldsymbol{p} = \boldsymbol{0}$ are substituted. The matrix

$$\boxed{\boldsymbol{A} = \overset{+}{\boldsymbol{p}}\overset{-}{\boldsymbol{p}}^T = \overset{-}{\boldsymbol{p}}^T\overset{+}{\boldsymbol{p}}} \tag{32}$$

is a 4×4 quaternion transformation in four-space. Since the transformations $\overset{+}{\boldsymbol{p}}$ and $\overset{-}{\boldsymbol{p}}^T$ are orthonormal, the norms of $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are identical. Also, from (31) we can see that the scalar part of $\boldsymbol{\alpha}$ after transformation is not changed, and the vector part of $\boldsymbol{\alpha}'$ is rotated by a transformation in terms of a unit axis $\boldsymbol{u}$ and an angle of θ. Therefore, the transformed quaternion is not stretched, and the transformation of the scalar and vector parts is independent.

With the aid of quaternion algebra, finite rotations in space may be dealt with in a simple and elegant manner. If $\boldsymbol{\alpha}$ is a vector quaternion, the equation

$$\boldsymbol{\alpha}' = \boldsymbol{p}\otimes\boldsymbol{\alpha}\otimes\boldsymbol{p}^* \tag{33}$$

is, in fact, an alternative statement of the Euler theorem that a general rotation in space can be achieved by a single rotation θ about an axis $\boldsymbol{u}$. The *rotational transformation matrix* $\boldsymbol{A}$ for a spatial vector can be obtained by taking the lower right submatrix of (31) directly:

$$\boldsymbol{A} = \overset{+}{\boldsymbol{E}^p}\overset{-}{\boldsymbol{E}^p}^T = (e_0^2 - \boldsymbol{e}^T\boldsymbol{e})\boldsymbol{U} + 2(\boldsymbol{e}\boldsymbol{e}^T + e_0\tilde{\boldsymbol{e}}). \tag{34}$$

Directly from Euler's theorem, the matrix $\boldsymbol{A}$ can be derived as

$$\boldsymbol{A} = (\cos\theta)\boldsymbol{U} + (1 - \cos\theta)\boldsymbol{u}\boldsymbol{u}^T + (\sin\theta)\tilde{\boldsymbol{u}}. \tag{35}$$

This matrix $\boldsymbol{A}$ has been given by Rodrigues (1840) [43], Gibbs (1960) [44], Wittenburg (1977) [37], Suh and Radcliffe (1978) [45], and Goldstein (1981) [46] and has been used by many researchers such as Paul (1981) [47] and Wang and Ravani (1985) [48]. Applying some trigonometric identities and substituting $e_0 = \cos\frac{\theta}{2}$ and $\boldsymbol{e} = (\sin\frac{\theta}{2})\boldsymbol{u}$, we can prove that the $\boldsymbol{A}$'s in (34) and (35) are identical.

Using Euler parameters to define a rotation matrix $\boldsymbol{A}$ is not unique. From (33), it is obvious that we can use $-\boldsymbol{p}$ to describe the same rotation as

$$\boldsymbol{\alpha}' = (-\boldsymbol{p})\otimes\boldsymbol{\alpha}\otimes(-\boldsymbol{p})^*.$$

Therefore, the Euler parameters define a two-to-one mapping from parameters to a rotation and provide a nonsingular one-to-two mapping from a rotation matrix back to its corresponding Euler parameters.

Obtaining the Euler parameters that specify the inverse rotation $\boldsymbol{A}^{-1}$ is trivial. If $\boldsymbol{A}$ is defined by the unit quaternion $\boldsymbol{p}$, then $\boldsymbol{A}^{-1}$ is specified by the conjugate of $\boldsymbol{p}$: $\boldsymbol{p}^*$. This can be verified when we write (33) as $\boldsymbol{\alpha} = (\boldsymbol{p}^*)\otimes\boldsymbol{\alpha}'\otimes(\boldsymbol{p}^*)^*$.

V. Angular Velocity

Consider a set of Euler parameters $\boldsymbol{p}_{i0}$ specifying the orientation of the coordinate system $o_i(x_i, y_i, z_i)$ with respect to the coordinate system $O_0(X_0, Y_0, Z_0)$ and a vector quaternion $\vec{r}$ (or a spatial vector) whose projections onto the systems o_i and O_0 are $\boldsymbol{r}^i$ and $\boldsymbol{r}^0$, respectively. The operations

$$\boldsymbol{r}^0 = \boldsymbol{p}_{i0}\otimes\boldsymbol{r}^i\otimes\boldsymbol{p}_{i0}^* \tag{36}$$

and

$$\boldsymbol{r}^i = \boldsymbol{p}_{i0}^*\otimes\boldsymbol{r}^0\otimes\boldsymbol{p}_{i0} \tag{37}$$

which transform $\boldsymbol{r}^i$ to $\boldsymbol{r}^0$ and $\boldsymbol{r}^0$ to $\boldsymbol{r}^i$ are pure rotations. Differentiating (36)

$$\dot{\boldsymbol{r}}^0 = \dot{\boldsymbol{p}}_{i0}\otimes\boldsymbol{r}^i\otimes\boldsymbol{p}_{i0}^* + \boldsymbol{p}_{i0}\otimes\boldsymbol{r}^i\otimes\dot{\boldsymbol{p}}_{i0}^* + \boldsymbol{p}_{i0}\otimes\dot{\boldsymbol{r}}^i\otimes\boldsymbol{p}_{i0}^*$$

and substituting (37) gives

$$\dot{\boldsymbol{r}}^0 = \dot{\boldsymbol{p}}\otimes\boldsymbol{p}^*\otimes\boldsymbol{r}^0 + \boldsymbol{r}^0\otimes\boldsymbol{p}\otimes\dot{\boldsymbol{p}}^* + \boldsymbol{p}\otimes\dot{\boldsymbol{r}}^i\otimes\boldsymbol{p}^*. \tag{38}$$

Note that the subscripts "$i0$" are dropped for simplicity and convenience. Since $\boldsymbol{p}\otimes\boldsymbol{p}^* = \boldsymbol{p}^*\otimes\boldsymbol{p} = 1$ and therefore $\dot{\boldsymbol{p}}\otimes\boldsymbol{p}^* + \boldsymbol{p}\otimes\dot{\boldsymbol{p}}^* = \dot{\boldsymbol{p}}^*\otimes\boldsymbol{p} + \boldsymbol{p}^*\otimes\dot{\boldsymbol{p}} = 0$, we can identify and define

$$\boxed{\boldsymbol{\alpha} = \dot{\boldsymbol{p}}\otimes\boldsymbol{p}^* = -\boldsymbol{p}\otimes\dot{\boldsymbol{p}}^*} \quad \text{and} \quad \boxed{\boldsymbol{\beta} = \dot{\boldsymbol{p}}^*\otimes\boldsymbol{p} = -\boldsymbol{p}^*\otimes\dot{\boldsymbol{p}}.} \tag{39}$$

Since $\boldsymbol{\alpha}$ is a vector quaternion (see Appendix A, Section A), we substitute $\boldsymbol{\alpha}$ into (38) and expand the equation using (8):

$$\begin{aligned}\dot{\boldsymbol{r}}^0 &= \boldsymbol{\alpha}\otimes\boldsymbol{r}^0 - \boldsymbol{r}^0\otimes\boldsymbol{\alpha} + \boldsymbol{p}\otimes\dot{\boldsymbol{r}}^i\otimes\boldsymbol{p}^* \\ &= (-\boldsymbol{a}\cdot\boldsymbol{r}^0 + \boldsymbol{a}\times\boldsymbol{r}^0) - (-\boldsymbol{r}^0\cdot\boldsymbol{a} + \boldsymbol{r}^0\times\boldsymbol{a}) + \boldsymbol{p}\otimes\dot{\boldsymbol{r}}^i\otimes\boldsymbol{p}^* \\ &= 2\boldsymbol{a}\times\boldsymbol{r}^0 + \boldsymbol{p}\otimes\dot{\boldsymbol{r}}^i\otimes\boldsymbol{p}^*\end{aligned} \tag{40}$$

or, in matrix form

$$\dot{\boldsymbol{r}}^0 = 2\tilde{\boldsymbol{a}}\boldsymbol{r}^0 + \boldsymbol{A}\dot{\boldsymbol{r}}^i. \tag{41}$$

Comparing (41) with the differentiation of $\boldsymbol{r}^0 = \boldsymbol{A}\boldsymbol{r}^i$:

$$\begin{aligned}\dot{\boldsymbol{r}}^0 &= \dot{\boldsymbol{A}}\boldsymbol{r}^i + \boldsymbol{A}\dot{\boldsymbol{r}}^i \\ &= \tilde{\boldsymbol{\Omega}}^0\boldsymbol{A}\boldsymbol{r}^i + \boldsymbol{A}\dot{\boldsymbol{r}}^i \\ &= \tilde{\boldsymbol{\Omega}}^0\boldsymbol{r}^0 + \boldsymbol{A}\dot{\boldsymbol{r}}^i\end{aligned} \tag{42}$$

we recognize that $2\boldsymbol{a}$ is, in fact, the angular velocity in three-space, and $2\boldsymbol{\alpha}$ is the angular velocity in four-space. Thus,

$$\boxed{\boldsymbol{\Omega}_{i0}^{0} = 2\boldsymbol{\alpha} = 2\dot{\boldsymbol{p}}_{i0} \otimes \boldsymbol{p}_{i0}^{*} = -2\boldsymbol{p}_{i0} \otimes \dot{\boldsymbol{p}}_{i0}^{*}} \tag{43}$$

or, in matrix form

$$\boxed{\boldsymbol{\Omega}_{i0}^{0} = 2\begin{bmatrix} \boldsymbol{p}_{i0}^{T}\dot{\boldsymbol{p}}_{i0} \\ \overset{+}{\boldsymbol{E}}{}_{i0}^{p}\dot{\boldsymbol{p}}_{i0} \end{bmatrix} = 2\begin{bmatrix} 0 \\ \overset{+}{\boldsymbol{E}}{}_{i0}^{p}\dot{\boldsymbol{p}}_{i0} \end{bmatrix} = 2\begin{bmatrix} 0 \\ \boldsymbol{a} \end{bmatrix}.} \tag{44}$$

From (44), we can see that the quaternion angular velocity is a combination of the normality constraint differentiated once and a spatial vector angular velocity $2\overset{+}{\boldsymbol{E}}{}^{p}\dot{\boldsymbol{p}}$. The superscript "0" indicates that the angular velocity has its projection in the system O_0.

The angular-velocity matrix can be derived directly from matrix manipulation of the first two terms of equation (40) (see Appendix A, Section A, for $\overset{+}{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\alpha}}$)

$$\boxed{\tilde{\boldsymbol{\Omega}}_{i0}^{0} = \overset{+}{\boldsymbol{\alpha}} - \bar{\boldsymbol{\alpha}} = \begin{bmatrix} 0 & \boldsymbol{0}^{T} \\ \boldsymbol{0} & 2\dot{\overset{+}{\boldsymbol{E}}}{}_{i0}^{p}\overset{+}{\boldsymbol{E}}{}_{i0}^{p\,T} \end{bmatrix}.} \tag{45}$$

In three-space, the angular-velocity matrix is $2\dot{\overset{+}{\boldsymbol{E}}}{}^{p}\overset{+}{\boldsymbol{E}}{}^{p\,T}$. To verify (45),

$$\begin{aligned} \tilde{\boldsymbol{\Omega}}_{i0}^{0} &= \overset{+}{\boldsymbol{\alpha}} - \bar{\boldsymbol{\alpha}} = \dot{\overset{+}{\boldsymbol{p}}}\overset{+}{\boldsymbol{p}}{}^{T} - \bar{\boldsymbol{p}}^{T}\dot{\bar{\boldsymbol{p}}} = \dot{\overset{+}{\boldsymbol{p}}}\overset{+}{\boldsymbol{p}}{}^{T} + \dot{\bar{\boldsymbol{p}}}^{T}\bar{\boldsymbol{p}} \\ &= \dot{\overset{+}{\boldsymbol{p}}}(\bar{\boldsymbol{p}}^{T}\bar{\boldsymbol{p}})\overset{+}{\boldsymbol{p}}{}^{T} + \dot{\bar{\boldsymbol{p}}}^{T}(\overset{+}{\boldsymbol{p}}\overset{+}{\boldsymbol{p}}{}^{T})\bar{\boldsymbol{p}} = \dot{\overset{+}{\boldsymbol{p}}}(\bar{\boldsymbol{p}}^{T}\bar{\boldsymbol{p}})\overset{+}{\boldsymbol{p}}{}^{T} + \overset{+}{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}^{T}\bar{\boldsymbol{p}}\overset{+}{\boldsymbol{p}}{}^{T} \\ &= (\dot{\overset{+}{\boldsymbol{p}}}\bar{\boldsymbol{p}}^{T} + \overset{+}{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}^{T})(\bar{\boldsymbol{p}}\overset{+}{\boldsymbol{p}}{}^{T}) = \dot{\boldsymbol{A}}\boldsymbol{A}^{T} \end{aligned} \tag{46}$$

where the formulas given in (13) and (A7) are applied.

In a similar way, differentiating (37) and substituting (36) and $\boldsymbol{\beta}$ (see Appendix A, Section B, for $\boldsymbol{\beta}$), we get

$$\begin{aligned} \dot{\boldsymbol{r}}^{i} &= \boldsymbol{\beta} \otimes \boldsymbol{r}^{i} - \boldsymbol{r}^{i} \otimes \boldsymbol{\beta} + \boldsymbol{p}^{*} \otimes \dot{\boldsymbol{r}}^{0} \otimes \boldsymbol{p} \\ &= 2\boldsymbol{b} \times \boldsymbol{r}^{0} + \boldsymbol{p}^{*} \otimes \dot{\boldsymbol{r}}^{0} \otimes \boldsymbol{p} \end{aligned} \tag{47}$$

or, in matrix form

$$\dot{\boldsymbol{r}}^{i} = 2\tilde{\boldsymbol{b}}\boldsymbol{r}^{i} + \boldsymbol{A}^{T}\dot{\boldsymbol{r}}^{0}. \tag{48}$$

Rewriting (42) as

$$\dot{\boldsymbol{r}}^{i} = -\tilde{\boldsymbol{\Omega}}^{i}\boldsymbol{r}^{0} + \boldsymbol{A}^{T}\dot{\boldsymbol{r}}^{0}$$

and comparing with (42), we get the angular velocity $\boldsymbol{\Omega}_{i0}^{i}$ of system o_i relative to system O_0, which is resolved in the system o_i as

$$\boxed{\boldsymbol{\Omega}_{i0}^{i} = -2\boldsymbol{\beta} = -2\dot{\boldsymbol{p}}_{i0}^{*} \otimes \boldsymbol{p}_{i0} = 2\boldsymbol{p}_{i0}^{*} \otimes \dot{\boldsymbol{p}}_{i0}} \tag{49}$$

or, in matrix form

$$\boxed{\boldsymbol{\Omega}_{i0}^{i} = 2\begin{bmatrix} \boldsymbol{p}_{i0}^{T}\dot{\boldsymbol{p}}_{i0} \\ \bar{\boldsymbol{E}}_{i0}^{p}\dot{\boldsymbol{p}}_{i0} \end{bmatrix} = 2\begin{bmatrix} 0 \\ \bar{\boldsymbol{E}}_{i0}^{p}\dot{\boldsymbol{p}}_{i0} \end{bmatrix} = 2\begin{bmatrix} 0 \\ -\boldsymbol{b} \end{bmatrix}.} \tag{50}$$

The angular-velocity matrix can be found as

$$\boxed{\tilde{\boldsymbol{\Omega}}_{i0}^{i} = \bar{\boldsymbol{\beta}} - \overset{+}{\boldsymbol{\beta}} = \begin{bmatrix} 0 & \boldsymbol{0}^{T} \\ \boldsymbol{0} & 2\bar{\boldsymbol{E}}_{i0}^{p}\dot{\bar{\boldsymbol{E}}}{}_{i0}^{p\,T} \end{bmatrix}.} \tag{51}$$

To verify (51),

$$\begin{aligned} \tilde{\boldsymbol{\Omega}}_{i0}^{i} &= \bar{\boldsymbol{\beta}} - \overset{+}{\boldsymbol{\beta}} = \bar{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}^{T} - \dot{\overset{+}{\boldsymbol{p}}}{}^{T}\overset{+}{\boldsymbol{p}} = \bar{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}^{T} + \overset{+}{\boldsymbol{p}}{}^{T}\dot{\overset{+}{\boldsymbol{p}}} \\ &= \bar{\boldsymbol{p}}(\overset{+}{\boldsymbol{p}}{}^{T}\overset{+}{\boldsymbol{p}})\dot{\bar{\boldsymbol{p}}}^{T} + \overset{+}{\boldsymbol{p}}{}^{T}(\bar{\boldsymbol{p}}\bar{\boldsymbol{p}}^{T})\dot{\overset{+}{\boldsymbol{p}}} = (\bar{\boldsymbol{p}}\overset{+}{\boldsymbol{p}}{}^{T})\overset{+}{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}^{T} + (\bar{\boldsymbol{p}}\overset{+}{\boldsymbol{p}}{}^{T})\dot{\overset{+}{\boldsymbol{p}}}\bar{\boldsymbol{p}}^{T} \\ &= (\bar{\boldsymbol{p}}\overset{+}{\boldsymbol{p}}{}^{T})(\overset{+}{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}^{T} + \dot{\overset{+}{\boldsymbol{p}}}\bar{\boldsymbol{p}}^{T}) = \boldsymbol{A}^{T}\dot{\boldsymbol{A}} \end{aligned} \tag{52}$$

where the properties (13) and (A13) are used. In (46) and (52), the angular-velocity matrices are shown to agree with Poisson's equations as given by Pipe (1963) [49].

The angular velocities $\boldsymbol{\Omega}_{i0}^{0}$ and $\boldsymbol{\Omega}_{i0}^{i}$ can be related. From (43),

$$\boldsymbol{\Omega}_{i0}^{0} = 2\bar{\boldsymbol{p}}^{T}\dot{\boldsymbol{p}} = (\bar{\boldsymbol{p}}^{T}\overset{+}{\boldsymbol{p}})(2\overset{+}{\boldsymbol{p}}{}^{T}\dot{\boldsymbol{p}}) = \boldsymbol{A}_{i0}\boldsymbol{\Omega}_{i0}^{i}. \tag{53}$$

Also, equating (46) with (52), we can find

$$\boxed{\tilde{\boldsymbol{\Omega}}_{i0}^{0} = \boldsymbol{A}_{i0}\tilde{\boldsymbol{\Omega}}_{i0}^{i}\boldsymbol{A}_{i0}^{T}.} \tag{54}$$

If the systems O_0 and o_i represent the inertial and the ith body-fixed coordinate system, respectively, (53) and (54) describe the transformations of the quantities pertaining to the angular velocities in the inertial system and the ith body-fixed system. The representation of angular velocities in (44) and (50) is not unique; they can be represented in different forms and are given in Appendix A, Section C.

VI. Angular Acceleration

The angular acceleration $\dot{\boldsymbol{\Omega}}_{i0}^{0}$ of system o_i relative to system O_0, which has its projection on system O_0, is derived by differentiating (43) directly

$$\boxed{\begin{aligned} \dot{\boldsymbol{\Omega}}_{i0}^{0} &= 2\dot{\boldsymbol{\alpha}} = 2(\ddot{\boldsymbol{p}}_{i0} \otimes \boldsymbol{p}_{i0}^{*} + \dot{\boldsymbol{p}}_{i0} \otimes \dot{\boldsymbol{p}}_{i0}^{*}) \\ &= -2(\boldsymbol{p}_{i0} \otimes \ddot{\boldsymbol{p}}_{i0}^{*} + \dot{\boldsymbol{p}}_{i0} \otimes \dot{\boldsymbol{p}}_{i0}^{*}) \end{aligned}} \tag{55}$$

or, in matrix form

$$\begin{aligned} \dot{\boldsymbol{\Omega}}_{i0}^{0} &= 2\left(\bar{\boldsymbol{p}}_{i0}^{T}\ddot{\boldsymbol{p}}_{i0} + \dot{\bar{\boldsymbol{p}}}_{i0}^{T}\dot{\boldsymbol{p}}_{i0}\right) \\ &= 2\left(\begin{bmatrix} \boldsymbol{p}_{i0}^{T} \\ \overset{+}{\boldsymbol{E}}{}_{i0}^{p} \end{bmatrix}[\ddot{\boldsymbol{p}}_{i0}] + \begin{bmatrix} \dot{\boldsymbol{p}}_{i0}^{T} \\ \dot{\overset{+}{\boldsymbol{E}}}{}_{i0}^{p} \end{bmatrix}[\dot{\boldsymbol{p}}_{i0}]\right) \\ &= 2\begin{bmatrix} \boldsymbol{p}_{i0}^{T}\ddot{\boldsymbol{p}}_{i0} + \dot{\boldsymbol{p}}_{i0}^{T}\dot{\boldsymbol{p}}_{i0} \\ \overset{+}{\boldsymbol{E}}{}_{i0}^{p}\ddot{\boldsymbol{p}}_{i0} \end{bmatrix} = 2\begin{bmatrix} 0 \\ \overset{+}{\boldsymbol{E}}{}_{i0}^{p}\ddot{\boldsymbol{p}}_{i0} \end{bmatrix} \end{aligned} \tag{56}$$

where $\dot{\overset{+}{\boldsymbol{E}}}{}^{p}\dot{\boldsymbol{p}} = 0$ can be verified easily.

Similarly, the angular acceleration $\dot{\boldsymbol{\Omega}}^{i}_{i0}$, which is resolved in the system o_i, can be derived by differentiating (49)

$$\boxed{\begin{aligned}\dot{\boldsymbol{\Omega}}^{i}_{i0} &= -2\dot{\boldsymbol{\beta}} = -2(\ddot{\boldsymbol{p}}^{*}_{i0} \otimes \boldsymbol{p}_{i0} + \dot{\boldsymbol{p}}^{*}_{i0} \otimes \dot{\boldsymbol{p}}_{i0}) \\ &= 2(\boldsymbol{p}^{*}_{i0} \otimes \ddot{\boldsymbol{p}}_{i0} + \dot{\boldsymbol{p}}^{*}_{i0} \otimes \dot{\boldsymbol{p}}_{i0})\end{aligned}} \tag{57}$$

or, in matrix form

$$\begin{aligned}\dot{\boldsymbol{\Omega}}^{i}_{i0} &= 2(\overset{+}{\boldsymbol{p}}{}^{T}_{i0}\ddot{\boldsymbol{p}}_{i0} + \dot{\overset{+}{\boldsymbol{p}}}{}^{T}_{i0}\dot{\boldsymbol{p}}_{i0}) \\ &= 2\left(\begin{bmatrix}\boldsymbol{p}^{T}_{i0} \\ \\ \bar{\boldsymbol{E}}^{p}_{i0}\end{bmatrix}[\ddot{\boldsymbol{p}}_{i0}] + \begin{bmatrix}\dot{\boldsymbol{p}}^{T}_{i0} \\ \\ \dot{\bar{\boldsymbol{E}}}{}^{p}_{i0}\end{bmatrix}[\dot{\boldsymbol{p}}_{i0}]\right) \\ &= 2\begin{bmatrix}\boldsymbol{p}^{T}_{i0}\ddot{\boldsymbol{p}}_{i0} + \dot{\boldsymbol{p}}^{T}_{i0}\dot{\boldsymbol{p}}_{i0} \\ \\ \bar{\boldsymbol{E}}^{p}_{i0}\ddot{\boldsymbol{p}}_{i0}\end{bmatrix} = 2\begin{bmatrix}0 \\ \\ \bar{\boldsymbol{E}}^{p}_{i0}\ddot{\boldsymbol{p}}_{i0}\end{bmatrix}\end{aligned} \tag{58}$$

where $\dot{\bar{\boldsymbol{E}}}{}^{p}\dot{\boldsymbol{p}} = 0$ can be verified directly.

The angular accelerations $\dot{\boldsymbol{\Omega}}^{0}_{i0}$ and $\dot{\boldsymbol{\Omega}}^{i}_{i0}$ are related by the rotational transformation matrix. This can be shown by manipulating (56):

$$\begin{aligned}\dot{\boldsymbol{\Omega}}^{0}_{i0} &= 2\left\{\bar{\boldsymbol{p}}^{T}\ddot{\boldsymbol{p}} + \dot{\bar{\boldsymbol{p}}}{}^{T}\dot{\boldsymbol{p}}\right\} = 2\left\{\bar{\boldsymbol{p}}^{T}(\overset{+}{\boldsymbol{p}}\overset{+}{\boldsymbol{p}}{}^{T})\ddot{\boldsymbol{p}} + \dot{\bar{\boldsymbol{p}}}{}^{T}(\bar{\boldsymbol{p}}\bar{\boldsymbol{p}}^{T})\dot{\boldsymbol{p}}\right\} \\ &= 2\left\{(\bar{\boldsymbol{p}}^{T}\overset{+}{\boldsymbol{p}})(\overset{+}{\boldsymbol{p}}{}^{T}\ddot{\boldsymbol{p}}) - \bar{\boldsymbol{p}}^{T}\dot{\bar{\boldsymbol{p}}}\dot{\overset{+}{\boldsymbol{p}}}\boldsymbol{p}^{*}\right\} \\ &= 2\left\{(\bar{\boldsymbol{p}}^{T}\overset{+}{\boldsymbol{p}})(\overset{+}{\boldsymbol{p}}{}^{T}\ddot{\boldsymbol{p}}) - \bar{\boldsymbol{p}}^{T}\dot{\overset{+}{\boldsymbol{p}}}\dot{\bar{\boldsymbol{p}}}\boldsymbol{p}^{*}\right\} \\ &= 2\left\{(\bar{\boldsymbol{p}}^{T}\overset{+}{\boldsymbol{p}})(\overset{+}{\boldsymbol{p}}{}^{T}\ddot{\boldsymbol{p}}) - \bar{\boldsymbol{p}}^{T}\dot{\overset{+}{\boldsymbol{p}}}\overset{+}{\boldsymbol{p}}{}^{T}\dot{\boldsymbol{p}}\right\} \\ &= 2\left\{(\bar{\boldsymbol{p}}^{T}\overset{+}{\boldsymbol{p}})(\overset{+}{\boldsymbol{p}}{}^{T}\ddot{\boldsymbol{p}}) + (\bar{\boldsymbol{p}}^{T}\overset{+}{\boldsymbol{p}})(\dot{\overset{+}{\boldsymbol{p}}}{}^{T}\dot{\boldsymbol{p}})\right\} \\ &= (\bar{\boldsymbol{p}}^{T}\overset{+}{\boldsymbol{p}})\left\{2(\overset{+}{\boldsymbol{p}}{}^{T}\ddot{\boldsymbol{p}} + \dot{\overset{+}{\boldsymbol{p}}}{}^{T}\dot{\boldsymbol{p}})\right\} = \boldsymbol{A}_{i0}\dot{\boldsymbol{\Omega}}^{i}_{i0}\end{aligned} \tag{59}$$

where the properties (A6) and (A7) are used.

From (56) and (58), we see that the angular acceleration is a vector quaternion because the scalar component is zero. In addition, the angular acceleration in quaternion space is a combination of the normality constraint differentiated twice and a spatial vector. Various representations for acceleration are given in Appendix A, Section D.

VII. Angular Momentum

In the inertial coordinate system $O_0(X_0, Y_0, Z_0)$, the general motion of rotation of a rigid body i of inertia tensor $\boldsymbol{I}^{0}$, acted upon by a system of forces, is defined by $\boldsymbol{T}^{0}_{i0} = \dot{\boldsymbol{H}}^{0}_{i0}$ where $\boldsymbol{T}^{0}_{i0}$ is the resultant moment (sum of the external moments or torque) applied to the body, and $\boldsymbol{H}^{0}_{i0}$ is the angular momentum about the moment center. The angular momentum of body i which is resolved in the inertial reference frame O_0 is written as $\boldsymbol{H}^{0}_{i0} = \hat{\boldsymbol{I}}^{0}\boldsymbol{\Omega}^{0}_{i0}$ in quaternion space, or

$$\boxed{\boldsymbol{H}^{0}_{i0} = 2\hat{\boldsymbol{I}}^{0}_{i0}\boldsymbol{\alpha} = 2\hat{\boldsymbol{I}}^{0}_{i0}(\dot{\boldsymbol{p}}_{i0} \otimes \boldsymbol{p}^{*}_{i0}) = -2\hat{\boldsymbol{I}}^{0}_{i0}(\boldsymbol{p}_{i0} \otimes \dot{\boldsymbol{p}}^{*})} \tag{60}$$

after substituting angular velocity from (43). In matrix form, (60) can be written as

$$\begin{aligned}\boldsymbol{H}^{0} &= 2\hat{\boldsymbol{I}}^{0}\bar{\boldsymbol{p}}^{T}\dot{\boldsymbol{p}} = 2\begin{bmatrix}k & \boldsymbol{0}^{T} \\ \boldsymbol{0} & \boldsymbol{I}^{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{p}^{T}\dot{\boldsymbol{p}} \\ \overset{+}{\boldsymbol{E}}{}^{p}\dot{\boldsymbol{p}}\end{bmatrix} \\ &= \begin{bmatrix}2k(\boldsymbol{p}^{T}\dot{\boldsymbol{p}}) \\ 2\boldsymbol{I}^{0}\overset{+}{\boldsymbol{E}}{}^{p}\dot{\boldsymbol{p}}\end{bmatrix} = \begin{bmatrix}0 \\ 2\boldsymbol{I}^{0}\overset{+}{\boldsymbol{E}}{}^{p}\dot{\boldsymbol{p}}\end{bmatrix}\end{aligned} \tag{61}$$

where $k = \mathrm{Trace}(\boldsymbol{I}^{0})/2$ (see Appendix B for details). The matrices $\hat{\boldsymbol{I}}^{0}$ and $\boldsymbol{I}^{0}$ are the inertia matrices in four- and three-space, respectively, and the superscript "0" indicates that the inertia matrices are given with respect to the reference frame O_0. We can see, from (61), that the angular momentum is a vector quaternion. Note that a subscript "$i0$" should be attached to the physical quantities in this section; however, it is dropped to unclutter the equations.

The resultant moment $\boldsymbol{T}^{0}$, which is the rate of change of the angular momentum, is also a vector quaternion. Differentiating $\boldsymbol{H}^{0}$ once and substituting $\dot{\boldsymbol{I}}^{0}$ (see Appendix C), we have

$$\boldsymbol{T}^{0} = \hat{\boldsymbol{I}}^{0}\dot{\boldsymbol{\Omega}}^{0} + \tilde{\boldsymbol{\Omega}}^{0}\hat{\boldsymbol{I}}^{0}\boldsymbol{\Omega}^{0} \tag{62}$$

in quaternion space, or

$$\boxed{\boldsymbol{T}^{0} = 2\hat{\boldsymbol{I}}^{0}\dot{\boldsymbol{\alpha}} + 2\left[\boldsymbol{\alpha} \otimes (\hat{\boldsymbol{I}}^{0}\boldsymbol{\alpha}) - (\hat{\boldsymbol{I}}^{0}\boldsymbol{\alpha}) \otimes \boldsymbol{\alpha}\right]} \tag{63}$$

where $\boldsymbol{\alpha}$ is given in (39). One particular representation of $\boldsymbol{T}^{0}$ in quaternion space is written as

$$\begin{aligned}\boldsymbol{T}^{0} =& 2\hat{\boldsymbol{I}}^{0}(\ddot{\boldsymbol{p}} \otimes \boldsymbol{p}^{*} + \dot{\boldsymbol{p}} \otimes \dot{\boldsymbol{p}}^{*}) + 2\Big\{(\dot{\boldsymbol{p}} \otimes \boldsymbol{p}^{*}) \otimes \left[\hat{\boldsymbol{I}}^{0}(\dot{\boldsymbol{p}} \otimes \boldsymbol{p}^{*})\right] \\ &- \left[\hat{\boldsymbol{I}}^{0}(\dot{\boldsymbol{p}} \otimes \boldsymbol{p}^{*})\right] \otimes (\dot{\boldsymbol{p}} \otimes \boldsymbol{p}^{*})\Big\}\end{aligned} \tag{64}$$

or, in matrix form

$$\begin{aligned}\boldsymbol{T}^{0} &= 2\hat{\boldsymbol{I}}^{0}(\bar{\boldsymbol{p}}^{T}\ddot{\boldsymbol{p}} + \dot{\bar{\boldsymbol{p}}}{}^{T}\dot{\boldsymbol{p}}) + 2(\dot{\overset{+}{\boldsymbol{p}}}\overset{+}{\boldsymbol{p}}{}^{T} - \bar{\boldsymbol{p}}^{T}\dot{\bar{\boldsymbol{p}}})\hat{\boldsymbol{I}}^{0}(\bar{\boldsymbol{p}}^{T}\dot{\boldsymbol{p}}) \\ &= 2\begin{bmatrix}k & \boldsymbol{0}^{T} \\ \\ \boldsymbol{0} & \boldsymbol{I}^{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{p}^{T}\ddot{\boldsymbol{p}} + \dot{\boldsymbol{p}}^{T}\dot{\boldsymbol{p}} \\ \\ \overset{+}{\boldsymbol{E}}{}^{p}\ddot{\boldsymbol{p}}\end{bmatrix} \\ &\quad + 2\begin{bmatrix}0 & \boldsymbol{0}^{T} \\ \\ 0 & 2\dot{\overset{+}{\boldsymbol{E}}}{}^{p}\overset{+}{\boldsymbol{E}}{}^{p\,T}\end{bmatrix}\begin{bmatrix}k & \boldsymbol{0}^{T} \\ \boldsymbol{0} & \boldsymbol{I}^{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{p}^{T}\dot{\boldsymbol{p}} \\ \overset{+}{\boldsymbol{E}}{}^{p}\dot{\boldsymbol{p}}\end{bmatrix} \\ &= \begin{bmatrix}2k(\boldsymbol{p}^{T}\ddot{\boldsymbol{p}} + \dot{\boldsymbol{p}}^{T}\dot{\boldsymbol{p}}) \\ \\ 2\boldsymbol{I}^{0}\overset{+}{\boldsymbol{E}}{}^{p}\ddot{\boldsymbol{p}} + 4\dot{\overset{+}{\boldsymbol{E}}}{}^{p}\overset{+}{\boldsymbol{E}}{}^{p\,T}\boldsymbol{I}^{0}\overset{+}{\boldsymbol{E}}{}^{p}\dot{\boldsymbol{p}}\end{bmatrix} \\ &= \begin{bmatrix}0 \\ \\ 2\boldsymbol{I}^{0}\overset{+}{\boldsymbol{E}}{}^{p}\ddot{\boldsymbol{p}} + 4\dot{\overset{+}{\boldsymbol{E}}}{}^{p}\overset{+}{\boldsymbol{E}}{}^{p\,T}\boldsymbol{I}^{0}\overset{+}{\boldsymbol{E}}{}^{p}\dot{\boldsymbol{p}}\end{bmatrix}.\end{aligned} \tag{65}$$

From (65), we see that the resultant torque T^0 is a vector quaternion, and it is simply a combination of the normality constraint of Euler parameters (30) twice differentiated and a spatial torque. Alternatively, we may interpret that (65) is the generalized Euler's equations of motion of rotation in three-space associated with the acceleration constraint of Euler parameters.

The moment T^i_{i0} of body i resolved in the system o_i can be derived directly from the relation $T^0 = A_{i0}T^i$ to be

$$\boldsymbol{T}^i = \hat{\boldsymbol{I}}^i\dot{\boldsymbol{\Omega}}^i + \tilde{\boldsymbol{\Omega}}^i\hat{\boldsymbol{I}}^i\boldsymbol{\Omega}^i. \tag{66}$$

Substituting $\boldsymbol{\Omega}^i$, $\tilde{\boldsymbol{\Omega}}^i$, and $\dot{\boldsymbol{\Omega}}^i$, we can express the moment in quaternion space as

$$\boxed{\boldsymbol{T}^i = -2\hat{\boldsymbol{I}}^i\dot{\boldsymbol{\beta}} + 2\left[\boldsymbol{\beta}\otimes(\hat{\boldsymbol{I}}^i\boldsymbol{\beta}) - (\hat{\boldsymbol{I}}^i\boldsymbol{\beta})\otimes\boldsymbol{\beta}\right]} \tag{67}$$

where $\boldsymbol{\beta}$ is given in (39). One particular representation of $\boldsymbol{T}^i$ in quaternion space is

$$\begin{aligned}\boldsymbol{T}^i =& 2\hat{\boldsymbol{I}}^i(\boldsymbol{p}^*\otimes\ddot{\boldsymbol{p}} + \dot{\boldsymbol{p}}^*\otimes\dot{\boldsymbol{p}}) + 2\Big\{(\boldsymbol{p}^*\otimes\dot{\boldsymbol{p}})\otimes\left[\hat{\boldsymbol{I}}^i(\boldsymbol{p}^*\otimes\dot{\boldsymbol{p}})\right]\\ &- \left[\hat{\boldsymbol{I}}^i(\boldsymbol{p}^*\otimes\dot{\boldsymbol{p}})\right]\otimes(\boldsymbol{p}^*\otimes\dot{\boldsymbol{p}})\Big\}\end{aligned} \tag{68}$$

or, in matrix form

$$\begin{aligned}\boldsymbol{T}^i &= 2\hat{\boldsymbol{I}}^i(\overset{+}{\boldsymbol{p}}{}^T\ddot{\boldsymbol{p}} + \dot{\overset{+}{\boldsymbol{p}}}{}^T\dot{\boldsymbol{p}}) + 2(\bar{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}{}^T - \dot{\overset{+}{\boldsymbol{p}}}{}^T\overset{+}{\boldsymbol{p}})\hat{\boldsymbol{I}}^i(\overset{+}{\boldsymbol{p}}{}^T\dot{\boldsymbol{p}})\\ &= 2\begin{bmatrix}k & 0^T\\ 0 & I^i\end{bmatrix}\begin{bmatrix}\boldsymbol{p}^T\ddot{\boldsymbol{p}} + \dot{\boldsymbol{p}}^T\dot{\boldsymbol{p}}\\ \bar{E}^p\ddot{\boldsymbol{p}}\end{bmatrix}\\ &\quad + 2\begin{bmatrix}0 & 0^T\\ 0 & 2\bar{E}^p\dot{\bar{E}}{}^{pT}\end{bmatrix}\begin{bmatrix}k & 0^T\\ 0 & I^i\end{bmatrix}\begin{bmatrix}\boldsymbol{p}^T\dot{\boldsymbol{p}}\\ \bar{E}^p\dot{\boldsymbol{p}}\end{bmatrix}\\ &= \begin{bmatrix}2k(\boldsymbol{p}^T\ddot{\boldsymbol{p}} + \dot{\boldsymbol{p}}^T\dot{\boldsymbol{p}})\\ 2I^i\bar{E}^p\ddot{\boldsymbol{p}} + 4\bar{E}^p\dot{\bar{E}}{}^{pT}I^i\bar{E}^p\dot{\boldsymbol{p}}\end{bmatrix}\\ &= \begin{bmatrix}0\\ 2I^i\bar{E}^p\ddot{\boldsymbol{p}} + 4\bar{E}^p\dot{\bar{E}}{}^{pT}I^i\bar{E}^p\dot{\boldsymbol{p}}\end{bmatrix}.\end{aligned} \tag{69}$$

The bottom vector equations of (69) are Euler's equations of motion of rotation in terms of Euler parameters and their derivatives in three-space.

VIII. Nonsingularity

In quaternion space, the scalars, vectors, and quaternions are unified. Expressing a spatial vector in quaternion space provides us with elegant properties for manipulating equations. Observe that the cross product of two spatial vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ has the property that $+\tilde{\boldsymbol{a}}\boldsymbol{b} = -\tilde{\boldsymbol{b}}\boldsymbol{a}$; the similarity to (12), $\overset{+}{\boldsymbol{\alpha}}\,\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}\,\boldsymbol{\alpha}$, can be recognized immediately. However, in vector space the matrices $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ are skew-symmetric (singular), whereas in quaternion space the matrices $\overset{+}{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}}$ are orthogonal (nonsingular), and they are orthonormal if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are normalized; that is, $(\overset{+}{\boldsymbol{\alpha}})^{-1} = (\overset{+}{\boldsymbol{\alpha}})^T$ and $(\bar{\boldsymbol{\beta}})^{-1} = (\bar{\boldsymbol{\beta}})^T$. A direct application of this property to robot kinematic equations was presented by Chou and Kamel in 1988 and 1991 [32], [33].

Applying the above property to the equations of angular velocity yields the desired form of kinematic differential equations for Euler parameters. In quaternion space, the invertibility of angular velocities can be identified immediately as

$$\begin{cases}\dot{\boldsymbol{p}} = \frac{1}{2}\bar{\boldsymbol{p}}\boldsymbol{\Omega}^0\\ \dot{\boldsymbol{p}} = \frac{1}{2}\overset{+}{\boldsymbol{p}}\boldsymbol{\Omega}^i\end{cases} \quad\text{and}\quad \begin{cases}\dot{\boldsymbol{p}}^* = -\frac{1}{2}\overset{+}{\boldsymbol{p}}{}^T\boldsymbol{\Omega}^0\\ \dot{\boldsymbol{p}}^* = -\frac{1}{2}\bar{\boldsymbol{p}}{}^T\boldsymbol{\Omega}^i.\end{cases} \tag{70}$$

In three-space, the invertibility property remains, and they become

$$\begin{cases}\dot{\boldsymbol{p}} = \frac{1}{2}(\overset{+}{E}{}^p)^T\boldsymbol{\Omega}^0\\ \dot{\boldsymbol{p}} = \frac{1}{2}(\bar{E}^p)^T\boldsymbol{\Omega}^i\end{cases} \quad\text{and}\quad \begin{cases}\dot{\boldsymbol{p}}^* = -\frac{1}{2}(\bar{E}^{p*})^T\boldsymbol{\Omega}^0\\ \dot{\boldsymbol{p}}^* = -\frac{1}{2}(\overset{+}{E}{}^{p*})^T\boldsymbol{\Omega}^i.\end{cases} \tag{71}$$

For angular acceleration, the terms of the second-order derivative of $\boldsymbol{p}$ can be expressed explicitly as

$$\begin{gathered}\begin{cases}\ddot{\boldsymbol{p}} = \frac{1}{2}\bar{\boldsymbol{p}}\dot{\boldsymbol{\Omega}}^0 - (\bar{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}{}^T)\dot{\boldsymbol{p}}\\ \ddot{\boldsymbol{p}} = \frac{1}{2}\overset{+}{\boldsymbol{p}}\dot{\boldsymbol{\Omega}}^i - (\overset{+}{\boldsymbol{p}}\dot{\overset{+}{\boldsymbol{p}}}{}^T)\dot{\boldsymbol{p}}\end{cases}\\ \text{and}\quad \begin{cases}\ddot{\boldsymbol{p}}^* = -\frac{1}{2}\overset{+}{\boldsymbol{p}}{}^T\dot{\boldsymbol{\Omega}}^0 - (\overset{+}{\boldsymbol{p}}{}^T\dot{\overset{+}{\boldsymbol{p}}})\dot{\boldsymbol{p}}^*\\ \ddot{\boldsymbol{p}}^* = -\frac{1}{2}\bar{\boldsymbol{p}}{}^T\dot{\boldsymbol{\Omega}}^i - (\bar{\boldsymbol{p}}{}^T\dot{\bar{\boldsymbol{p}}})\dot{\boldsymbol{p}}^*.\end{cases}\end{gathered} \tag{72}$$

Similarly, in three-space the invertibility retains as

$$\begin{gathered}\begin{cases}\ddot{\boldsymbol{p}} = \frac{1}{2}(\overset{+}{E}{}^p)^T\dot{\boldsymbol{\Omega}}^0 - (\dot{\boldsymbol{p}}^T\dot{\boldsymbol{p}})\boldsymbol{p}\\ \ddot{\boldsymbol{p}} = \frac{1}{2}(\bar{E}^p)^T\dot{\boldsymbol{\Omega}}^i - (\dot{\boldsymbol{p}}^T\dot{\boldsymbol{p}})\boldsymbol{p}\end{cases}\\ \text{and}\quad \begin{cases}\ddot{\boldsymbol{p}}^* = -\frac{1}{2}(\bar{E}^{p*})^T\dot{\boldsymbol{\Omega}}^0 - (\dot{\boldsymbol{p}}^{*T}\dot{\boldsymbol{p}}^*)\boldsymbol{p}^*\\ \ddot{\boldsymbol{p}}^* = -\frac{1}{2}(\overset{+}{E}{}^{p*})^T\dot{\boldsymbol{\Omega}}^i - (\dot{\boldsymbol{p}}^{*T}\dot{\boldsymbol{p}}^*)\boldsymbol{p}^*.\end{cases}\end{gathered} \tag{73}$$

When the dynamic equations of motion are developed using Euler equations, the invertibility property of (65) and (69) is useful; in four-space, we have

$$\begin{cases}\ddot{\boldsymbol{p}} = \frac{1}{2}\bar{\boldsymbol{p}}(\hat{\boldsymbol{I}}^0)^{-1}\boldsymbol{T}^0 - (\bar{\boldsymbol{p}}\dot{\bar{\boldsymbol{p}}}{}^T)\dot{\boldsymbol{p}} - \bar{\boldsymbol{p}}(\hat{\boldsymbol{I}}^0)^{-1}(\overset{+}{\boldsymbol{\alpha}} - \bar{\boldsymbol{\alpha}})\hat{\boldsymbol{I}}^0(\bar{\boldsymbol{p}}{}^T\dot{\boldsymbol{p}})\\ \ddot{\boldsymbol{p}} = \frac{1}{2}\overset{+}{\boldsymbol{p}}(\hat{\boldsymbol{I}}^i)^{-1}\boldsymbol{T}^i - (\overset{+}{\boldsymbol{p}}\dot{\overset{+}{\boldsymbol{p}}}{}^T)\dot{\boldsymbol{p}} - \overset{+}{\boldsymbol{p}}(\hat{\boldsymbol{I}}^i)^{-1}(\bar{\boldsymbol{\beta}} - \overset{+}{\boldsymbol{\beta}})\hat{\boldsymbol{I}}^i(\overset{+}{\boldsymbol{p}}{}^T\dot{\boldsymbol{p}}).\end{cases} \tag{74}$$

In three-space, they are

$$\begin{cases}\ddot{\boldsymbol{p}} = \frac{1}{2}(\overset{+}{E}{}^p)^T(I^0)^{-1}\boldsymbol{T}^0 - (\dot{\boldsymbol{p}}^T\dot{\boldsymbol{p}})\boldsymbol{p}\\ \qquad -2(\overset{+}{E}{}^p)^T(I^0)^{-1}(\dot{\overset{+}{E}}{}^p)(\overset{+}{E}{}^p)^T\boldsymbol{I}^0(\overset{+}{E}{}^p)\dot{\boldsymbol{p}}\\ \ddot{\boldsymbol{p}} = \frac{1}{2}(\bar{E}^p)^T(I^i)^{-1}\boldsymbol{T}^i - (\dot{\boldsymbol{p}}^T\dot{\boldsymbol{p}})\boldsymbol{p}\\ \qquad -2(\bar{E}^p)^T(I^i)^{-1}(\dot{\bar{E}}{}^p)(\bar{E}^p)^T\boldsymbol{I}^i(\bar{E}^p)\dot{\boldsymbol{p}}.\end{cases} \tag{75}$$

Using different $\boldsymbol{\alpha}$ and $\dot{\boldsymbol{\alpha}}$, we can derive similar equations, in four-space, for the conjugate Euler parameters as

$$\begin{cases} \ddot{\mathbf{p}}^* = -\frac{1}{2}\overset{+}{\mathbf{p}}^T(\hat{I}^0)^{-1}T^0 - (\overset{+}{\mathbf{p}}^T\dot{\overset{+}{\mathbf{p}}})\dot{\mathbf{p}}^* \\ \qquad -\overset{+}{\mathbf{p}}^T(\hat{I}^0)^{-1}(\overset{+}{\boldsymbol{\alpha}} - \bar{\boldsymbol{\alpha}})\hat{I}^0(\overset{+}{\mathbf{p}}\dot{\mathbf{p}}^*) \\ \ddot{\mathbf{p}}^* = -\frac{1}{2}\bar{\mathbf{p}}^T(\hat{I}^i)^{-1}T^i - (\bar{\mathbf{p}}^T\dot{\bar{\mathbf{p}}})\dot{\mathbf{p}}^* \\ \qquad -\bar{\mathbf{p}}^T(\hat{I}^i)^{-1}(\bar{\beta} - \overset{+}{\beta})\hat{I}^i(\bar{\mathbf{p}}\dot{\mathbf{p}}^*) \end{cases} \tag{76}$$

or in three-space

$$\begin{cases} \ddot{\mathbf{p}}^* = -\frac{1}{2}(\bar{E}^{p*})^T(I^0)^{-1}T^0 - (\dot{\mathbf{p}}^{*T}\dot{\mathbf{p}}^*)\mathbf{p}^* \\ \qquad -2(\bar{E}^{p*})^T(I^0)^{-1}(\dot{\bar{E}}^{p*})(\bar{E}^{p*})^T I^0(\bar{E}^{p*})\dot{\mathbf{p}}^* \\ \ddot{\mathbf{p}}^* = -\frac{1}{2}(\overset{+}{E}^{p*})^T(I^i)^{-1}T^i - (\dot{\mathbf{p}}^{*T}\dot{\mathbf{p}}^*)\mathbf{p}^* \\ \qquad -2(\overset{+}{E}^{p*})^T(I^i)^{-1}(\overset{+}{E}^{p*})(\dot{\overset{+}{E}}^{p*})^T I^i(\overset{+}{E}^{p*})\dot{\mathbf{p}}^*. \end{cases} \tag{77}$$

The invertibility property demonstrated in (74) to (77) is useful for applying the Newton–Euler formulation to deriving the equations of motion of mechanical systems. Incorporating the second-order differentiation of the normality constraint of Euler parameters with Euler's equations, we can obtain nonsingular coefficient matrices for $\ddot{\mathbf{p}}$ and $\ddot{\mathbf{p}}^*$: $2\bar{\mathbf{p}}^T\hat{I}^0$, $2\overset{+}{\mathbf{p}}^T\hat{I}^i$, $2\overset{+}{\mathbf{p}}\hat{I}^0$, and $2\bar{\mathbf{p}}\hat{I}^i$. This property has been applied to an n-body pendulum dynamics simulation in order to design a linear computational-cost scheme by Chou in 1989 [50].

Besides the above orthogonality property, the extraction of a unit quaternion from any given rotational transformation matrix does not produce a singular case. In contrast to any other set of three generalized coordinates for describing a rotation, there is no critical case in which the inverse formulas are singular. This fact has been demonstrated by Klumpp [51], Horn [36], and Wittenburg [37].

The complexity in (75) and (77) can be reduced when we apply the properties given in (A4), (A5), (A11), and (A12). For example, the third terms of the right-hand side in (75) and (77) can be rewritten as

$$\begin{cases} -2\left[(\overset{+}{E}^p)^T(I^0)^{-1}(\overset{+}{E}^p)\right]\left[(\dot{\overset{+}{E}}^p)^T I^0(\dot{\overset{+}{E}}^p)\right]\mathbf{p} \\ +2\left[(\bar{E}^p)^T(I^i)^{-1}(\bar{E}^p)\right]\left[(\dot{\bar{E}}^p)^T I^i(\dot{\bar{E}}^p)\right]\mathbf{p} \\ -2\left[(\bar{E}^{p*})^T(I^0)^{-1}(\bar{E}^{p*})\right]\left[(\dot{\bar{E}}^{p*})^T I^0(\dot{\bar{E}}^{p*})\right]\mathbf{p}^* \\ +2\left[(\overset{+}{E}^{p*})^T(I^i)^{-1}(\overset{+}{E}^{p*})\right]\left[(\dot{\overset{+}{E}}^{p*})^T I^i(\dot{\overset{+}{E}}^{p*})\right]\mathbf{p}^*. \end{cases} \tag{78}$$

The triple products are symmetric matrices. When we compute this portion of the equation, we can minimize its computational cost.

IX. Conclusions

This paper is devoted to the introduction of quaternion algebra, generalized properties of quaternion multiplications and divisions, and to the derivation of quaternion kinematic and dynamic differential equations for the physical quantities pertaining to the motion of rotation such as angular velocity, acceleration, and momentum. These physical quantities are spatial vectors and are also shown to be vector quaternions.

The Euler parameters are a unit quaternion and are represented by a normalized vector of four real numbers. They are one of the minimal sets of parameters capable of defining a nonsingular mapping between the parameters and their corresponding rotation matrix. As we can see from the first few sections in this paper, the quaternion algebra provides many mathematical properties for manipulating equations involving rotations. One example is that the derivation of the inverse rotation is trivial when unit quaternions are used. Another example is that the combination of successive rotations can be greatly simplified using quaternion multiplications directly without using 3×3 rotation matrices [52]. The generalized commutative properties of quaternion multiplication are very useful for manipulating kinematic equations derived from successive rotations in space. One application given by Chou [52] is for the derivation of partial derivatives of a sequence of rotations written in terms of a sequence of unit quaternions. Furthermore, quaternion algebra is potentially useful for systems involving very large numbers of rotations in space to represent all the unit quaternions as a graph for a computer data base. A fundamental study in this direction was initiated by Chou [52].

Using Euler parameters to define finite rotations not only possesses mathematical beauty but also retains the physical significance of rotations in Cartesian space. From (29), we can see that the mathematical parameters are in fact their physical counterparts of a rotational axis $\mathbf{u}$ and a rotation angle θ that are used in the Euler theorem regarding finite rotations. That is, taking the vector part of a unit quaternion and normalizing it, we can find the rotational axis right away, and from the first parameter we can obtain the angle of rotation.

Since the quaternion space includes the spatial vector space, quaternion algebra is applicable to spatial vectors. This is true when we first show that the spatial vectors regarding rotations such as angular velocities, accelerations, and momentums are vector quaternions followed by deriving very simple and compact differential expressions for them in quaternion space. From these kinematic and dynamic differential equations, we can further see the nonsingularity properties that the Euler parameters provide us. Since the Euler parameters are well defined for any rotation, their relationship to angular velocity, acceleration, and momentum is always nonsingular. That is, the highest order term of Euler parameters in any kinematic or dynamic equation can be expressed explicitly in closed form without any singularity problem.

On the surface, quaternion algebra seems a cumbersome tool for analyzing physical phenomena that are three-dimensional in nature and can be thus described more elegantly using three-dimensional vector calculus. This can be observed from the equations for vectors such as (44), (50), (56), (58), (61), (65), and (69) in which there is always a zero redundant element, and from the equations for matrices such as (45) and (51) in which there is a redundant zero row and column. Although we did not prove that other physical quantities such as linear displacement, velocity, acceleration, and force are vector quaternions, we can expect that they are. That, again, demonstrates that when using quaternions to represent physical quantities, we have to carry the redundancy in our equations all the time.

Although directly using four-vector equations such as (44), (50), (56), (58), (61), (65), and (69) is impractical, it is very practical to derive these equations in four-space before we obtain their three-space equations because the manipulation of these equations in four-space is facilitated by quaternion algebra. One recent application of this methodology to camera calibrations was initiated by Chou [34], [35]. In this work, all the physical quantities were represented as quaternions during the manipulation of equations. As a result, the original equation of interest was able to be transformed into a more elegant and applicable form for the rest of the research.

In this paper, we not only show that the rotational physical quantities are vector quaternions, but also emphasize the fundamental quaternion algebra which will be very useful for many applications related to rotations in space.

Appendix A
Angular Velocity and Acceleration

A. Alpha

From (39), we have $\boldsymbol{\alpha} = \dot{\boldsymbol{p}} \otimes \boldsymbol{p}^* = -\boldsymbol{p} \otimes \dot{\boldsymbol{p}}^*$. In matrix form, it is

$$\boldsymbol{\alpha} = \dot{\overset{+}{\boldsymbol{p}}} \boldsymbol{p}^* = \overset{-}{\boldsymbol{p}}^T \dot{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p}^T \dot{\boldsymbol{p}} \\ \overset{+}{E}{}^p \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} 0 \\ \overset{+}{E}{}^p \dot{\boldsymbol{p}} \end{bmatrix}. \tag{A1}$$

We see that the quaternion $\boldsymbol{\alpha}$ is a vector quaternion since $a_0 = 0$. Using the properties given in (16) and (27), we can find

$$\begin{aligned} \overset{+}{\boldsymbol{\alpha}} = \dot{\overset{+}{\boldsymbol{p}}} \overset{+}{\boldsymbol{p}}{}^T &= \begin{bmatrix} \dot{\boldsymbol{p}}^{*T} \\ \dot{\overset{-}{E}}{}^{p*} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}^* & (\overset{-}{E}{}^{p*})^T \end{bmatrix} \\ &= \begin{bmatrix} \dot{\boldsymbol{p}}^{*T} \boldsymbol{p}^* & [(\overset{-}{E}{}^{p*}) \dot{\boldsymbol{p}}^*]^T \\ -(\overset{-}{E}{}^{p*}) \dot{\boldsymbol{p}}^* & (\dot{\overset{-}{E}}{}^{p*})(\overset{-}{E}{}^{p*})^T \end{bmatrix} \\ &= \begin{bmatrix} \dot{\boldsymbol{p}}^T \boldsymbol{p} & -[(\overset{+}{E}{}^p) \dot{\boldsymbol{p}}]^T \\ (\overset{+}{E}{}^p) \dot{\boldsymbol{p}} & (\dot{\overset{+}{E}}{}^p)(\overset{+}{E}{}^p)^T \end{bmatrix}. \end{aligned} \tag{A2}$$

Similarly, using the properties given in (17) and (28), we can find

$$\begin{aligned} \overset{-}{\boldsymbol{\alpha}} = \overset{-}{\boldsymbol{p}}{}^T \dot{\overset{-}{\boldsymbol{p}}} &= \begin{bmatrix} \boldsymbol{p}^T \\ \overset{+}{E}{}^p \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}} & (\dot{\overset{+}{E}}{}^p)^T \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{p}^T \dot{\boldsymbol{p}} & -[(\overset{+}{E}{}^p) \dot{\boldsymbol{p}}]^T \\ (\overset{+}{E}{}^p) \dot{\boldsymbol{p}} & -(\dot{\overset{+}{E}}{}^p)(\overset{+}{E}{}^p)^T \end{bmatrix} \end{aligned} \tag{A3}$$

where the entries of $\overset{-}{E}{}^{p*}$ are made by $\boldsymbol{p}^*$. The identities

$$\boxed{(\overset{-}{E}{}^{p*}) \dot{\boldsymbol{p}}^* = -(\dot{\overset{-}{E}}{}^{p*}) \boldsymbol{p}^* = (\dot{\overset{+}{E}}{}^p) \boldsymbol{p} = -(\overset{+}{E}{}^p) \dot{\boldsymbol{p}}} \tag{A4}$$

$$\boxed{\begin{aligned} (\dot{\overset{-}{E}}{}^{p*})(\overset{-}{E}{}^{p*})^T &= -(\overset{-}{E}{}^{p*})(\dot{\overset{-}{E}}{}^{p*})^T \\ &= (\dot{\overset{+}{E}}{}^p)(\overset{+}{E}{}^p)^T = -(\overset{+}{E}{}^p)(\dot{\overset{+}{E}}{}^p)^T \end{aligned}} \tag{A5}$$

and $\boldsymbol{p}^T \dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}^T \boldsymbol{p} = \dot{\boldsymbol{p}}^{*T} \boldsymbol{p}^* = \boldsymbol{p}^{*T} \dot{\boldsymbol{p}}^* = 0$ can be proved easily and are used in the derivation of (A2) and (A3).

Since the representation of $\boldsymbol{\alpha}$ is not unique, the representations of $\overset{+}{\boldsymbol{\alpha}}$ and $\overset{-}{\boldsymbol{\alpha}}$ are also not unique. Using the formulas developed in (18) and (19), we obtain the following properties:

$$\boxed{\overset{+}{\boldsymbol{\alpha}} = \dot{\overset{+}{\boldsymbol{p}}} \overset{+}{\boldsymbol{p}}{}^T = -\overset{+}{\boldsymbol{p}} \dot{\overset{+}{\boldsymbol{p}}}{}^T} \tag{A6}$$

and

$$\boxed{\overset{-}{\boldsymbol{\alpha}} = \overset{-}{\boldsymbol{p}}{}^T \dot{\overset{-}{\boldsymbol{p}}} = -\dot{\overset{-}{\boldsymbol{p}}}{}^T \overset{-}{\boldsymbol{p}}.} \tag{A7}$$

B. Beta

From (39), we have $\boldsymbol{\beta} = \dot{\boldsymbol{p}}^* \otimes \boldsymbol{p} = -\boldsymbol{p}^* \otimes \dot{\boldsymbol{p}}$. In matrix form, it is

$$\boldsymbol{\beta} = -\overset{+}{\boldsymbol{p}}{}^T \dot{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p}^T \dot{\boldsymbol{p}} \\ -\overset{-}{E}{}^p \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\overset{-}{E}{}^p \dot{\boldsymbol{p}} \end{bmatrix}. \tag{A8}$$

We can observe also that the quaternion $\boldsymbol{\beta}$ is a vector quaternion since $b_0 = 0$. Using the properties given in (16) and (27), we can find

$$\begin{aligned} \overset{+}{\boldsymbol{\beta}} = (\dot{\overset{+}{\boldsymbol{p}}})^T \overset{+}{\boldsymbol{p}} &= \begin{bmatrix} \dot{\boldsymbol{p}}^T \\ \dot{\overset{-}{E}}{}^p \end{bmatrix} \begin{bmatrix} \boldsymbol{p} & (\overset{-}{E}{}^p)^T \end{bmatrix} \\ &= \begin{bmatrix} \dot{\boldsymbol{p}}^T \boldsymbol{p} & [(\overset{-}{E}{}^p) \dot{\boldsymbol{p}}]^T \\ (\overset{-}{E}{}^p) \dot{\boldsymbol{p}} & -(\overset{-}{E}{}^p)(\overset{-}{E}{}^p)^T \end{bmatrix}. \end{aligned} \tag{A9}$$

Similarly, using the properties given in (17) and (28), we can find

$$\overset{-}{\boldsymbol{\beta}} = \overset{-}{\boldsymbol{p}} (\dot{\overset{-}{\boldsymbol{p}}})^T = \begin{bmatrix} \boldsymbol{p}^{*T} \\ \overset{+}{E}{}^{p*} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}}^* & (\dot{\overset{+}{E}}{}^{p*})^T \end{bmatrix}$$

$$
= \begin{bmatrix} \boldsymbol{p}^{*T}\dot{\boldsymbol{p}}^{*} & [(\dot{\overset{+}{\boldsymbol{E}}}{}^{p*})\boldsymbol{p}^{*}]^{T} \\ (\overset{+}{\boldsymbol{E}}{}^{p*})\dot{\boldsymbol{p}}^{*} & (\overset{+}{\boldsymbol{E}}{}^{p*})(\dot{\overset{+}{\boldsymbol{E}}}{}^{p*})^{T} \end{bmatrix}
= \begin{bmatrix} \dot{\boldsymbol{p}}^{T}\boldsymbol{p} & [(\overset{-}{\boldsymbol{E}}{}^{p})\dot{\boldsymbol{p}}]^{T} \\ -(\overset{-}{\boldsymbol{E}}{}^{p})\dot{\boldsymbol{p}} & (\overset{-}{\boldsymbol{E}}{}^{p})(\dot{\overset{-}{\boldsymbol{E}}}{}^{p})^{T} \end{bmatrix}. \tag{A10}
$$

The identities

$$
\boxed{(\overset{+}{\boldsymbol{E}}{}^{p*})\dot{\boldsymbol{p}}^{*} = -(\dot{\overset{+}{\boldsymbol{E}}}{}^{p*})\boldsymbol{p}^{*} = (\dot{\overset{-}{\boldsymbol{E}}}{}^{p})\boldsymbol{p} = -(\overset{-}{\boldsymbol{E}}{}^{p})\dot{\boldsymbol{p}}} \tag{A11}
$$

and

$$
\boxed{\begin{aligned} (\dot{\overset{+}{\boldsymbol{E}}}{}^{p*})(\overset{+}{\boldsymbol{E}}{}^{p*})^{T} &= -(\overset{+}{\boldsymbol{E}}{}^{p*})(\dot{\overset{+}{\boldsymbol{E}}}{}^{p*})^{T} \\ &= (\dot{\overset{-}{\boldsymbol{E}}}{}^{p})(\overset{-}{\boldsymbol{E}}{}^{p})^{T} = -(\overset{-}{\boldsymbol{E}}{}^{p})(\dot{\overset{-}{\boldsymbol{E}}}{}^{p})^{T} \end{aligned}} \tag{A12}
$$

can be proved easily and are used in deriving (A9) and (A10).

Since the representation of $\boldsymbol{\beta}$ is not unique, the representations of $\overset{+}{\boldsymbol{\beta}}$ and $\overset{-}{\boldsymbol{\beta}}$ are also not unique. Using the formulas developed in (16) and (17), we obtain the following properties:

$$
\boxed{\overset{+}{\boldsymbol{\beta}} = \dot{\overset{+}{\boldsymbol{p}}}{}^{T}\overset{+}{\boldsymbol{p}} = -\overset{+}{\boldsymbol{p}}{}^{T}\dot{\overset{+}{\boldsymbol{p}}}} \tag{A13}
$$

and

$$
\boxed{\overset{-}{\boldsymbol{\beta}} = \overset{-}{\boldsymbol{p}}\dot{\overset{-}{\boldsymbol{p}}}{}^{T} = -\dot{\overset{-}{\boldsymbol{p}}}\overset{-}{\boldsymbol{p}}{}^{T}.} \tag{A14}
$$

C. Various Representations of Angular Velocity

From (43), the angular velocity $\boldsymbol{\Omega}_{i0}^{0}$ can be expressed in two different ways: $2\dot{\boldsymbol{p}}_{i0} \otimes \boldsymbol{p}_{i0}^{*} = -2\boldsymbol{p}_{i0} \otimes \dot{\boldsymbol{p}}_{i0}^{*}$. If we manipulate these two quaternion representations using the multiplicative commutativity given in (12) and the matrices in (27) and (28), we can easily obtain the following identities for expressing $\boldsymbol{\Omega}_{i0}^{0}$ in three-space:

$$
\boxed{\boldsymbol{\Omega}_{i0}^{0} = 2(\overset{+}{\boldsymbol{E}}{}^{p})\dot{\boldsymbol{p}} = -2(\dot{\overset{+}{\boldsymbol{E}}}{}^{p})\boldsymbol{p} = 2(\dot{\overset{-}{\boldsymbol{E}}}{}^{p*})\boldsymbol{p}^{*} = -2(\overset{-}{\boldsymbol{E}}{}^{p*})\dot{\boldsymbol{p}}^{*}.} \tag{A15}
$$

Similarly, we can express $\boldsymbol{\Omega}_{i0}^{i}$ in three-space as:

$$
\boxed{\boldsymbol{\Omega}_{i0}^{i} = 2(\overset{-}{\boldsymbol{E}}{}^{p})\dot{\boldsymbol{p}} = -2(\dot{\overset{-}{\boldsymbol{E}}}{}^{p})\boldsymbol{p} = 2(\dot{\overset{+}{\boldsymbol{E}}}{}^{p*})\boldsymbol{p}^{*} = -2(\overset{+}{\boldsymbol{E}}{}^{p*})\dot{\boldsymbol{p}}^{*}.} \tag{A16}
$$

Equations (A15) and (A16) can be obtained directly from (A4) and (A11) also. The angular-velocity matrix for $\boldsymbol{\Omega}_{i0}^{0}$, in three-space, can be obtained directly using (A5) as

$$
\boxed{\begin{aligned} \tilde{\boldsymbol{\Omega}}_{i0}^{0} &= 2(\dot{\overset{-}{\boldsymbol{E}}}{}^{p*})(\overset{-}{\boldsymbol{E}}{}^{p*})^{T} = -2(\overset{-}{\boldsymbol{E}}{}^{p*})(\dot{\overset{-}{\boldsymbol{E}}}{}^{p*})^{T} \\ &= 2(\dot{\overset{+}{\boldsymbol{E}}}{}^{p})(\overset{+}{\boldsymbol{E}}{}^{p})^{T} = -2(\overset{+}{\boldsymbol{E}}{}^{p})(\dot{\overset{+}{\boldsymbol{E}}}{}^{p})^{T}. \end{aligned}} \tag{A17}
$$

For $\boldsymbol{\Omega}_{i0}^{i}$, in three-space, they are obtained from (A12) as

$$
\boxed{\begin{aligned} \tilde{\boldsymbol{\Omega}}_{i0}^{i} &= 2(\dot{\overset{+}{\boldsymbol{E}}}{}^{p*})(\overset{+}{\boldsymbol{E}}{}^{p*})^{T} = -2(\overset{+}{\boldsymbol{E}}{}^{p*})(\dot{\overset{+}{\boldsymbol{E}}}{}^{p*})^{T} \\ &= 2(\dot{\overset{-}{\boldsymbol{E}}}{}^{p})(\overset{-}{\boldsymbol{E}}{}^{p})^{T} = -2(\overset{-}{\boldsymbol{E}}{}^{p})(\dot{\overset{-}{\boldsymbol{E}}}{}^{p})^{T}. \end{aligned}} \tag{A18}
$$

D. Various Representations of Angular Acceleration

Similar to angular velocity, the representation of angular acceleration is not unique. For $\dot{\boldsymbol{\Omega}}_{i0}^{0}$, applying the multiplicative commutativity property given in (12) and the properties in (27) and (28) to (55), we can obtain

$$
\boxed{\dot{\boldsymbol{\Omega}}_{i0}^{0} = 2(\overset{+}{\boldsymbol{E}}{}^{p})\ddot{\boldsymbol{p}} = -2(\ddot{\overset{+}{\boldsymbol{E}}}{}^{p})\boldsymbol{p} = 2(\ddot{\overset{-}{\boldsymbol{E}}}{}^{p*})\boldsymbol{p}^{*} = -2(\overset{-}{\boldsymbol{E}}{}^{p*})\ddot{\boldsymbol{p}}^{*}.} \tag{A19}
$$

Similarly, we can express $\dot{\boldsymbol{\Omega}}_{i0}^{i}$ in three-space as

$$
\boxed{\dot{\boldsymbol{\Omega}}_{i0}^{i} = 2(\overset{-}{\boldsymbol{E}}{}^{p})\ddot{\boldsymbol{p}} = -2(\ddot{\overset{-}{\boldsymbol{E}}}{}^{p})\boldsymbol{p} = 2(\ddot{\overset{+}{\boldsymbol{E}}}{}^{p*})\boldsymbol{p}^{*} = -2(\overset{+}{\boldsymbol{E}}{}^{p*})\ddot{\boldsymbol{p}}^{*}.} \tag{A20}
$$

Appendix B
Inertia Matrix in Four-Space

Consider a rigid body and a coordinate system $O_0(X_0, Y_0, Z_0)$. Let the vector $\boldsymbol{r} = [r_x, r_y, r_z]^T$ specify the position of an infinitesimal mass dm in the body. The moments of inertia of masses with respect to the X_0, Y_0, and Z_0 axes are defined as

$$
I_{xx} = \int (r_y^2 + r_z^2)dm
$$

$$
I_{yy} = \int (r_z^2 + r_x^2)dm
$$

and

$$
I_{zz} = \int (r_x^2 + r_y^2)dm
$$

respectively, and the products of inertia of masses are defined as

$$
I_{xy} = \int (r_x r_y)dm
$$

$$
I_{yz} = \int (r_y r_z)dm
$$

and

$$I_{zx} = \int (r_z r_x) dm$$

where the integrations are carried out over the whole mass of the body [53].

The inertia matrix in three-space is defined as [46], [54]

$$\boldsymbol{I}^0 = \int (\boldsymbol{r}^T \boldsymbol{r} \boldsymbol{U} - \boldsymbol{r}\boldsymbol{r}^T) dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \quad \text{(B1)}$$

where $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, and $I_{yz} = I_{zy}$.

In four-space, the vector quaternion $\boldsymbol{r} = [0, r_x, r_y, r_z]^T$ specifies the position of mass dm, and the inertia matrix is defined as

$$\begin{aligned} \hat{\boldsymbol{I}}^0 &= \int (\boldsymbol{r}^T \boldsymbol{r} \boldsymbol{U} - \boldsymbol{r}\boldsymbol{r}^T) dm = \int (\overset{+}{E}{}^{r\,T} \overset{+}{E}{}^{r}) dm \\ &= \int (\overset{-}{E}{}^{r\,T} \overset{-}{E}{}^{r}) dm \end{aligned} \quad \text{(B2)}$$

where

$$\overset{+}{E}{}^{r\,T} \overset{+}{E}{}^{r} = \begin{bmatrix} r_x^2 + r_y^2 + r_z^2 & 0 & 0 & 0 \\ 0 & r_y^2 + r_z^2 & -r_x r_y & -r_x r_z \\ 0 & -r_y r_x & r_x^2 + r_z^2 & -r_y r_z \\ 0 & -r_z r_x & -r_z r_y & r_x^2 + r_y^2 \end{bmatrix}.$$

Since [47]

$$\int x^2 dm = \frac{1}{2}(I_{yy} - I_{xx} + I_{zz})$$

$$\int y^2 dm = \frac{1}{2}(I_{xx} - I_{yy} + I_{zz})$$

$$\int z^2 dm = \frac{1}{2}(I_{xx} + I_{yy} - I_{zz})$$

therefore

$$\int (x^2 + y^2 + z^2) dm = \frac{1}{2}(I_{xx} + I_{yy} + I_{zz}). \quad \text{(B3)}$$

Defining

$$k = \frac{1}{2}(I_{xx} + I_{yy} + I_{zz}) = \frac{1}{2}\text{Trace}(\boldsymbol{I}^0) \quad \text{(B4)}$$

we obtain the inertia matrix in four-space as

$$\hat{\boldsymbol{I}}^0 = \begin{bmatrix} k & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{I}^0 \end{bmatrix}. \quad \text{(B5)}$$

Appendix C
Differentiation of Inertia Matrix

The inertia matrix $\boldsymbol{I}^0$, referred to the inertial coordinate system $O_0(X_0, Y_0, Z_0)$, and the inertia matrix $\boldsymbol{I}^i$, referred to the body-fixed coordinate system $o_i(x_i, y_i, z_i)$ of body i, can be related by [45], [55]

$$\boldsymbol{I}^0 = \boldsymbol{A}_{i0} \boldsymbol{I}^i \boldsymbol{A}_{i0}^T \quad \text{(C1)}$$

where $\boldsymbol{A}_{i0}$ is the rotational transformation matrix of system o_i relative to system O_0. Differentiating (C1) once, we have

$$\dot{\boldsymbol{I}}^0 = \dot{\boldsymbol{A}}_{i0} \boldsymbol{I}^i \boldsymbol{A}_{i0}^T + \boldsymbol{A}_{i0} \boldsymbol{I}^i \dot{\boldsymbol{A}}_{i0}^T + \boldsymbol{A}_{i0} \dot{\boldsymbol{I}}^i \boldsymbol{A}_{i0}^T. \quad \text{(C2)}$$

Substituting (46), the Poisson's equation, gives

$$\dot{\boldsymbol{I}}^0 = \tilde{\boldsymbol{\Omega}}^0 \boldsymbol{A}_{i0} \boldsymbol{I}^i \boldsymbol{A}_{i0}^T + \boldsymbol{A}_{i0} \boldsymbol{I}^i \boldsymbol{A}_{i0}^T \tilde{\boldsymbol{\Omega}}^{0T} \quad \text{(C3)}$$

since $\dot{\boldsymbol{I}}^i = 0$. Substituting (C1) into (C3) gives

$$\dot{\boldsymbol{I}}^0 = \tilde{\boldsymbol{\Omega}}^0 \boldsymbol{I}^0 - \boldsymbol{I}^0 \tilde{\boldsymbol{\Omega}}^0. \quad \text{(C4)}$$

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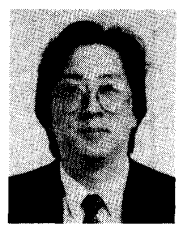

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