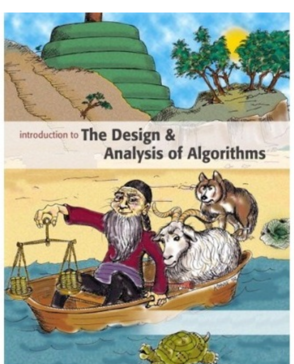
Introduction to

# *Algorithm Design and Analysis*

## [8] *logn* search

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# In the last class…

- **Selection – warm up**
  - Max and min
  - Second largest
- **Selection – Rank k (median)**
  - Expected linear time
  - Worst-case linear time
- **Adversary argument**
  - Lower bound

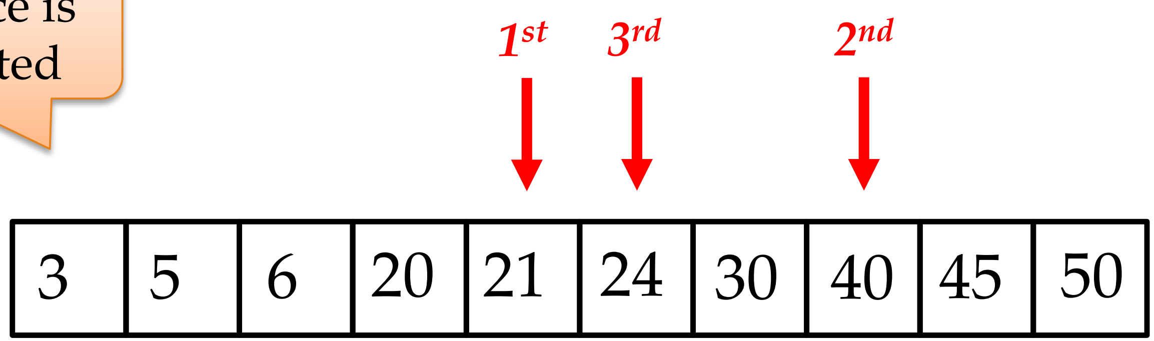
# *logn* search

- **Essential of the *searching* problem**
  - How to *organize the data* to enable efficient search
  - *logn* search
    - Each search cuts off half of the search space
    - How to organize the data to enable *logn* search
- ***logn* search techniques**
  - Binary search over *sorted* sequences
  - *Balanced* Binary Search Tree (BST)

# Binary Search by Example

- **Binary search for "24"**
  - Divide the search space
  - Cut off half the space after each search

The sequence is already sorted




Pseudo code in p.129 [Baase01]

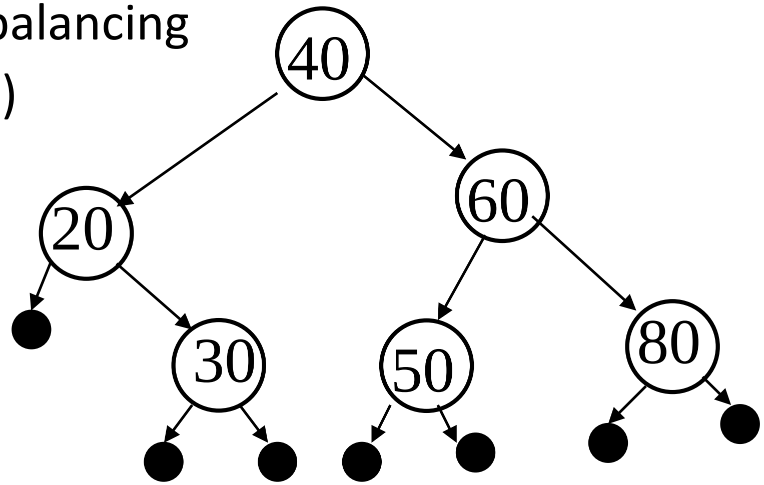
# Balanced Binary Search Tree

- **Binary search tree (BST)**
  - Definitions and basic operations
- **Definition of Red-Black Tree (RBT)**
  - Black height
- **RBT operations**
  - Insertion into a red-black tree
  - Deletion from a red-black tree

# Binary Search Tree Revisited

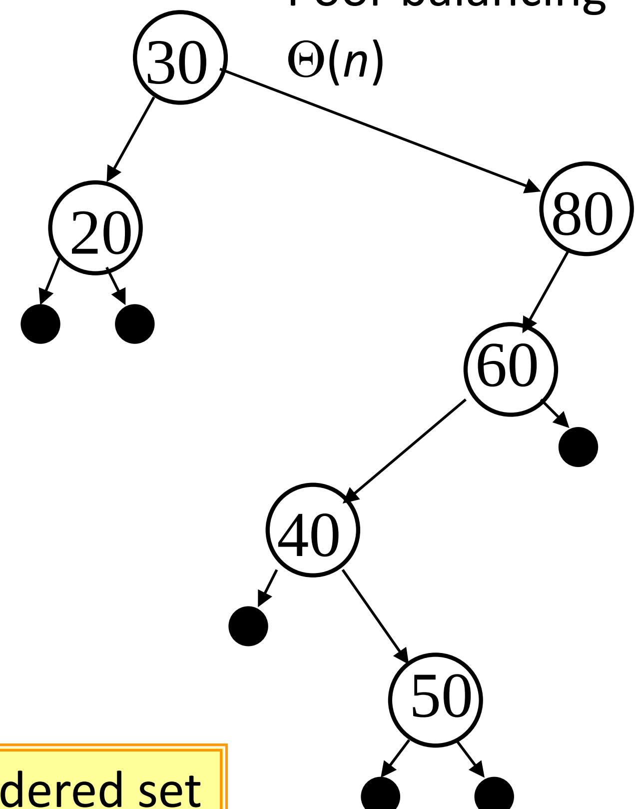
Good balancing

$\Theta(\log n)$




Poor balancing

$\Theta(n)$

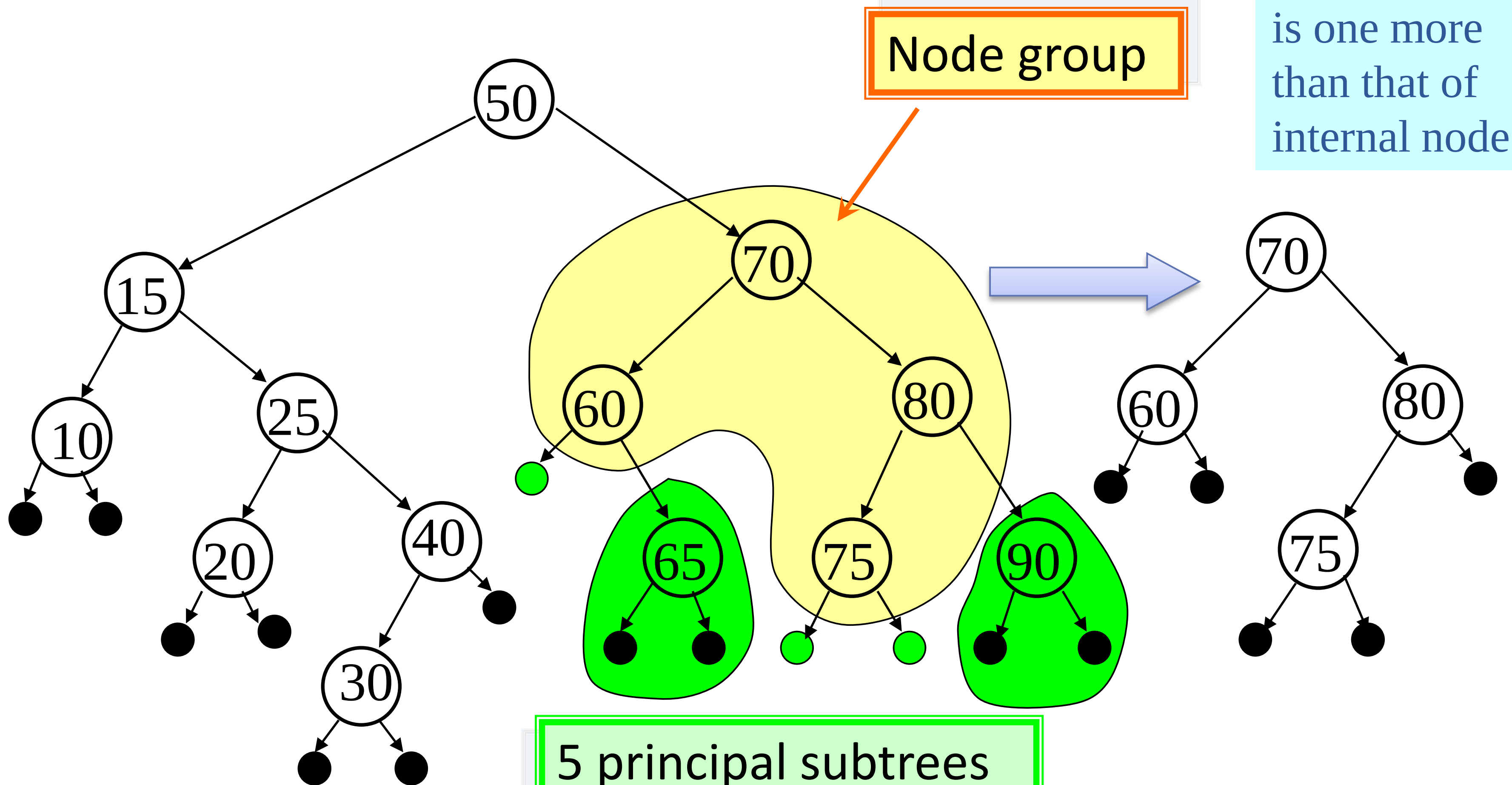



***In a properly drawn tree, pushing forward to get the ordered list.***

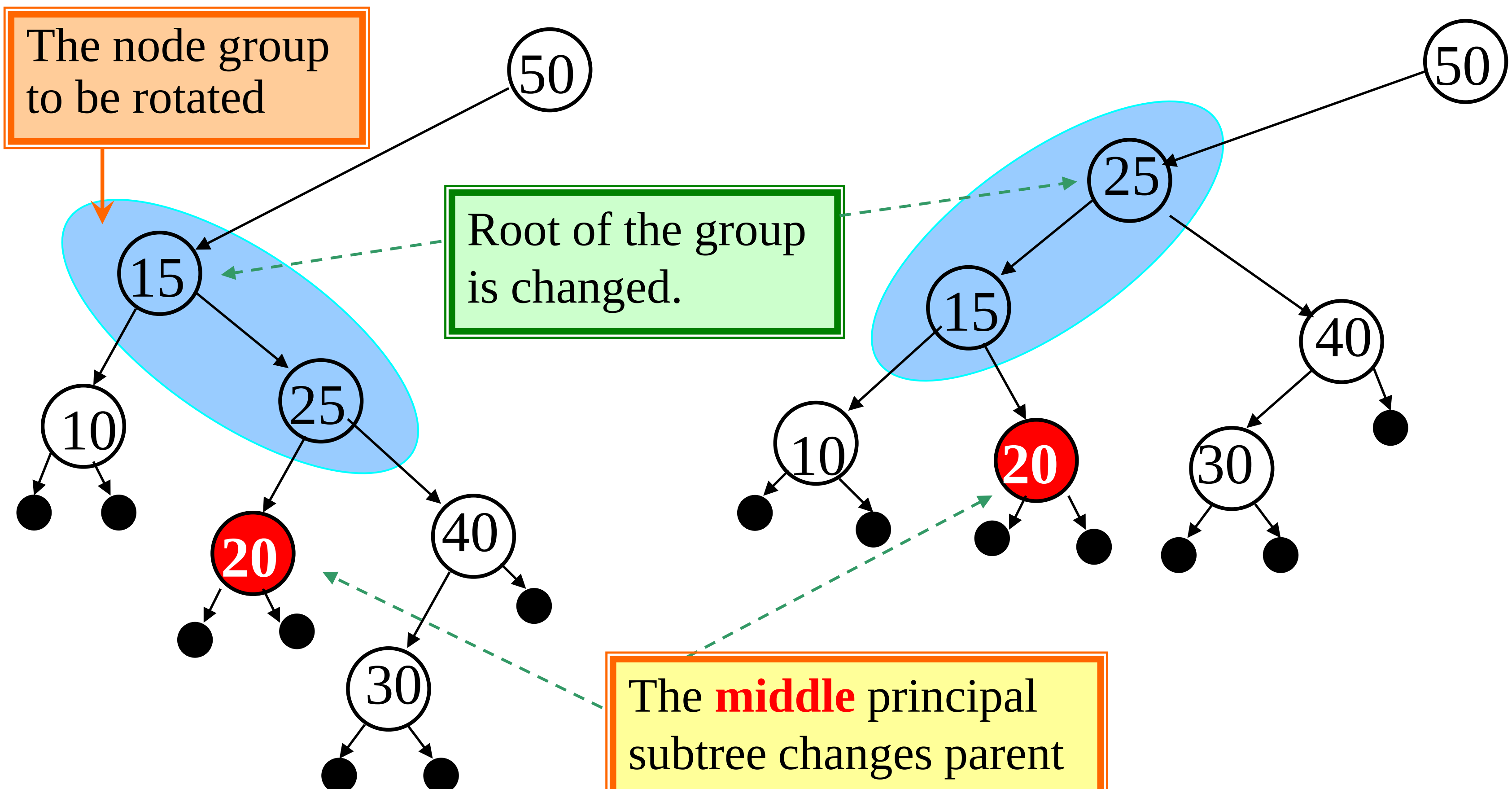
- Each node has a key, belonging to a linear ordered set
- An inorder traversal produces a sorted list of the keys

# Node Group

As in 2-tree, the number of external node is one more than that of internal node



# Balancing by Rotation



# Red-Black Tree: the Definition

- **If *T* is a binary tree in which each node has a color, red or black, and all external nodes are black, then *T* is a red-black tree if and only if:**
  - [*Color constraint*] No red node has a red child
  - [*Black height constraint*] The **black length** of all external paths from a given node *u* is the same (the black height of *u*)
  - The root is black.
- ***Almost*-red-black tree(ARB tree)**
  - Root is red, satisfying the other constraints.

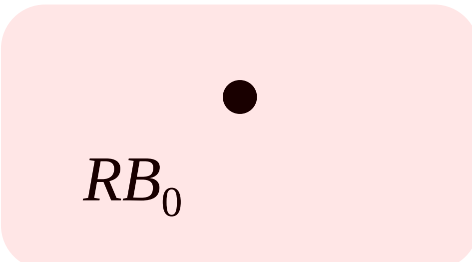
Balancing is under control

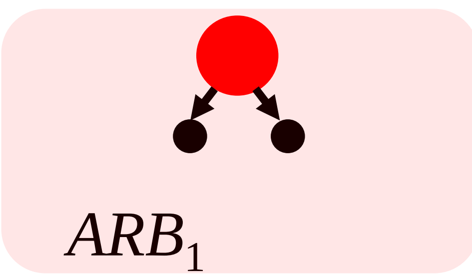
# Recursive Definition of RBT

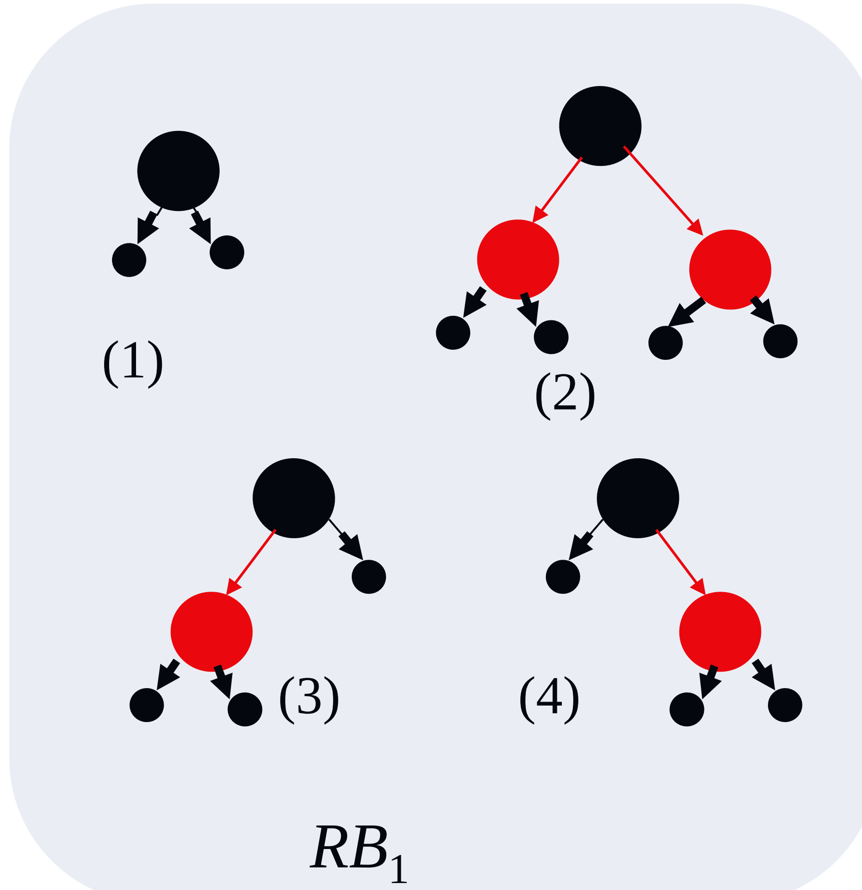
**(A red-black tree of black height *h* is denoted as $RB_h$)**

- **Definition:**
  - An external node is an $RB_0$ tree, and the node is black.
  - A binary tree is an $ARB_h$ ($h \geq 1$) tree if: ← No $ARB_0$
    - Its root is red, and
    - Its left and right subtrees are each an $RB_{h-1}$ tree.
  - A binary tree is an $RB_h$ ($h \geq 1$) tree if:
    - Its root is black, and
    - Its left and right subtrees are each either an $RB_{h-1}$ tree or an $ARB_h$ tree.

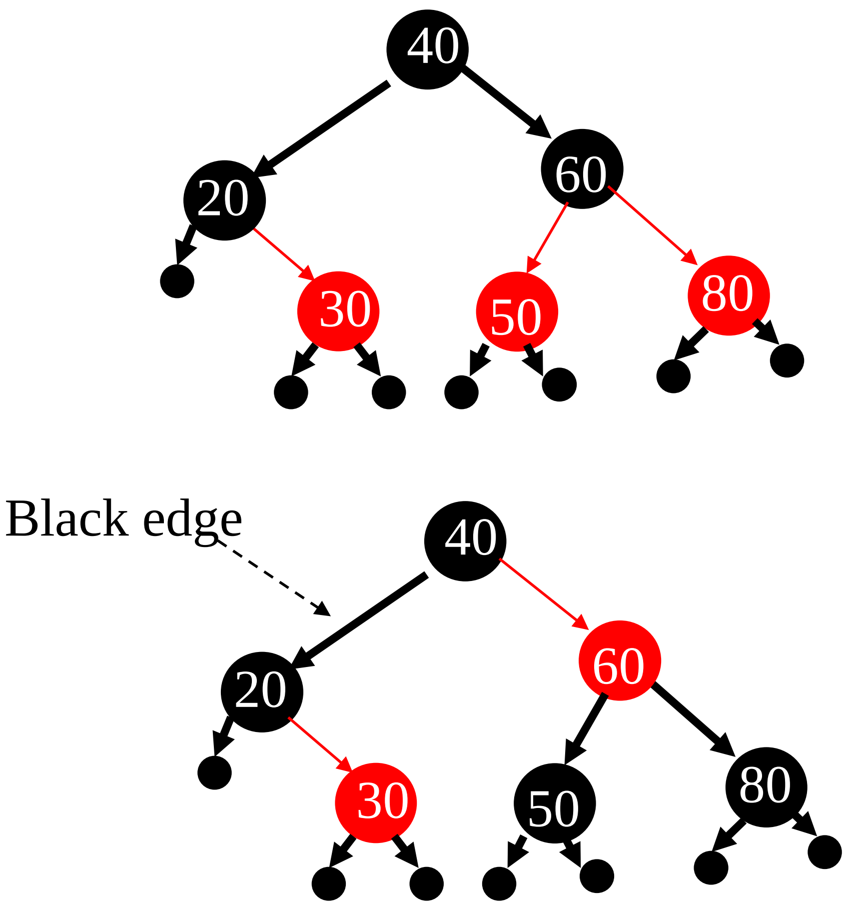
# $RB_i$ and $ARB_i$

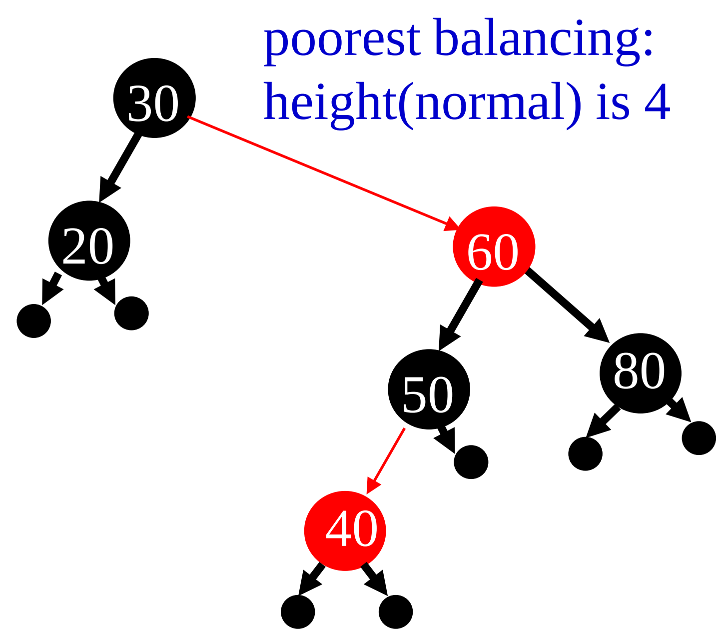


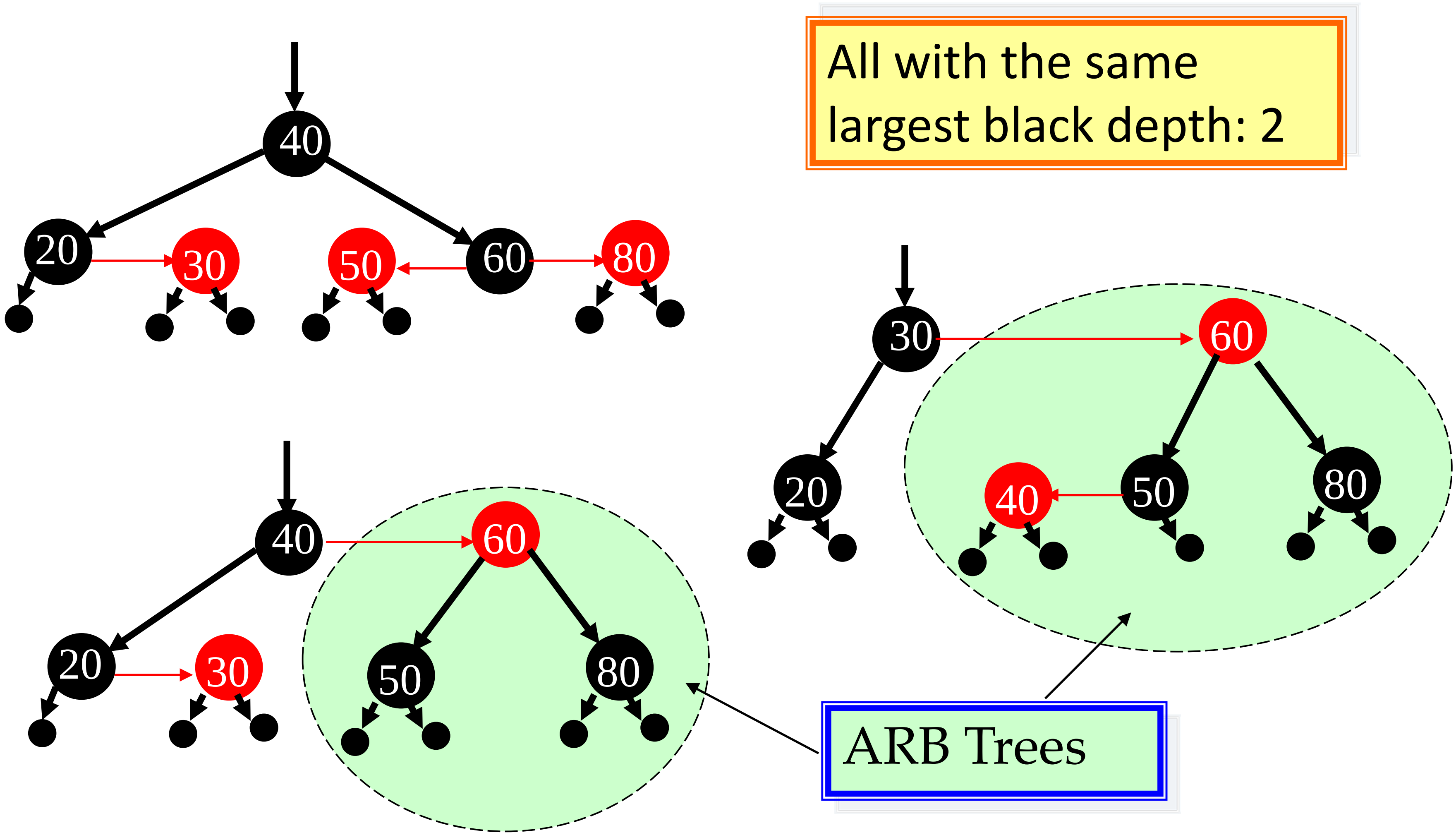



# Red-Black Tree with 6 Nodes



# Black-depth Convention



# Properties of Red-Black Tree

- **The black height of any $RB_h$ tree or $ARB_h$ tree is well-defind and is $h$.**
- **Let $T$ be an $RB_h$ tree, then:**
  - $T$ has at least $2^h$-1 internal black nodes.
  - $T$ has at most $4^h$-1 internal nodes.
  - The depth of any black node is at most twice its black depth.
- **Let $A$ be an $\mathrm{ARB_h}$ tree, then:**
  - $A$ has at least $2^h$-2 internal black nodes.
  - $A$ has at most $(4^h)/2$-1 internal nodes.
  - The depth of any black node is at most twice its black depth.

# Well-Defined Black Height

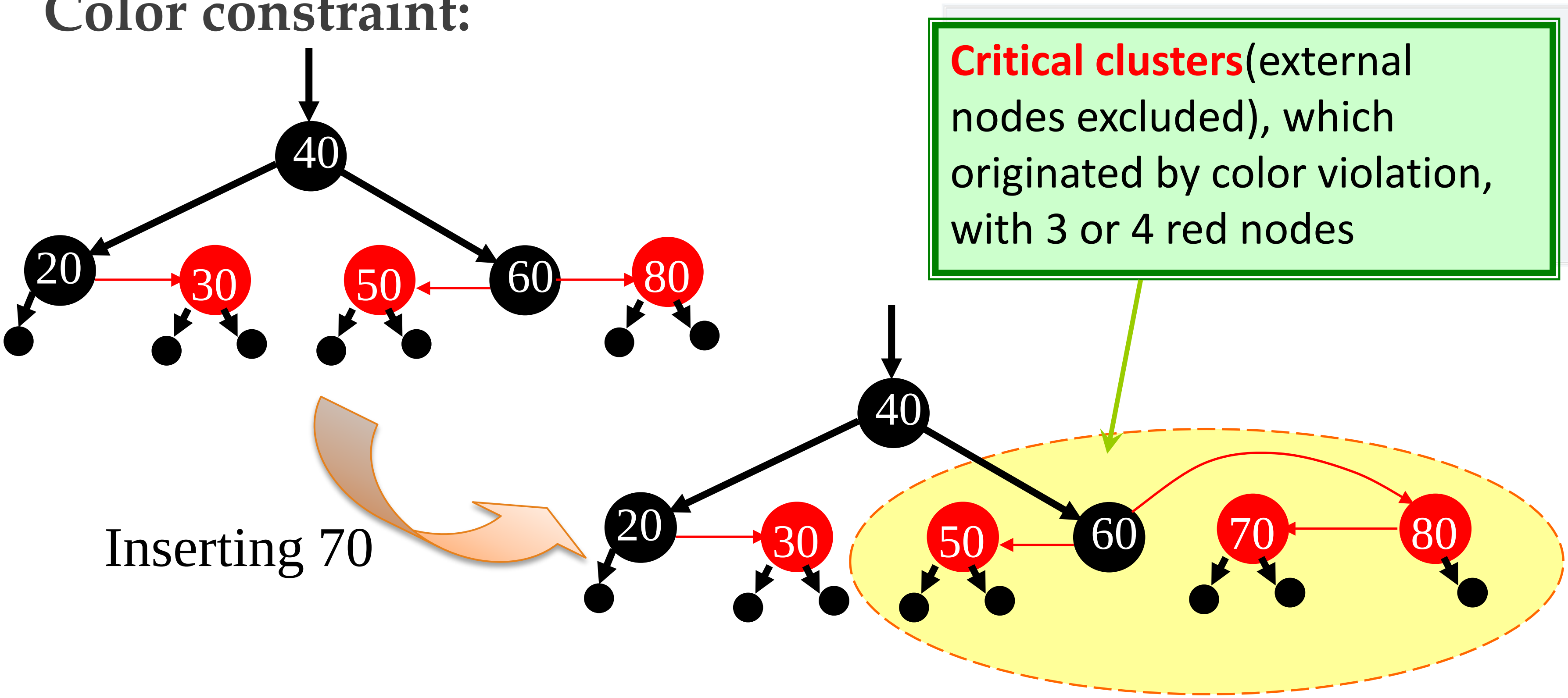
- That "the black height of any $RB_h$ tree or $ARB_h$ tree is well defind" means *the black length of all external paths from the root is the same*.
- Proof: induction on $h$
- Base case: $h$=0, that is $RB_0$ (there is no $ARB_0$)
- In $ARB_{h+1}$, its two subtrees are both $RB_h$. Since the root is red, the black length of all external paths from the root is $h$, that's the same as its two subtrees.
- In $RB_{h+1}$:
  - Case 1: two subtrees are $RB_h$'s
  - Case 2: two subtrees are $ARB_{h+1}$'s
  - Case 3: one subtree is an $RB_h$(black height=$h$), and the another is an $ARB_{h+1}$(black height=$h$+1)

# Bound on Depth of Node in RBTree

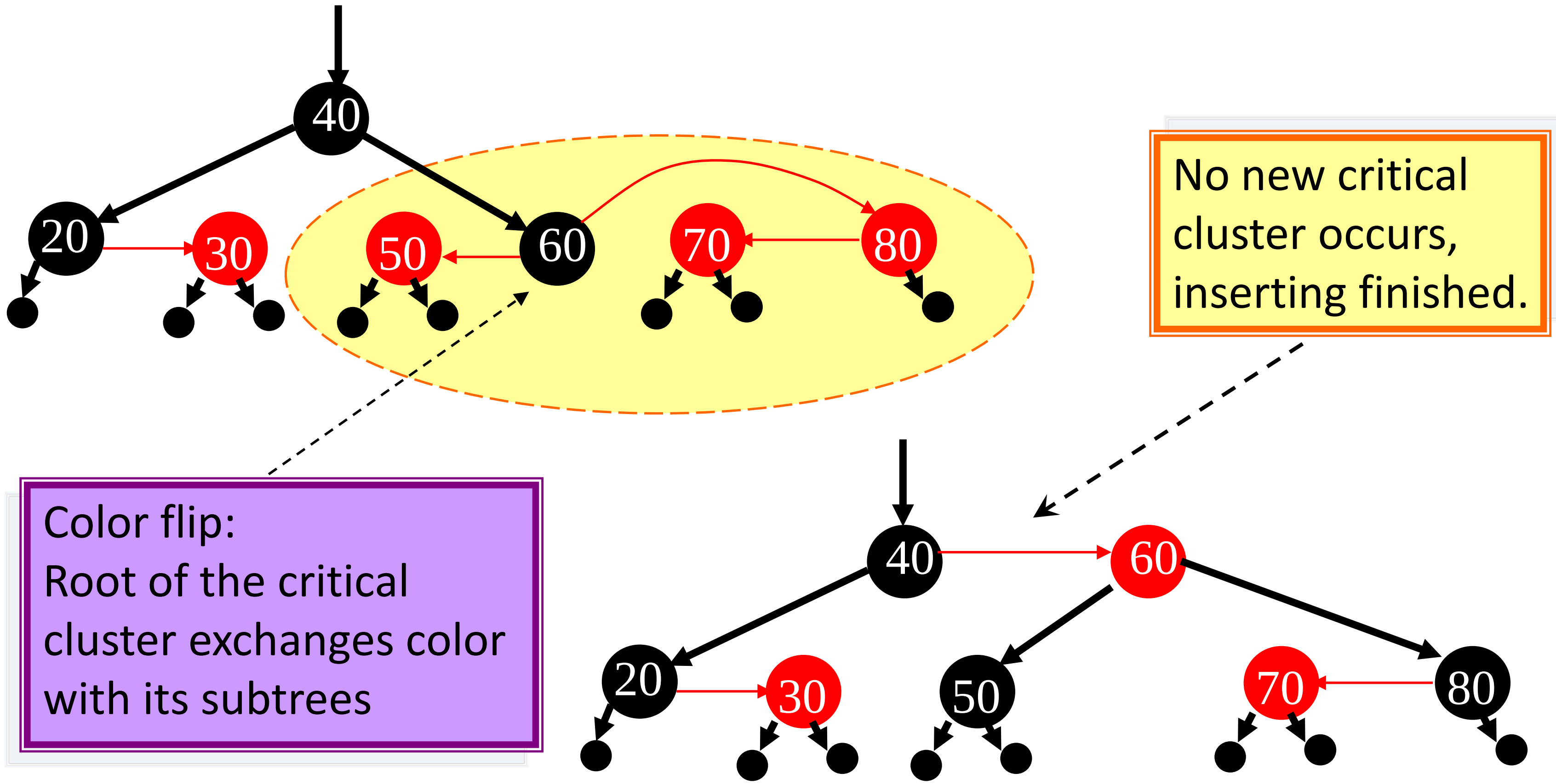
- **Let $T$ be a red-black tree with $n$ internal nodes. Then no node has black depth greater than 2log($n$+1), which means that the height of $T$ in the usual sense is at most 2log($n$+1).**
  - Proof:
  - Let $h$ be the black height of $T$. The number of internal nodes, $n$, is at least the number of internal black nodes, which is at least $2^h$-1, so $h \leq \log(n+1)$. The node with greatest depth is some external node. All external nodes are with black depth $h$. So, the depth is at most $2h$.

# Influences of Insertion to an RBT

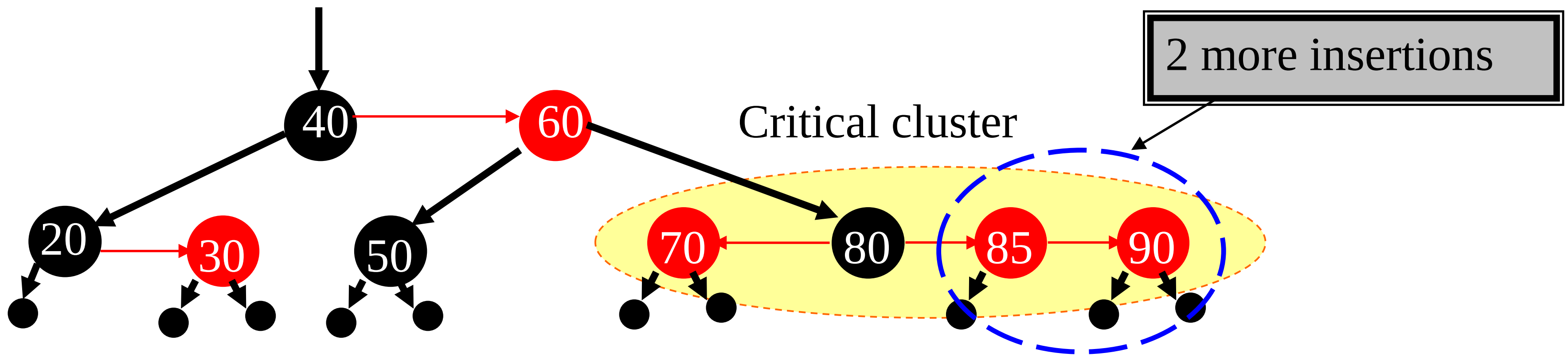
- **Black height constraint:**
  - No violation ***if*** inserting a red node.
- **Color constraint:**



# Repairing 4-node Critical Cluster



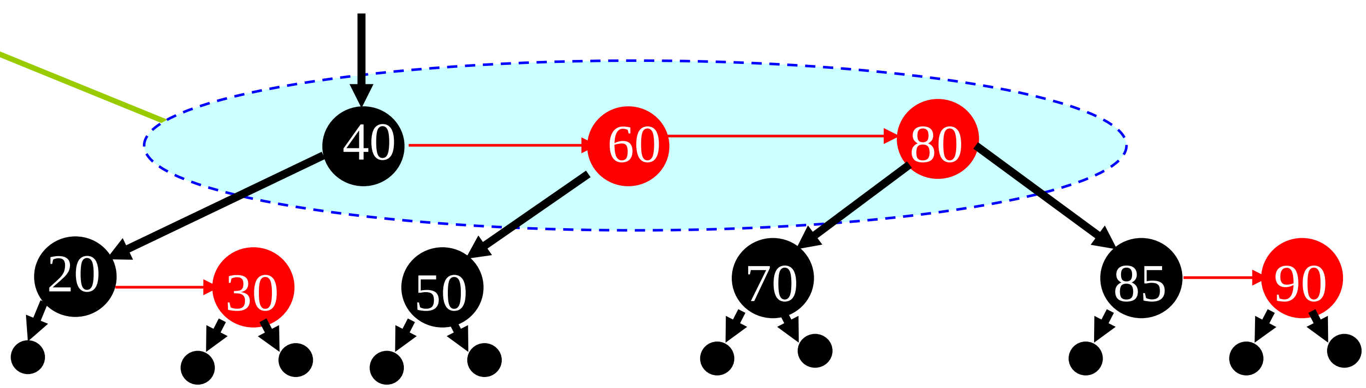
# Repairing 4-node Critical Cluster



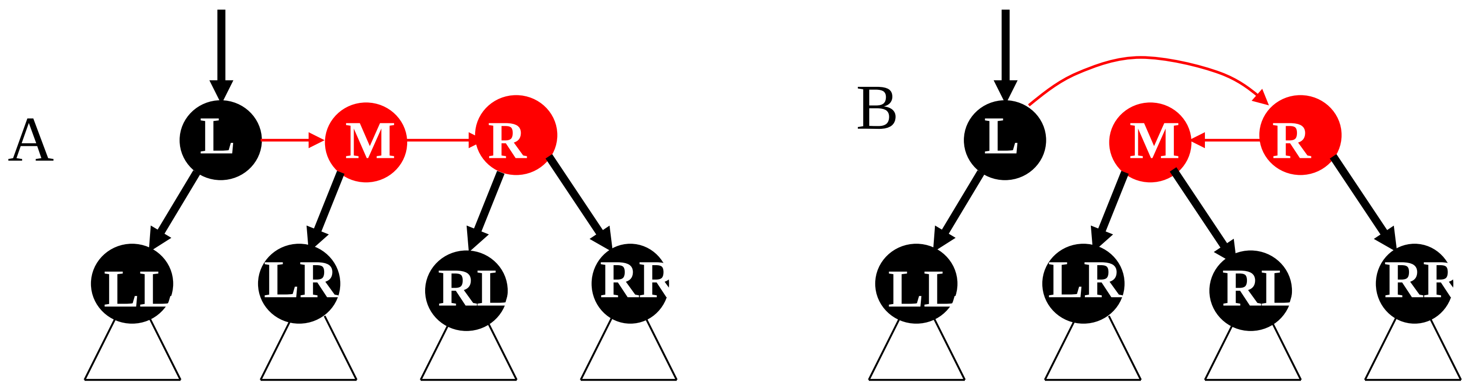

New critical cluster with 3 nodes.

Color flip doesn't work,

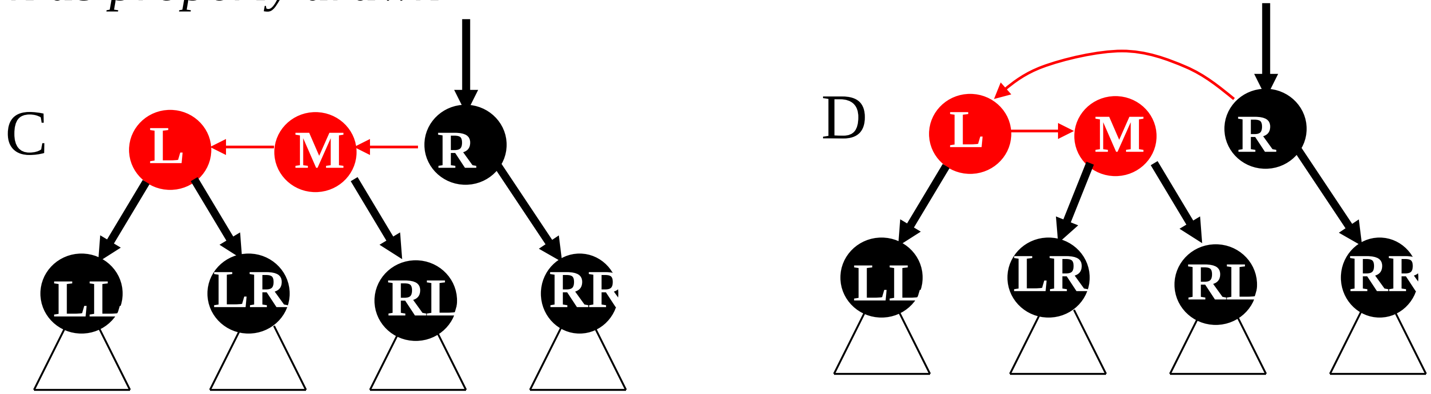
**Why?**



# Patterns of 3-node Critical Cluster



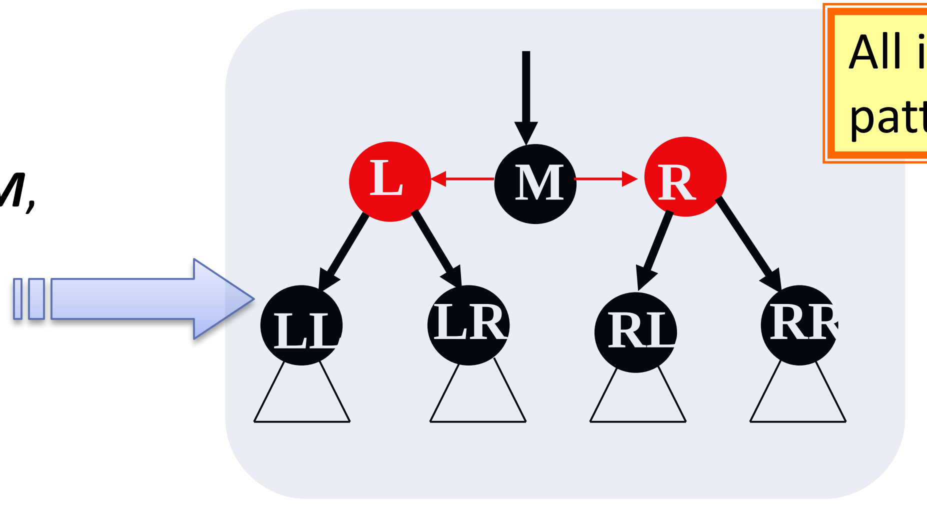

*Shown as properly drawn*



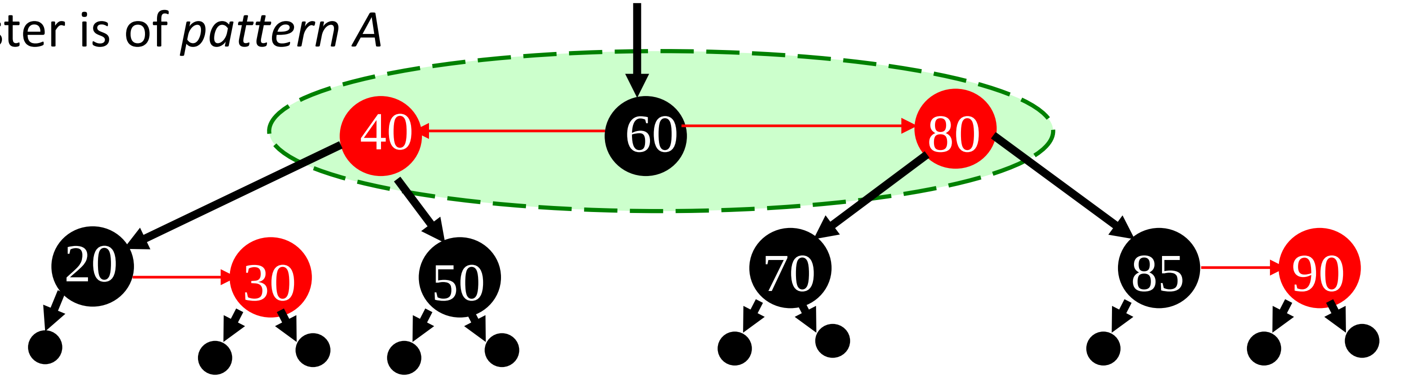
# Repairing 3-Node Critical Cluster

All into one pattern

Root of the critical cluster is changed to ***M***, and the parentship is adjusted accordingly




The incurred critical cluster is of *pattern A*



# Implementing Insertion: Class

```
class RBtree
        Element root;
        RBtree leftSubtree;
        RBtree rightSubtree;
        int color; /* red, black */

        static class InsReturn
                public RBtree newTree;
                public int status /* ok, rbr, brb, rrb, brr */
```

***Color pattern***

# Implementing Insertion: Procedure

```
RBtree rbtInsert (RBtree oldRBtree, Element newNode)
    InsReturn ans
    If (ans.newTr
        ans.newTre
    return ans.ne
```

*the wrapper*

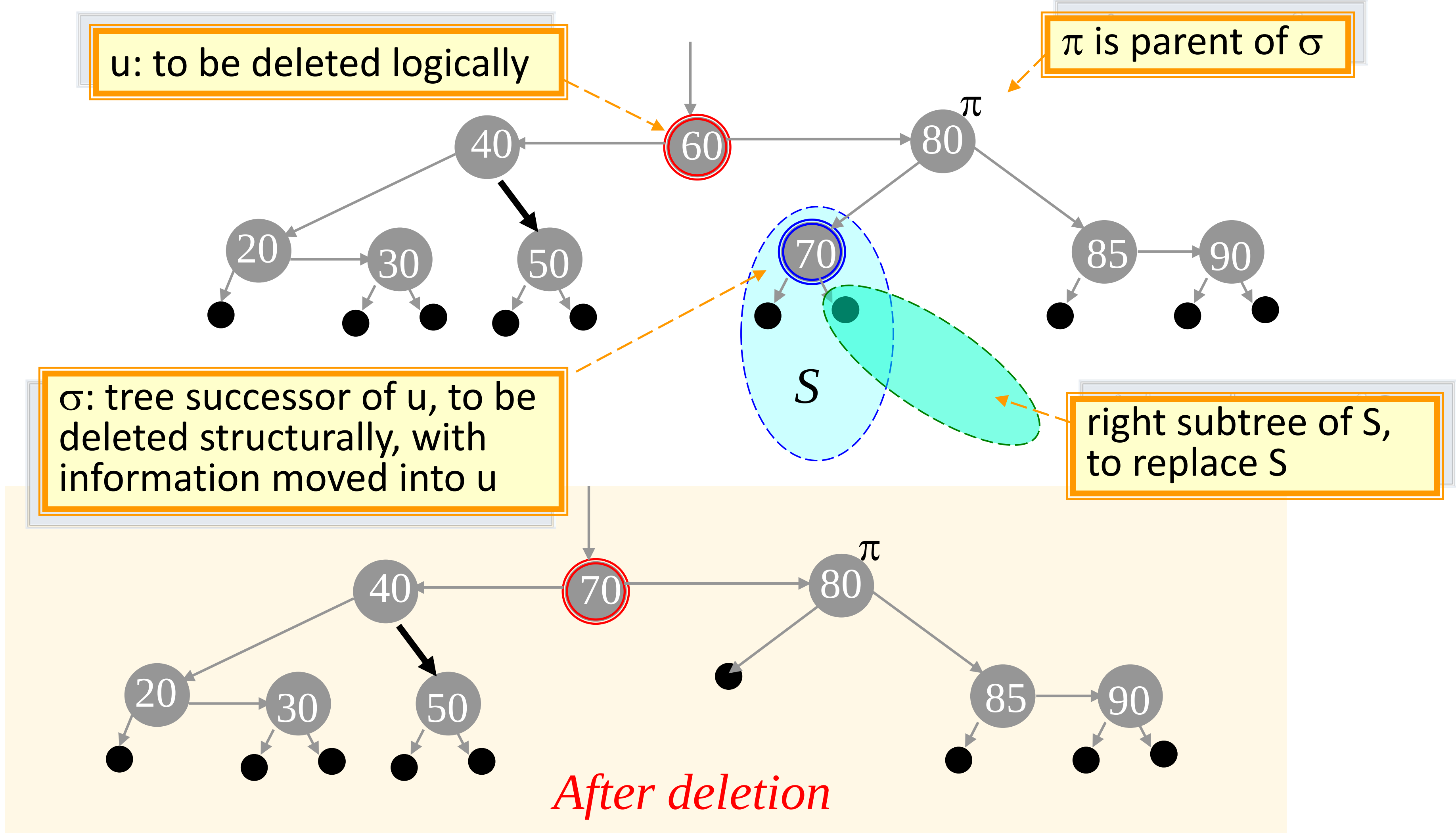
```
InsReturn rbtIns(RBtree oldRBtree, Element newNode)
    InsReturn ans, ansLeft, ansRight;
    if (oldRBtree = nil) then <Inserting simply>;
    else
        if (newNode.key <oldRBtree.root.key)
            ansLeft = rbtIns (oldRBtree.leftSubtree, newNode);
            ans = repairLeft(oldRBtree, ansLeft);
        else
            ansRight = rbtIns(oldRBtree.rightSubtree, newNode);
            ans = repairRight(oldRBtree, ansRight);
    return ans
```

*the recursive function*

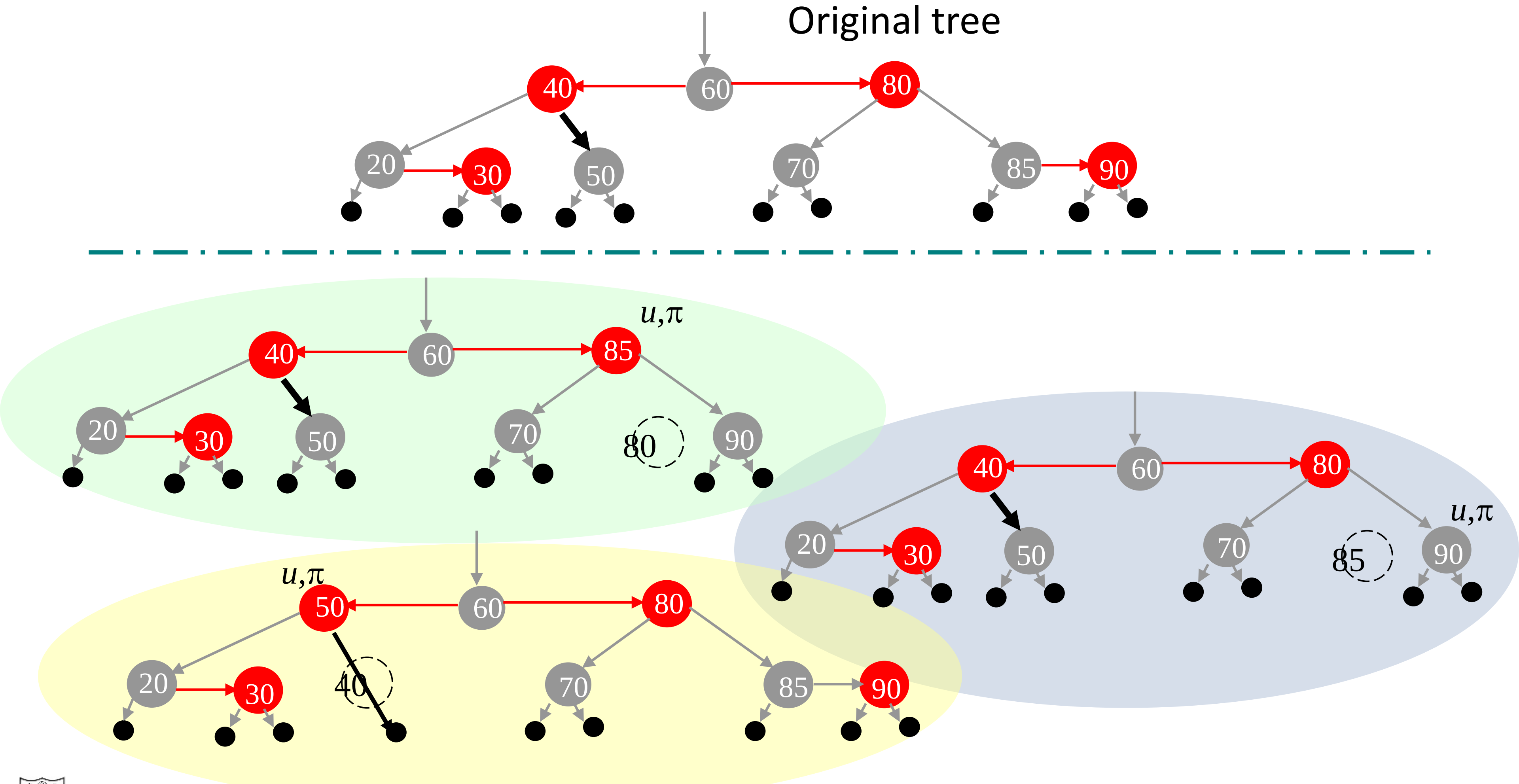
# Correctness of Insertion

- **If the parameter oldRBtree of rbtIns is an $RB_h$ tree or an $ARB_{h+1}$ tree(which is true for the recursive calls on rbtIns), then the newTree and status fields returned are one of the following combinations:**
  - Status=ok, and newTree is an $RB_h$ or an $ARB_{h+1}$ tree,
  - Status=rbr, and newTree is an $RB_h$,
  - Status=brb, and newTree is an $ARB_{h+1}$ tree,
  - Status=rrb, and newTree.color=red, newTree.leftSubtree is an $ARB_{h+1}$ tree and newTree.rightSubtree is an $RB_h$ tree,
  - Status=brr, and newTree.color=red, newTree.rightSubtree is an $ARB_{h+1}$ tree and newTree.leftSubtree is an $RB_h$ tree
- **For those cases with red root, the color will be changed to black, with other constraints satisfied by repairing subroutines.**

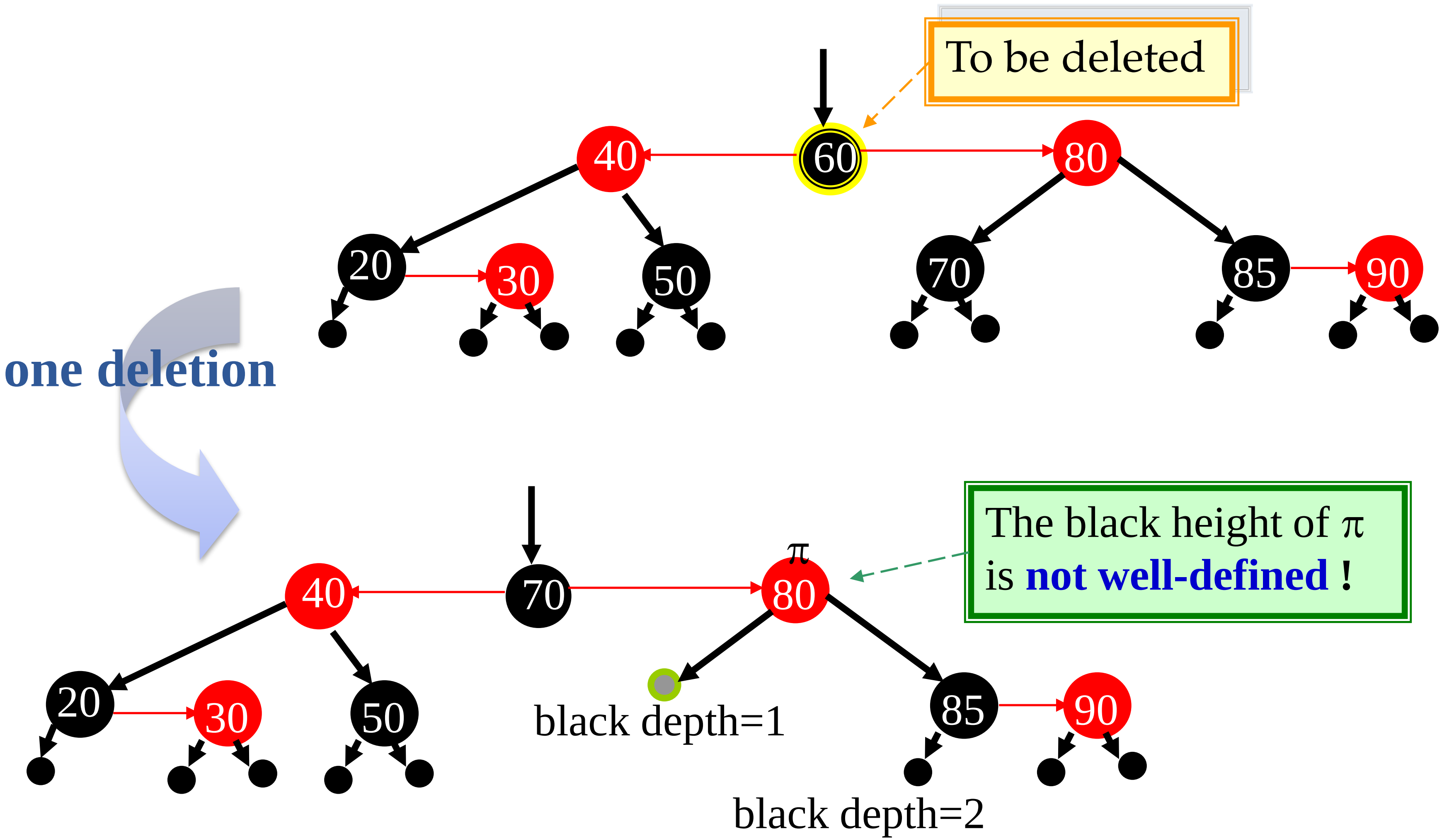
# Deletion: Logical and Structural



# Deletion from RBT - Examples



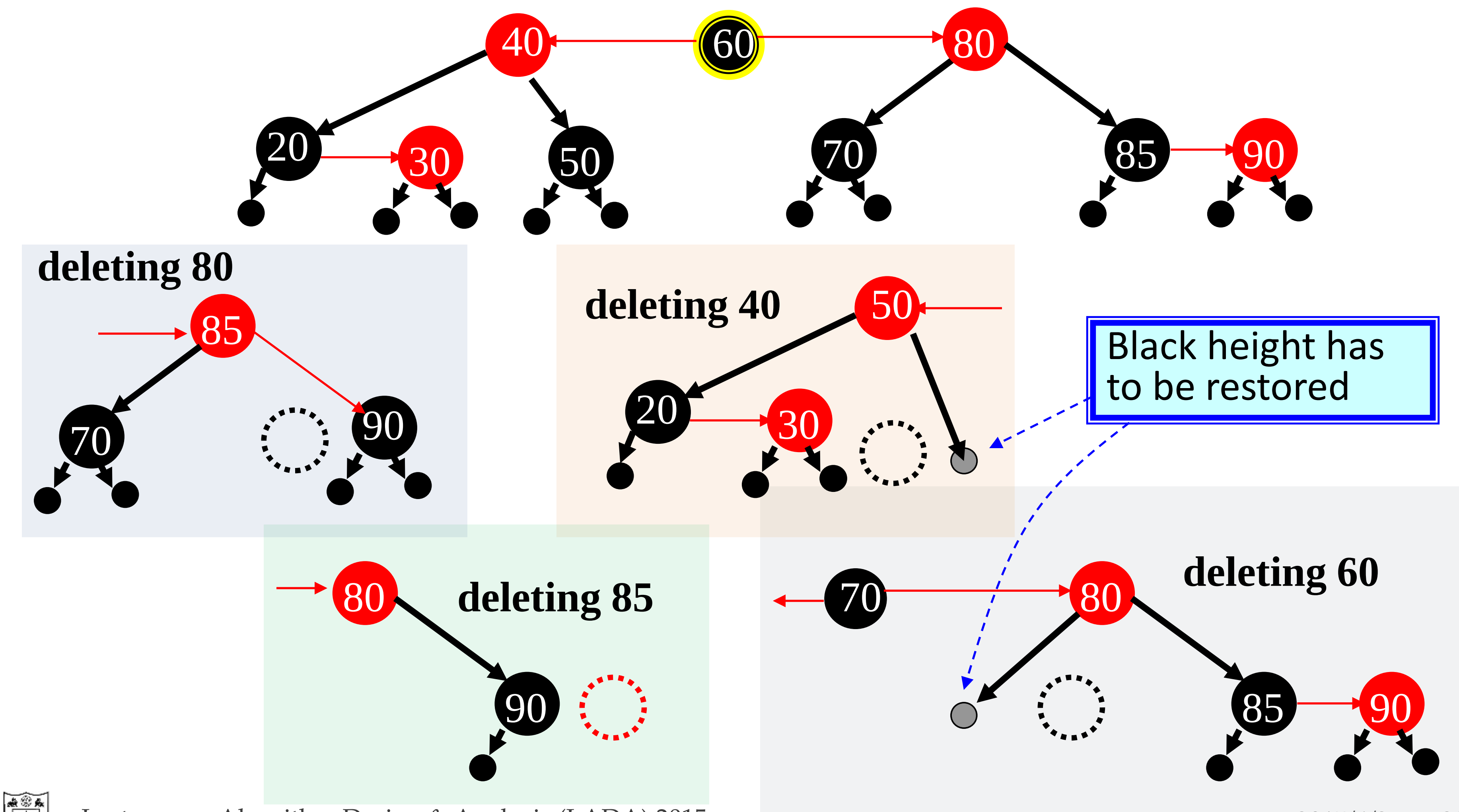
# Deletion in RBT



# Procedure of Red-Black Deletion

1. Do a standard BST search to locate the node to be logically deleted, call it *u*
2. If the right child of *u* is an external node, identify *u* as the node to be structurally deleted.
3. If the right child of *u* is an internal node, find the tree successor of *u* , call it σ, copy the key and information from σ to *u*. (color of *u* not changed) Identify σ as the node to be deleted structurally.
4. Carry out the structural deletion and repair any imbalance of black height.

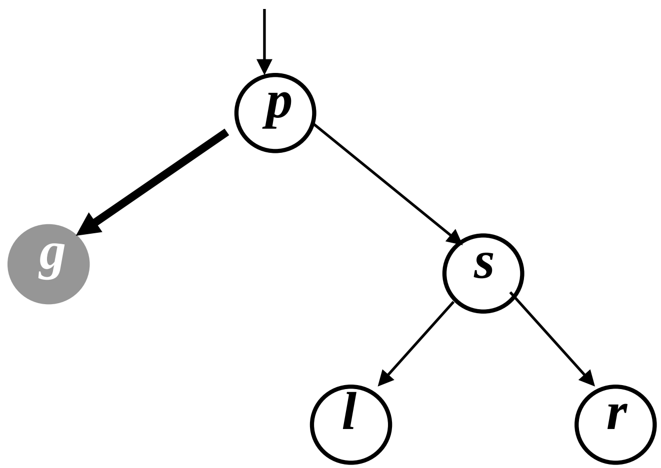
# Imbalance of Black Height



# Analysis of Black Imbalance

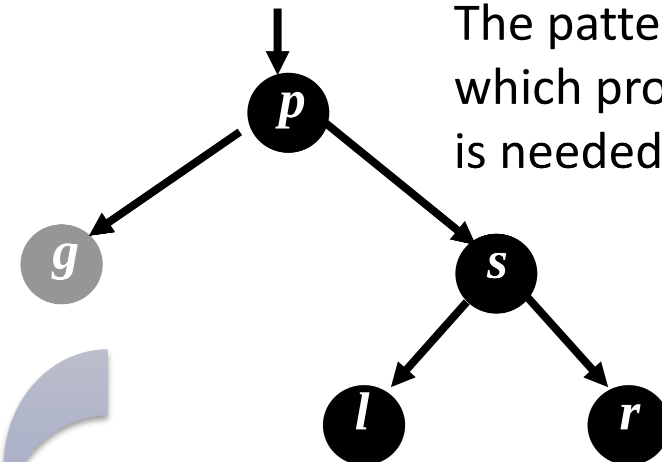
- **The imbalance occurs when:**
  - A black node is deleted structurally, and
  - Its right subtree is black (external)
- **The result is:**
  - An $RB_{h\text{-}1}$ occupies the position of an $RB_h$ as required by its parent, coloring it as a "gray" node.
- **Solution:**
  - Find a red node and turn it black as locally as possible.
  - The gray color might propagate up the tree.

# Propagation of Gray Node

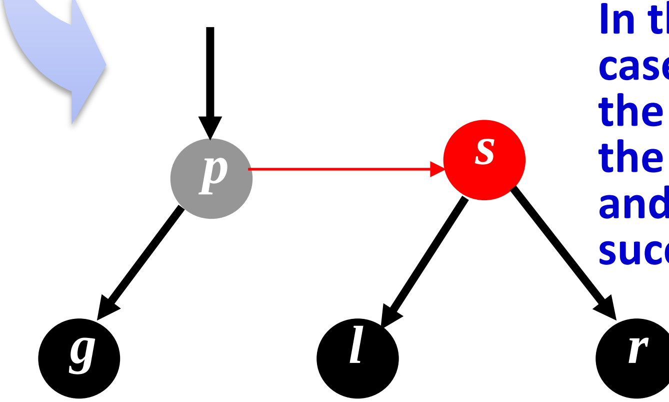



Map of the vicinity of ***g***, the gray node

**g**-subtree gets well-defined black height, but that is less than that required by its parent

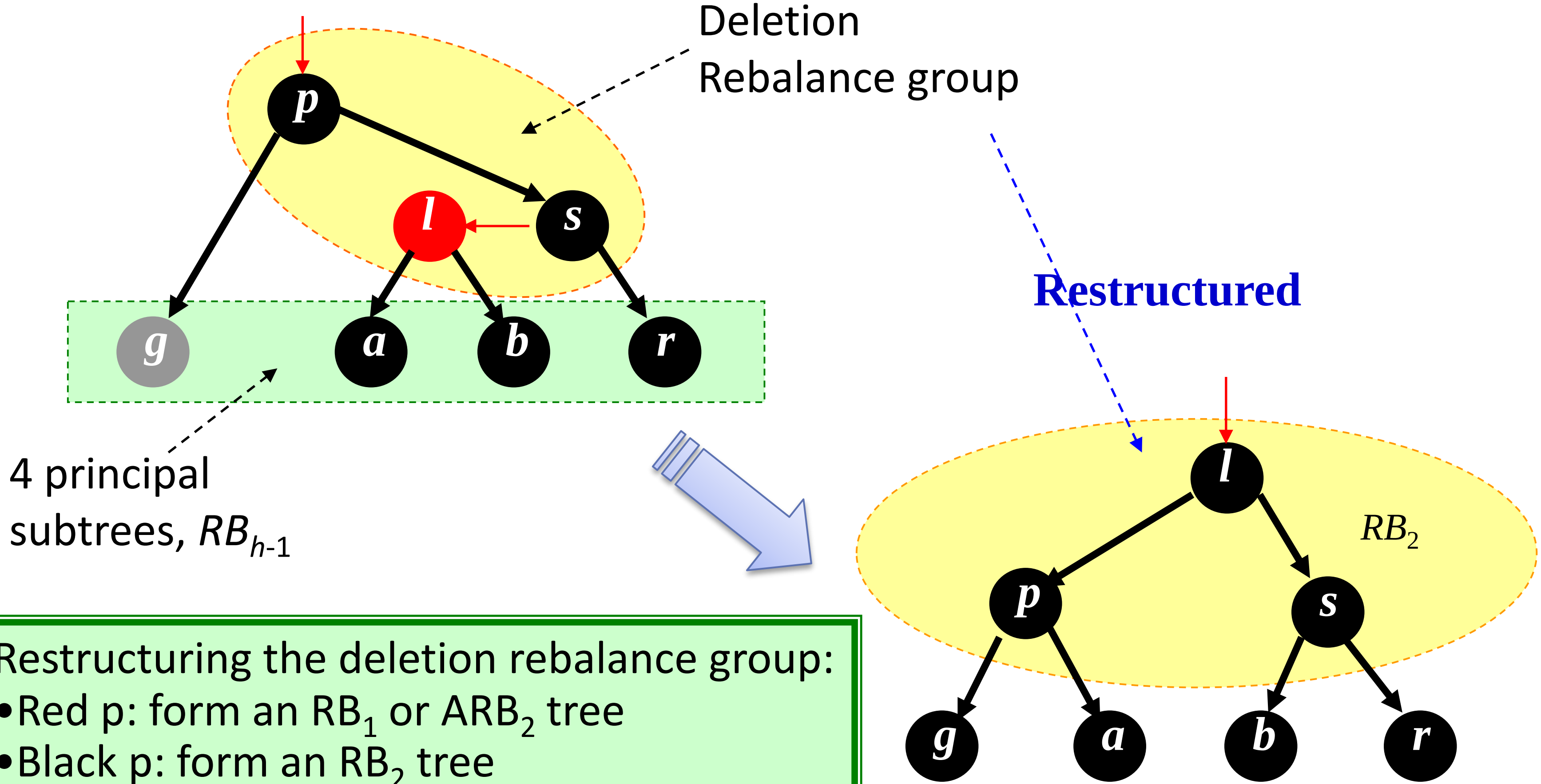



The pattern for which propagation is needed


**In the worst case, up to the root of the tree, and successful**

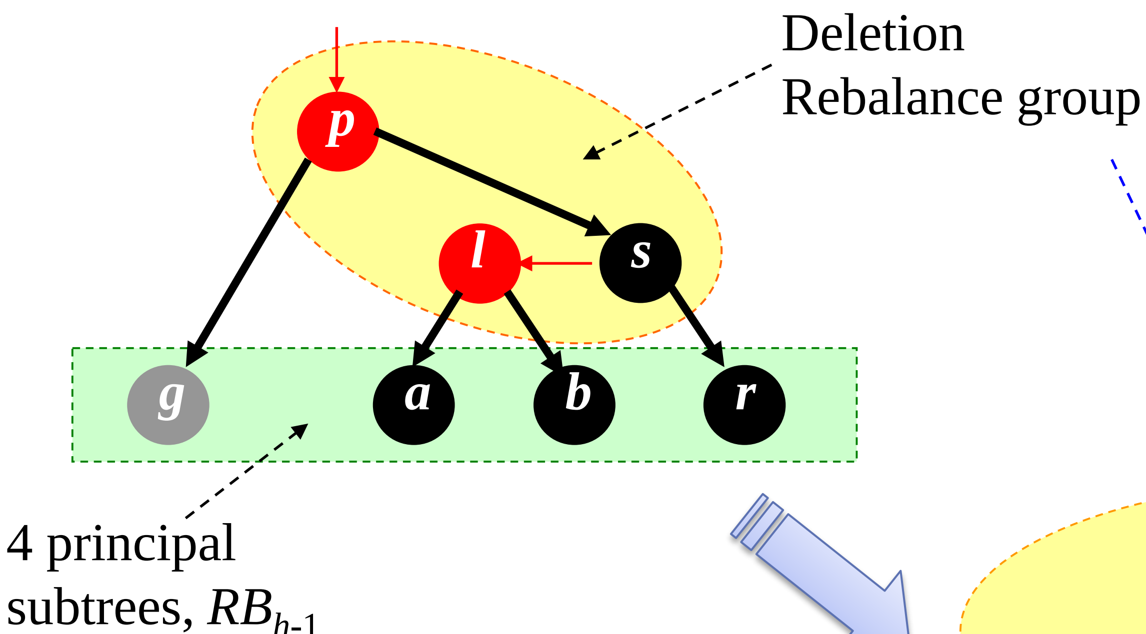
# Repairing without Propagation

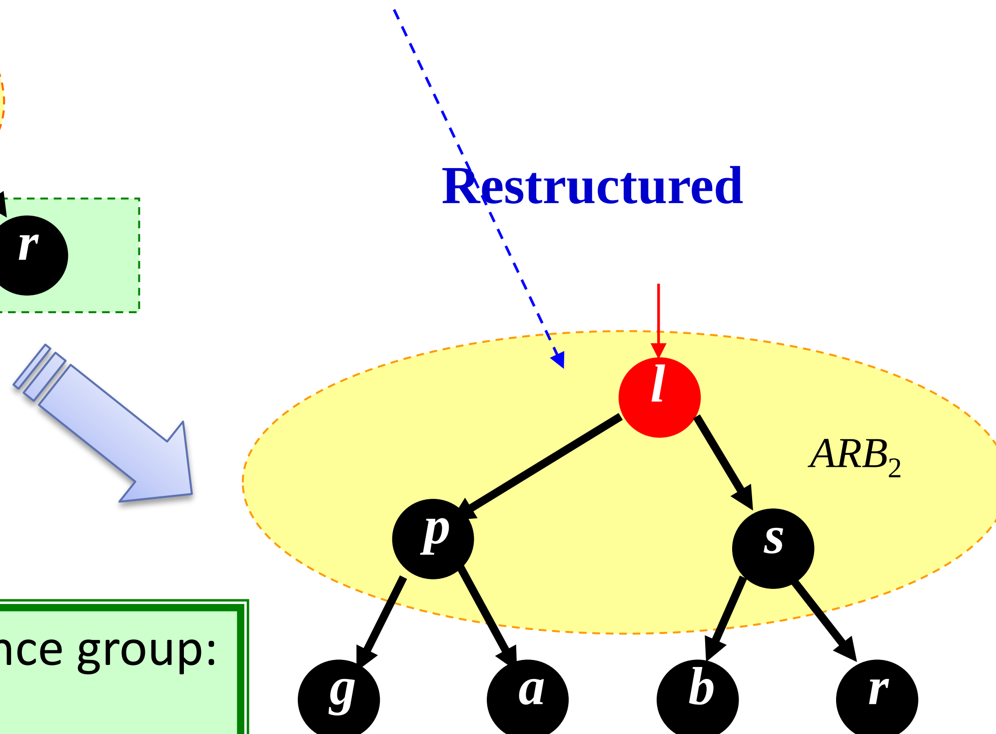



Restructuring the deletion rebalance group:

- Red p: form an $RB_1$ or $ARB_2$ tree
- Black p: form an $RB_2$ tree

# Repairing without Propagation




Restructuring the deletion rebalance group:
- Red p: form an $RB_1$ or $ARB_2$ tree
- Black p: form an $RB_2$ tree

# Complexity of Operations on RBT

- **With reasonable implementation**
  - A new node can be inserted correctly in a red-black tree with $n$ nodes in $\Theta(\log n)$ time in the worst case.
  - Repairs for deletion do $O(1)$ structural changes, but may do $O(\log n)$ color changes.

# *Thank you!*

# *Q & A*

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