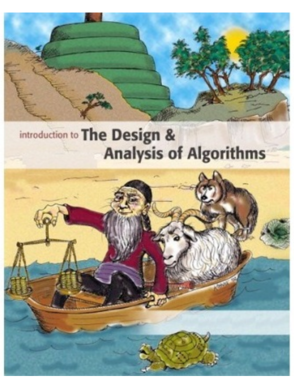
Introduction to

Algorithm Design and Analysis

[6] MergeSort



Yu Huang

http://cs.nju.edu.cn/yuhuang
Institute of Computer Software
Nanjing University

In the last class…

- **Heap**
  - FixHeap
  - ConstructHeap

- **HeapSort**
  - Complexity
  - Accelerated HeapSort

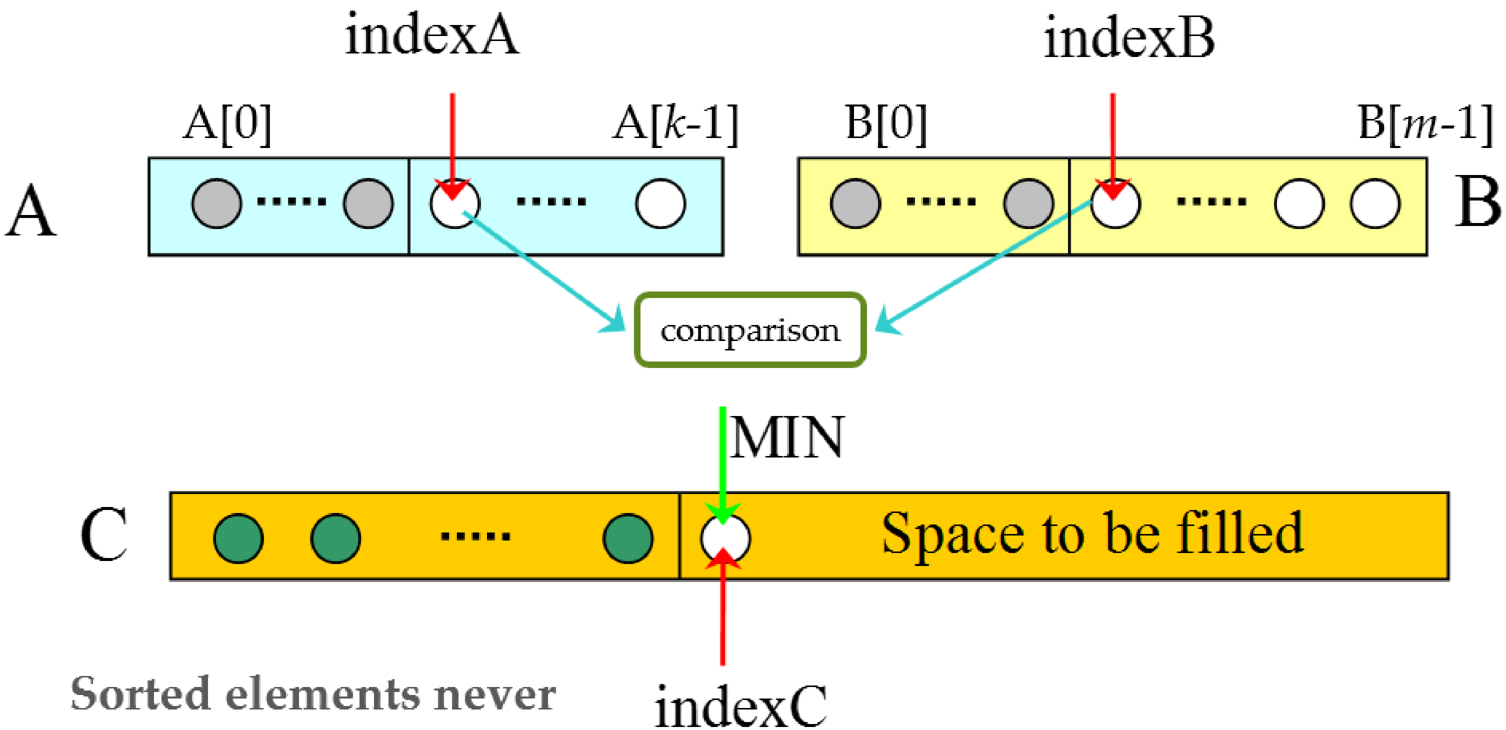
MergeSort

- **MergeSort**
  - Worst-case analysis of MergeSort
- **Lower Bounds for *comparison-based sorting***
  - Worst-case
  - Average-case

MergeSort: the Strategy

- **Easy division**
  - No comparison is done during the division
  - Minimizing the size difference between the divided subproblems
- **Merging two sorted subranges**
  - Using *Merge*

Merging Sorted Arrays



Sorted elements never examined again

Merge: the Specification

- **Input**
  - o Array A with k elements and B with m elements, whose keys are in non-decreasing order
- **Output**
  - o Array C containing $n = k + m$ elements from A and B in non-decreasing order
  - o C is passed in and the algorithm fills it

Merge: Recursive Version

merge(A,B,C)

if (A is empty)

  rest of C = rest of B

else if (B is empty)

  rest of C = rest of A

else

  if (first of $A \leq$ first of B)

    first of C =first of A

    merge(rest of A, B, rest of C)

  else

    first of C =first of B

    merge(A, rest of B, rest of C)

return

Base cases

Worst Case Complexity of Merge

- **Observations**
  - Worst case is that the last comparison is conducted between A[k-1] and B[m-1]
    - After each comparison, one element is inserted into Array C, ***at least***.
    - After entering Array C, an element will never be compared again
    - After the last comparison, at least two elements (the two just compared) have not yet been moved to Array C. So at most n-1 comparisons are done.
- **In worst case, *n-1* comparisons are done, where $n=k+m$**

Optimality of Merge

- **Any algorithm to merge two sorted arrays, each containing $k=m=n/2$ entries, by comparison of keys, does at least n-1 comparisons in the worst case.**
  - Choose keys so that:

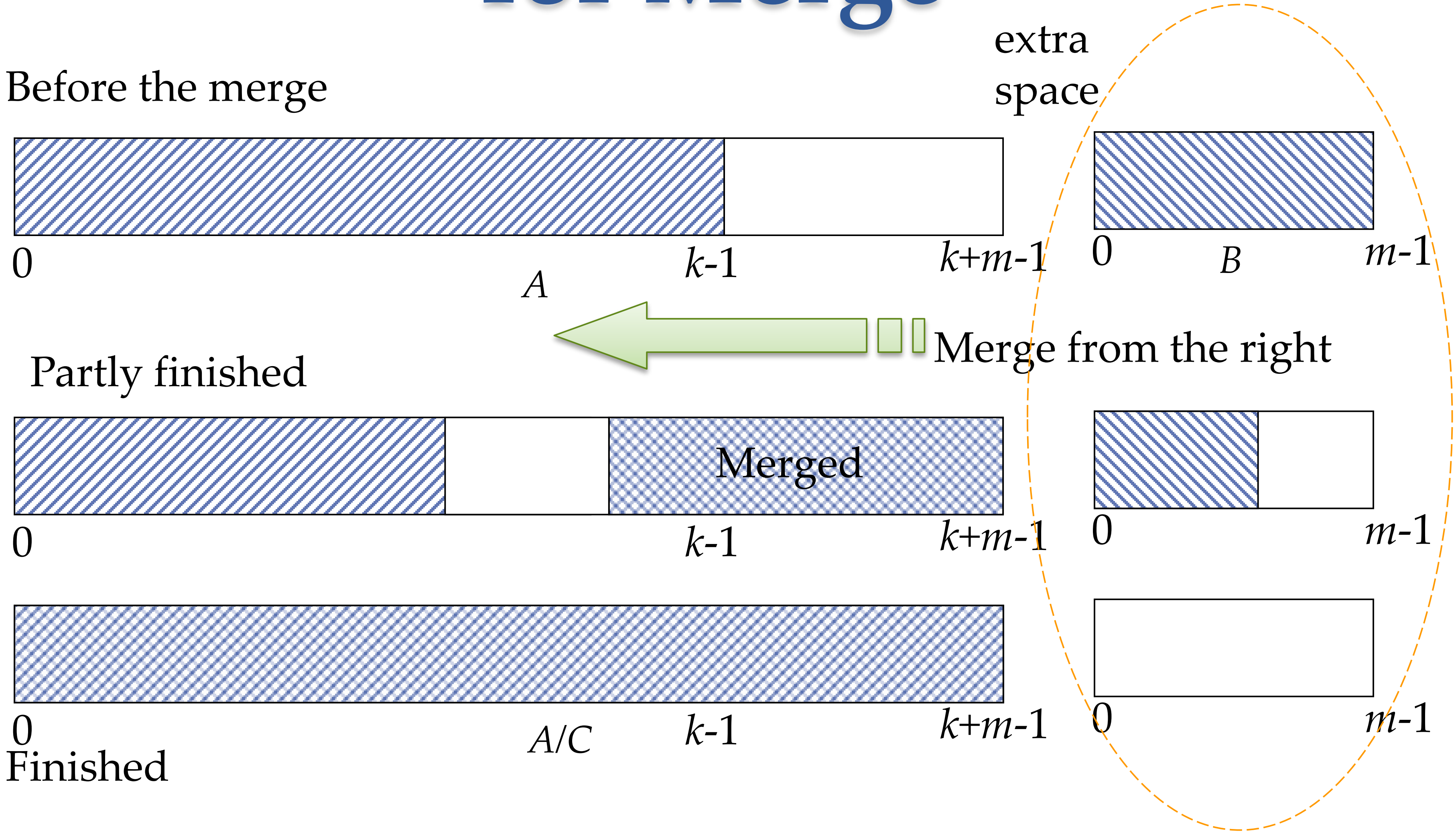
    $$b_0<a_0<b_1< a_1<\ldots<b_i<a_i<b_{i+1},\ldots,<b_{m-1}<a_{k-1}$$
  - Then the algorithm must compare a_i with b_i for every i in $[0,m-1]$, and must compare a_i with b_{i+1} for every i in $[0, m-2]$, so, there are $n-1$ comparisons.

Valid for $|k-m|\leq 1$, as well.

Space Complexity of Merge

- A algorithm is "in space", if the extra space it has to use is in $\Theta(1)$
- Merge *is not* a algorithm "in space", since it need enough extra space to store the merged sequence during the merging process.

Overlapping Arrays for Merge



MergeSort

- **Input: Array *E* and indexes *first*, and *last*, such that the elements of *E*[*i*] are defined for *first*≤*i*≤*last*.**
- **Output: E[*first*],…,E[*last*] is a sorted rearrangement of the same elements.**
- **Procedure**

```
void mergeSort(Element[] E, int first, int last)
  if (first<last)
    int mid=(first+last)/2;
    mergeSort(E, first, mid);
    mergeSort(E, mid+1, last);
    merge(E, first, mid, last)
  return
```

Analysis of MergeSort

- **The recurrence equation for Mergesort**
  - $W(n)=W(\lfloor n/2 \rfloor)+W(\lceil n/2 \rceil)+n-1$
  - $W(1)=0$

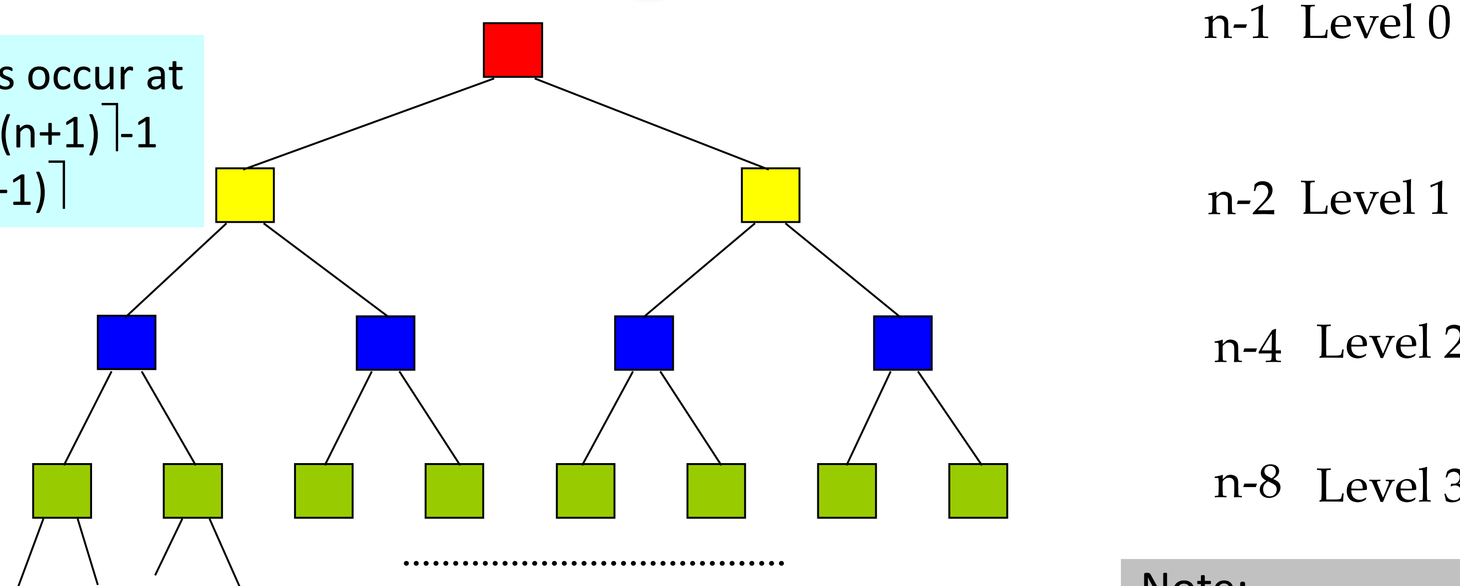
Where n=last-first+1, the size of range to be sorted

- **The *Master Theorem* applies for the equation, so:**

$$W(n)\in\Theta(n\log n)$$

Recursion Tree for Mergesort

Base cases occur at depth $\lceil \lg(n+1) \rceil - 1$ and $\lceil \lg(n+1) \rceil$



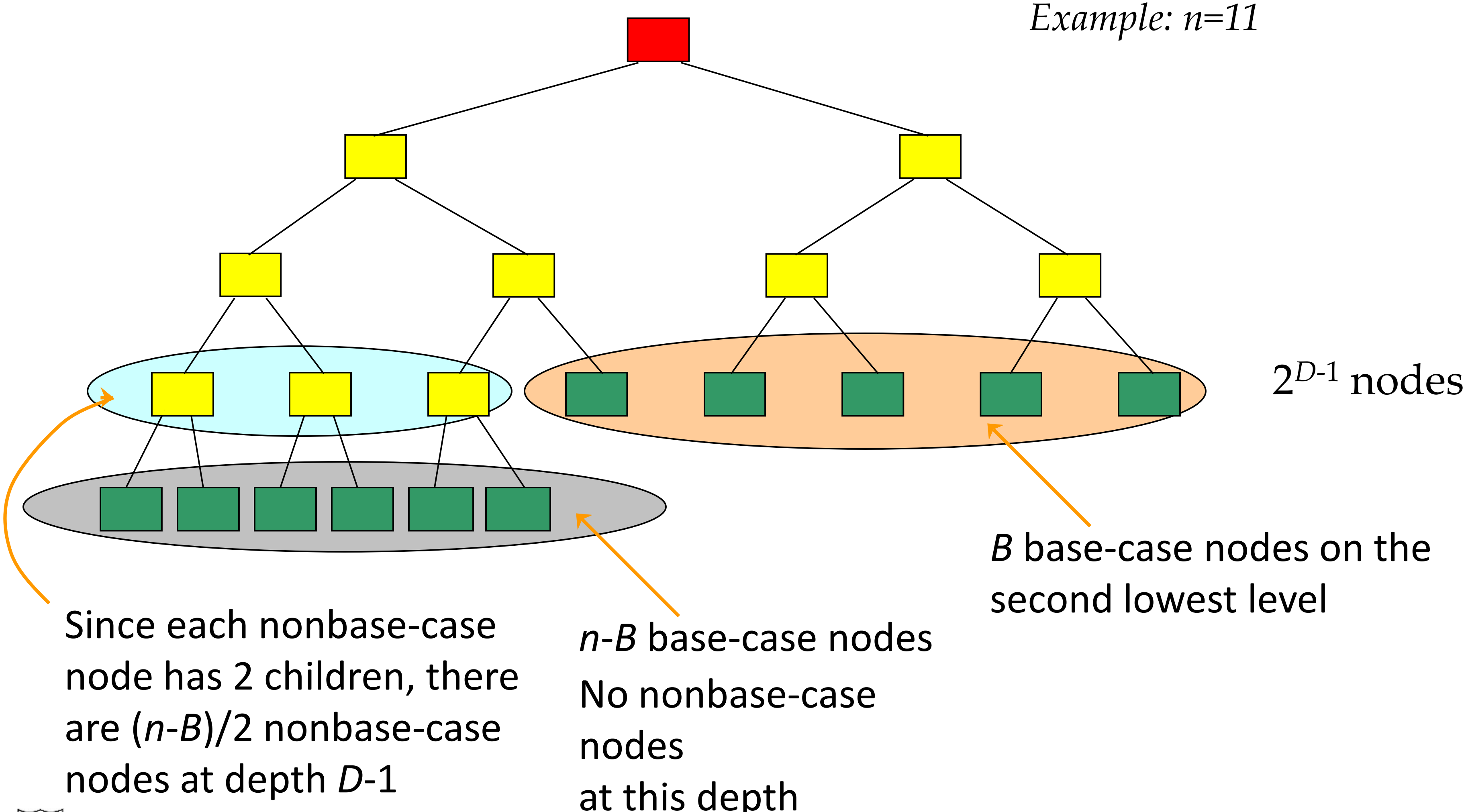
Note:
nonrecursive costs on level k is n-2^k for all level without basecase node

T(n)	n-1
T(n/2)	n/2-1
T(n/4)	n/4-1
T(n/8)	n/8-1

$k/2$ may be $\lceil k/2 \rceil$ or $\lfloor k/2 \rfloor$

Non-complete Recursion Tree

Example: n=11



Number of Comparison of MergeSort

- **The maximum depth *D* of the recursive tree is $\lceil \log(n+1) \rceil$.**
- **Let *B* base case nodes on depth *D*-1, and *n*-*B* on depth *D*, (Note: base case node has nonrecursive cost 0).**
- **(*n*-*B*)/2 nonbase case nodes at depth *D*-1, each has nonrecursive cost 1.**
- **So:**

$$W(n) = \sum_{d=0}^{D-2}(n - 2^d) + \frac{n-B}{2} = n(D-1) \quad (2^{D-1} - 1) + \frac{n-B}{2}$$

$$Since\ (2^D - 2B) + B = n, that\ is\ B = 2^D - n$$

$$So, W(n) = nD - 2^D + 1$$

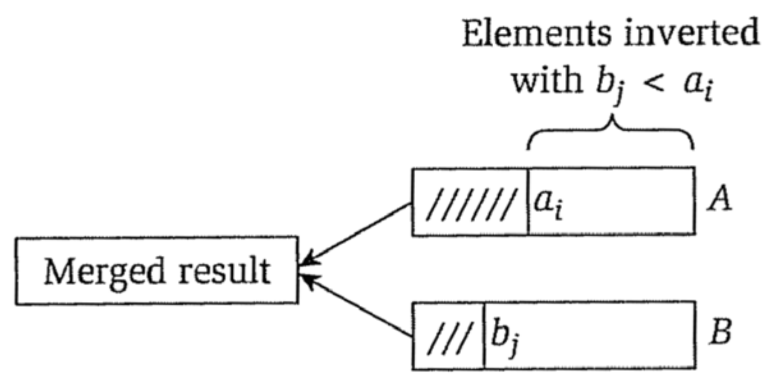
$$Let\ \frac{2^D}{n} = 1 + \frac{B}{n} = \alpha, then\ 1 \leqslant \alpha < 2, \quad D = logn + log\alpha$$

$$So, W(n) = nlogn - (\alpha - log\alpha)n + 1$$

- **$\lceil n\log(n)-n+1 \rceil \leq$ number of comparison $\leq \lceil n\log(n)-0.914n \rceil$**

The MergeSort D&C

- **Counting the number of inversions**
  - o Brute force: O(n^2)
  - o Divide & conquer: O($nlogn$)
- **MergeSort as the carrier**
  - o If $a_i < b_i$
    - Increase counter by the number of elements remaining in A



The MergeSort D&C

- **Max-sum subsequence**
- **Finding the *frequent* element**
- **Find the nearest two points on the plane**
- **Counting inversions**
- **…**

Just evenly divide

↓

Linear-time combination

↓

T(n)=2T(n/2)+O(n)

↓

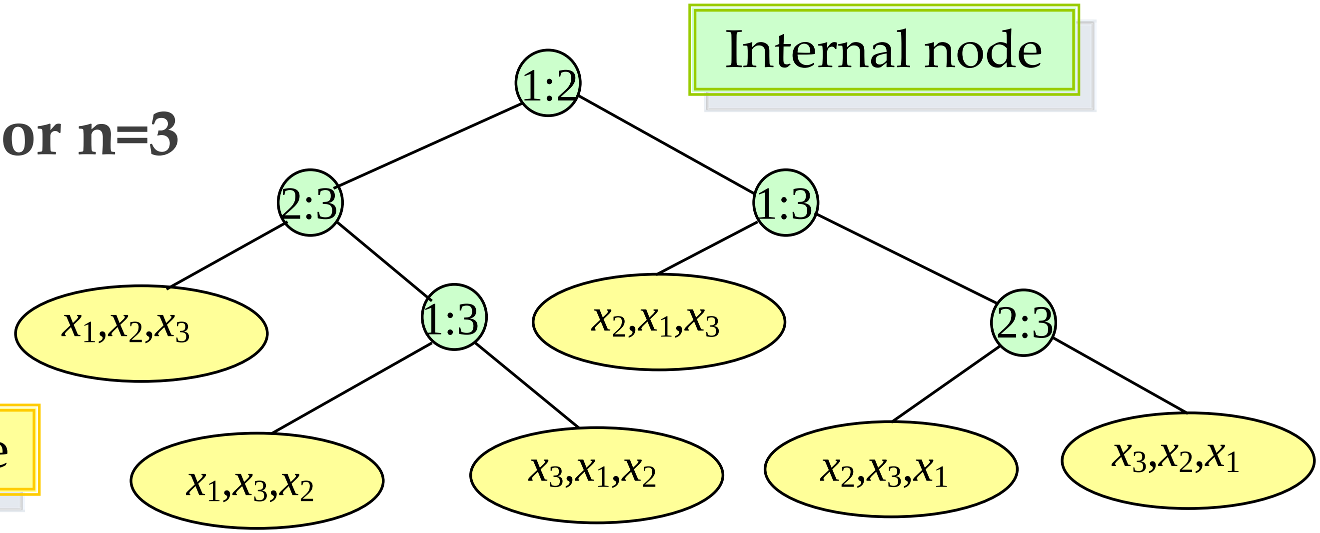
T(n)=O(nlogn)

Lower Bounds for Comparison-based Sorting

- **Upper bound, e.g., worst-case cost**
  - For any possible input, the cost of the specific algorithm A is no more than the *upper bound*
    - $Max\{Cost(i) \mid i\ is\ an\ input\}$
- **Lower bound, e.g., comparison-based sorting**
  - For any possible (comparison-based) sorting algorithm A, the worst-case cost is no less than the *lower bound*
    - Min{*Worst-case*(a) | a is an algorithm}

Decision Tree for Sorting

A example for n=3



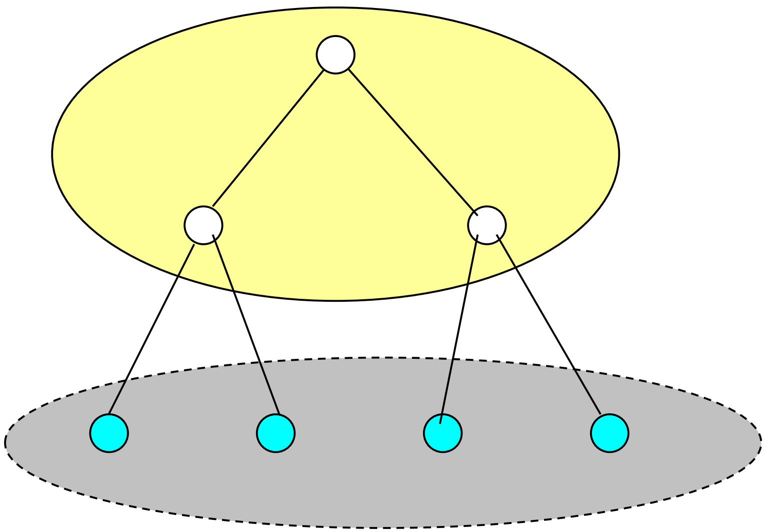
External node

- Decision tree is a 2-tree (Assuming no same keys)
- The action of Sort on a particular input corresponds to following on path in its decision tree from the root to a leaf associated to the specific output

2-Tree

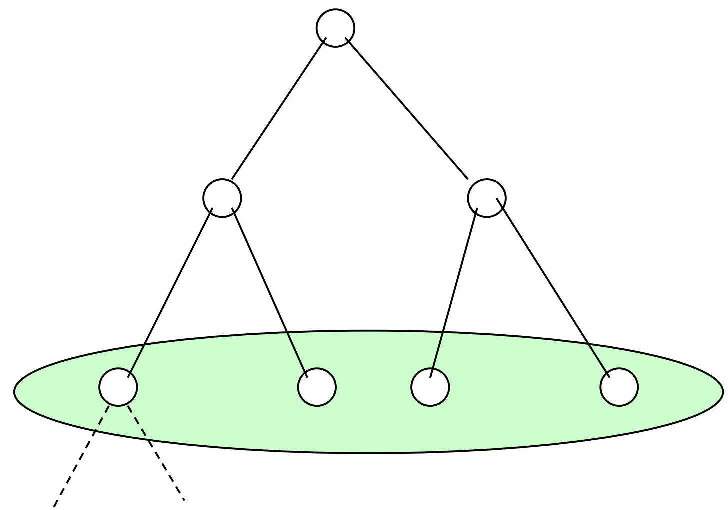
- **2-Tree**

internal nodes



external nodes
no child any type

- **Common Binary Tree**



Both left and right children of
these nodes are empty tree

Characterizing the Decision Tree

- **For a sequence of n distinct elements, there are n! different permutation**
  - So, the decision tree has at least n! leaves, and exactly n! leaves can be reached from the root.
  - So, for the purpose of lower bounds evaluation, we use trees with exactly n! leaves.
- **The number of comparison done in the *worst case* is the height of the tree.**
- **The *average* number of comparison done is the average of the lengths of all paths from the root to a leaf.**

Lower Bound for Worst Case

- ***Theorem*: Any algorithm to sort n items by comparisons of keys must do at least $\lceil \log n! \rceil$, or approximately $\lceil n\log n - 1.443n \rceil$, key comparisons in the worst case.**
  - Note: Let $L=n!$, which is the number of leaves, then $L \le 2^h$, where h is the height of the tree, that is $h \ge \lceil \log L \rceil = \lceil \log n! \rceil$
    - Lemma: let L be the number of leaves in a binary tree and h be its height. Then $L \le 2^h$
  - For the asymptotic behavior:

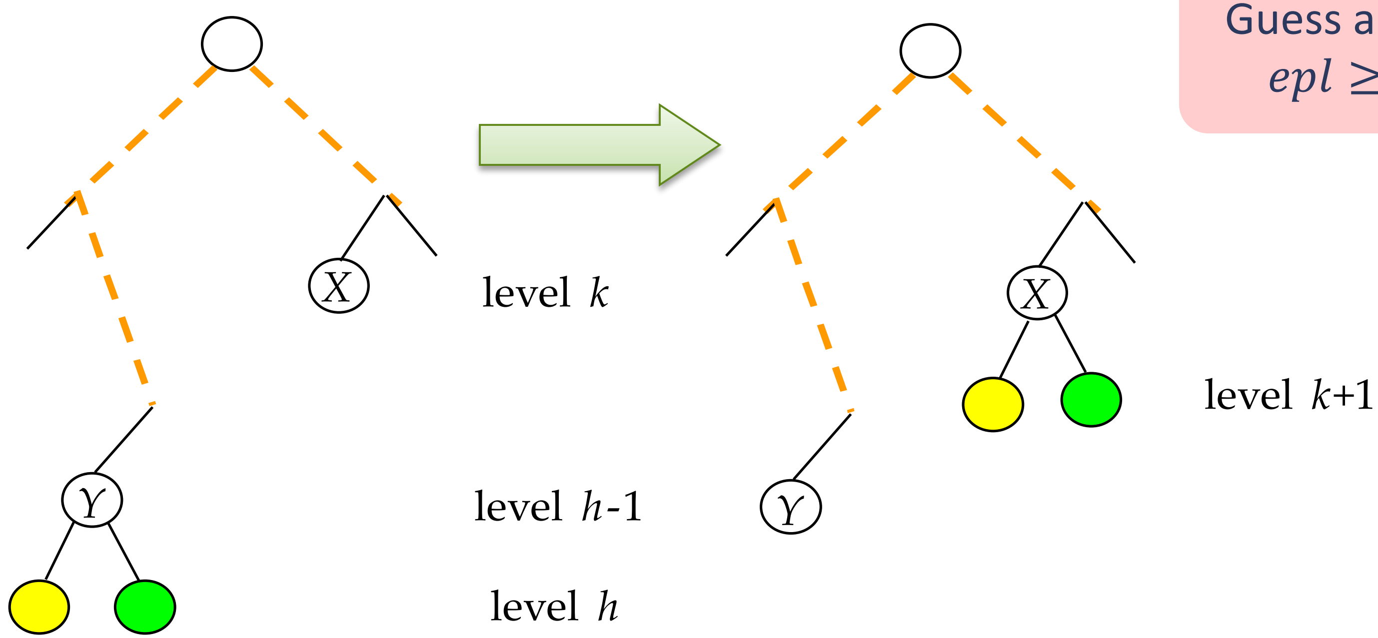
$$\log(n!) \geq \log(n(n-1)\cdots(\lceil \frac{n}{2} \rceil)) \geq \log(\frac{n}{2})^{\frac{n}{2}} = \frac{n}{2}\log(\frac{n}{2}) \in \Theta(n \log n)$$

External Path Length(EPL)

- **EPL – sum of path length to every leaf**
  - The EPL t is recursively defined as follows:
  - [Base case] 0 for a single external node
  - [Recursion] *t* is non-leaf with sub-trees *L* and *R*, then the sum of:
    - the external path length of *L*;
    - the number of external node of *L*;
    - the external path length of *R*;
    - the number of external node of *R*;

More Balanced 2-tree, Less EPL

Guess and prove:
$epl \geq L \log L$



Assuming that *h*-*k*>1, when calculating *epl*, *h*+*h*+*k* is replaced by (*h*-1)+2(*k*+1). The net change in *epl* is *k*-*h*+1<0, that is, the *epl* decreases.

Properties of EPL

- Let t be a 2-tree, then the *epl* of t is the sum of the paths from the root to each external node.
- $epl \geq m\log(m)$, where m is the number of external nodes in t
  - o $epl = epl_L + epl_R + m \geq m_L\log(m_L) + m_R\log(m_R) + m$,
    - note $f(\mathrm{x})+f(\mathrm{y}) \geq 2f((\mathrm{x}+\mathrm{y})/2)$ for $\mathrm{f}(x)=x\log x$
  - o so,

$$epl \geq 2((m_L+m_R)/2)\log((m_L+m_R)/2)+m$$
$$= m(\log(m)-1)+m = m\log m.$$

p. 116, Lemma 3.7 of [Baase01]

Lower Bound for Average Behavior

- **Since a decision tree with L leaves is a 2-tree, the average path length from the root to a leaf is $\frac{epl}{L}$.**
  - Recall that $epl \geq L\log(L)$.

- **Theorem: The average number of comparison done by an algorithm to sort n items by comparison of keys is at least $\log(n!)$, which is about $n\log n$-1.443n.**

MergeSort Has Optimal Average Performance

- The average number of comparisons done by an algorithm to sort n items by comparison of keys is at least about $n\log n$-1.443n
- The worst complexity of MergeSort is in $\Theta(n\log n)$
- So, MergeSort is optimal as for its average performance

Thank you!

Q & A

Yu Huang
http://cs.nju.edu.cn/yuhuang